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MULTIPLICITY OF HIGH ENERGY SOLUTIONS FOR FRACTIONAL SCHRÖDINGER-POISSON SYSTEMS WITH CRITICAL FREQUENCY

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Abstract. In this article we study the fractional Schrödinger-Poisson system

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = \phi|u|^{2_s^*-3}u, \quad x \in \mathbb{R}^3,$$
$$(-\Delta)^s \phi = |u|^{2_s^*-1}, \quad x \in \mathbb{R}^3,$$

where $s \in (1/2, 1)$, $\varepsilon > 0$ is a parameter, $2_s^* = 6/(3-2s)$ is the critical Sobolev exponent, $V \in L^{\frac{3}{2s}}(\mathbb{R}^3)$ is a nonnegative function which may be zero in some region of $\mathbb{R}^3$. By means of variational methods, we present the number of high energy bound states with the topology of the zero set of V for small ε.

1. Introduction

In the past decades, the nonlinear Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + V(x)u + K(x)\phi u = f(x,u), \quad x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.1}$$

has been the interesting object for many researcher. As a model describing the interaction of a charge particle with an electromagnetic field, it arises in mathematical physics context [5], and is usually known as Schrödinger-Poisson system. In the pioneering paper [5], Benci and Fortunato studied the eigenvalue problem for (1.1) in bounded domain $\Omega \subset \mathbb{R}^3$ by using the variational methods. After that, the existence and multiplicity of solutions for Schrödinger-Poisson system (1.1) under variant assumptions on V, K and f, had been widely investigated by numerous authors and there have developed many effective methods to deal with equations or systems with nonlocal terms, we refer the interested readers to see [1, 2, 8, 24, 29, 37, 38] and the references therein.

For the Schrödinger-Poisson system with a nonlocal critical term

$$\begin{gathered} -\Delta u + V(x)u - K(x)\phi|u|^3 u = f(x,u), \quad x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)|u|^5, \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.2}$$

Liu [23] obtained the existence of positive solutions by using mountain pass theorem and the concentration-compactness principle. Li and He [25] studied the existence

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and multiplicity of positive solutions for (1.2) by using the variational methods. Azzollini, d'Avenia, Vaira [3] studied the existence and nonexistence results of positive and sign-changing solutions for (1.2) on bounded domains. Li, Li and Shi [19, 20] considered positive solutions to the another Schrödinger-Poisson-type systems with critically growing nonlocal term

$$\begin{gathered} -\Delta u + bu + q\phi|u|^3 u = f(u), \quad x \in \mathbb{R}^3, \\ -\Delta \phi = |u|^5, \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.3}$$

and obtained the existence of positive solutions to (1.3). Guo [15] obtained two positive bound state solutions of (1.3) with $b = 0, q = -1, f(u) = \lambda Q(x)|u|^{p-2}u$, a sublinear term, by decomposition the Nehari manifold and fine estimates.

In the setting of the fractional Laplacian, system (1.1), or (1.2) becomes the fractional Schrödinger-Poisson type system. It is a fundamental equation in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes [7, 17, 18]. By using a perturbation approach, Zhang, do Ó and Squassina [33] considered the existence and the asymptotical behaviors of positive solutions to the fractional Schrödinger-Poisson system

$$\begin{gathered} (-\Delta)^s u + V(x)u + K(x)\phi u = f(x,u), \quad x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = K(x)u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{1.4}$$

with $V(x) = 0$ and $K(x) = \lambda > 0$, a parameter, and a general subcritical or critical nonlinearity f. Teng [31] analyzed the existence of ground state solutions of (1.4) with $K(x) = 1$ and $f(x,u) = \mu|u|^{q-1}u + |u|^{2_s^*-2}u, q \in (2, 2_s^*)$, by combining Pohozaev-Nehari manifold, arguments of Brezis-Nirenberg type, the monotonicity trick and global compactness Lemma. Murcia and Siciliano [27] studied the semiclassical state of the system

$$\begin{gathered} \varepsilon^{2s}(-\Delta)^s u + V(x)u + K(x)\phi u = f(u), \quad x \in \mathbb{R}^N, \\ \varepsilon^{\theta}(-\Delta)^{\alpha/2}\phi = \gamma_\alpha u^2, \quad x \in \mathbb{R}^N. \end{gathered} \tag{1.5}$$

and established the multiplicity of positive solutions that concentrate on the minima of $V(x)$ as $\varepsilon \to 0$ by the Ljusternik-Schnirelmann category theory. Liu and Zhang [22] studied multiplicity and concentration of solutions of (1.5) when the nonlinearity f is critical growth. Recently, Yang, Yu and Zhao [32] considered the fractional Schrödinger-Poisson system with critical exponent

$$\begin{gathered} \varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) + |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^3, \\ \varepsilon^{2t}(-\Delta)^t \phi = u^2, \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.6}$$

where $f(u) = \lambda|u|^{p-2}u, \lambda > 0, \frac{4s+2t}{s+t} < p < 2_s^*$, and the potential V satisfies the following hypotheses introduced by del Pino and Felmer [11]:

(A1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) > 0$.

(A2) There exists a bounded open set $\Lambda \subset \mathbb{R}^3$ such that

$$V_0 := \inf_{\Lambda} V < \min_{\partial\Lambda} V \quad \text{and} \quad M = \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

Using penalization techniques and concentration-compactness principle, the authors obtained a positive ground state solution for $\varepsilon > 0$ small, and they showed that these ground state solutions concentrate around a local minimum of V as $\varepsilon \to 0$. Later, Ambrosio [4] obtained the multiplicity and concentration of positive solutions to

(1.6) by the penalization techniques and Ljusternik-Schnirelmann theory. Recently, Chen, Li and Peng [10] obtained multiple higher e nergy solutions of (1.6) by a global compactness lemma and Lusternik-Schnirelman theory, where $f(u) \equiv 0$. For more related results for (1.6), we refer to [14, 34] and references therein.

In the fractional scenario, (1.2) takes the form

$$
\begin{gathered}
(-\Delta)^s u + V(x)u - K(x)\phi|u|^{2_s^*-3}u = f(x,u), \quad x \in \mathbb{R}^3, \\
(-\Delta)^s \phi = K(x)|u|^{2_s^*-1}, \quad x \in \mathbb{R}^3,
\end{gathered} \tag{1.7}
$$

and we note that there are only two papers that deals with fractional Schrödinger-Poisson system with nonlocal critical term after a bibliography review. Feng [12] proved the existence of mountain pass type solution of (1.7) and extended the main results of [23] to the fractional Lapalcian case. He [16] considered the fractional Schrödinger-Poisson system with doubly critical exponents

$$
\begin{gathered}
(-\Delta)^s u + V(x)u - \phi|u|^{2_s^*-3}u = |u|^{2_s^*-2}u + f(u), \quad x \in \mathbb{R}^3, \\
(-\Delta)^s \phi = |u|^{2_s^*-1}, \quad x \in \mathbb{R}^3,
\end{gathered} \tag{1.8}
$$

and proved the existence of a mountain pass solution by employing the concentration-compactness principle and mountain pass theorem.

The novelty of (1.7) and (1.8) is that the second equation is nonlocal critical growth and driven by nonlocal operators, which make the study of problem (1.7) and (1.8) more interesting and challenging. As we observed that, the previous results on the existence and multiplicity of solutions for systems (1.7) and (1.8), were mainly focused on the existence of ground state solutions under the condition $(A1)$. The multiplicity of semiclassical states for (1.7) with both critical frequency $(V(x) \geq 0, \not\equiv 0, x \in \mathbb{R}^3)$ and critical growth has not been considered before. The purpose of this paper is to fill this gap. Concretely speaking, we study the fractional Schrödinger-Poisson system

$$
\begin{gathered}
\varepsilon^{2s}(-\Delta)^s u + V(x)u = \phi|u|^{2_s^*-3}u, \quad x \in \mathbb{R}^3, \\
(-\Delta)^s \phi = |u|^{2_s^*-1}, \quad x \in \mathbb{R}^3,
\end{gathered} \tag{1.9}
$$

where the potential $V(x)$ satisfies the assumption

- (A3) $V \in L^{\frac{3}{2s}}(\mathbb{R}^3), V(x) \geq 0$ on $\mathbb{R}^3$, and the set $M = \{x \in \mathbb{R}^3 : \ V(x) = 0\}$ is nonempty and bounded.

Recall that, if Y is a closed subset of a topological space X, the Lusternik-Schnirelmann category $\mathrm{cat}_X(Y)$ is the least number of closed and contractible sets in X, which cover Y. For any fixed $\tau > 0$. Denote $M_\tau = \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) \leq \tau\}$. Now, we state our main results.

Theorem 1.1. *Let* (A3) *be satisfied. Then, for any $\tau > 0$, there exist $\varepsilon_\tau > 0$ such that for any $\varepsilon \in (0, \varepsilon_\tau)$, then system* (1.9) *has at least* $\mathrm{cat}_{M_\tau}(M)$ *high energy semiclassical states in* $D^{s,2}(\mathbb{R}^3)$.

The proof of Theorem 1.1 is of variational. From the technical point of view, the appearance of the double non-localities from nonlocal critical convolution term and the nonlocal operator in system (1.9) make the bounded (PS) sequences could not converge. It is difficult for us to check the $(PS)_c$ condition since the nodal solutions of (1.9) do not possess the double energy characteristics, as we known the double energy property plays a key role in proving the main result in [10, 34]. To overcome these difficulties, we shall employ some idea from [13] and establish a new global compactness lemma in the fractional case, and some estimates become more subtle and delicate to be established. We shall construct two barycenter functions, and use Ljusternik-Schnirelmann category theory to obtain the desired results. As far as we know, the multiplicity of high energy solutions for system (1.9) has not been studied in the literature.

This article is organized as follows: In Section 2 we give some preliminary results. In Section 3, we introduce the limit problem and prove some useful lemmas. In Section 4, we give a global compactness lemma which describes the behavior of Palais-Smale sequences, and we regain the compactness if the functional energy lies in a suitable interval. In Section 5, we first define two barycenter functions and present some estimations; after these preparations, we complete the proof of Theorem 1.1 by means of the Ljusternik-Schnirelmann category theory.

2. Preliminaries

We recall that the fractional Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined as

$$D^{s,2}(\mathbb{R}^3) = \{u \in L^{2_s^*}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx < \infty\},$$

with the norm [26, 28]

$$\|u\|^2 := \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx = \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3-2s}}\, dx\, dy.$$

The embedding $D^{s,2}(\mathbb{R}^3) \to L^{2_s^*}(\mathbb{R}^3)$ is continuous and there exists a best constant $S > 0$ such that

$$S = \inf_{u \in D^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\right)^{2/2_s^*}}. \tag{2.1}$$

From the Lax-Milgram theorem, we have that, for given $u \in D^{s,2}(\mathbb{R}^3)$, there exists a unique solution $\phi = \phi_u \in D^{s,2}(\mathbb{R}^3)$ satisfying $(-\Delta)^s \phi_u = |u|^{2_s^*-1}$ in a weak sense. The function ϕ_u is represented by

$$\phi_u(x) = C_s \int_{\mathbb{R}^3} \frac{|u(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dy, \quad x \in \mathbb{R}^3,$$

where $C_s = \pi^{-\frac{3}{2}} 2^{-2s}\Gamma(3-2s)(\Gamma(s))^{-1}$, see [16, 12] for instance. Then function ϕ_u has the following properties.

Lemma 2.1. *For each $u \in D^{s,2}(\mathbb{R}^3)$, the function ϕ_u has the following properties:*

(i) *$\phi_{\theta u} = \theta^{2_s^*-1}\phi_u$ for all $\theta > 0$.*

(ii) *For each $u \in D^{s,2}(\mathbb{R}^3)$, one has $\|\phi_u\| \le S^{-1/2}|u|_{2_s^*}^{2_s^*-1}$ and*

$$\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \le S^{-1/2}\Big(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\Big)^{(2_s^*-1)/2_s^*} \|\phi_u\| \le S^{-1}|u|_{2_s^*}^{2(2_s^*-1)}.$$

(iii) *If $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^3)$, and $u_n \to u$ a.e. in $\mathbb{R}^3$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\mathbb{R}^3)$; and $\phi_{u_n} - \phi_{u_n-u} - \phi_u \to 0$ in $D^{s,2}(\mathbb{R}^3)$.*

(iv) *If $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^3)$, and $u_n \to u$ a.e. in $\mathbb{R}^3$, then*

$$\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} \phi_{u_n-u}|u_n-u|^{2_s^*-1}dx - \int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx \to 0, \tag{2.2}$$

and

$$\phi_{u_n}|u_n|^{2_s^*-3}u_n - \phi_{u_n-u}|u_n-u|^{2_s^*-3}(u_n-u) - \phi_u|u|^{2_s^*-3}u \rightharpoonup 0, \tag{2.3}$$

in $(D^{s,2}(\mathbb{R}^3))^*$, *where* $(D^{s,2}(\mathbb{R}^3))^*$ *is the dual space of* $D^{s,2}(\mathbb{R}^3)$.

Proof. Item (i) is obvious from the definition of ϕ_u.

(ii) For each $u \in D^{s,2}(\mathbb{R}^3)$, we can deduce that

$$\begin{aligned}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}\phi_u|^2 dx &= \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \\ &\le \Big(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\Big)^{(2_s^*-1)/2_s^*} \Big(\int_{\mathbb{R}^3} |\phi_u|^{2_s^*} dx\Big)^{1/2_s^*} \\ &\le S^{-1/2}\Big(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\Big)^{(2_s^*-1)/2_s^*} \|\phi_u\|,\end{aligned} \tag{2.4}$$

and so

$$\|\phi_u\| \le S^{-1/2}\Big(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\Big)^{(2_s^*-1)/2_s^*} = S^{-1/2}|u|_{2_s^*}^{2_s^*-1}.$$

Therefore,

$$\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \le S^{-1/2}\Big(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\Big)^{(2_s^*-1)/2_s^*} \|\phi_u\| \le S^{-1}|u|_{2_s^*}^{2(2_s^*-1)}.$$

(iii) From the Sobolev embedding, one has $u_n \rightharpoonup u \in L^{2_s^*}(\mathbb{R}^3)$. Then $|u_n|^{2_s^*-1} \rightharpoonup |u|^{2_s^*-1}$ in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$. Using Brezis-Lieb lemma [6], we have that

$$|u_n|^{2_s^*-1} - |u_n-u|^{2_s^*-1} - |u|^{2_s^*-1} \to 0 \quad \text{in } L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3). \tag{2.5}$$

Therefore, for any $v \in D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$, we obtain

$$(\phi_{u_n}, v) = \int_{\mathbb{R}^3} |u_n|^{2_s^*-1} v\, dx \to \int_{\mathbb{R}^3} |u|^{2_s^*-1} v\, dx = (\phi_u, v),$$

which reveals that $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\mathbb{R}^3)$. Since for every $w \in D^{s,2}(\mathbb{R}^3)$.

$$\begin{aligned}|\langle \phi_{u_n} - \phi_{v_n} - \phi_u, w\rangle| &= \Big|\int_{\mathbb{R}^3} w(|u_n|^{2_s^*-1} - |v_n|^{2_s^*-1} - |u|^{2_s^*-1})\Big| \\ &\le |w|_{2_s^*}\big||u_n|^{2_s^*-1} - |v_n|^{2_s^*-1} - |u|^{2_s^*-1}\big|_{2_s^*/(2_s^*-1)},\end{aligned}$$

then $\phi_{u_n} - \phi_{u_n-u} - \phi_u \to 0$ in $D^{s,2}(\mathbb{R}^3)$, which implies the assertion.

(iv) Set $v_n = u_n - u$, then $v_n \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ and $v_n \to 0$ a.e. in $\mathbb{R}^3$. By item (iii) we have $\phi_{v_n} \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^3)$. Since $u_n \to u$ a.e. in $\mathbb{R}^3$, and $u_n \rightharpoonup u \in L^{2_s^*}(\mathbb{R}^3)$, we infer that $|u_n|^{2_s^*-1} \rightharpoonup |u|^{2_s^*-1}$ in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$, and so, (2.5) holds. Consequently, by the weak convergence of $\{u_n\}$ and Hölder inequality, as $n \to \infty$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx \\ &= \int_{\mathbb{R}^3} [\phi_{u_n} - \phi_{v_n} - \phi_u]|u_n|^{2_s^*-1}dx + \int_{\mathbb{R}^3} \phi_{v_n}[|u_n|^{2_s^*-1} - |u_n-u|^{2_s^*-1} - |u|^{2_s^*-1}]dx \\ &\quad + \int_{\mathbb{R}^3} \phi_{v_n}|u|^{2_s^*-1}dx + \int_{\mathbb{R}^3} \phi_u[|u_n|^{2_s^*-1} - |u|^{2_s^*-1}]dx \to 0,\end{aligned}$$

which implies (2.2). To prove (2.3), we have by item (iii) that $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\mathbb{R}^3)$, which yields $\phi_{u_n} \rightharpoonup \phi_u$ in $L^{2_s^*}(\mathbb{R}^3)$. On the other hand, by virtue of $u_n \to u$ a.e. in $\mathbb{R}^3$ and

$$\int_{\mathbb{R}^3} |\phi_{u_n}|u_n|^{2_s^*-3}u_n|^{\frac{2_s^*}{2_s^*-1}} dx \le \Big(\int_{\mathbb{R}^3} |\phi_{u_n}|^{2_s^*} dx\Big)^{\frac{1}{2_s^*-1}} \Big(\int_{\mathbb{R}^3} |u_n|^{2_x^*} dx\Big)^{\frac{2_s^*-2}{2_s^*-1}} dx \le C.$$

From [36, Proposition 5.4.7], we see that

$$\phi_{u_n}|u_n|^{2_s^*-3}u_n \rightharpoonup \phi_u|u|^{2_s^*-3}u, \phi_{v_n}|v_n|^{2_s^*-3}v_n \rightharpoonup 0$$

in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$. Hence, for any $\varphi \in D^{s,2}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} [\phi_{u_n}|u_n|^{2_s^*-3}u_n - \phi_{u_n-u}|u_n-u|^{2_s^*-3}(u_n-u)]\varphi\, dx \to \int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-3}u\varphi dx,$$

and (2.3) follows. □

Lemma 2.2. *If $N \ge 3$, and $V(\in L^{\frac{N}{2s}}(\mathbb{R}^N)$, then the functional*

$$\mathcal{H}(u) = \int_{\mathbb{R}^N} V(x)u^2 dx$$

is weakly continuous in $D^{s,2}(\mathbb{R}^N)$.

Proof. It is similar to the proof of [35, Lemma 2.13], we sketch it here for convenience. The functional $\mathcal{H}$ is well defined by the Sobolev space and Hölder inequalities. Assume that $u_n \rightharpoonup u$ weakly in $D^{s,2}(\mathbb{R}^N)$. Since the Sobolev embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous, we have $u_n \rightharpoonup u$ weakly in $L^{2_s^*}(\mathbb{R}^N)$, and so, $u_n^2 \rightharpoonup u^2$ weakly in $L^{\frac{N}{N-2s}}(\mathbb{R}^N)$. Note that $V \in L^{\frac{N}{2s}}(\mathbb{R}^N) = (L^{\frac{N}{N-2s}}(\mathbb{R}^N))^*$, thus,

$$\int_{\mathbb{R}^N} V(x)u^2 dx \to \int_{\mathbb{R}^N} V(x)u^2 dx \quad \text{as } n \to \infty,$$

which implies the conclusion. □

Finally, we recall that the Hardy-Littlewood-Sobolev inequality.

Proposition 2.3 ([21])**.** *Let $t, r > 1$ and $0 < \alpha < n$ with $1/t + \alpha/n + 1/r = 2$, $f \in L^t(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. There exists a sharp constant $C(t, n, \alpha, r)$ independent of f, h such that*

$$\iint_{\mathbb{R}^{2n}} \frac{f(x)h(y)}{|x-y|^\alpha}\, dx\, dy \le C(t, n, \alpha, r)|f|_t|h|_r. \tag{2.6}$$

If $t = r = \frac{2N}{2N-\alpha}$, then

$$C(t, N, \alpha, r) = C(N, \alpha) = \pi^{\alpha/2} \frac{\Gamma(\frac{\pi}{2} - \frac{\alpha}{2})}{\Gamma(N - \frac{\alpha}{2})} \Big\{\frac{\Gamma(\frac{\pi}{2})}{\Gamma(N)}\Big\}^{-1+\frac{\alpha}{N}}.$$

In this case there is equality in (2.6) *if and only if* $f \equiv (\text{constant})h$ *and*

$$h(x) = A(\gamma^2 + |x-a|^2)^{-\frac{2N-\alpha}{2}}$$

for some $A \in \mathbb{C}$, $0 \ne \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

3. Limit problem

To study (1.9), we need to introduce the limit equation. First, we introduce the system

$$\begin{gathered}(-\Delta)^s u=\phi|u|^{2_s^*-3}u, \quad x\in\mathbb{R}^3,\\ (-\Delta)^s\phi=|u|^{2_s^*-1}, \quad x\in\mathbb{R}^3.\end{gathered}\tag{3.1}$$

It is well known that the best embedding constant S is achieved at the function [30]:

$$U(x)=\frac{\beta_1}{(1+|x|^2)^{(3-2s)/2}}, \quad \beta_1=\Big(\frac{S^{3/(2s)}\Gamma(3)}{\pi^{3/2}\Gamma(3/2)}\Big)^{\frac{3-2s}{6}},$$

and $U(x)$ is a ground state solution of the equation

$$(-\Delta)^s u=|u|^{2_s^*-2}u, \quad x\in\mathbb{R}^3.\tag{3.2}$$

Moreover, by the invariance of scaling and translation, we known that the function

$$U_{\delta,z_0}(x):=\delta^{-\frac{3-2s}{2}}U\big(\frac{x-z_0}{\delta}\big)$$

solves (3.2), and satisfies

$$\|U_{\delta,z_0}\|^2=|U_{\delta,z_0}|_{2_s^*}^{2_s^*}=S^{\frac{3}{2s}}.\tag{3.3}$$

The following observation is useful in the energy estimation of the functionals below.

Lemma 3.1. *Assume that u,ϕ are positive solutions of* (3.1)*, then*

$$u(x)=\phi(x)=U_{\delta,z_0}(x) \quad \textit{for some } z_0\in\mathbb{R}^3,\ \textit{and } \delta>0.$$

Proof. Let u and ϕ be a pair of positive solution to (3.1). Then we have

$$(-\Delta)^s(u-\phi)=(\phi-u)|u|^{2_s^*-2},\ x\in\mathbb{R}^3.$$

Multiplying this equation by $u-\phi$, and integrating by part, we obtain

$$\int_{\mathbb{R}^3}|(-\Delta)^{s/2}(u-\phi)|^2dx+\int_{\mathbb{R}^3}(u-\phi)^2|u|^{2_s^*-2}dx=0.$$

Whence, we can conclude $u=\phi$. Furthermore, u satisfies (3.1), and the conclusion follows. □

The functional of (1.9) is

$$I_\varepsilon(u)=\frac{\varepsilon^{2s}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx+\frac12\int_{\mathbb{R}^3}V(x)u^2dx-\frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx,$$

and introduce the limit equation of (1.9) as

$$\begin{gathered}\varepsilon^{2s}(-\Delta)^s u=\phi|u|^{2_s^*-3}u, \quad x\in\mathbb{R}^3,\\ (-\Delta)^s\phi=|u|^{2_s^*-1}, \quad x\in\mathbb{R}^3.\end{gathered}\tag{3.4}$$

whose energy functional is denoted by

$$J_\varepsilon(u)=\frac{\varepsilon^{2s}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx-\frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx.$$

By Lemma 3.1, it is easy to see that $u=\varepsilon^{\frac{3-2s}{4}}U_{\delta,x_0}$ is the ground state of (3.4) and

$$J_\varepsilon(u)=\frac{\varepsilon^{\frac{3+2s}{2}}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}U_{\delta,x_0}|^2dx-\frac{\varepsilon^{\frac{3+2s}{2}}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{U_{\delta,x_0}}|U_{\delta,x_0}|^{2_s^*-1}dx$$

$$
\begin{aligned}
&= \varepsilon^{\frac{3+2s}{2}}\Big(\frac{1}{2}-\frac{1}{2(2_s^*-1)}\Big)S^{\frac{3}{2s}}\\
&= \frac{2s}{3+2s}\varepsilon^{\frac{3+2s}{2}}S^{\frac{3}{2s}}.
\end{aligned}
$$

For each $\varepsilon > 0$, we define the Nehari manifolds of $I_\varepsilon, J_\varepsilon$ as follows

$$
\begin{gathered}
N_\varepsilon = \{u \in D^{s,2}(\mathbb{R}^3)\backslash\{0\} : I_\varepsilon'(u)u = 0\},\\
N_\varepsilon^\infty = \{u \in D^{s,2}(\mathbb{R}^3)\backslash\{0\} : J_\varepsilon'(u)u = 0\},\\
m_\varepsilon := \inf_{u\in N_\varepsilon} I_\varepsilon(u), \quad m_\varepsilon^\infty := \inf_{u\in N_\varepsilon^\infty} J_\varepsilon(u).
\end{gathered}
$$

Lemma 3.2. *Let* (A3) *hold. Then* $m_\varepsilon = \frac{2s}{3+2s}\varepsilon^{\frac{3+2s}{2}}S^{\frac{3}{2s}}$.

Proof. For any $u \in N_\varepsilon$, by Lemma 2.1, we have

$$
\begin{aligned}
\varepsilon^{2s}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx &\le \varepsilon^{2s}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx + \int_{\mathbb{R}^3}V(x)u^2dx\\
&= \int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx \le S^{-1}|u|_{2_s^*}^{2(2_s^*-1)}\\
&\le S^{-2_s^*}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)^{2_s^*-1},
\end{aligned}
\tag{3.5}
$$

which implies that

$$
\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx = \|u\|^2 + \int_{\mathbb{R}^3}V(x)u^2dx \ge \|u\|^2 \ge \varepsilon^{\frac{3-2s}{2}}S^{\frac{3}{2s}}. \tag{3.6}
$$

Hence,

$$
\begin{aligned}
I_\varepsilon(u) &= I_\varepsilon(u) - \frac{1}{2(2_s^*-1)}I_\varepsilon'(u)u\\
&= \frac{2s}{3+2s}\varepsilon^{2s}\|u\|^2 + \frac{2s}{3+2s}\int_{\mathbb{R}^3}V(x)u^2dx\\
&\ge \frac{2s}{3+2s}\varepsilon^{\frac{3+2s}{2}}S^{\frac{3}{2s}},
\end{aligned}
\tag{3.7}
$$

which implies that

$$
m_\varepsilon \ge \frac{2s}{3+2s}\varepsilon^{\frac{3+2s}{2}}S^{\frac{3}{2s}}.
$$

Now, consider $v_n(x) = \varepsilon^{\frac{3-2s}{4}}w(x-z_n) \in N_\varepsilon^\infty$, where w is a positive solution of (3.1) centered at zero, and $z_n \in \mathbb{R}^3$ satisfies $|z_n| \to \infty$. Let $t_n > 0$ be such that $t_nv_n \in N_\varepsilon$. Using $v_n \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^3)$, and Lemma 2.2, we have

$$
\int_{\mathbb{R}^3}V(x)v_n^2dx \to 0.
$$

Then $t_n \to 1$. Hence,

$$
m_\varepsilon \le I_\varepsilon(t_nv_n) = I_\varepsilon(v_n) + o_n(1) = J_\varepsilon(v_n) + o_n(1) = \frac{2s}{3+2s}\varepsilon^{\frac{3+2s}{2}}S^{\frac{3}{2s}} + o_n(1),
$$

which implies that $m_\varepsilon = \frac{2s}{3+2s}\varepsilon^{\frac{3+2s}{2}}S^{\frac{3}{2s}}$. □

Lemma 3.3. *Let* (A3) *hold. Then* $m_\varepsilon = m_\varepsilon^\infty$ *holds, and* m_ε *is not attained.*

Proof. For $u \in D^{s,2}(\mathbb{R}^N)$, we define the function

$$
\begin{aligned}
&\gamma(t)\\
&= \langle I'_\varepsilon(tu), tu\rangle\\
&= t^2\varepsilon^{2s}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx + t^2\int_{\mathbb{R}^3}V(x)u^2dx - t^{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx\\
&= at^2 - bt^{22^*_{\alpha,s}},
\end{aligned} \tag{3.8}
$$

where

$$
a = \int_{\mathbb{R}^3}(\varepsilon^{2s}|(-\Delta)^{s/2}u|^2 + V(x)u^2)dx, \quad b = \int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx.
$$

It is easy to see that there exist unique $t(u) > 0$ and $s(u) > 0$ such that $t(u)u \in N_\varepsilon, s(u)u \in N_\varepsilon^\infty$, and

$$
I_\varepsilon(t(u)u) = \max_{t>0} I_\varepsilon(tu), \quad J_\varepsilon(s(u)u) = \max_{t>0} J_\varepsilon(tu).
$$

For any $u \in N_\varepsilon$,

$$
\begin{aligned}
m_\infty &\le I_\infty(s(u)u)\\
&\le \frac{\varepsilon^{2s}}{2}\|s(u)u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V(x)|s(u)u|^2dx - \frac{|s(u)|^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx\\
&= I_\varepsilon(s(u)u) \le I_\varepsilon(u),
\end{aligned} \tag{3.9}
$$

which implies that

$$
m_\varepsilon^\infty \le m_\varepsilon. \tag{3.10}
$$

Assume that w is a positive solution of (3.4) centered at origin, $\{z_n\} \subset \mathbb{R}^N$ satisfying $|z_n| \to \infty$ as $n \to \infty$, $w_n(x) = \varepsilon^{\frac{3-2s}{4}}w(x - z_n)$ and $t_n = t(w_n)$. It is clear that $w_n \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$, and by Lemma 2.2, one has

$$
\int_{\mathbb{R}^3}V(x)w_n^2dx \to 0. \tag{3.11}
$$

Now, we have that

$$
I_\varepsilon(t_nw_n) = \frac{t_n^2\varepsilon^{2s}}{2}\|w_n\|^2 + \frac{t_n^2}{2}o_n(1) - \frac{|t_n|^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{w_n}|w_n|^{2_s^*-1}dx. \tag{3.12}
$$

Noting that $w_n \in N_\varepsilon^\infty$ and $t_nw_n \in N_\varepsilon$, we obtain

$$
\varepsilon^{2s}\|w_n\|^2 = \int_{\mathbb{R}^3}\phi_{w_n}|w_n|^{2_s^*-1}dx, \tag{3.13}
$$

$$
\varepsilon^{2s}t_n^2\|w_n\|^2 + t_n^2\int_{\mathbb{R}^3}V(x)w_n^2dx = t^{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{w_n}|w_n|^{2_s^*-1}dx. \tag{3.14}
$$

By (3.13) and (3.14) we have

$$
\varepsilon^{2s}(t_n^{2(2_s^*-2)} - 1)\|w_n\|^2 = \int_{\mathbb{R}^3}V(x)w_n^2dx = o_n(1), \tag{3.15}
$$

which means that $t_n \to 1$ as $n \to \infty$. By (3.12), we see that $\lim_{n\to\infty} I_\varepsilon(u_n) = m_\varepsilon^\infty$. Therefore, we obtain $m_\varepsilon \le m_\varepsilon^\infty$, and so $m_\varepsilon = m_\varepsilon^\infty$.

Next, we prove that m_ε cannot be achieved. Suppose by contradiction that, there exists some $\widetilde{u} \in N_\varepsilon$ such that $I_\varepsilon(\widetilde{u}) = m_\varepsilon = m_\varepsilon^\infty$. Then, from $s(\widetilde{u})\widetilde{u} \in N_\varepsilon^\infty$, we have

$$
m_\varepsilon^\infty \le J_\varepsilon(s(\widetilde{u})\widetilde{u})
$$

$$
\begin{aligned}
&\le \frac{\varepsilon^{2s}}{2}\|s(\widetilde{u})\widetilde{u}\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|s(\widetilde{u})\widetilde{u}|^2 dx - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^N} \phi_{s(\widetilde{u})\widetilde{u}}|s(\widetilde{u})\widetilde{u}|^{2_s^*-1}dx \\
&= I(s(\widetilde{u})\widetilde{u}) \\
&\le I(\widetilde{u}) = m_\varepsilon^\infty.
\end{aligned}
$$

We infer that

$$
\int_{\mathbb{R}^N} V(x)|s(\widetilde{u})\widetilde{u}|^2 dx = 0,\ s(\widetilde{u}) = 1 \Rightarrow \widetilde{u} \equiv 0 \quad \text{on } \mathbb{R}^3\backslash M.
$$

Hence, $\widetilde{u} \in N_\varepsilon^\infty, J_\varepsilon(\widetilde{u}) = m_\varepsilon^\infty$. Thus, by Lemma 3.1, $\widetilde{u}(x) = \varepsilon^{\frac{3-2s}{4}} U_{\delta,z_0}(x) > 0, \forall x \in \mathbb{R}^3$, for some $\delta > 0$, $z_0 \in \mathbb{R}^3$, which leads to a contradicts to $\widetilde{u}(x) \equiv 0$ on $\mathbb{R}^3\backslash M$. $\square$

Corollary 3.4. *Assume that $\{u_n\}$ is a Palais-Smale sequence of I_ε constrained on N_ε, then $\{u_n\}$ is a Palais-Smale sequence of I_ε. If u is a critical point of I_ε constrained on N_ε, then u must be a critical point of I_ε.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence of I_ε constrained on N_ε, then there exists $\lambda_n \in \mathbb{R}$ such that

$$
o_n(1) = I'_\varepsilon(u_n) - \lambda_n Q'_\varepsilon(u_n),
$$

where $Q_\varepsilon(u) = I'_\varepsilon(u)u$. Note that $\{u_n\}$ is bounded in $D^{s,2}(\mathbb{R}^3)$. Thus, one has

$$
o_n(1) = I'_\varepsilon(u_n)u_n - \lambda_n Q'_\varepsilon(u_n)u_n = -\lambda_n Q'_\varepsilon(u_n)u_n,
$$

On the other hand, since $2_s^* > 2$, then by (3.6), we derive that

$$
\begin{aligned}
&Q'_\varepsilon(u_n)u_n \\
&= 2\varepsilon^{2s}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx + 2\int_{\mathbb{R}^3} V(x)u_n^2 dx - 2(2_s^*-1)\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-1}dx \\
&= [2(2-2_s^*)]\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-1}dx \\
&\le [2(2-2_s^*)]\varepsilon^{\frac{3-2s}{2}} S^{\frac{3}{2s}} < 0.
\end{aligned}
\tag{3.16}
$$

Thus, $\lambda_n \to 0$ as $n \to \infty$. Moreover, by the boundedness of $\{u_n\}$, we assert that $\{Q'_\varepsilon(u_n)\}$ is bounded. Hence, $I'_\varepsilon(u_n) \to 0$ as $n \to \infty$. If u is a critical point of I_ε constrained on N_ε, then there exists $\lambda \in \mathbb{R}$ such that $I'_\varepsilon(u) = \lambda Q'_\varepsilon(u)$. Therefore,

$$
0 = Q_\varepsilon(u) = I'_\varepsilon(u)u - \lambda Q'_\varepsilon(u)u.
$$

By the same calculation as for (3.16), one finds that

$$
Q'_\varepsilon(u)u \le [2(2-2_s^*)]\varepsilon^{\frac{3-2s}{2}} S^{\frac{3}{2s}}.
$$

Thus, $\lambda = 0$, and then, $I'_\varepsilon(u) = 0$. $\square$

4. Global compactness

The following global compactness lemma plays a key role in proving the compactness of the (PS) sequences.

Lemma 4.1. *Under condition* (A3)*, for each $\varepsilon > 0$, suppose that $\{u_n\} \subset D^{s,2}(\mathbb{R}^3)$ is a Palais-Smale sequence of I_ε at level c. Then, replacing u_n if necessary, with a subsequence, there exist a number $k \in \mathbb{N}$, sequences of points $x_n^1, \dots, x_n^k \in \mathbb{R}^3$ and radii $r_n^1, \dots, r_n^k$ such that:*

(1) $u_n^0 \equiv u_n \rightharpoonup u^0$ *in* $D^{s,2}(\mathbb{R}^3)$*;*
(2) $u_n^j \equiv (u_n^{j-1} - u^{j-1})_{r_n^j, x_n^j} \rightharpoonup u^j$ *in* $D^{s,2}(\mathbb{R}^3)$*,* $j = 1, 2, \dots, k$*;*
(3) $\|u_n\|^2 \to \sum_{j=0}^k \|u^j\|^2$*;*
(4) $I_\varepsilon(u_n) \to I_\varepsilon(u^0) + \sum_{j=1}^k J_\varepsilon(u^j)$*,*

as $n \to \infty$ *where* u^0 *is a solution of* (1.9) *and* $u^j, 1 \le j \le k$*, are the nontrivial solutions of* (3.4)*. Moreover, we agree that in the case* $k = 0$ *the above holds without* u^j*.*

Proof. Note that $\{u_n\}$ is a $(PS)_c$ sequence for I_ε; then, we can prove that $\{u_n\}$ is bounded in $D^{s,2}(\mathbb{R}^3)$. Without loss of generality, we may assume that $u_n \rightharpoonup u_0$ in $D^{s,2}(\mathbb{R}^3)$ as $n \to \infty, u_n \to u_0$ a.e. in $\mathbb{R}^3$. Moreover, $I'_\varepsilon(u^0) = 0$. In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, we obtain

$$I'_\varepsilon(u_n)\varphi = \varepsilon^{2s} \int_{\mathbb{R}^3} (-\Delta)^{s/2} u_n (-\Delta)^{s/2} \varphi dx + \int_{\mathbb{R}^3} V(x) u_n \varphi dx - \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-3} u_n \varphi.$$

From Lemma 2.1 we can see that $\phi_{u_n} \rightharpoonup \phi_{u^0}$ in $D^{s,2}(\mathbb{R}^3)$ and so $\phi_{u_n} \rightharpoonup \phi_{u^0}$ in $L^{2_s^*}(\mathbb{R}^3)$. Therefore,

$$\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_{u^0}) |u^0|^{2_s^*-3} u^0 \varphi \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

Since $u_n \to u$ a.e. in $\mathbb{R}^3$ and by Hölder inequality, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} \left|\phi_{u_n}(|u_n|^{2_s^*-3} u_n - |u^0|^{2_s^*-3} u^0)\right|^{\frac{2_s^*}{2_s^*-1}} dx \\ &\le C\Big(|\phi_{u_n}|_{2_s^*}^{\frac{2_s^*}{2_s^*-1}} |u_n|_{2_s^*}^{\frac{2_s^*(2_s^*-2)}{2_s^*-1}} + |\phi_{u_n}|_{2_s^*}^{\frac{2_s^*}{2_s^*-1}} |u^0|_{2_s^*}^{\frac{2_s^*(2_s^*-2)}{2_s^*-1}}\Big) \le C, \end{aligned} \tag{4.2}$$

and so $\phi_{u_n}(|u_n|^{2_s^*-3} u_n - |u^0|^{2_s^*-3} u) \rightharpoonup 0$ in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$, and thus

$$\int_{\mathbb{R}^3} \phi_{u_n}(|u_n|^{2_s^*-3} u_n - |u^0|^{2_s^*-3} u^0) \varphi dx \to 0 \quad \text{as } n \to \infty, \tag{4.3}$$

which together with (4.1) implies

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-3} u_n \varphi \, dx \to \int_{\mathbb{R}^3} \phi_{u^0} |u^0|^{2_s^*-3} u \varphi \, dx \quad \text{as } n \to \infty. \tag{4.4}$$

By (4.4) and the weak convergence of $u_n \rightharpoonup u^0$ in $D^{s,2}(\mathbb{R}^3)$, we have

$$I'_\varepsilon(u^0)\varphi = \lim_{n\to\infty} I'_\varepsilon(u_n)\varphi = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3). \tag{4.5}$$

Therefore, $I'_\varepsilon(u^0) = 0$ and u is a critical point of I_ε by density of $C_0^\infty(\mathbb{R}^3)$ in $D^{s,2}(\mathbb{R}^3)$. Let

$$v_n^1(x) = u_n(x) - u^0(x).$$

By Brezis-Lieb Lemma [6] and Lemma 2.1 (iv), one can easily obtain

$$\|v_n^1\|^2 = \|u_n^1\|^2 - \|u^0\|^2 + o_n(1), \tag{4.6}$$

$$I_\varepsilon(v_n^1) = I_\varepsilon(u_n^1) - I_\varepsilon(u^0) + o_n(1), \tag{4.7}$$

$$I'_\varepsilon(v_n^1) = I'_\varepsilon(u_n^1) - I'_\varepsilon(u^0) + o_n(1). \tag{4.8}$$

Note that $v_n^1 \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^3)$. Then, by Lemma 2.2,

$$\int_{\mathbb{R}^3} V(x)|v_n^1|^2 dx = o_n(1), \ \int_{\mathbb{R}^3} V(x) v_n^1 \varphi dx = o_n(1)\|\varphi\|,$$

for each $\varphi \in D^{s,2}(\mathbb{R}^3)$. Therefore,

$$\begin{gathered} J_\varepsilon(v_n^1) = I_\varepsilon(v_n^1) + o_n(1) = I_\varepsilon(u_n) - I_\varepsilon(u) + o_n(1), \\ J'_\varepsilon(v_n^1) = I'_\varepsilon(v_n^1) + o_n(1) = o_n(1). \end{gathered} \tag{4.9}$$

If $v_n^1 \to 0$ in $D^{s,2}(\mathbb{R}^3)$, then the proof is complete. If $v_n^1 \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^3)$, then there exists $\zeta > 0$ such that

$$J_\varepsilon(v_n^1) > \zeta > 0. \tag{4.10}$$

We assert that there exist two sequences $\{r_n\} \subset \mathbb{R}^+$ and $\{y_n\} \subset \mathbb{R}^3$ such that

$$g_n = (v_n^1)_{r_n,y_n} \rightharpoonup g \neq 0, \quad \text{in } D^{s,2}(\mathbb{R}^3), \tag{4.11}$$

where $(v_n^1)_{r_n,y_n} = r_n^{\frac{3-2s}{2}} v_n^1(r_n x + y_n)$. In fact, by (4.9) we have

$$\varepsilon^{2s}\|v_n^1\|^2 = \int_{\mathbb{R}^3} \phi_{v_n^1}|v_n^1|^{2_s^*-1}dx + o_n(1), \tag{4.12}$$

$$J_\varepsilon(v_n^1) = \frac{2s}{3+2s}\int_{\mathbb{R}^3} \phi_{v_n^1}|v_n^1|^{2_s^*-1}dx + o_n(1). \tag{4.13}$$

From (4.10), Lemma 2.1(ii) and the boundedness of $\{u_n\}$, we derive that

$$0 < d_1 < |v_n^1|_{2_s^*}^{2_s^*-1} < D_1, \tag{4.14}$$

for some $d_1, D_1 > 0$. We introductive a Lévy concentration function

$$Q_n(r) := \sup_{x\in\mathbb{R}^3}\int_{B_r(z)} |v_n^1|^{2_s^*}dx.$$

Since $Q_n(0) = 0$ and $Q_n(\infty) > d_1^{\frac{6}{3+2s}}$, we can assume that there exist sequences $\{r_n\} \subset \mathbb{R}^+$ and $\{y_n\} \subset \mathbb{R}^3$ such that

$$\sup_{x\in\mathbb{R}^3}\int_{B_r(z)} |v_n^1|^{2_s^*}dx = \int_{B_{r_n}(y_n)} |v_n^1|^{2_s^*}dx = b,$$

where

$$0 < b < \min\Big\{d_1^{\frac{6}{3+2s}}, \Big(\frac{S}{2C_sD_1}\Big)^{\frac{6}{6s-3}}\Big\}.$$

We define $g_n = (v_n^1)_{r_n,y_n}$, without loss of generality, we may suppose that $g_n \rightharpoonup g$ in $D^{s,2}(\mathbb{R}^3)$, and $g_n \to g$ a.e. in $\mathbb{R}^3$. Direct calculations show that

$$\sup_{z\in\mathbb{R}^3}\int_{B_1(z)} |g_n(x)|^{2_s^*}dx = \int_{B_1(0)} |g_n(x)|^{2_s^*}dx = \int_{B_{r_n}(y_n)} |v_n^1|^{2_s^*}dx = b, \tag{4.15}$$

$$\|v_n^1\|^2 = \|g_n\|^2, \quad |v_n^1|_{2_s^*} = |g_n|_{2_s^*}, \tag{4.16}$$

$$\int_{\mathbb{R}^3} \phi_{g_n}|g_n|^{2_s^*-1}dx = \int_{\mathbb{R}^3} \phi_{v_n^1}|v_n^1|^{2_s^*-1}dx\,. \tag{4.17}$$

Based on the above two properties, we have

$$J_\varepsilon(g_n) = J_\varepsilon(v_n^1) = J_\varepsilon(u_n) - J_\varepsilon(u^0) + o_n(1), \tag{4.18}$$

$$J'_\varepsilon(g_n) = J'_\varepsilon(v_n^1) = o_n(1). \tag{4.19}$$

If $g = 0$, then $g_n \to 0$ in $L^2_{\rm loc}(\mathbb{R}^3)$. Assume that $\psi \in C_0^\infty(\mathbb{R}^3)$ satisfying supp$\psi \subset B_1(y^*)$ for some $y^* \in \mathbb{R}^3$, and $\nabla\psi(x)| \le C$ for $x \in \mathbb{R}^3$. Note that

$$
\begin{aligned}
\|\psi g_n\|^2 &= \iint_{\mathbb{R}^6} \frac{|\psi(x)g_n(x) - \psi(y)g_n(y)|^2}{|x-y|^{3+2s}}\,dx\,dy \\
&= \iint_{\mathbb{R}^6} \frac{(g_n(x)-g_n(y))(\psi^2(x)g_n(x) - \psi^2(y)g_n(y))}{|x-y|^{3+2s}}\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 g_n(x)g_n(y)}{|x-y|^{3+2s}}\,dx\,dy.
\end{aligned}
\tag{4.20}
$$

By a direct computation, we have

$$
\begin{aligned}
&\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 g_n(x)g_n(y)}{|x-y|^{3+2s}}\,dx\,dy \\
&\le \Big(\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 g_n^2(x)}{|x-y|^{3+2s}}\,dx\,dy\Big)^{1/2} \\
&\quad \times \Big(\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 g_n^2(x)}{|x-y|^{3+2s}}\,dx\,dy\Big)^{1/2}.
\end{aligned}
\tag{4.21}
$$

We next show that

$$
\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 g_n^2(x)}{|x-y|^{3+2s}}\,dx\,dy = \iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 g_n^2(y)}{|x-y|^{3+2s}}\,dx\,dy = o_n(1).
$$

In fact, we have

$$
\begin{aligned}
&\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 g_n^2(x)}{|x-y|^{3+2s}}\,dx\,dy \\
&= \int_{B_1(y^*)}\int_{B_1(y^*)} \frac{|\psi(x)-\psi(y)|^2 g_n^2(x)}{|x-y|^{3+2s}}\,dx\,dy \\
&\quad + 2\int_{B_1(y^*)}\int_{B_1^c(y^*)} \frac{|\psi(x)-\psi(y)|^2 g_n^2(x)}{|x-y|^{3+2s}}\,dx\,dy \\
&\le \int_{B_1(y^*)} g_n^2(x)dx \int_{B_1(y^*)} \frac{|\nabla\psi(y+\theta(x-y))|^2|x-y|^2}{|x-y|^{3+2s}} dy \\
&\quad + C\int_{B_1(y^*)} g_n^2(x)dx \int_{B_1^c(y^*)} \frac{1}{|x-y|^{3+2s}} dy \\
&\le C_1 \int_{B_1(y^*)} g_n^2(x)dx \int_0^2 \frac{r^2}{r^{1+2s}} dr + C\int_{B_1(y^*)} g_n^2(x)dx \int_1^\infty \frac{r^2}{r^{3+2s}} dr \\
&\le C_2 \int_{B_1(y^*)} g_n^2(x)\,dx \to 0
\end{aligned}
\tag{4.22}
$$

as $n \to \infty$, in view of $g_n \to 0$ in $L^2_{\rm loc}(\mathbb{R}^3)$, where $\theta = \theta(y) \in (0,1)$. Similarly, we have

$$
\iint_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2 g_n^2(y)}{|x-y|^{3+2s}}\,dx\,dy \to 0, \quad \text{as } n \to \infty. \tag{4.23}
$$

By (4.20)–(4.23), we have

$$
\|\psi g_n\|^2 = \iint_{\mathbb{R}^6} \frac{(g_n(x)-g_n(y))(\psi^2(x)g_n(x) - \psi^2(y)g_n(y))}{|x-y|^{3+2s}}\,dx\,dy + o_n(1). \tag{4.24}
$$

Furthermore, based on the above results and Proposition 2.3, we have

$$
\begin{aligned}
\varepsilon^{2s}S|\psi g_n|_{2_s^*}^2 &\le \varepsilon^{2s}\|\psi g_n\|^2\\
&= \varepsilon^{2s}\iint_{\mathbb{R}^6}\frac{(g_n(x)-g_n(y))(\psi^2(x)g_n(x)-\psi^2(y)g_n(y))}{|x-y|^{3+2s}}\,dx\,dy + o_n(1)\\
&= \varepsilon^{2s}\int_{\mathbb{R}^3}\phi_{g_n}|g_n|^{2_s^*-1}\psi^2dx + o_n(1)\\
&\le \varepsilon^{2s}C_s|g_n|_{2_s^*}^{2_s^*-1}\Big(\int_{\mathbb{R}^3}(|g_n|^{2_s^*-1}\psi^2)^{\frac{6}{3+2s}}dx\Big)^{\frac{3+2s}{6}} + o_n(1)\\
&= \varepsilon^{2s}C_s|g_n|_{2_s^*}^{2_s^*-1}\Big(\int_{\mathbb{R}^3}|g_n|^{\frac{6s-3}{3-2s}\frac{6}{3+2s}}|g_n\psi|^{\frac{12}{3+2s}}dx\Big)^{\frac{3+2s}{6}} + o_n(1)\\
&\le \varepsilon^{2s}C_s|g_n|_{2_s^*}^{2_s^*-1}\Big(\int_{\mathbb{R}^3}|\psi g_n|^{2_s^*}dx\Big)^{\frac{3-2s}{3}}\Big(\int_{B_1(y^*)}|g_n|^{2_s^*}dx\Big)^{\frac{6s-3}{6}} + o_n(1)\\
&\le \varepsilon^{2s}C_sD_1b^{\frac{6s-3}{6}}\Big(\int_{\mathbb{R}^3}|\psi g_n|^{2_s^*}dx\Big)^{\frac{3-2s}{3}} + o_n(1)\\
&< \frac{1}{2}\varepsilon^{2s}S|\psi g_n|_{2_s^*}^2 + o_n(1),
\end{aligned}
$$

which implies that $g_n \to 0$ in $L^{2_s^*}_{\rm loc}(\mathbb{R}^3)$, contradicting with (4.15). Therefore, $g \neq 0$. By (4.9) and the weakly sequential continuity of I'_ε, we know $J'_\varepsilon(g) = 0$, hence the sequences $\{g_n\}, \{r_n^1\}, \{y_n^1\}$ are the wanted sequences. By iteration, we obtain sequences $v_n^j = u_n^{j-1} - u^{j-1}$, $j \ge 2$, and the re-scaled functions $u_n^j = (v_n^j)_{r_n^j, y_n^j} \rightharpoonup u^j$ in $D^{s,2}(\mathbb{R}^3)$, where u^j is a nontrivial solution to (3.4). Furthermore, by (4.6),(4.9) and (4.18), we obtain

$$
\begin{aligned}
\|u_n^j\|^2 = \|v_n^j\|^2 &= \|u_n^{j-1}\|^2 - \|u^{j-1}\|^2 + o_n(1) = \dots\\
&= \|u_n\|^2 - \sum_{i=0}^{j-1}\|u^i\|^2 + o_n(1),
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
J_\varepsilon(u_n^j) = J_\varepsilon(v_n^j) &= J_\varepsilon(u_n^{j-1}) - J_\varepsilon(u^{j-1}) + o_n(1) = \dots\\
&= I_\varepsilon(u_n) - I_\varepsilon(u^0) - \sum_{i=1}^{j-1}J_\varepsilon(u^i).
\end{aligned}
\tag{4.26}
$$

Moreover, as

$$
\begin{aligned}
0 = J'_\varepsilon(u^j)u^j &= \varepsilon^{2s}\|u^j\|^2 - \int_{\mathbb{R}^3}\phi_{u^j}|u^j|^{2_s^*-1}dx\\
&\ge \varepsilon^{2s}\|u^j\|^2 - S^{-1}|u^j|_{2_s^*}^{2(2_s^*-1)}\\
&\ge \|u^j\|^2[\varepsilon^{2s} - S^{-2_s^*}(\|u^j\|^2)^{2_s^*-2}],
\end{aligned}
\tag{4.27}
$$

we have

$$
\|u^j\|^2 \ge \varepsilon^{\frac{3-2s}{2}}S^{\frac{3}{2s}},
$$

and the iteration must terminate at some index $k \ge 0$. □

Corollary 4.2. *Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε with $c \in (m_\varepsilon, 2^{\frac{6s-3}{4s}}m_\varepsilon)$. Then for each $\varepsilon > 0$, $\{u_n\}$ is relatively compact in $D^{s,2}(\mathbb{R}^3)$.*

Proof. From Lemma 4.1, it follows that there exist a number $k \in \mathbb{N}$, a solution u^0 of (1.9) and solutions $u^1, \dots, u^k$ of (3.4), such that

$$\|u_n\|^2 \to \sum_{j=0}^{k} \|u^j\|^2; I_\varepsilon(u_n) \to I_\varepsilon(u^0) + \sum_{j=1}^{k} J_\varepsilon(u^j).$$

By Lemma 3.3, if $u^0 \neq 0$, then $I(u^0) > m_\varepsilon$. On the other hand, for each nontrivial solution u^j of (3.4), if u^j is positive or negative, we have that

$$J_\varepsilon(u^j) = \frac{2s}{3+2s}\varepsilon^{\frac{3+2s}{2}} S^{\frac{3}{2s}} = m_\varepsilon.$$

If u^j changes its sign, then, from the proof of [13, Proposition 3.2], we have for all $t^+, t^- > 0$,

$$J_\varepsilon(t^+(u^j)^+)^{\frac{4s}{6s-3}} + J_\varepsilon(t^-(u^j)^-)^{\frac{4s}{6s-3}} \leq J_\varepsilon(u^j)^{\frac{4s}{6s-3}}.$$

Fixing $t^+, t^- > 0$ such that $J'_\varepsilon(t^\pm(u^j)^\pm)(t^\pm(u^j)^\pm) = 0$, it follows that

$$J_\varepsilon(t^\pm(u^j)^\pm) \geq \frac{2s}{3+2s}\varepsilon^{\frac{3+2s}{2}} S^{\frac{3}{2s}},$$

which implies that

$$J_\varepsilon(u^j) \geq 2^{\frac{6s-3}{4s}} \frac{2s}{3+2s}\varepsilon^{\frac{3+2s}{2}} S^{\frac{3}{2s}} = 2^{\frac{6s-3}{4s}} m_\varepsilon.$$

Since $c \in (m_\varepsilon, 2^{\frac{6s-3}{4s}} m_\varepsilon)$, we must have $k = 0$, and so $u_n \to u^0$ in $D^{s,2}(\mathbb{R}^3)$. $\square$

5. Proof of Theorem 1.1

In this section, we are devoted to showing the multiplicity of high energy semi-classical states. For small $\tau > 0$, we may choose $\rho = \rho(\tau) > 0$ such that $M_\tau \subset B_\rho(0)$. Let

$$\chi(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases} \tag{5.1}$$

We define $\beta : N_\varepsilon \to \mathbb{R}^3$ and $\gamma : N_\varepsilon \to \mathbb{R}^+$ by

$$\beta(u) = \frac{1}{\varepsilon^{\frac{3-2s}{2}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x)|(-\Delta)^{s/2}u|^2 dx,$$
$$\gamma(u) = \frac{1}{\varepsilon^{\frac{3-2s}{2}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |\chi(x) - \beta(u)|(-\Delta)^{s/2}u|^2 dx.$$

It is easy to see that for any $U_{\delta,z} \in D^{s,2}(\mathbb{R}^3)$, there exists a unique $t_{\delta,z}$ in $(0, +\infty)$ such that

$$\Phi_{\delta,z}(x) := t_{\delta,z}\varepsilon^{\frac{3-2s}{4}} U_{\delta,z}(x) \in N_\varepsilon. \tag{5.2}$$

We also introduce the set

$$\Lambda = \Lambda(\rho, \delta_1, \delta_1) = \{(x, \delta) \in \mathbb{R}^3 \times \mathbb{R} : |x| < \rho/2, \delta_1 < \delta < \delta_2\}. \tag{5.3}$$

A direct computation yields that, for any fixed $z \in \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} V(x)U^2_{\delta,z}(x)dx \to 0 \quad \text{as } \delta \to 0.$$

Then, for each $\varepsilon > 0$ and any fixed $z \in \mathbb{R}^3$, we see that $\lim_{\delta\to 0} t_{\delta,z} = 1$. Hence, for every $\varepsilon > 0$, there exist $\delta_1 = \delta_1(\varepsilon)$ and $\delta_2 = \delta_2(\varepsilon)$ with $\delta_1 < \delta_2$ and $\delta_1, \delta_2 \to 0$ as $\varepsilon \to 0$, such that

$$\sup\{I_\varepsilon(_{\delta,z}) : (z,\delta) \in \Lambda\} < \varepsilon^{\frac{3+2s}{2}}\Big(\frac{2s}{3+2s} S^{\frac{3}{2s}} + h(\varepsilon)\Big), \tag{5.4}$$

where $h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Lemma 5.1. $\lim_{\delta\to 0} \gamma(\Phi_{\delta,z}) = 0$ *uniformly for* $|z| \le \rho/2$.

Proof. For $0 < 2\xi < \rho$, by $t_{\delta,z} = 1 + o_\delta(1)$, we have

$$\begin{aligned}
\gamma(\Phi_{\delta,z}) &= \frac{1}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |\chi(x) - \beta(\Phi_{\delta,z})| (-\Delta)^{s/2} U_{\delta,z}|^2 dx + o_\delta(1) \\
&= \frac{1}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3 \setminus B_\xi(z)} |\chi(x) - \beta(\Phi_{\delta,z})| (-\Delta)^{s/2} U_{\delta,z}|^2 dx \\
&\quad + \frac{1}{S^{\frac{3}{2s}}} \int_{B_\xi(z)} |\chi(x) - \beta(\Phi_{\delta,z})| (-\Delta)^{s/2} U_{\delta,z}|^2 dx + o_\delta(1) \\
&:= I_1 + I_2 + o_\delta(1).
\end{aligned}$$

Note that

$$\begin{aligned}
I_1 &= \frac{1}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3 \setminus B_\xi(0)} |\chi(x) - \beta(\Phi_{\delta,0})| (-\Delta)^{s/2} U_{\delta,0}|^2 dx \\
&\le C\rho \int_{\mathbb{R}^3 \setminus B_\xi(0)} |(-\Delta)^{s/2} U_{\delta,0}|^2 dx \to 0
\end{aligned}$$

as $\delta \to 0$.

For I_2, we have

$$\begin{aligned}
\beta(\Phi_{\delta,z}) &= \beta(\varepsilon^{\frac{3-2s}{4}} U_{\delta,z}) + o_\delta(1) \\
&= \frac{1}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{s/2} U_{\delta,z}(x)|^2 dx + o_\delta(1) \\
&= z + \frac{\delta^{3+2s}}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} [\chi(\delta x + z) - z] |(-\Delta)^{s/2} U_{1,0}|^2 dx + o_\delta(1) \\
&= z + o_\delta(1).
\end{aligned} \tag{5.5}$$

Consequently,

$$\begin{aligned}
S^{\frac{3}{2s}} I_2 &\le \int_{B_\xi(z)} |\chi(x) - \chi(z)| |(-\Delta)^{s/2} U_{\delta,z}|^2 dx \\
&\quad + \int_{B_\xi(z)} |\chi(z) - \beta(\Phi_{\delta,z})| |(-\Delta)^{s/2} U_{\delta,z}|^2 dx \\
&\le 2 \int_{B_\xi(z)} |x - z| |(-\Delta)^{s/2} U_{\delta,z}|^2 dx + 2\xi S^{\frac{3}{2s}} \\
&\quad + \int_{B_\xi(z)} |\chi(z) - z| |(-\Delta)^{s/2} U_{\delta,z}|^2 dx + o_\delta(1) \le 4\xi S^{\frac{3}{2s}} + o_\delta(1).
\end{aligned}$$

where we have used [9, Lemma 2], which says

$$\chi(x) - \chi(z) \le 2|x - z| + 2\xi, \quad x \in B_\xi(z).$$

Since $\xi > 0$ is arbitrary, we have $\lim_{\delta\to 0} \gamma(\Phi_{\delta,z}) = 0$, uniformly for $|z| \le \rho/2$. $\square$

We now define a set $\widetilde{N}_\varepsilon \subset N_\varepsilon$ by

$$\begin{aligned}\widetilde{N}_\varepsilon = \Big\{ & u \in N_\varepsilon : \frac{2s}{3+2s}\varepsilon^{\frac{3+2s}{2}} S^{\frac{3}{2s}} < I_\varepsilon(u) < \varepsilon^{\frac{3+2s}{2}}\Big(\frac{2s}{3+2s} S^{\frac{3}{2s}} + h(\varepsilon)\Big), \\ & (\beta(u), \gamma(u)) \in \Lambda \Big\},\end{aligned}$$

where Λ is given by (5.3). According to Lemma 5.1, we can modify $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ such that $\widetilde{N}_\varepsilon \neq \emptyset$ for $\varepsilon > 0$ small.

Lemma 5.2. $\lim_{\varepsilon\to 0}\sup_{u\in\widetilde{N}_\varepsilon} \operatorname{dist}(\beta(u), M_\tau) = 0$, *for any* $\tau > 0$.

Proof. Let $u_n \in \widetilde{N}_{\varepsilon_n}$ be such that

$$\operatorname{dist}(\beta(u_n), M_\tau) = \sup_{u\in\widetilde{N}_{\varepsilon_n}} \operatorname{dist}(\beta(u), M_\tau) + o_n(1),$$

and assume that $\varepsilon_n \to 0$. It suffices to find a sequence $z_n \in M_\tau$ such that

$$\beta(u_n) = z_n + o_n(1). \tag{5.6}$$

Since $u_n \in N_{\varepsilon_n}$, by (3.6) we see that $\|u_n\|^2 \geq \varepsilon_n^{\frac{3-2s}{2}} S^{\frac{3}{2s}}$. Hence,

$$\begin{aligned}&\frac{2s}{3+2s}\varepsilon_n^{\frac{3+2s}{2}} S^{\frac{3}{2s}} \\ &\leq \frac{2s}{3+2s}\varepsilon_n^{2s}\|u_n\|^2 \\ &\leq \frac{2s}{3+2s}\varepsilon_n^{2s}\|u_n\|^2 + \frac{2s}{3+2s}\int_{\mathbb{R}^3} V(x)u_n^2 dx \\ &= I_{\varepsilon_n}(u_n) - \frac{1}{2(2_s^*-1)} I'_{\varepsilon_n}(u_n)u_n \\ &< \varepsilon_n^{\frac{3+2s}{2}}\Big(\frac{2s}{3+2s} S^{\frac{3}{2s}} + h(\varepsilon_n)\Big).\end{aligned} \tag{5.7}$$

Setting $w_n := \varepsilon^{\frac{2s-3}{4}} u_n$, we have from (5.7) that

$$\|w_n\|^2 \to \frac{2s}{3+2s} S^{\frac{3}{2s}} \quad \text{and} \quad \varepsilon_n^{-2s}\int_{\mathbb{R}^3} V(x)w_n^2 dx \to 0 \quad \text{as } n \to \infty. \tag{5.8}$$

Using $u_n \in N_{\varepsilon_n}$ again, we have

$$\int_{\mathbb{R}^3} |(-\Delta)^{s/2} w_n|^2 dx + \varepsilon_n^{-2s}\int_{\mathbb{R}^3} V(x)w_n^2 dx = \int_{\mathbb{R}^3} \phi_{w_n}|w_n|^{2_s^*-1} dx, \tag{5.9}$$

which implies that $\{w_n\}$ is a PS sequence for I_1. It follows from Lemmas 3.3 and 4.1 with $\varepsilon = 1$ that there exist a number $k \in \mathbb{N}$, sequences of points $x_n^1, \dots, x_n^k \in \mathbb{R}^3$ and radii $r_n^1, \dots, r_n^k$ such that:

(1) $w_n^0 \equiv w_n \rightharpoonup w^0$ in $D^{s,2}(\mathbb{R}^3)$;
(2) $w_n^j \equiv (w_n^{j-1} - w^{j-1})_{r_n^j, x_n^j} \rightharpoonup w^j$ in $D^{s,2}(\mathbb{R}^3)$, $j = 1, 2, \dots, k$;
(3) $\|w_n\|^2 \to \sum_{j=0}^k \|w^j\|^2$;
(4) $I_1(w_n) \to I_1(w^0) + \sum_{j=1}^k J_1(w^j)$,

as $n \to \infty$ where u^0 is a solution of (1.9) and $u^j, 1 \leq j \leq k$, are the nontrivial solutions of (3.4). If $w^0 \neq 0$, by Lemma 3.3 we know that

$$I_1(w^0) > \frac{2s}{3+2s} S^{\frac{3}{2s}},$$

which contradicts with the above conclusion (4) since $J_1(w^j) > \frac{2s}{3+2s}S^{\frac{3}{2s}}$ and $I_1(w_n) \to \frac{2s}{3+2s}S^{\frac{3}{2s}}$. Therefore, $w^0 = 0$. Moreover, using

$$J_1(w^j) > \frac{2s}{3+2s}S^{\frac{3}{2s}} \quad \text{and} \quad I_1(w_n) \to \frac{2s}{3+2s}S^{\frac{3}{2s}},$$

again, we must have $k = 1$ and w^1 is a ground state of (3.4) with $I_1(w^1) = \frac{2s}{3+2s}S^{\frac{3}{2s}}$. So, there exist $\delta_1 > 0$ and $z_1 \in \mathbb{R}^3$ such that $w^1 = U_{\delta_1,z_1}$, and there exists $(r_n^1, x_n^1) \in \mathbb{R}+ \times\mathbb{R}^3$ such that $\|(w_n)_{r_n^1,x_n^1} - w^1\| \to 0$. Consequently, there exist a sequence of points $\{z_n\} \subset \mathbb{R}^3$ and a sequence of $\{\sigma_n\} \subset (0, +\infty)$ such that $\|h_n\| := \|w_n - U_{\sigma_n,z_n}\| \to 0$, where $z_n = x_n^1 + r_n^1 z_1$ and $\sigma_n = r_n^1\delta_1$. We claim that

$$\sigma_n \to 0 \text{ and } \{z_n\} \text{ is bounded.} \tag{5.10}$$

Indeed, denoting $\Psi_{\sigma_n,z_n} = \varepsilon_n^{\frac{3-2s}{4}} U_{\sigma_n,z_n}$ by $h_n \to 0$ in $D^{s,2}(\mathbb{R}^3)$, one has

$$\begin{aligned}
\beta(u_n) &= \frac{1}{\varepsilon^{\frac{3-2s}{2}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x)|(-\Delta)^{s/2}u_n|^2dx \\
&= \frac{1}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x)|(-\Delta)^{s/2}w_n|^2dx \\
&= \frac{1}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x)|(-\Delta)^{s/2}U_{\sigma_n,z_n}|^2dx + o_n(1) \\
&= \frac{1}{\varepsilon^{\frac{3-2s}{2}} S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x)|(-\Delta)^{s/2}\Psi_{\sigma_n,z_n}|^2dx + o_n(1) \\
&= \beta(\Psi_{\sigma_n,z_n}) + o_n(1).
\end{aligned} \tag{5.11}$$

From $u_n \in \widetilde{N}_\varepsilon$ we may assume $\beta(\Psi_{\sigma_n,y_n}) \subset B_{\rho/2}(0)$. If $\sigma_n \to \infty$, then we know that for each $R > 0$, by [28, Proposition 2.2], we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\varepsilon_n^{\frac{3-2s}{2}}} \int_{B_R(0)} |(-\Delta)^{s/2}\Psi_{\sigma_n,z_n}|^2dx &= \lim_{n\to\infty} \int_{B_R(0)} |(-\Delta)^{s/2}U_{\sigma_n,z_n}|^2dx \\
&\le \lim_{n\to\infty} \int_{B_R(0)} |\nabla U_{\sigma_n,z_n}|^2dx = 0.
\end{aligned}$$

Using this fact and the definition of the mapping γ, we obtain

$$\begin{aligned}
\gamma(\Psi_{\sigma_n,z_n}) &= \frac{\varepsilon^{-\frac{3-2s}{2}}}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |\chi(x) - \beta(\Psi_{\sigma_n,z_n})|(-\Delta)^{s/2}\Psi_{\sigma_n,z_n}|^2dx \\
&\ge \frac{\varepsilon^{-\frac{3-2s}{2}}}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |\chi(x)||(-\Delta)^{s/2}\Psi_{\sigma_n,z_n}|^2dx - \beta(\Psi_{\sigma_n,z_n}) \\
&\ge \frac{\rho\varepsilon^{-\frac{3-2s}{2}}}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3\setminus B_R(0)} |(-\Delta)^{s/2}\Psi_{\sigma_n,z_n}|^2dx - \frac{\rho}{2} \\
&= \frac{\rho\varepsilon^{-\frac{3-2s}{2}}}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} |(-\Delta)^{s/2}\Psi_{\sigma_n,z_n}|^2dx - \frac{\rho}{2} + o_n(1) \\
&= \frac{\rho}{2} + o_n(1).
\end{aligned} \tag{5.12}$$

Since $\|h_n\| := \|u_n - \Psi_{\sigma_n,z_n}\| \to 0$, one has $\gamma(u_n) = \gamma(\Psi_{\sigma_n,z_n}) + o_n(1)$. So, $\gamma(u_n) > \frac{\rho}{2} + o_n(1)$. However, from $u_n \in \widetilde{N}_{\varepsilon_n}$, we obtain

$$\delta_1(\varepsilon_n) < \gamma(u_n) < \delta_2(\varepsilon_n), \tag{5.13}$$

where $\delta_i(\varepsilon_n) \to 0$, $i = 1,2$ as $n \to \infty$. This leads to a contradiction, and so, $\{\sigma_n\}$ is bounded. Now we assume that $\sigma_n \to \bar{\sigma} \geq 0$ as $n \to \infty$. If $\bar{\sigma} > 0$, then we must have that $|z_n| \to \infty$. Otherwise, U_{σ_n,z_n} would converge strongly in $D^{s,2}(\mathbb{R}^3)$, and so would w_n. Consequently, I_1 possesses nontrivial minimizer on N_1, which is impossible by Lemma 3.3. Then, for every $R > 0$, the fact that $\lim_{n\to\infty} |z_n| = \infty$ implies that

$$\lim_{n\to\infty} \int_{B_R(0)} |(-\Delta)^{s/2} U_{\sigma_n,z_n}|^2 dx = 0.$$

Hence, one can similarly obtain the estimation (5.12), a contradiction to (5.13). The proof of the boundedness of the sequence $\{z_n\}$ is similar, and it is omitted here. Hence, (5.10) holds.

Now, we may assume that $z_n \to z^*$ and $\sigma_n \to 0$. By choosing subsequences of $\{\sigma_n\}$ and $\{\varepsilon_n\}$, still denoted by $\{\sigma_n\}$ and $\{\varepsilon_n\}$ such that $\frac{\sigma_{n_i}}{\varepsilon_{n_i}} = o_{n_i}(1)$ as $n_i \to \infty$, we may replace $\{\sigma_{n_i}\}$ by $\{\varepsilon_{n_i}\}$ and relabel $\{\varepsilon_{n_i}\}$ by $\{\varepsilon_n\}$. Define $v_n(x) = \varepsilon_n^{\frac{3-2s}{4}} w_n(\varepsilon_n x + z_n)$. Then $v_n \to U_{1,0}$ in $D^{s,2}(\mathbb{R}^3)$, and we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} V(\varepsilon_n x + z_n) v_n(x)^2 dx = \lim_{n\to\infty} \varepsilon_n^{-2s} \int_{\mathbb{R}^3} V(x) w_n(x)^2 dx = 0;$$

which implies that $\int_{\mathbb{R}^3} V(z^*) U_{1,0}^2(x) dx = 0$. Therefore, $V(z^*) = 0$ and $z^* \in M$. Furthermore, $z_n \in M_\tau$ for large n. Then by (5.11), we obtain

$$\begin{aligned}
\beta(u_n) &= \frac{1}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} \chi(x) |(-\Delta)^{s/2} U_{\sigma_n,z_n}|^2 dx + o_n(1) \\
&= \frac{1}{S^{\frac{3}{2s}}} \int_{\mathbb{R}^3} [\chi(\varepsilon_n x + z_n) - z_n] |(-\Delta)^{s/2} U_{1,0}|^2 dx + z_n + o_n(1).
\end{aligned}$$

Since $\varepsilon_n x + z_n \to z^* \in M$, we deduce that $\beta(u_n) = z_n + o_n(1)$, hence the sequence $\{z_n\}$ is what we need. Consequently, (5.6) is true and we complete the proof. □

Proof of Theorem 1.1. For any $\tau > 0$, set small $\varepsilon = \varepsilon_\tau > 0$. Then $\Phi : [\delta_1, \delta_2] \times M \to \widetilde{N}_\varepsilon$ given by $\Phi(\delta, z) = \Phi_{\delta,z}$ is well defined and by Lemma 5.2, we have $\beta(\widetilde{N}_\varepsilon) \subset M_\tau$. From (5.5), we obtain

$$\beta(\Phi_{\delta,z}) = z + o_\delta(1) \quad \text{uniformly in } z \in M.$$

For $\delta \in [\delta_1, \delta_2]$, we denote $\beta(\Phi_{\delta,z}) = z + \mu(z)$ for $z \in M$, where $|\mu(z)| < \tau/2$ uniformly for $z \in M$. Define $H(t, (\delta, z)) := (\delta, z + (1-t)\mu(z))$. It is easy to see that $H : [0,1] \times [\delta_1, \delta_2] \times M \to [\delta_1, \delta_2] \times M_\tau$ is continuous. Obviously,

$$H(0, (\delta, z)) = (\delta, \beta(\Phi_{\delta,z})),\ H(1, (\delta, z)) = (\delta, z).$$

Therefore,

$$\Theta(\delta, z) := (\delta, \beta(\Phi_{\delta,z}))) : [\delta_1, \delta_2] \times M \to [\delta_1, \delta_2] \times M_\tau$$

is homotopic to the inclusion mapping $Id : [\delta_1, \delta_2] \times M \to [\delta_1, \delta_2] \times M_\tau$. Thus, we have

$$\mathrm{cat}(\widetilde{N}_\varepsilon) \geq \mathrm{cat}_{[\delta_1,\delta_2]\times M_\tau}([\delta_1, \delta_2] \times M) = \mathrm{cat}_{M_\tau}(M).$$

By Corollaries 3.4 and 4.2, the functional I_ε satisfies the $(PS)_c$ condition on $\widetilde{N}_\varepsilon$. Hence, the Ljusternik-Schnirelman theory of critical points implies that I_ε has at least $\mathrm{cat}_{M_\tau}(M)$ solutions. □

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References

[1] A. Ambrosetti; On Schrödinger-Poisson systems, *Milan J. Math.* **76**(2008), 257-274.

[2] A. Ambrosetti, D. Ruiz; Multiple bound states for the Schrödinger-Poisson equation, *Commun. Contemp. Math.* **10** (2008), 1-14.

[3] A. Azzollini, P. d'Avenia, G. Vaira; Generalized Schrödinger-Newton system in dimension $N \geq 3$: critical case, *J. Math. Anal. Appl.* **449** (2017), 531-552.

[4] V. Ambrosio; Multiplicity and concentration results for a class of critical fractional Schrödinger-Poisson systems via penalization method, *Communications in Contemporary Mathematics* **22** (2020), No. 01, 1850078.

[5] V. Benci, D. Fortunato; An eigenvalue problem for the Schrödinger-Maxwell equations, *Topol. Methods Nonlinear Anal.* **11** (1998), 283-293.

[6] H. Brézis, E. Lieb; A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88** (1983), 486-490.

[7] L. A. Caffarelli, L. Silvestre; An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), 1245-1260.

[8] G. Cerami, G. Vaira; Positive solutions for some non-autonomous Schrödinger-Poisson systems, *J. Differential Equations* **248** (2010), 521-543.

[9] J. Chabrowski, J. Yang; Multiple semiclassical solutions of the Schrödinger equation involving a critical Sobolev exponent, *Portugal. Math.* **57** (2000), 273-284.

[10] M. Chen, Q. Li, S. Peng; Bound states for fractional Schrödinger-Poisson system with critical exponent, *Discrete Conti. Dyn. Syst. S* **14** (2021), 1819-1835.

[11] M. Del Pino, P. L. Felmer; Local Mountain Pass for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differential Equations* **4** (1996), 121-137.

[12] X. Feng; Nontrivial solution for Schrodinger-Poisson equations involving the fractional Laplacian with critical exponent, *RACSAM* **115**(2021), No. 10. https://doi.org/10.1007/s13398-020-00953-w.

[13] M. Ghimenti, J. van Schaftingen; Nodal solutions for the Choquard equation, *J. Funct. Anal.* **271** (2016), 107-135.

[14] L. Guo, Q. Li; Multiple bound state solutions for fractional Choquard equation with Hardy-Littlewood-Sobolev critical exponent, *J. Math. Phys.* **61**(2020), 121501, doi: 10.1063/5.0013475.

[15] Z. Guo; Multiple solutions for Schrödinger-Poisson systems with critical nonlocal term, *Topol. Methods Nonlinear Anal.* **54** (2019), 495-513.

[16] X. He; Positive solutions for fractional Schrödinger-Poisson systems with doubly critical exponents, *Appl. Math. Lett.* **120** (2021), 107190.

[17] N. Laskin; Fractional quantum mechanics and Lévy path integrals, *Phys. Lett. A* **268** (2000), 298-305.

[18] N. Laskin; Fractional Schrödinger equation, *Phys. Rev. E* **66** (2002), 056108.

[19] F. Li, Y. Li, J. Shi; Existence and multiplicity of positive solutions to Schrödinger-Poisson type systems with critical nonlocal term, *Calc. Var. Partial Differential Equations* **56**, 134 (2017), https://doi.org/10.1007/s00526-017-1229-2.

[20] F. Li, Y. Li, J. Shi; Existence of positive solutions to Schrödinger-Poisson type systems with critical exponent, *Commun. Contemp. Math.* **16** (2014), 1450036.

[21] E. Lieb, M. Loss; *Analysis*, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, 2001.

[22] Z. Liu, J. Zhang; Multiplicity and concentration of positive solutions for the fractional Schrödinger-Poisson systems with critical growth, *ESAIM: Control, optimization and calculus of variations* **23**(2017), 1515-1542.

[23] H. Liu; Positive solutions of an asymptotically periodic Schrödinger-Poisson system with critical exponent, *Nonlinear Anal. RWA* **32** (2016), 198-212.

[24] L. Liu, K. Teng; Ground state and multiple solutions for critical fractional Schrodinger-Poisson equations with perturbation terms, *Electron. J. Differential Equations*, 2021 (2021), no. 07, 1-21.

[25] N. Li, X. He; Existence and multiplicity results for some Schrödinger-Poisson system with critical growth, *J. Math. Anal. Appl.* **488** (2020), No. 2, Article ID 124071, 35 p.

[26] G. Molica Bisci, V. Radulescu, R. Servadei; Variational Methods for Nonlocal Fractional Problems, Cambridge University Press, 162 Cambridge, 2016.

[27] E. Murcia, G. Siciliano; Positive semiclassical states for a fractional Schrödinger-Poisson system, *Differential Integral Equations* **30** (2017), 231-258.
[28] E. Di Nezza, G. Palatucci, E. Valdinoci; Hithiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012), 521-573.
[29] D. Ruiz; The Schrödinger-Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.* **237** (2006), 655-674.
[30] R. Servadei, E. Valdinoci; The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.*, **367** (2015), 67-102.
[31] K. Teng; Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent, *J. Differential Equations* **261**(2016), 3061-3106.
[32] Z. Yang, Y. Yu, F. Zhao; Concentration behavior of ground state solutions for a fractional Schrödinger-Poisson system involving critical exponent, *Commun. Contemp. Math.* **21**(2018), 1850027, 46 pages.
[33] J. Zhang, J. M. do Ó, M. Squassina; Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity, *Adv. Nonlinear Stud.* **16** (2016), 15-30.
[34] H. Zhang, F. Zhang; Multiplicity of semiclassical states for fractional Schrödinger equations with critical frequency, *Nonlinear Anal. TMA* **190** (2020), 11599.
[35] M. Willem; Minimax Theorems, Birkhäuser, Boston, 1996.
[36] M. Willem; Functional Analysis. Fundamentals and Applications, Cornerstones, vol. 14. New York: Birkhäuser, Springer, 2013.
[37] X. Zhang, S. Ma, Q. Xie; Bound state solutions of Schrödinger-Poisson system with critical exponent, *Discret. Contn. Dyn. Syst.* **37** (2017), 605-625.
[38] X. Zhong, C. Tang; Ground state sign-changing solutions for a Schrödinger-Poisson system with a critical nonlinearity in $\mathbb{R}^3$, *Nonlinear Anal. RWA* **39** (2018), 166-184.

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