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ON PERIODIC LINEAR NEUTRAL DELAY DIFFERENTIAL AND DIFFERENCE EQUATIONS

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Abstract. This article concerns the behavior of the solutions to periodic linear neutral delay differential equations as well as to periodic linear neutral delay difference equations. Some new results are obtained via two appropriate distinct roots of the corresponding (so called) characteristic equation.

1. Introduction

Motivated by the old but very interesting asymptotic and stability results for delay differential equations due to Driver [6, 7] and to Driver, Sasser and Slater [9], a number of articles has been published during the last few years, which are concerned with the asymptotic behavior (and, more general, the behavior) and the stability for delay differential equations, neutral delay differential equations and (neutral or non-neutral) integrodifferential equations with unbounded delay as well as for delay difference equations (with discrete or continuous variable), neutral delay difference equations and (neutral or non-neutral) Volterra difference equations with infinite delay. See [1, 4, 5, 10, 11, 16, 17, 18, 19, 20], [22]–[34], [38]; for some related results, see [2, 8, 12, 13, 21, 36, 37].

Recently, the authors [30] obtained some results concerning the behavior of the solutions to autonomous linear delay differential equations as well as to autonomous linear neutral delay differential equations; these results are essentially motivated by a result due to Driver [6, Theorem 2]. In [31], the authors continued the work in [30] to a wide class of autonomous linear neutral delay differential equations (and, especially, delay differential equations) with infinitely many distributed type delays. The authors' paper [28] contains the discrete analogues of the results given in [30] for autonomous linear delay difference equations as well as for autonomous linear neutral delay difference equations; a result of the same type for autonomous linear delay difference equations with continuous variable is also contained in [28]. The study in [28, 30, 31] was continued by the authors in [33] to linear neutral integrodifferential equations with unbounded delay, and, especially, to linear (non-neutral) integrodifferential equations with unbounded delay; the discrete analogues of the results in [33] for linear neutral (and, especially, non-neutral) Volterra difference

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equations with infinite delay have been presented by the authors in [34]. It must be noted that the results in [28, 30, 31, 33, 34] are obtained via two distinct roots of the corresponding characteristic equation.

A further continuation of the study in [28, 30, 31, 33, 34] was, very recently, presented by the authors in [35]; the paper [35] contains some results on the behavior of the solutions to *periodic* linear delay differential equations as well as to *periodic* linear delay difference equations, which are derived by the use of two (appropriate) distinct roots of the corresponding (so called) characteristic equation.

It is the subject of this paper to present a study analogous to the one in [35] for the behavior of the solutions to *periodic* linear *neutral* delay differential equations as well as to *periodic* linear *neutral* delay difference equations. Section 2 is devoted to the case of differential equations, and Section 3 is concerned with the case of difference equations. It is remarkable that an application of the main results of the present work to the special case of (non-neutral) periodic linear delay differential equations or to the special case of (non-neutral) periodic linear delay difference equations leads to the main results of the previous authors' paper [35], *under some additional hypotheses*; but, these (additional) hypotheses are not needed for the main results in [35] to hold. So, although the differential and difference equations treated in [35] can be considered as special cases of the ones studied in this paper, the main results in [35] cannot be obtained as corollaries of those given here.

Finally, we note that some considerable difficulty arises in the attempt to extend the results of this paper to the more general case of periodic linear neutral delay differential equations with several delays (such as those studied by the authors in [23]) as well as of periodic linear neutral delay difference equations with several delays (such as the ones treated by the authors in [29]).

2. On the behavior of the solutions to periodic linear neutral delay differential equations

This section is concerned with the behavior of the solutions of the linear neutral delay differential equation

$$[x(t) + cx(t-\sigma)]' = a(t)x(t) + b(t)x(t-\tau), \tag{2.1}$$

where c *is a real number,* a *and* b *are continuous real-valued functions on the interval* $[0,\infty)$*, and* σ *and* τ *are positive real numbers. The function* b *is assumed to be not identically zero on* $[0,\infty)$. Moreover, *it will be supposed that the coefficients* a *and* b *are periodic functions with a common period* $T > 0$ *and that there exist positive integers* ℓ *and* m *such that*

$$\sigma = \ell T \quad \textit{and} \quad \tau = mT.$$

Consider the positive real number

$$r = \max\{\sigma, \tau\}.$$

As usual, a continuous real-valued function x defined on the interval $[-r,\infty)$ is said to be a *solution* of the neutral delay differential equation (2.1) if the function $x(t) + cx(t-\sigma)$ is continuously differentiable for $t \ge 0$ and x satisfies (2.1) for all $t \ge 0$.

Together with the neutral delay differential equation (2.1), it is customary to specify an *initial condition* of the form

$$x(t) = \phi(t) \quad \text{for } -r \le t \le 0, \tag{2.2}$$

where *the initial function* ϕ *is a given continuous real-valued function on the interval* $[-r, 0]$.

Equations (2.1) and (2.2) constitute an *initial value problem* (IVP, for short). It is well-known (see, for example, Diekmann *et al.* [3], Hale [14], or Hale and Verduyn Lunel [15]) that there exists a unique solution x of the neutral delay differential equation (2.1) which satisfies the initial condition (2.2); this unique solution x will be called the *solution* of the initial value problem (2.1), (2.2) or, more briefly, the *solution* of the IVP (2.1), (2.2).

Throughout this section, we shall use the notation

$$A = \frac{1}{T}\int_0^T a(t)dt \quad \text{and} \quad B = \frac{1}{T}\int_0^T b(t)dt.$$

We note that A and B are real constants. Also, it must be noted that $B \neq 0$ in the case where the coefficient b is assumed to be of one sign on the interval $[0, \infty)$.

Along with the neutral delay differential equation (2.1), one associates the equation

$$\lambda(1 + ce^{-\lambda\sigma}) = A + Be^{-\lambda\tau}, \tag{2.3}$$

which will be called the *characteristic equation* of (2.1) (see the authors' paper [23]).

In the sequel, by $\widetilde{a}$ and $\widetilde{b}$ we shall denote the T-periodic extensions of the coefficients a and b, respectively, on the interval $[-r, \infty)$. Moreover, for any real number λ, by f_λ we will denote the continuous real-valued function defined on the interval $[-r, \infty)$ as follows

$$f_\lambda(t) = \widetilde{a}(t) + \widetilde{b}(t)e^{-\lambda\tau} \quad \text{for } t \geq -r.$$

Theorem 2.1 below has been proved by the authors in [23] for more general periodic linear neutral delay differential equations with several delays. This theorem is closely related to the main result (Theorem 2.4 below) of this section and constitutes a fundamental asymptotic result for the solutions of the neutral delay differential equation (2.1). In order to state Theorem 2.1, we introduce the notation

$$\widehat{B} = \frac{1}{T}\int_0^T |b(t)|dt.$$

Clearly, $\widehat{B}$ is a positive constant. It is obvious that $|B| \leq \widehat{B}$. Moreover, we note that $|B| = \widehat{B}$ in the case where the coefficient b is assumed to be of constant sign on the interval $[0, \infty)$.

Theorem 2.1. *Let λ_0 be a real root of the characteristic equation* (2.3)*, and set*

$$\rho_{\lambda_0} = 1 + ce^{-\lambda_0\sigma} \tag{2.4}$$

and

$$\widehat{F}_{\lambda_0} = \frac{1}{T}\int_0^T |f_{\lambda_0}(t)|dt.$$

Assume that the root λ_0 has the property

$$|c|(|\rho_{\lambda_0}| + \widehat{F}_{\lambda_0}\sigma)e^{-\lambda_0\sigma} + |\rho_{\lambda_0}|\widehat{B}\tau e^{-\lambda_0\tau} < |\rho_{\lambda_0}|. \tag{2.5}$$

Define

$$\gamma_{\lambda_0} = c(1 - \lambda_0\sigma)e^{-\lambda_0\sigma} + B\tau e^{-\lambda_0\tau}. \tag{2.6}$$

(Note that Property (2.5) *guarantees that $\rho_{\lambda_0} > 0$ and $1 + \gamma_{\lambda_0} > 0$.)*

Then the solution x of the IVP (2.1), (2.2) *satisfies*

$$\lim_{t\to\infty}\left\{x(t)\exp\left[-\frac{1}{\rho_{\lambda_0}}\int_0^t f_{\lambda_0}(u)du\right]\right\} = \frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}}, \tag{2.7}$$

where

$$\begin{aligned} L_{\lambda_0}(\phi) = \phi(0) + c\Big\{&\phi(-\sigma) \\ &- \frac{1}{\rho_{\lambda_0}}e^{-\lambda_0\sigma}\int_{-\sigma}^0 f_{\lambda_0}(s)\phi(s)\exp\left[-\frac{1}{\rho_{\lambda_0}}\int_0^s f_{\lambda_0}(u)du\right]ds\Big\} \\ &+ e^{-\lambda_0\tau}\int_{-\tau}^0 \widetilde{b}(s)\phi(s)\exp\left[-\frac{1}{\rho_{\lambda_0}}\int_0^s f_{\lambda_0}(u)du\right]ds. \end{aligned} \tag{2.8}$$

In the main result (Theorem 2.4 below) of this section, *it is supposed that $c \le 0$ and that b is nonpositive on the interval* $[0,\infty)$. The hypothesis that b is nonpositive on $[0,\infty)$ together with the assumptions that b is not identically zero on $[0,\infty)$ and that b is a T-periodic function imply that *the constant B is always negative.* Furthermore, we notice that Theorem 2.4 is obtained by the use of *two real roots λ_0 and λ_1, $\lambda_0 \ne \lambda_1$, of the characteristic equation* (2.3)*; for the root λ_0 it is assumed that $\rho_{\lambda_0} \ne 0$ and*

$$\rho_{\lambda_0}\left[a(t)+b(t)e^{-\lambda_0\tau}\right] \le 0 \quad \textit{for } t \ge 0, \tag{2.9}$$

where ρ_{λ_0} is defined by (2.4). Under these assumptions for the real root λ_0, we obviously have

$$\frac{1}{\rho_{\lambda_0}}\left[a(t)+b(t)e^{-\lambda_0\tau}\right] \le 0 \quad \text{for } t \ge 0,$$

which gives

$$\frac{1}{\rho_{\lambda_0}}\left\{\left[\frac{1}{T}\int_0^T a(t)dt\right] + \left[\frac{1}{T}\int_0^T b(t)dt\right]e^{-\lambda_0\tau}\right\} \le 0,$$

i.e.,

$$\frac{1}{\rho_{\lambda_0}}(A + Be^{-\lambda_0\tau}) \le 0.$$

So, by (2.4) and the fact that λ_0 is a root of (2.3), it follows immediately that *the root λ_0 is always nonpositive.*

After the above observations, we give a lemma (Lemma 2.2 below) concerning the real roots of the characteristic equation (2.3). This lemma is a special case of a more general lemma due to the authors [30].

Lemma 2.2. *Suppose that $c \le 0$ and $B < 0$.*

(I) Let λ_0 be a nonpositive real root of the characteristic equation (2.3)*, and let γ_{λ_0} be defined by* (2.6)*. Then*

$$1+\gamma_{\lambda_0} > 0$$

if (2.3) *has another real root less than λ_0, and*

$$1+\gamma_{\lambda_0} < 0$$

if (2.3) *has another nonpositive real root greater than λ_0.*

(II) If $A = 0$, then $\lambda = 0$ is not a root of the characteristic equation (2.3).

(III) Assume that $A = 0$ and that $c \ge -1$. Then the characteristic equation (2.3) *has no positive real roots.*

(IV) Assume that $A+B \le 0$ and $c+B\tau \ge -1$. Then the characteristic equation (2.3) *has no positive real roots.*

(V) Assume that $A+B \le 0$, $Ar < 1$, and

$$(1-Ar)ce^{-(A-\frac{1}{r})\sigma} + Bre^{-(A-\frac{1}{r})\tau} > -1.$$

Then: (i) $\lambda = A - \frac{1}{r}$ is not a root of the characteristic equation (2.3). *(ii) In the interval $(A-\frac{1}{r}, 0]$,* (2.3) *has a unique root. (iii) In the interval $(-\infty, A-\frac{1}{r})$,* (2.3) *has a unique root.*

As we have previously noted, in Theorem 2.4 (the main result of this section) we use two suitable distinct real roots of the characteristic equation (2.3). We will give here a lemma (Lemma 2.3 below), which plays a crucial role in proving Theorem 2.4, although this lemma is rather technical.

Lemma 2.3. *Let λ_0 and λ_1, $\lambda_0 \ne \lambda_1$, be two real roots of the characteristic equation* (2.3) *with $\rho_{\lambda_0} \ne 0$ and $\rho_{\lambda_1} \ne 0$, where ρ_{λ_0} is defined by* (2.4) *and ρ_{λ_1} is defined in an analogous way, i.e.,*

$$\rho_{\lambda_1} = 1 + ce^{-\lambda_1\sigma}. \tag{2.10}$$

Then, for each $t \ge 0$, we have

$$\begin{aligned} 1 = &-ce^{-\lambda_1\sigma} + ce^{-\lambda_0\sigma}\frac{1}{\rho_{\lambda_0}}\int_{t-\sigma}^{t} f_{\lambda_0}(s)\exp\left\{\int_s^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds \\ &- e^{-\lambda_0\tau}\int_{t-\tau}^{t} \widetilde{b}(s)\exp\left\{\int_s^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds. \end{aligned} \tag{2.11}$$

Proof. First of all, we will establish some equalities needed below. It is obvious that

$$f_{\lambda_0}(t) - f_{\lambda_1}(t) = \widetilde{b}(t)\left(e^{-\lambda_0\tau} - e^{-\lambda_1\tau}\right) \quad \text{for } t \ge -r. \tag{2.12}$$

Also, in view of (2.4) and (2.10), we have

$$\rho_{\lambda_0} - \rho_{\lambda_1} = c\left(e^{-\lambda_0\sigma} - e^{-\lambda_1\sigma}\right). \tag{2.13}$$

Furthermore, the T-periodicity of the coefficients a and b implies that the functions f_{λ_0} and f_{λ_1} are also T-periodic. So, by taking into account the fact that $\tau = mT$, we obtain for $t \ge 0$,

$$\begin{aligned} \int_{t-\tau}^{t} f_{\lambda_0}(u)du &= \int_0^{\tau} f_{\lambda_0}(u)du \\ &= \left[\frac{1}{\tau}\int_0^{\tau} f_{\lambda_0}(u)du\right]\tau \\ &= \left[\frac{1}{T}\int_0^{T} f_{\lambda_0}(u)du\right]\tau \\ &= \left\{\left[\frac{1}{T}\int_0^{T} a(u)du\right] + \left[\frac{1}{T}\int_0^{T} b(u)du\right]e^{-\lambda_0\tau}\right\}\tau \\ &= (A + Be^{-\lambda_0\tau})\tau. \end{aligned}$$

Thus, because of (2.4) and the fact that λ_0 is a root of (2.3), it follows that

$$\frac{1}{\rho_{\lambda_0}}\int_{t-\tau}^{t} f_{\lambda_0}(u)du = \lambda_0\tau \quad \text{for every } t \ge 0. \tag{2.14}$$

In a similar way, by using (2.10) and the fact that λ_1 is also a root of (2.3), we can see that

$$\frac{1}{\rho_{\lambda_1}}\int_{t-\tau}^{t} f_{\lambda_1}(u)du = \lambda_1\tau \quad \text{for every } t \ge 0. \tag{2.15}$$

Moreover, by taking again into account the fact that the function f_{λ_0} is T-periodic, for any positive integer ν, we get

$$\begin{aligned}
\int_0^{\nu T} f_{\lambda_0}(u)du &= \left[\frac{1}{\nu T}\int_0^{\nu T} f_{\lambda_0}(u)du\right](\nu T) \\
&= \left[\frac{1}{T}\int_0^{T} f_{\lambda_0}(u)du\right](\nu T) \\
&= (A + Be^{-\lambda_0\tau})(\nu T).
\end{aligned}$$

Hence, as ρ_{λ_0} is given by (2.4) and λ_0 is a root of (2.3), we have

$$\frac{1}{\rho_{\lambda_0}}\int_0^{\nu T} f_{\lambda_0}(u)du = \nu\,(\lambda_0 T) \quad (\nu = 1, 2, \dots). \tag{2.16}$$

In a similar manner, by taking into account the fact that f_{λ_1} is a T-periodic function and using (2.10) and the fact that λ_1 is also a root of (2.3), one can verify that

$$\frac{1}{\rho_{\lambda_1}}\int_0^{\nu T} f_{\lambda_1}(u)du = \nu\,(\lambda_1 T) \quad (\nu = 1, 2, \dots). \tag{2.17}$$

Now, let us fix a point $t \ge 0$. We will show that, for this fixed point t, equality (2.11) holds true.

By using (2.12) and (2.13), we obtain

$$\begin{aligned}
&\int_{t-\tau}^{t} \widetilde{b}(s)\exp\left\{\int_s^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds \\
&= \frac{1}{e^{-\lambda_0\tau} - e^{-\lambda_1\tau}}\int_{t-\tau}^{t} \widetilde{b}(s)\left(e^{-\lambda_0\tau} - e^{-\lambda_1\tau}\right)\exp\left\{\int_s^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds \\
&= \frac{1}{e^{-\lambda_0\tau} - e^{-\lambda_1\tau}}\int_{t-\tau}^{t} [f_{\lambda_0}(s) - f_{\lambda_1}(s)]\exp\left\{\int_s^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds \\
&= \frac{1}{e^{-\lambda_0\tau} - e^{-\lambda_1\tau}}\int_{t-\tau}^{t} \left\{(\rho_{\lambda_0} - \rho_{\lambda_1})\frac{f_{\lambda_0}(s)}{\rho_{\lambda_0}} + \rho_{\lambda_1}\left[\frac{f_{\lambda_0}(s)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(s)}{\rho_{\lambda_1}}\right]\right\}\times \\
&\quad \times \exp\left\{\int_s^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds \\
&= c\frac{e^{-\lambda_0\sigma} - e^{-\lambda_1\sigma}}{e^{-\lambda_0\tau} - e^{-\lambda_1\tau}}\cdot\frac{1}{\rho_{\lambda_0}}\int_{t-\tau}^{t} f_{\lambda_0}(s)\exp\left\{\int_s^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds \\
&\quad + \frac{1}{e^{-\lambda_0\tau} - e^{-\lambda_1\tau}}\rho_{\lambda_1}\int_{t-\tau}^{t} \left[\frac{f_{\lambda_0}(s)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(s)}{\rho_{\lambda_1}}\right]\exp\left\{\int_s^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds.
\end{aligned}$$

But, by (2.14) and (2.15), we have

$$\begin{aligned}
&\int_{t-\tau}^{t} \left[\frac{f_{\lambda_0}(s)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(s)}{\rho_{\lambda_1}}\right]\exp\left\{\int_s^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds \\
&= -\left(\exp\left\{\int_s^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}\right)_{s=t-\tau}^{s=t}
\end{aligned}$$

$$
\begin{aligned}
&= -1 + \exp\Big\{\int_{t-\tau}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\} \\
&= -1 + \exp\Big[\frac{1}{\rho_{\lambda_0}}\int_{t-\tau}^{t} f_{\lambda_0}(u)du - \frac{1}{\rho_{\lambda_1}}\int_{t-\tau}^{t} f_{\lambda_1}(u)du\Big] \\
&= -1 + e^{\lambda_0\tau - \lambda_1\tau} = -1 + e^{(\lambda_0-\lambda_1)\tau}.
\end{aligned}
$$

So, it holds

$$
\begin{aligned}
&\int_{t-\tau}^{t}\widetilde{b}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds \\
&= c\frac{e^{-\lambda_0\sigma} - e^{-\lambda_1\sigma}}{e^{-\lambda_0\tau} - e^{-\lambda_1\tau}}\cdot\frac{1}{\rho_{\lambda_0}}\int_{t-\tau}^{t} f_{\lambda_0}(s) \\
&\quad\times\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds - \frac{1}{e^{-\lambda_0\tau}}\rho_{\lambda_1}.
\end{aligned}
\tag{2.18}
$$

Furthermore, in view of (2.10), equality (2.11) becomes

$$
\begin{aligned}
\rho_{\lambda_1} &= ce^{-\lambda_0\sigma}\frac{1}{\rho_{\lambda_0}}\int_{t-\sigma}^{t} f_{\lambda_0}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds \\
&\quad - e^{-\lambda_0\tau}\int_{t-\tau}^{t}\widetilde{b}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds.
\end{aligned}
$$

Hence, by using (2.18), we see that (2.11) can equivalently be written as follows

$$
\begin{aligned}
0 &= ce^{-\lambda_0\sigma}\frac{1}{\rho_{\lambda_0}}\int_{t-\sigma}^{t} f_{\lambda_0}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds \\
&\quad - e^{-\lambda_0\tau}c\frac{e^{-\lambda_0\sigma} - e^{-\lambda_1\sigma}}{e^{-\lambda_0\tau} - e^{-\lambda_1\tau}}\cdot\frac{1}{\rho_{\lambda_0}}\int_{t-\tau}^{t} f_{\lambda_0}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds.
\end{aligned}
$$

It is clear that this equality holds if

$$
\begin{aligned}
&\frac{e^{-\lambda_0\sigma}}{e^{-\lambda_0\sigma} - e^{-\lambda_1\sigma}}\int_{t-\sigma}^{t} f_{\lambda_0}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds \\
&= \frac{e^{-\lambda_0\tau}}{e^{-\lambda_0\tau} - e^{-\lambda_1\tau}}\int_{t-\tau}^{t} f_{\lambda_0}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&\frac{1}{1-e^{(\lambda_0-\lambda_1)\sigma}}\int_{t-\sigma}^{t} f_{\lambda_0}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds \\
&= \frac{1}{1-e^{(\lambda_0-\lambda_1)\tau}}\int_{t-\tau}^{t} f_{\lambda_0}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds.
\end{aligned}
\tag{2.19}
$$

So, the proof of the lemma can be accomplished by proving that (2.19) holds. It suffices to show that

$$
\begin{aligned}
&\frac{1}{1-e^{(\lambda_0-\lambda_1)\sigma}}\int_{t-\sigma}^{t} f_{\lambda_0}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds \\
&= \frac{1}{1-e^{(\lambda_0-\lambda_1)T}}\int_{t-T}^{t} f_{\lambda_0}(s)\exp\Big\{\int_{s}^{t}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds
\end{aligned}
\tag{2.20}
$$

and

$$\begin{aligned}
&\frac{1}{1-e^{(\lambda_0-\lambda_1)\tau}}\int_{t-\tau}^{t} f_{\lambda_0}(s)\exp\left\{\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds\\
&=\frac{1}{1-e^{(\lambda_0-\lambda_1)T}}\int_{t-T}^{t} f_{\lambda_0}(s)\exp\left\{\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds.
\end{aligned}\tag{2.21}$$

Next, we will establish (2.20). By taking into account the fact that the functions f_{λ_0} and f_{λ_1} are T-periodic and that $\sigma=\ell T$ and using (2.16) and (2.17), we obtain

$$\begin{aligned}
&\int_{t-\sigma}^{t} f_{\lambda_0}(s)\exp\left\{\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds\\
&=\sum_{i=1}^{\ell}\int_{t-iT}^{t-(i-1)T} f_{\lambda_0}(s)\exp\left\{\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds\\
&=\sum_{i=1}^{\ell}\int_{t-T}^{t} f_{\lambda_0}(s-(i-1)T)\exp\left\{\int_{s-(i-1)T}^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds\\
&=\sum_{i=1}^{\ell}\int_{t-T}^{t} f_{\lambda_0}(s)\exp\left\{\int_{s-(i-1)T}^s\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right.\\
&\quad\left.+\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds\\
&=\sum_{i=1}^{\ell}\int_{t-T}^{t} f_{\lambda_0}(s)\exp\left\{\left[\frac{1}{\rho_{\lambda_0}}\int_{s-(i-1)T}^s f_{\lambda_0}(u)du-\frac{1}{\rho_{\lambda_1}}\int_{s-(i-1)T}^s f_{\lambda_1}(u)du\right]\right.\\
&\quad\left.+\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds\\
&=\sum_{i=1}^{\ell}\int_{t-T}^{t} f_{\lambda_0}(s)\exp\left\{\left[\frac{1}{\rho_{\lambda_0}}\int_0^{(i-1)T} f_{\lambda_0}(u)du-\frac{1}{\rho_{\lambda_1}}\int_0^{(i-1)T} f_{\lambda_1}(u)du\right]\right.\\
&\quad\left.+\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds\\
&=\sum_{i=1}^{\ell}\int_{t-T}^{t} f_{\lambda_0}(s)\exp\Big\{[(i-1)(\lambda_0T)-(i-1)(\lambda_1T)]\\
&\quad\left.+\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds\\
&=\left[\sum_{i=1}^{\ell}e^{(i-1)(\lambda_0-\lambda_1)T}\right]\int_{t-T}^{t} f_{\lambda_0}(s)\exp\left\{\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds\\
&=\frac{1-e^{\ell(\lambda_0-\lambda_1)T}}{1-e^{(\lambda_0-\lambda_1)T}}\int_{t-T}^{t} f_{\lambda_0}(s)\exp\left\{\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds\\
&=\frac{1-e^{(\lambda_0-\lambda_1)\sigma}}{1-e^{(\lambda_0-\lambda_1)T}}\int_{t-T}^{t} f_{\lambda_0}(s)\exp\left\{\int_s^t\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds.
\end{aligned}$$

This proves (2.20).

Finally, by taking into account the fact that f_{λ_0} and f_{λ_1} are T-periodic functions and that $\tau = mT$ and using again (2.16) and (2.17), we can follow the same procedure to show that (2.21) is also satisfied.

The proof of the lemma is now complete. □

Now, we are in a position to present the main result of this section, i.e., the following theorem.

Theorem 2.4. *Suppose that $c \leq 0$ and that b is nonpositive on the interval $[0, \infty)$. Let λ_0 be a real root of the characteristic equation (2.3) with $\rho_{\lambda_0} \neq 0$ and satisfying (2.9), where ρ_{λ_0} is defined by (2.4), and assume that*

$$1 + \gamma_{\lambda_0} \neq 0,$$

where γ_{λ_0} is defined by (2.6). Let also λ_1, $\lambda_1 \neq \lambda_0$, be another real root of (2.3) with $\rho_{\lambda_1} \neq 0$, where ρ_{λ_1} is defined by (2.10).

Then the solution x of the IVP (2.1), (2.2) satisfies

$$\begin{aligned} U_1(\lambda_0, \lambda_1; \phi) &\leq \left\{ x(t) - \frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} \exp\left[\frac{1}{\rho_{\lambda_0}} \int_0^t f_{\lambda_0}(u)du \right] \right\} \exp\left[-\frac{1}{\rho_{\lambda_1}} \int_0^t f_{\lambda_1}(u)du \right] \\ &\leq U_2(\lambda_0, \lambda_1; \phi) \quad \text{for all } t \geq 0, \end{aligned}$$

where $L_{\lambda_0}(\phi)$ is defined by (2.8), and

$$\begin{aligned} U_1(\lambda_0, \lambda_1; \phi) = \min_{-r \leq t \leq 0} \Big(& \left\{ \phi(t) - \frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} \exp\left[\frac{1}{\rho_{\lambda_0}} \int_0^t f_{\lambda_0}(u)du \right] \right\} \times \\ & \times \exp\left[-\frac{1}{\rho_{\lambda_1}} \int_0^t f_{\lambda_1}(u)du \right] \Big), \end{aligned}$$

$$\begin{aligned} U_2(\lambda_0, \lambda_1; \phi) = \max_{-r \leq t \leq 0} \Big(& \left\{ \phi(t) - \frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} \exp\left[\frac{1}{\rho_{\lambda_0}} \int_0^t f_{\lambda_0}(u)du \right] \right\} \times \\ & \times \exp\left[-\frac{1}{\rho_{\lambda_1}} \int_0^t f_{\lambda_1}(u)du \right] \Big). \end{aligned}$$

Note: The constant B is negative and the root λ_0 is necessarily nonpositive; hence, by Part (I) of Lemma 2.2, we always have $1+\gamma_{\lambda_0} \neq 0$ if λ_1 is also nonpositive.

It is remarkable that the double inequality in the conclusion of the above theorem can equivalently be written as follows

$$\begin{aligned} & U_1(\lambda_0, \lambda_1; \phi) \exp\left\{ \int_0^t \left[\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}} - \frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} \right] du \right\} \\ & \leq x(t) \exp\left[-\frac{1}{\rho_{\lambda_0}} \int_0^t f_{\lambda_0}(u)du \right] - \frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}} \\ & \leq U_2(\lambda_0, \lambda_1; \phi) \exp\left\{ \int_0^t \left[\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}} - \frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} \right] du \right\} \quad \text{for } t \geq 0. \end{aligned}$$

Hence, if the roots λ_0 and λ_1 of the characteristic equation (2.3) are such that

$$\lim_{t \to \infty} \int_0^t \left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}} \right] du = \infty,$$

then the solution x of the IVP (2.1), (2.2) satisfies (2.7). Furthermore, we see that the double inequality in the conclusion of Theorem 2.4 is equivalent to

$$\begin{aligned}
&U_1(\lambda_0,\lambda_1;\phi)\exp\Big[\frac{1}{\rho_{\lambda_1}}\int_0^t f_{\lambda_1}(u)du\Big]+\frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}}\exp\Big[\frac{1}{\rho_{\lambda_0}}\int_0^t f_{\lambda_0}(u)du\Big]\\
&\le x(t)\\
&\le U_2(\lambda_0,\lambda_1;\phi)\exp\Big[\frac{1}{\rho_{\lambda_1}}\int_0^t f_{\lambda_1}(u)du\Big]+\frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}}\exp\Big[\frac{1}{\rho_{\lambda_0}}\int_0^t f_{\lambda_0}(u)du\Big]
\end{aligned}$$

for $t\ge 0$.

Proof of Theorem 2.4. Let x be the solution of the IVP (2.1), (2.2), and consider the function y defined by

$$y(t)=x(t)\exp\Big[-\frac{1}{\rho_{\lambda_0}}\int_0^t f_{\lambda_0}(u)du\Big]\quad \text{for } t\ge -r.$$

Furthermore, let us define

$$z(t)=y(t)-\frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}}\quad \text{for } t\ge -r.$$

As it has been shown by the authors [23] (for more general periodic linear neutral delay differential equations with several delays), the fact that x satisfies (2.1) for $t\ge 0$ is equivalent to the fact that z satisfies

$$\begin{aligned}
&z(t)+ce^{-\lambda_0\sigma}z(t-\sigma)\\
&=ce^{-\lambda_0\sigma}\frac{1}{\rho_{\lambda_0}}\int_{t-\sigma}^t f_{\lambda_0}(s)z(s)ds-e^{-\lambda_0\tau}\int_{t-\tau}^t \widetilde{b}(s)z(s)ds\quad \text{for } t\ge 0.
\end{aligned}\tag{2.22}$$

(Note that in [23] we have $\rho_{\lambda_0}>0$, but it suffices to have $\rho_{\lambda_0}\ne 0$.)

Next, we introduce the function

$$w(t)=z(t)\exp\Big\{\int_0^t\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}\quad \text{for } t\ge -r.$$

By using this function, we can immediately see that (2.22) takes the equivalent form

$$\begin{aligned}
&w(t)+ce^{-\lambda_0\sigma}w(t-\sigma)\exp\Big\{\int_{t-\sigma}^t\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}\\
&=ce^{-\lambda_0\sigma}\frac{1}{\rho_{\lambda_0}}\int_{t-\sigma}^t f_{\lambda_0}(s)w(s)\exp\Big\{\int_s^t\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds\\
&\quad -e^{-\lambda_0\tau}\int_{t-\tau}^t \widetilde{b}(s)w(s)\exp\Big\{\int_s^t\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}}-\frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds\quad \text{for } t\ge 0.
\end{aligned}\tag{2.23}$$

In view of the fact that the coefficients a and b are T-periodic functions, it follows that the functions f_{λ_0} and f_{λ_1} are also T-periodic. So, by taking into account the fact that $\sigma=\ell T$ and following the same procedure as in proving (2.14) and (2.15) (in the proof of Lemma 2.3), we can show that

$$\frac{1}{\rho_{\lambda_0}}\int_{t-\sigma}^t f_{\lambda_0}(u)du=\lambda_0\sigma\quad \text{for } t\ge 0$$

and

$$\frac{1}{\rho_{\lambda_1}}\int_{t-\sigma}^{t} f_{\lambda_1}(u)du = \lambda_1\sigma \quad \text{for } t \geq 0.$$

Consequently,

$$\exp\left\{\int_{t-\sigma}^{t}\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\} = e^{(\lambda_0-\lambda_1)\sigma} \quad \text{for every } t \geq 0.$$

By using this fact, we see that (2.23) can equivalently be written as follows

$$\begin{aligned} & w(t) + ce^{-\lambda_1\sigma}w(t-\sigma) \\ & = ce^{-\lambda_0\sigma}\frac{1}{\rho_{\lambda_0}}\int_{t-\sigma}^{t} f_{\lambda_0}(s)w(s)\exp\left\{\int_{s}^{t}\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds \\ & \quad - e^{-\lambda_0\tau}\int_{t-\tau}^{t}\widetilde{b}(s)w(s)\exp\left\{\int_{s}^{t}\left[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\right]du\right\}ds \quad \text{for } t \geq 0. \end{aligned} \tag{2.24}$$

Combining the definitions of the functions y, z and w, we have

$$w(t) = \left\{x(t) - \frac{L_{\lambda_0}(\phi)}{1+\gamma_{\lambda_0}}\exp\left[\frac{1}{\rho_{\lambda_0}}\int_0^t f_{\lambda_0}(u)du\right]\right\}\exp\left[-\frac{1}{\rho_{\lambda_1}}\int_0^t f_{\lambda_1}(u)du\right]$$

for $t \geq -r$. Thus, by taking into account the initial condition (2.2) and the definitions of the constants $U_1(\lambda_0, \lambda_1; \phi)$ and $U_2(\lambda_0, \lambda_1; \phi)$, we can immediately conclude that the double inequality in the conclusion of our theorem can be written in the equivalent form

$$\min_{-r\leq s\leq 0} w(s) \leq w(t) \leq \max_{-r\leq s\leq 0} w(s) \quad \text{for all } t \geq 0.$$

What we have to prove is that the last double inequality is valid. We will confine our attention in establishing that

$$w(t) \geq \min_{-r\leq s\leq 0} w(s) \quad \text{for every } t \geq 0. \tag{2.25}$$

In a similar way, one can show that

$$w(t) \leq \max_{-r\leq s\leq 0} w(s) \quad \text{for every } t \geq 0.$$

It remains to prove (2.25). For this purpose, let us consider an arbitrary real number K with $K < \min_{-r\leq s\leq 0} w(s)$. Then we obviously have

$$w(t) > K \quad \text{for } -r \leq t \leq 0. \tag{2.26}$$

We claim that

$$w(t) > K \quad \text{for all } t \geq 0. \tag{2.27}$$

Otherwise, in view of (2.26), there exists a point $t_0 > 0$ so that

$$w(t) > K \quad \text{for } -r \leq t < t_0, \quad \text{and} \quad w(t_0) = K. \tag{2.28}$$

We notice that it is supposed that $c \leq 0$. Also, we observe that, as $\rho_{\lambda_0} \neq 0$, the hypothesis (2.9) can be written in the form $\frac{1}{\rho_{\lambda_0}}f_{\lambda_0}(t) \leq 0$ for $t \geq 0$. So, since the function f_{λ_0} is T-periodic and r is a multiple of the period T, we always have

$$\frac{1}{\rho_{\lambda_0}}f_{\lambda_0}(t) \leq 0 \quad \text{for every } t \geq -r.$$

Furthermore, since the function b is T-periodic and $\tau = mT$, the assumption that b is not identically zero on $[0, \infty)$ means that $\widetilde{b}$ is not identically zero on the interval

$[t_0-\tau, t_0)$, while the hypothesis that b is nonpositive on $[0,\infty)$ means that $\widetilde{b}$ is nonpositive on $[t_0-\tau, t_0)$. Hence, the function $\widetilde{b}$ is nonpositive, but not identically zero, on the interval $[t_0-\tau, t_0)$. We also have $\widetilde{b}(t_0) \le 0$. Now, by using (2.28) and taking into account the above observations, from (2.24) we obtain

$$\begin{aligned}
K = w(t_0) &= -ce^{-\lambda_1\sigma} w(t_0-\sigma) \\
&\quad + ce^{-\lambda_0\sigma}\frac{1}{\rho_{\lambda_0}}\int_{t_0-\sigma}^{t_0} f_{\lambda_0}(s)w(s)\exp\Big\{\int_s^{t_0}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds \\
&\quad - e^{-\lambda_0\tau}\int_{t_0-\tau}^{t_0} \widetilde{b}(s)w(s)\exp\Big\{\int_s^{t_0}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds \\
&> K\Big(-ce^{-\lambda_1\sigma} + ce^{-\lambda_0\sigma}\frac{1}{\rho_{\lambda_0}}\int_{t_0-\sigma}^{t_0} f_{\lambda_0}(s)\exp\Big\{\int_s^{t_0}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds \\
&\quad - e^{-\lambda_0\tau}\int_{t_0-\tau}^{t_0} \widetilde{b}(s)\exp\Big\{\int_s^{t_0}\Big[\frac{f_{\lambda_0}(u)}{\rho_{\lambda_0}} - \frac{f_{\lambda_1}(u)}{\rho_{\lambda_1}}\Big]du\Big\}ds\Big).
\end{aligned}$$

Thus, using (2.11) for $t = t_0$, we arrive at the contradiction $K > K$. This contradiction establishes our claim, i.e., (2.27) holds. Since (2.27) is satisfied for all real numbers K such that $K < \min_{-r\le s\le 0} w(s)$, it follows that (2.25) is always fulfilled.

The proof of the theorem is complete. □

Before closing this section, we will consider two special cases, namely the non-neutral case and the autonomous case.

Consider, first, the periodic linear (*non-neutral*) delay differential equation

$$x'(t) = a(t)x(t) + b(t)x(t-\tau). \tag{2.29}$$

Equation (2.29) can be obtained (as a special case) from (2.1) *by taking* $c = 0$ *and* $\sigma = \tau$. As it concerns the (non-neutral) delay differential equation (2.29), we have the number τ in place of r. The characteristic equation of (2.29) is

$$\lambda = A + Be^{-\lambda\tau}. \tag{2.30}$$

By applying Theorem 2.4 to the (non-neutral) delay differential equation (2.29), we are led to [35, Theorem 2.3] in the previous authors' paper, *under the additional hypotheses that the root* λ_0 *of the characteristic equation* (2.30) *is such that* $a(t) + b(t)e^{-\lambda_0\tau} \le 0$ *for* $t \ge 0$ (and so λ_0 is always nonpositive) *and that* λ_0 *satisfies* $1 + B\tau e^{-\lambda_0\tau} \ne 0$. (Note that we always have $1 + B\tau e^{-\lambda_0\tau} \ne 0$ if the other root λ_1 of (2.30) is also nonpositive.) But, these (additional) hypotheses are not needed for [35, Theorem 2.3] to hold. Hence, [35, Theorem 2.3] cannot be obtained as a corollary from Theorem 2.4, the main result of this section.

Next, let us consider the *autonomous* linear neutral delay differential equation

$$[x(t) + cx(t-\sigma)]' = ax(t) + bx(t-\tau), \tag{2.31}$$

where c, a *and* $b \ne 0$ *are real numbers, and* σ *and* τ *are positive real constants.* The characteristic equation of (2.31) is the following one

$$\lambda(1 + ce^{-\lambda\sigma}) = a + be^{-\lambda\tau}. \tag{2.32}$$

The constant coefficients a and b of the autonomous neutral delay differential equation (2.31) can be considered as T-periodic functions, for any real number $T > 0$. We observe that the hypothesis that the root λ_0 of the characteristic equation (2.32) satisfies (2.9) is equivalent to the hypothesis that λ_0 is nonpositive. After these observations, it is not difficult to apply Theorem 2.4 to the special case of the autonomous linear neutral delay differential equation (2.31). The result obtained by such an application is a special case of a more general result given by the authors in [30] (for autonomous linear neutral delay differential equations *with several delays*), *under the assumption that there exist* $T > 0$ *and positive integers* ℓ *and* m *with* $\sigma = \ell T$ *and* $\tau = mT$*, and the hypothesis that the roots* λ_0 *and* λ_1 *of the characteristic equation* (2.32) *satisfy* $1 + ce^{-\lambda_0 \sigma} \neq 0$ *and* $1 + ce^{-\lambda_1 \sigma} \neq 0$. Note that these restrictions are not necessary for the special case of the more general result in [30] to be valid. Such restrictions are not imposed in the corresponding result in [30] concerning more general autonomous linear neutral delay differential equations.

3. On the behavior of the solutions to periodic linear neutral delay difference equations

This section is devoted to the study of the behavior of the solutions of the linear neutral delay difference equation

$$\Delta\left(x_n + cx_{n-\sigma}\right) = a(n)x_n + b(n)x_{n-\tau}, \tag{3.1}$$

where c *is a real number,* $(a(n))_{n\geq 0}$ *and* $(b(n))_{n\geq 0}$ *are sequences of real numbers, and* σ *and* τ *are positive integers. It is supposed that the sequence* $(b(n))_{n\geq 0}$ *is not identically zero.* Moreover, *it will be assumed that the coefficients* $(a(n))_{n\geq 0}$ *and* $(b(n))_{n\geq 0}$ *are periodic sequences with a common period* T *(where* T *is a positive integer) and that there exist positive integers* ℓ *and* m *such that*

$$\sigma = \ell T \quad \textit{and} \quad \tau = mT.$$

Let us consider the positive integer r defined by

$$r = \max\{\sigma, \tau\}.$$

A *solution* of the neutral delay difference equation (3.1) is a sequence of real numbers $(x_n)_{n\geq -r}$, which satisfies (3.1) for all $n \geq 0$.

With the neutral delay difference equation (3.1), one associates an *initial condition* of the form

$$x_n = \phi_n \quad \text{for } n = -r, \dots, 0, \tag{3.2}$$

where *the initial values* ϕ_n $(n = -r, \dots, 0)$ *are given real numbers.* For convenience, we will use the notation $\phi = (\phi_n)_{n=-r}^{0}$.

Equations (3.1) and (3.2) constitute an *initial value problem* (IVP, for short). It is clear that there exists exactly one solution $(x_n)_{n\geq -r}$ of the neutral delay difference equation (3.1) which satisfies the initial condition (3.2); we shall call this unique solution $(x_n)_{n\geq -r}$ the *solution* of the initial value problem (3.1), (3.2) or, more briefly, the *solution* of the IVP (3.1), (3.2).

With the neutral delay difference equation (3.1), we associate the equation

$$\left[\lambda(1 + c\lambda^{-\sigma})\right]^T = \prod_{k=0}^{T-1}\left[1 + c\lambda^{-\sigma} + a(k) + b(k)\lambda^{-\tau}\right]; \tag{3.3}$$

this equation will be called the *characteristic equation* of (3.1) (see the authors' paper [29]).

Now, we shall introduce certain notation, which will be used throughout this section without any further mention.

By $(\widetilde{a}(n))_{n\geq -r}$ and $(\widetilde{b}(n))_{n\geq -r}$ we will denote the T-periodic extensions of the coefficients $(a(n))_{n\geq 0}$ and $(b(n))_{n\geq 0}$, respectively. (Clearly, r is a multiple of the period T.)

We consider positive roots λ of the characteristic equation (3.3) with the following property:

$$1 + c\lambda^{-\sigma} \neq 0. \tag{3.4}$$

We immediately observe that a positive root λ of (3.3) with the property (3.4) satisfies

$$\lambda^T = \prod_{k=0}^{T-1} \Big\{1 + \frac{1}{1 + c\lambda^{-\sigma}} \big[a(k) + b(k)\lambda^{-\tau}\big]\Big\}.$$

Furthermore, for any positive root λ of the characteristic equation (3.3) with the property (3.4), by $(h_\lambda(n))_{n\geq -r}$ we shall denote the sequence of real numbers defined as follows

$$h_\lambda(n) = 1 + \frac{1}{1 + c\lambda^{-\sigma}}[\widetilde{a}(n) + \widetilde{b}(n)\lambda^{-\tau}] \quad \text{for } n \geq -r.$$

Since the sequences $(a(n))_{n\geq 0}$ and $(b(n))_{n\geq 0}$ are T-periodic, it follows immediately that, for each positive root λ of (3.3) with the property (3.4), the sequence $(h_\lambda(n))_{n\geq -r}$ is also T-periodic.

By the use of the above notation, we have

$$\lambda^T = \prod_{k=0}^{T-1} h_\lambda(k)$$

for each positive root λ of the characteristic equation (3.3) with the property (3.4). This fact will be used quite frequently in the sequel without any specific mention.

We will make use of positive roots λ of the characteristic equation (3.3) with the property (3.4) and the additional property: *If* $T > 1$, *then*

$$h_\lambda(k) \equiv 1 + \frac{1}{1 + c\lambda^{-\sigma}}[a(k) + b(k)\lambda^{-\tau}] > 0 \quad (k = 1, \dots, T-1). \tag{3.5}$$

(It must be noted that (3.5) holds by itself when $T = 1$.)

The following simple result will be kept in mind in what follows:

If λ is a positive root of the characteristic equation (3.3) *with the properties* (3.4) *and* (3.5), *then*

$$h_\lambda(n) > 0 \quad \textit{for all } n \geq -r.$$

This result has been established by the authors in [29] for the case of more general periodic linear neutral delay difference equations with several delays. Note that in [29] we have used positive roots λ of the characteristic equation (3.3) with the property

$$1 + c\lambda^{-\sigma} > 0 \tag{3.6}$$

in place of the property (3.4) considered here. But, the above result remains valid with (3.4) instead of (3.6).

Let us introduce another notation. If λ is a positive root of the characteristic equation (3.3) with the properties (3.4) and (3.5), then $(H_\lambda(n))_{n\geq -r}$ will stand for the sequence of positive real numbers defined by

$$H_\lambda(n) = \begin{cases} \prod_{k=0}^{n-1} h_\lambda(k) & \text{for } n \geq 0 \\ \big[\prod_{k=n}^{-1} h_\lambda(k)\big]^{-1} & \text{for } n = -r, \dots, 0. \end{cases}$$

Note that, here and in the sequel, we use the usual convention $\prod_{k=0}^{-1} = 1$.

A basic asymptotic criterion for the solutions of the neutral delay difference equation (3.1) is the following theorem, which is closely related to the main result (Theorem 3.3 below) of this section.

Theorem 3.1. *Let λ_0 be a positive root of the characteristic equation* (3.3) *with the properties* (3.4) *and* (3.5) *with λ_0 instead of λ, and the following property:*

$$|c|\lambda_0^{-\sigma}\left[1+\sum_{s=0}^{\sigma-1}\left|1-\frac{1}{h_{\lambda_0}(s)}\right|\right]+\lambda_0^{-\tau}\sum_{s=0}^{\tau-1}\frac{1}{h_{\lambda_0}(s)}|b(s)|<1. \tag{3.7}$$

Set

$$\beta_{\lambda_0}=c\lambda_0^{-\sigma}\left\{1-\sum_{s=0}^{\sigma-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\right\}+\lambda_0^{-\tau}\sum_{s=0}^{\tau-1}\frac{1}{h_{\lambda_0}(s)}b(s). \tag{3.8}$$

(Note that Property (3.7) *guarantees that $1+\beta_{\lambda_0}>0$.)*

Then the solution $(x_n)_{n\geq -r}$ of the IVP (3.1), (3.2) *satisfies*

$$\lim_{n\to\infty}\frac{x_n}{H_{\lambda_0}(n)}=\frac{M_{\lambda_0}(\phi)}{1+\beta_{\lambda_0}}, \tag{3.9}$$

where

$$\begin{aligned} M_{\lambda_0}(\phi)=\phi_0+c\left\{\phi_{-\sigma}-\lambda_0^{-\sigma}\sum_{s=-\sigma}^{-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\left[\prod_{k=s}^{-1}h_{\lambda_0}(k)\right]\phi_s\right\}\\ +\lambda_0^{-\tau}\sum_{s=-\tau}^{-1}\left[\prod_{k=s+1}^{-1}h_{\lambda_0}(k)\right]\widetilde{b}(s)\phi_s. \end{aligned} \tag{3.10}$$

The above theorem has been established in the authors' paper [29] for more general periodic linear neutral delay difference equations with several delays. Note that in [29] we have assumed that the positive root λ_0 of the characteristic equation (3.3) has the property (3.6) with λ_0 instead of λ, while in Theorem 3.1 it is supposed that λ_0 has the property (3.4) with λ_0 instead of λ. But, if λ_0 has the properties (3.4) and (3.5) with λ_0 instead of λ, then (3.7) makes sense and it implies that $|c|\lambda_0^{-\sigma}<1$, which gives $1+c\lambda_0^{-\sigma}>0$, i.e., λ_0 has always the property (3.6) with λ_0 instead of λ.

The main result of this section, i.e., Theorem 3.3 below, is derived via two suitable distinct positive roots of the characteristic equation (3.3). Before stating and proving Theorem 3.3, we give a lemma (Lemma 3.2 below), which is rather technical, but it is crucial in proving Theorem 3.3.

Lemma 3.2. *Let λ_0 and λ_1, $\lambda_0\neq\lambda_1$, be two positive roots of the characteristic equation* (3.3) *with the properties* (3.4) *and* (3.5) *with λ_0 instead of λ as well as* (3.4) *and* (3.5) *with λ_1 instead of λ, respectively. Then, for each $n\geq 0$, we have*

$$\begin{aligned} 1=-c\lambda_1^{-\sigma}+c\lambda_0^{-\sigma}\sum_{s=n-\sigma}^{n-1}\left[1-\frac{1}{h_{\lambda_0}(s)}\right]\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\\ -\lambda_0^{-\tau}\sum_{s=n-\tau}^{n-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}(s)\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}. \end{aligned} \tag{3.11}$$

Proof. Set $\mu_{\lambda_0} = 1 + c\lambda_0^{-\sigma}$ and $\mu_{\lambda_1} = 1 + c\lambda_1^{-\sigma}$. We immediately see that

$$\mu_{\lambda_0}\left[h_{\lambda_0}(n) - 1\right] - \mu_{\lambda_1}\left[h_{\lambda_1}(n) - 1\right] = \widetilde{b}(n)\left(\lambda_0^{-\tau} - \lambda_1^{-\tau}\right) \quad \text{for } n \geq -r. \tag{3.12}$$

Also, it is obvious that

$$\mu_{\lambda_0} - \mu_{\lambda_1} = c\left(\lambda_0^{-\sigma} - \lambda_1^{-\sigma}\right). \tag{3.13}$$

Furthermore, we note that the sequences $(h_{\lambda_0}(n))_{n\geq -r}$ and $(h_{\lambda_1}(n))_{n\geq -r}$ are T-periodic. Thus, by taking into account the fact that $\tau = mT$, we obtain for $n \geq 0$

$$\prod_{k=n-\tau}^{n-1} h_{\lambda_0}(k) = \prod_{k=0}^{\tau-1} h_{\lambda_0}(k) = \left[\prod_{k=0}^{T-1} h_{\lambda_0}(k)\right]^m = \left(\lambda_0^T\right)^m = \lambda_0^{mT} = \lambda_0^{\tau}.$$

That is,

$$\prod_{k=n-\tau}^{n-1} h_{\lambda_0}(k) = \lambda_0^{\tau} \quad \text{for every } n \geq 0. \tag{3.14}$$

Similarly, we have

$$\prod_{k=n-\tau}^{n-1} h_{\lambda_1}(k) = \lambda_1^{\tau} \quad \text{for every } n \geq 0. \tag{3.15}$$

Moreover, for any positive integer ν, we get

$$\prod_{k=0}^{\nu T-1} h_{\lambda_0}(k) = \left[\prod_{k=0}^{T-1} h_{\lambda_0}(k)\right]^{\nu} = \left(\lambda_0^T\right)^{\nu} = \lambda_0^{\nu T}.$$

Consequently,

$$\prod_{k=0}^{\nu T-1} h_{\lambda_0}(k) = \lambda_0^{\nu T} \quad (\nu = 1, 2, \dots). \tag{3.16}$$

Analogously, we find

$$\prod_{k=0}^{\nu T-1} h_{\lambda_1}(k) = \lambda_1^{\nu T} \quad (\nu = 1, 2, \dots). \tag{3.17}$$

Now, let us consider an arbitrary but fixed integer $n \geq 0$. We shall show that, for this fixed integer n, equality (3.11) is fulfilled.

We take into account (3.12) and (3.13) to obtain

$$\begin{aligned}
&\sum_{s=n-\tau}^{n-1} \frac{1}{h_{\lambda_0}(s)}\widetilde{b}(s)\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\
&= \frac{1}{\lambda_0^{-\tau} - \lambda_1^{-\tau}} \sum_{s=n-\tau}^{n-1} \frac{1}{h_{\lambda_0}(s)}\widetilde{b}(s)\left(\lambda_0^{-\tau} - \lambda_1^{-\tau}\right)\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\
&= \frac{1}{\lambda_0^{-\tau} - \lambda_1^{-\tau}} \sum_{s=n-\tau}^{n-1} \frac{1}{h_{\lambda_0}(s)}\Big\{\mu_{\lambda_0}\left[h_{\lambda_0}(s) - 1\right] - \mu_{\lambda_1}\left[h_{\lambda_1}(s) - 1\right]\Big\}\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\
&= \frac{1}{\lambda_0^{-\tau} - \lambda_1^{-\tau}} \sum_{s=n-\tau}^{n-1} \frac{1}{h_{\lambda_0}(s)}\Big\{\left(\mu_{\lambda_0} - \mu_{\lambda_1}\right)\left[h_{\lambda_0}(s) - 1\right] + \mu_{\lambda_1}\left[h_{\lambda_0}(s) - h_{\lambda_1}(s)\right]\Big\}\times \\
&\quad \times \prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}
\end{aligned}$$

$$
\begin{aligned}
&= c\frac{\lambda_0^{-\sigma}-\lambda_1^{-\sigma}}{\lambda_0^{-\tau}-\lambda_1^{-\tau}}\sum_{s=n-\tau}^{n-1}\Big[1-\frac{1}{h_{\lambda_0}(s)}\Big]\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\\
&\quad-\frac{1}{\lambda_0^{-\tau}-\lambda_1^{-\tau}}\mu_{\lambda_1}\sum_{s=n-\tau}^{n-1}\Big[\frac{h_{\lambda_1}(s)}{h_{\lambda_0}(s)}-1\Big]\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}.
\end{aligned}
$$

But, using (3.14) and (3.15), we have

$$
\begin{aligned}
\sum_{s=n-\tau}^{n-1}\Big[\frac{h_{\lambda_1}(s)}{h_{\lambda_0}(s)}-1\Big]\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}
&=\sum_{s=n-\tau}^{n-1}\Bigg[\frac{1}{\frac{h_{\lambda_0}(s)}{h_{\lambda_1}(s)}}\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}-\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Bigg]\\
&=\sum_{s=n-\tau}^{n-1}\Big[\prod_{k=s+1}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}-\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Big]\\
&=\sum_{s=n-\tau}^{n-1}\Delta\Big[\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Big]\\
&=\Big[\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Big]_{s=(n-1)+1}-\Big[\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Big]_{s=n-\tau}\\
&=\prod_{k=n}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}-\prod_{k=n-\tau}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\\
&=1-\frac{\prod_{k=n-\tau}^{n-1}h_{\lambda_0}(k)}{\prod_{k=n-\tau}^{n-1}h_{\lambda_1}(k)}=1-\frac{\lambda_0^{\tau}}{\lambda_1^{\tau}}=1-\Big(\frac{\lambda_0}{\lambda_1}\Big)^{\tau}.
\end{aligned}
$$

Note that we have used the usual convention that $\prod_n^{n-1}=1$. Hence, it holds

$$
\begin{aligned}
&\sum_{s=n-\tau}^{n-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}(s)\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\\
&=c\frac{\lambda_0^{-\sigma}-\lambda_1^{-\sigma}}{\lambda_0^{-\tau}-\lambda_1^{-\tau}}\sum_{s=n-\tau}^{n-1}\Big[1-\frac{1}{h_{\lambda_0}(s)}\Big]\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}-\frac{1}{\lambda_0^{-\tau}}\mu_{\lambda_1}.
\end{aligned}\tag{3.18}
$$

Next, we observe that (3.11) can equivalently be written as follows

$$
\mu_{\lambda_1}=c\lambda_0^{-\sigma}\sum_{s=n-\sigma}^{n-1}\Big[1-\frac{1}{h_{\lambda_0}(s)}\Big]\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}-\lambda_0^{-\tau}\sum_{s=n-\tau}^{n-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}(s)\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}.
$$

So, in view of (3.18), equality (3.11) is equivalent to

$$
\begin{aligned}
0=&\,c\lambda_0^{-\sigma}\sum_{s=n-\sigma}^{n-1}\Big[1-\frac{1}{h_{\lambda_0}(s)}\Big]\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\\
&-\lambda_0^{-\tau}c\frac{\lambda_0^{-\sigma}-\lambda_1^{-\sigma}}{\lambda_0^{-\tau}-\lambda_1^{-\tau}}\sum_{s=n-\tau}^{n-1}\Big[1-\frac{1}{h_{\lambda_0}(s)}\Big]\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}.
\end{aligned}
$$

The above equality is satisfied if

$$
\frac{\lambda_0^{-\sigma}}{\lambda_0^{-\sigma}-\lambda_1^{-\sigma}}\sum_{s=n-\sigma}^{n-1}\Big[1-\frac{1}{h_{\lambda_0}(s)}\Big]\prod_{k=s}^{n-1}\frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}
$$

$$= \frac{\lambda_0^{-\tau}}{\lambda_0^{-\tau} - \lambda_1^{-\tau}} \sum_{s=n-\tau}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)},$$

i.e., if the following equality holds,

$$\begin{aligned} &\frac{1}{1 - \left(\frac{\lambda_0}{\lambda_1}\right)^{\sigma}} \sum_{s=n-\sigma}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\ &= \frac{1}{1 - \left(\frac{\lambda_0}{\lambda_1}\right)^{\tau}} \sum_{s=n-\tau}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}. \end{aligned} \tag{3.19}$$

In the rest of the proof, we will establish (3.19). To this end, it is sufficient to prove that

$$\begin{aligned} &\frac{1}{1 - \left(\frac{\lambda_0}{\lambda_1}\right)^{\sigma}} \sum_{s=n-\sigma}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\ &= \frac{1}{1 - \left(\frac{\lambda_0}{\lambda_1}\right)^{T}} \sum_{s=n-T}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\frac{1}{1 - \left(\frac{\lambda_0}{\lambda_1}\right)^{\tau}} \sum_{s=n-\tau}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\ &= \frac{1}{1 - \left(\frac{\lambda_0}{\lambda_1}\right)^{T}} \sum_{s=n-T}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}. \end{aligned} \tag{3.21}$$

To prove (3.20) and (3.21), we shall take into account the fact that the sequences $(h_{\lambda_0}(n))_{n\geq -r}$ and $(h_{\lambda_1}(n))_{n\geq -r}$ are T-periodic and that $\sigma = \ell T$ and $\tau = mT$ and we will use equalities (3.16) and (3.17). We confine ourselves to showing (3.20). Equality (3.21) can be established by an analogous procedure. We obtain

$$\begin{aligned} &\sum_{s=n-\sigma}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\ &= \sum_{i=1}^{\ell} \sum_{s=n-iT}^{n-1-(i-1)T} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\ &= \sum_{i=1}^{\ell} \sum_{s=n-T}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s-(i-1)T)}\Big] \prod_{k=s-(i-1)T}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\ &= \sum_{i=1}^{\ell} \sum_{s=n-T}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \Big[\prod_{k=s-(i-1)T}^{s-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Big] \Big[\prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Big] \\ &= \sum_{i=1}^{\ell} \sum_{s=n-T}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \frac{\prod_{k=s-(i-1)T}^{s-1} h_{\lambda_0}(k)}{\prod_{k=s-(i-1)T}^{s-1} h_{\lambda_1}(k)} \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\ell} \sum_{s=n-T}^{n-1} \left[1 - \frac{1}{h_{\lambda_0}(s)}\right] \frac{\prod_{k=0}^{(i-1)T-1} h_{\lambda_0}(k)}{\prod_{k=0}^{(i-1)T-1} h_{\lambda_1}(k)} \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\
&= \sum_{i=1}^{\ell} \sum_{s=n-T}^{n-1} \left[1 - \frac{1}{h_{\lambda_0}(s)}\right] \frac{\lambda_0^{(i-1)T}}{\lambda_1^{(i-1)T}} \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\
&= \left[\sum_{i=1}^{\ell} \left(\frac{\lambda_0}{\lambda_1}\right)^{(i-1)T}\right] \sum_{s=n-T}^{n-1} \left[1 - \frac{1}{h_{\lambda_0}(s)}\right] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\
&= \frac{1 - \left(\frac{\lambda_0}{\lambda_1}\right)^{\ell T}}{1 - \left(\frac{\lambda_0}{\lambda_1}\right)^{T}} \sum_{s=n-T}^{n-1} \left[1 - \frac{1}{h_{\lambda_0}(s)}\right] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\
&= \frac{1 - \left(\frac{\lambda_0}{\lambda_1}\right)^{\sigma}}{1 - \left(\frac{\lambda_0}{\lambda_1}\right)^{T}} \sum_{s=n-T}^{n-1} \left[1 - \frac{1}{h_{\lambda_0}(s)}\right] \prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}.
\end{aligned}
$$

So, we have proved that (3.20) holds.

The proof of our lemma is complete. □

Now, we proceed to establish the main result of this section, i.e., Theorem 3.3 below.

Theorem 3.3. *Suppose that* $c \le 0$ *and that* $(b(n))_{n\ge0}$ *is nonpositive. Let* λ_0 *be a positive root of the characteristic equation* (3.3) *with the properties* (3.4) *and* (3.5) *with* λ_0 *instead of* λ *and such that*

$$
\left(1 + c\lambda_0^{-\sigma}\right)\left[a(n) + b(n)\lambda_0^{-\tau}\right] \le 0 \quad \text{for } n \ge 0, \tag{3.22}
$$

and assume that $1 + \beta_{\lambda_0} \neq 0$*, where* β_{λ_0} *is defined by* (3.8)*. Let also* λ_1*,* $\lambda_1 \neq \lambda_0$*, be another positive root of* (3.3) *with the properties* (3.4) *and* (3.5) *with* λ_1 *instead of* λ*.*

Then the solution $(x_n)_{n\ge -r}$ *of the IVP* (3.1)*,* (3.2) *satisfies*

$$
V_1(\lambda_0, \lambda_1; \phi) \le \frac{1}{H_{\lambda_1}(n)}\left[x_n - \frac{M_{\lambda_0}(\phi)}{1 + \beta_{\lambda_0}} H_{\lambda_0}(n)\right] \le V_2(\lambda_0, \lambda_1; \phi) \quad \text{for all } n \ge 0,
$$

where $M_{\lambda_0}(\phi)$ *is defined by* (3.10)*, and*

$$
\begin{aligned}
V_1(\lambda_0, \lambda_1; \phi) &= \min_{n=-r,\dots,0} \left\{\frac{1}{H_{\lambda_1}(n)}\left[\phi_n - \frac{M_{\lambda_0}(\phi)}{1 + \beta_{\lambda_0}} H_{\lambda_0}(n)\right]\right\}, \\
V_2(\lambda_0, \lambda_1; \phi) &= \max_{n=-r,\dots,0} \left\{\frac{1}{H_{\lambda_1}(n)}\left[\phi_n - \frac{M_{\lambda_0}(\phi)}{1 + \beta_{\lambda_0}} H_{\lambda_0}(n)\right]\right\}.
\end{aligned}
$$

Let λ_0 *be a positive root of the characteristic equation* (3.3) *with the properties* (3.4) *and* (3.5) *with* λ_0 *instead of* λ*, and such that* (3.22) *holds.* Then we immediately see that (3.22) can equivalently be written as follows

$$
\frac{1}{1 + c\lambda_0^{-\sigma}}[\widetilde{a}(n) + \widetilde{b}(n)\lambda_0^{-\tau}] \le 0 \quad \text{for } n \ge -r.
$$

Hence, *inequality* (3.22) *is equivalent to*

$$
h_{\lambda_0}(n) \le 1 \quad \text{for } n \ge -r. \tag{3.23}
$$

Furthermore, it follows easily that *the root* λ_0 *is always less than or equal to* 1.

It must be noted that the double inequality in the conclusion of Theorem 3.3 is equivalent to

$$V_1(\lambda_0,\lambda_1;\phi)\frac{H_{\lambda_1}(n)}{H_{\lambda_0}(n)} \le \frac{x_n}{H_{\lambda_0}(n)} - \frac{M_{\lambda_0}(\phi)}{1+\beta_{\lambda_0}} \le V_2(\lambda_0,\lambda_1;\phi)\frac{H_{\lambda_1}(n)}{H_{\lambda_0}(n)} \text{ for } n \ge 0.$$

Thus, (3.9) is satisfied if the roots λ_0 and λ_1 of (3.3) are such that

$$\lim_{n\to\infty}\frac{H_{\lambda_0}(n)}{H_{\lambda_1}(n)} = \infty.$$

Moreover, we notice that another equivalent form of the double inequality in the conclusion of Theorem 3.3 is the following one

$$\begin{aligned} &V_1(\lambda_0,\lambda_1;\phi)H_{\lambda_1}(n) + \frac{M_{\lambda_0}(\phi)}{1+\beta_{\lambda_0}}H_{\lambda_0}(n) \\ &\le x_n \\ &\le V_2(\lambda_0,\lambda_1;\phi)H_{\lambda_1}(n) + \frac{M_{\lambda_0}(\phi)}{1+\beta_{\lambda_0}}H_{\lambda_0}(n) \quad \text{for } n \ge 0. \end{aligned}$$

Proof of Theorem 3.3. Consider the solution $(x_n)_{n\ge -r}$ of the IVP (3.1), (3.2), and set

$$y_n = \frac{x_n}{H_{\lambda_0}(n)} \quad \text{for } n \ge -r.$$

Define

$$z_n = y_n - \frac{M_{\lambda_0}(\phi)}{1+\beta_{\lambda_0}} \quad \text{for } n \ge -r.$$

It has been shown by the authors [29] (for more general periodic linear neutral delay difference equations with several delays) that $(x_n)_{n\ge -r}$ satisfies (3.1) for $n \ge 0$ if and only if $(z_n)_{n\ge -r}$ satisfies

$$z_n + c\lambda_0^{-\sigma} z_{n-\sigma} = c\lambda_0^{-\sigma}\sum_{s=n-\sigma}^{n-1}\Big[1-\frac{1}{h_{\lambda_0}(s)}\Big]z_s - \lambda_0^{-\tau}\sum_{s=n-\tau}^{n-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}(s)z_s \tag{3.24}$$

for $n \ge 0$. Note that in [29] it is assumed that λ_0 has the property (3.6) with λ_0 instead of λ, but it is sufficient to suppose that λ_0 has the property (3.4) with λ_0 instead of λ.

Next, we define

$$w_n = \frac{H_{\lambda_0}(n)}{H_{\lambda_1}(n)}z_n \quad \text{for } n \ge -r.$$

Then we see that (3.24) reduces to the equivalent equation

$$\begin{aligned} &w_n + c\lambda_0^{-\sigma}\frac{H_{\lambda_1}(n-\sigma)/H_{\lambda_1}(n)}{H_{\lambda_0}(n-\sigma)/H_{\lambda_0}(n)}w_{n-\sigma} \\ &= c\lambda_0^{-\sigma}\sum_{s=n-\sigma}^{n-1}\Big[1-\frac{1}{h_{\lambda_0}(s)}\Big]\frac{H_{\lambda_0}(n)/H_{\lambda_0}(s)}{H_{\lambda_1}(n)/H_{\lambda_1}(s)}w_s \\ &\quad - \lambda_0^{-\tau}\sum_{s=n-\tau}^{n-1}\frac{1}{h_{\lambda_0}(s)}\widetilde{b}(s)\frac{H_{\lambda_0}(n)/H_{\lambda_0}(s)}{H_{\lambda_1}(n)/H_{\lambda_1}(s)}w_s \quad \text{for } n \ge 0. \end{aligned} \tag{3.25}$$

By taking into account the fact that the sequence $(h_{\lambda_0}(n))_{n\geq -r}$ is T-periodic and that $\sigma = \ell T$, we obtain for $n \geq 0$

$$
\begin{aligned}
\frac{H_{\lambda_0}(n-\sigma)}{H_{\lambda_0}(n)} &= \begin{cases} \frac{\prod_{k=0}^{n-\sigma-1} h_{\lambda_0}(k)}{\prod_{k=0}^{n-1} h_{\lambda_0}(k)}, & \text{if } n \geq \sigma \\ \frac{\left[\prod_{k=n-\sigma}^{-1} h_{\lambda_0}(k)\right]^{-1}}{\prod_{k=0}^{n-1} h_{\lambda_0}(k)}, & \text{if } 0 \leq n \leq \sigma \end{cases} \\
&= \begin{cases} \frac{1}{\left[\prod_{k=0}^{n-1} h_{\lambda_0}(k)\right]\left[\prod_{k=0}^{n-\sigma-1} h_{\lambda_0}(k)\right]^{-1}}, & \text{if } n \geq \sigma \\ \frac{1}{\left[\prod_{k=n-\sigma}^{-1} h_{\lambda_0}(k)\right]\left[\prod_{k=0}^{n-1} h_{\lambda_0}(k)\right]}, & \text{if } 0 \leq n \leq \sigma \end{cases} \\
&= \Big[\prod_{k=n-\sigma}^{n-1} h_{\lambda_0}(k)\Big]^{-1} = \Big[\prod_{k=0}^{\sigma-1} h_{\lambda_0}(k)\Big]^{-1} \\
&= \Big\{\Big[\prod_{k=0}^{T-1} h_{\lambda_0}(k)\Big]^{\ell}\Big\}^{-1} = \Big[\prod_{k=0}^{T-1} h_{\lambda_0}(k)\Big]^{-\ell} \\
&= \left(\lambda_0^T\right)^{-\ell} = \lambda_0^{-\ell T} = \lambda_0^{-\sigma}.
\end{aligned}
$$

That is,

$$
\frac{H_{\lambda_0}(n-\sigma)}{H_{\lambda_0}(n)} = \lambda_0^{-\sigma} \quad \text{for all } n \geq 0. \tag{3.26}
$$

In a similar way, by using the fact that the sequence $(h_{\lambda_1}(n))_{n\geq -r}$ is T-periodic and that $\sigma = \ell T$, we find

$$
\frac{H_{\lambda_1}(n-\sigma)}{H_{\lambda_1}(n)} = \lambda_1^{-\sigma} \quad \text{for all } n \geq 0. \tag{3.27}
$$

Furthermore, for any integers n and s with $n \geq 0$ and $-r \leq s \leq n-1$, we get

$$
\begin{aligned}
\frac{H_{\lambda_0}(n)}{H_{\lambda_0}(s)} &= \begin{cases} \frac{\prod_{k=0}^{n-1} h_{\lambda_0}(k)}{\prod_{k=0}^{s-1} h_{\lambda_0}(k)}, & \text{if } 0 \leq s \leq n-1 \\ \frac{\prod_{k=0}^{n-1} h_{\lambda_0}(k)}{\left[\prod_{k=s}^{-1} h_{\lambda_0}(k)\right]^{-1}}, & \text{if } -r \leq s \leq 0 \end{cases} \\
&= \begin{cases} \frac{\prod_{k=0}^{n-1} h_{\lambda_0}(k)}{\prod_{k=0}^{s-1} h_{\lambda_0}(k)}, & \text{if } 0 \leq s \leq n-1 \\ \left[\prod_{k=s}^{-1} h_{\lambda_0}(k)\right]\left[\prod_{k=0}^{n-1} h_{\lambda_0}(k)\right], & \text{if } -r \leq s \leq 0. \end{cases}
\end{aligned}
$$

So, we have

$$
\frac{H_{\lambda_0}(n)}{H_{\lambda_0}(s)} = \prod_{k=s}^{n-1} h_{\lambda_0}(k) \quad \text{for } n \geq 0 \text{ and } -r \leq s \leq n-1. \tag{3.28}
$$

Similarly,

$$
\frac{H_{\lambda_1}(n)}{H_{\lambda_1}(s)} = \prod_{k=s}^{n-1} h_{\lambda_1}(k) \quad \text{for } n \geq 0 \text{ and } -r \leq s \leq n-1. \tag{3.29}
$$

In view of (3.26), (3.27), (3.28) and (3.29), equation (3.25) is written in the following equivalent form

$$\begin{aligned} w_n + c\lambda_1^{-\sigma} w_{n-\sigma} = c\lambda_0^{-\sigma} &\sum_{s=n-\sigma}^{n-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \Big[\prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Big] w_s \\ &- \lambda_0^{-\tau} \sum_{s=n-\tau}^{n-1} \frac{1}{h_{\lambda_0}(s)} \widetilde{b}(s) \Big[\prod_{k=s}^{n-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Big] w_s \quad \text{for } n \ge 0. \end{aligned} \tag{3.30}$$

By the definitions of $(y_n)_{n\ge -r}$, $(z_n)_{n\ge -r}$ and $(w_n)_{n\ge -r}$, it follows immediately that

$$w_n = \frac{1}{H_{\lambda_1}(n)} \Big[x_n - \frac{M_{\lambda_0}(\phi)}{1+\beta_{\lambda_0}} H_{\lambda_0}(n)\Big] \quad \text{for } n \ge -r.$$

Thus, in view of initial condition (3.2) and because of the definition of $V_1(\lambda_0, \lambda_1; \phi)$ and $V_2(\lambda_0, \lambda_1; \phi)$, we see that what we have to prove is that $(w_n)_{n\ge -r}$ satisfies

$$\min_{s=-r,\dots,0} w_s \le w_n \le \max_{s=-r,\dots,0} w_s \quad \text{for all } n \ge 0.$$

We will restrict ourselves to showing the left hand part of the above double inequality. The right hand part of this double inequality can be established by an analogous procedure. So, it remains to prove that

$$w_n \ge \min_{s=-r,\dots,0} w_s \quad \text{for every } n \ge 0. \tag{3.31}$$

To prove (3.31), it suffices to show that, for any real number K with $K < \min_{s=-r,\dots,0} w_s$, it holds

$$w_n > K \quad \text{for all } n \ge 0. \tag{3.32}$$

Let us consider an arbitrary real number K with $K < \min_{s=-r,\dots,0} w_s$. Then

$$w_n > K \quad \text{for } n = -r, \dots, 0. \tag{3.33}$$

Assume, for the sake of contradiction, that (3.32) is not valid. Then, because of (3.33), there exists an integer $n_0 > 0$ so that

$$w_n > K \quad \text{for } n = -r, \dots, n_0 - 1, \text{ and } w_{n_0} \le K. \tag{3.34}$$

We notice that we have supposed that $c \le 0$. Also, it follows from (3.23) that

$$1 - \frac{1}{h_{\lambda_0}(n)} \le 0 \quad \text{for all } n \ge -r.$$

Furthermore, as the coefficient sequence $(b(n))_{n\ge 0}$ is T-periodic and $\tau = mT$, the assumption that $(b(n))_{n\ge 0}$ is not identically zero means that $(\widetilde{b}(n))_{n=n_0-\tau}^{n_0-1}$ is not identically zero, and the hypothesis that $(b(n))_{n\ge 0}$ is nonpositive means that $(\widetilde{b}(n))_{n=n_0-\tau}^{n_0-1}$ is nonpositive. Hence, $(\widetilde{b}(n))_{n=n_0-\tau}^{n_0-1}$ is nonpositive, but not identically zero. Now, by using (3.34) and taking into account the above observations, from (3.30) we obtain

$$\begin{aligned} K &\ge w_{n_0} \\ &= -c\lambda_1^{-\sigma} w_{n_0-\sigma} + c\lambda_0^{-\sigma} \sum_{s=n_0-\sigma}^{n_0-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \Big[\prod_{k=s}^{n_0-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Big] w_s \\ &\quad - \lambda_0^{-\tau} \sum_{s=n_0-\tau}^{n_0-1} \frac{1}{h_{\lambda_0}(s)} \widetilde{b}(s) \Big[\prod_{k=s}^{n_0-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)}\Big] w_s \end{aligned}$$

$$
\begin{aligned}
&> K\Big\{-c\lambda_1^{-\sigma} + c\lambda_0^{-\sigma} \sum_{s=n_0-\sigma}^{n_0-1} \Big[1 - \frac{1}{h_{\lambda_0}(s)}\Big] \prod_{k=s}^{n_0-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \\
&\quad - \lambda_0^{-\tau} \sum_{s=n_0-\tau}^{n_0-1} \frac{1}{h_{\lambda_0}(s)} \widetilde{b}(s) \prod_{k=s}^{n_0-1} \frac{h_{\lambda_0}(k)}{h_{\lambda_1}(k)} \Big\}.
\end{aligned}
$$

So, we can use (3.11) for $n = n_0$ to arrive at the contradiction $K > K$. This contradiction implies that (3.32) holds true. We have thus proved that (3.31) is always satisfied.

The proof of the theorem is complete. □

Before closing this section and ending the paper, let us concentrate on the following two special cases.

Consider the special case of the periodic linear (*non-neutral*) delay difference equation

$$\Delta x_n = a(n)x_n + b(n)x_{n-\tau}. \tag{3.35}$$

This equation can be obtained (as a particular case) from (3.1) *by taking* $c = 0$ *and considering the delay* σ *to be chosen arbitrarily so that* $\sigma \le \tau$ (for example, σ can be chosen to be equal to τ). In the case considered, we have the integer τ instead of r. As it concerns the (non-neutral) delay difference equation (3.35), the characteristic equation (3.3) becomes

$$\lambda^T = \prod_{k=0}^{T-1} \left[1 + a(k) + b(k)\lambda^{-\tau}\right]. \tag{3.36}$$

We observe that (3.4) holds by itself, for any positive root λ of the characteristic equation (3.36). Moreover, it is not difficult to see that, if λ is a positive root of the characteristic equation (3.36) with the property (3.5), then every positive root λ^* of (3.36) with $\lambda^* > \lambda$ has also the property (3.5) with λ^* instead of λ, provided that $(b(n))_{n\ge 0}$ is nonpositive. An application of Theorem 3.3 to the (non-neutral) delay difference equation (3.35) leads to [35, Theorem 3.4] in the authors' paper, *under the additional assumptions that the root* λ_0 *of the characteristic equation* (3.36) *satisfies* $a(n) + b(n)\lambda_0^{-\tau} \le 0$ *for* $n \ge 0$ (which implies that λ_0 is always less than or equal to 1) *and that* $1 + \lambda_0^{-\tau} \sum_{s=0}^{\tau-1} \frac{1}{h_{\lambda_0}(s)} b(s) \ne 0$. These additional assumptions are not necessary for the validity of [35, Theorem 3.4]. Thus, [35, Theorem 3.4] is not a corollary of the main result of this section, i.e., Theorem 3.3.

Finally, let us consider the special case of the *autonomous* linear neutral delay difference equation

$$\Delta(x_n + cx_{n-\sigma}) = ax_n + bx_{n-\tau}, \tag{3.37}$$

where c, a *and* $b \ne 0$ *are real constants, and* σ *and* τ *are positive integers.* The constant coefficients a and b of (3.37) can be considered as T-periodic sequences of real numbers with $T = 1$. The assumption that there exist positive integers ℓ and m such that $\sigma = \ell T$ and $\tau = mT$ holds by itself. The characteristic equation of (3.37) is

$$(\lambda - 1)\left(1 + c\lambda^{-\sigma}\right) = a + b\lambda^{-\tau}. \tag{3.38}$$

We can immediately see that the hypothesis that the root λ_0 of the characteristic equation (3.38) is such that (3.22) holds is equivalent to the hypothesis that $\lambda_0 \le 1$. By applying Theorem 3.3 to the special case of the autonomous linear neutral delay difference equation (3.37), we can easily be led to a particular case of a more general result obtained by the authors in [28] (for autonomous linear neutral delay difference equations *with several delays*), *under the restriction that the roots* λ_0 *and* λ_1 *of the characteristic equation* (3.38) *are such that* $1 + c\lambda_0^{-\sigma} \neq 0$ *and* $1 + c\lambda_1^{-\sigma} \neq 0$. It must be noted that this restriction can be removed; indeed, the corresponding more general result in [28] holds without such a restriction.

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