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BIFURCATION OF CRITICAL PERIODS OF A QUINTIC SYSTEM

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Abstract. We investigate the critical period bifurcations of the system

$$\dot{x} = ix + x\bar{x}(ax^3 + bx^2\bar{x} + \bar{x}\bar{x}^2 + d\bar{x}^3)$$

studied in [6]. We prove that at most three critical periods can bifurcate from any nonlinear center of the system.

1. Introduction

Consider a system of ordinary differential equations on $\mathbb{R}^2$ of the form

$$\begin{aligned} \dot{u} &= -v + P(u, v), \\ \dot{v} &= u + Q(u, v), \end{aligned} \tag{1.1}$$

where u and v are real unknown functions and P and Q are polynomials without constant and linear terms. The singularity at the origin of system (1.1) is either a center or a focus. In a neighborhood of a center the so-called *period function* $T(r)$ gives the least period of the periodic solution passing through the point with coordinates $(u, v) = (r, 0)$ inside the period annulus of the center.

If $T(r)$ is constant in a neighbourhood of the origin, then the center at the origin is called isochronous. For a center that is not isochronous any value $r > 0$ for which $T'(r) = 0$ is called a critical period. The problem of critical period bifurcations is aimed on estimation of the number of critical periods that can arise near the center under small perturbations. It was investigated for the first time by Chicone and Jacobs [2] in 1989 for quadratic systems and some Hamiltonian systems. After that, many studies were devoted to the problem (see, e.g. [1, 5, 7, 10, 12, 15, 16, 17, 18, 19, 20, 21] and references given there). One of difficulties in investigations of this problem is that before studying the critical periods bifurcation for a polynomial system one should resolve the center problem for the system, that is, find all systems in the family with a center at the origin.

Studies of the center problem are usually simpler if one considers the problem in the complex setting. To perform a complexification we can make the substitution

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$x = u + iv$ obtaining from (1.1) the complex differential equation

$$\dot{x} = ix - \sum_{j+k=1}^{n-1} a_{jk} x^{j+1} \bar{x}^k. \tag{1.2}$$

Adjoining to (1.2) its complex conjugate and considering $\bar{a}_{jk}$ as a new parameter b_{kj} and $\bar{x}$ as a distinct unknown function y we obtain the system

$$\begin{aligned} \dot{x} &= ix - \sum_{j+k=1}^{n-1} a_{jk} x^{j+1} y^k = ix + \widetilde{P}(x,y), \\ \dot{y} &= -iy + \sum_{j+k=1}^{n-1} b_{kj} x^k y^{j+1} = -iy + \widetilde{Q}(x,y). \end{aligned} \tag{1.3}$$

This system is called the *complexification* of (1.1) and it is equivalent to (1.2) when $y = \bar{x}$ and $b_{kj} = \bar{a}_{jk}$.

By Poincaré-Lyapunov theorem system (1.1) has a center at the origin if and only if it admits in a neighbourhood of the origin an analytic first integral of the form

$$\Phi(u,v) = u^2 + v^2 + \text{h.o.t.},$$

which is equivalent to the existence of a first integral of the form

$$\Psi(x,\bar{x}) = x\bar{x} + \text{h.o.t.}$$

for system (1.2).

Thus, extending the notion of center from real systems to systems (1.3) it is said that complex system (1.3) has a center at the origin if in a neighbourhood of the origin it admits an analytic first integral of the form

$$\Psi(x,y) = xy + \sum_{j+k=3}^{\infty} \Psi_{jk} x^j y^k. \tag{1.4}$$

Since to each a_{jk} in the first equation of (1.3) corresponds the parameter b_{kj} in the second equation of (1.3), system (1.3) has 2ℓ parameters. We denote the ordered 2ℓ-tuple of the parameters of (1.3) by (a,b); that is,

$$(a,b) = (a_{p_1 q_1}, \dots, a_{p_\ell q_\ell}, b_{q_\ell, p_\ell}, \dots, b_{q_1 p_1}), \tag{1.5}$$

and we use the notation $\mathbb{C}[a,b]$ for the ring of polynomials in the variables $a_{p_1 q_1}$, $a_{p_2 q_2}$, $\dots$, $b_{q_1 p_1}$ over $\mathbb{C}$.

Recently, García, Llibre and Maza [6] studied limit cycle bifurcations near a center or a focus at the origin of the quintic system written in the complex form as the equation

$$\dot{x} = ix + x\bar{x}(ax^3 + bx^2\bar{x} + c\bar{x}x^2 + d\bar{x}^3),$$

which, in order to use the notation similar to the one in (1.2), we write as the complex equation

$$\dot{x} = i(x - a_{31}x^4\bar{x} - a_{22}x^3\bar{x}^2 - a_{13}x^2\bar{x}^3 - a_{04}x\bar{x}^4). \tag{1.6}$$

In this paper we study critical period bifurcations from the center at the origin of system (1.6). We first describe a way to compute the period function of system (1.2) using the normal form of its complexification (1.3). Then we prove that at most three critical periods can bifurcate from any nonlinear center of the system.

2. Preliminaries

To study critical period bifurcations of system (1.6) we have to compute a series expansion of the period function $T(r)$ of the system. One possibility is to pass to polar coordinates. This way is geometrically and theoretically straightforward, however it is not computationally efficient since one needs to compute integrals of trigonometric polynomials, and this is a difficult task in the case of polynomials of high degree.

Another possible computational approach relies on calculations of Poincaré-Dulac normal form of the complexification (1.3). We briefly remind it following to [14] and [5].

As it is well-known after a change of coordinates

$$\begin{aligned} x &= y_1 + \sum_{j+k\geq 2} h_1^{(j,k)} y_1^j y_2^k , \\ y &= y_2 + \sum_{j+k\geq 2} h_2^{(j,k)} y_1^j y_2^k , \end{aligned} \tag{2.1}$$

system (1.3) can be brought to the Poincaré-Dulac normal form

$$\begin{aligned} \dot{y}_1 &= y_1(i + \sum_{j=1}^{\infty} Y_1^{(j+1,j)} (y_1y_2)^j) = y_1(i + Y_1(y_1y_2)), \\ \dot{y}_2 &= y_2(-i + \sum_{j=1}^{\infty} Y_2^{(j,j+1)} (y_1y_2)^j) = y_2(-i + Y_2(y_1y_2)). \end{aligned} \tag{2.2}$$

The normal form (2.2) is not uniquely defined since the so-called resonant coefficient $h_1^{(j+1,j)}$ and $h_2^{(j,j+1)}$ in (2.1) can be chosen arbitrary. We will chose for all j ($j = 1, 2, \dots$) $h_1^{(j+1,j)} = h_2^{(j,j+1)} = 0$ (in such case the transformation (2.1) is called distinguished).

The coefficients $Y_1^{(j+1,j)}$ and $Y_2^{(j,j+1)}$ in (2.2) are polynomials of the ring $\mathbb{C}[a, b]$. Denote by $\mathcal{Y}$ the ideal generated by all coefficients of the normal form

$$\mathcal{Y} := \langle Y_1^{(j+1,j)},\ Y_2^{(j,j+1)} : j \in \mathbb{N}\rangle \subset \mathbb{C}[a, b], \tag{2.3}$$

and by $\mathcal{Y}_K$ the ideal generated by the first K pairs of the coefficients,

$$\mathcal{Y}_K := \langle Y_1^{(j+1,j)}, Y_2^{(j,j+1)} : j = 1, \dots, K\, \rangle.$$

The normal form of a particular system (a^*, b^*) with the fixed parameters is linear when all the coefficients of the normal form evaluated at (a^*, b^*) are equal to zero,

$$Y_1^{(j+1,j)}(a^*, b^*) = Y_2^{(j,j+1)}(a^*, b^*) = 0 \quad \text{for all } j \in \mathbb{N},$$

that is, when the point (a^*, b^*) belongs to the variety of the ideal $\mathcal{Y}$. The variety $\mathbf{V}(I)$ of a polynomial ideal I is the set of common zeros of all polynomials of the ideal. The variety $V_{\mathscr{L}} := \mathbf{V}(\mathcal{Y})$ is called the *linearizability variety* of system (1.3). As it is well known system (1.1) has an isochronous center at the origin if and only if the system is linearizable. Thus, the real systems (1.1), which parameters after the complexification are in $V_{\mathscr{L}}$, have isochronous centers at the origin.

For system (1.3) one can find a function (1.4) such that

$$[ix + \widetilde{P}(x, y)]\Psi_x(x, y) + [-iy + \widetilde{Q}(x, y)]\Psi_y(x, y) = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots ,$$

where g_{kk} is a polynomial in the coefficients of system (1.3). The polynomial g_{kk} is called the k-th *focus quantity*. Clearly, system (1.3) with fixed coefficients (a^*, b^*) has a center at the origin if and only if $g_{kk} \equiv 0$ for all $k \in \mathbb{N}$. We call the ideal

$$\mathscr{B} := \langle g_{kk} : k \in \mathbb{N} \rangle \subset \mathbb{C}[a, b]$$

the *Bautin ideal* of system (1.3). The variety of $\mathscr{B}$, $V_{\mathscr{C}} = \mathbf{V}(\mathscr{B})$, is called the *center variety*. We will also use the ideal generated by the first K focus quantities, which we denote

$$\mathscr{B}_K := \langle g_{kk} : k = 1, \dots, K \rangle \subset \mathbb{C}[a, b].$$

Let us denote

$$\begin{aligned} G &= Y_1 + Y_2, \\ H &= Y_1 - Y_2. \end{aligned}$$

It is easy to see that the origin is a center for (1.3) if and only if $G \equiv 0$, in which case H has purely imaginary coefficients and the distinguished normalizing transformation converges. We also define

$$\widetilde{H}(w) = -\frac{1}{2} i H(w).$$

When (1.3) is the complexification of a real system one can recover the real system by replacing every occurrence of y_2 by $\bar{y}_1$ in each equation of (2.2). In such case, performing the transformation $y_1 = re^{i\varphi}$ we obtain from (2.2) the equations for $\dot{r}$ and $\dot{\varphi}$ as follows:

$$\begin{aligned} \dot{r} &= \frac{1}{2r}(\dot{y}_1 \bar{y}_1 + y_1 \dot{\bar{y}}_1) = 0, \\ \dot{\varphi} &= \frac{i}{2r^2}(y_1 \dot{\bar{y}}_1 - \dot{y}_1 \bar{y}_1) = 1 + \widetilde{H}(r^2)\,. \end{aligned} \tag{2.4}$$

We write the function $\widetilde{H}$ as

$$\widetilde{H}(w) = \sum_{k=1}^{\infty} \widetilde{H}_{2k+1} w^k.$$

The integration of the second equation in (2.4) gives the least period of the periodic solution of (**??**)assing through the point with coordinates $(r, 0)$ as

$$T(r) = \frac{2\pi}{1 + \widetilde{H}(r^2)} = 2\pi\Big(1 + \sum_{k=1}^{\infty} p_{2k}(a, \bar{a}) r^{2k}\Big) \tag{2.5}$$

for some coefficients p_{2k}. The center at the origin of system (1.6) corresponding to a parameter a^* is isochronous if and only if $p_{2k}(a^*, \bar{a}^*) = 0$ for $k \geq 1$.

It is easy to see that p_{2k} are polynomials in the parameters a, $\bar{a}$ of system (1.2). We can extend the polynomial functions $p_{2k}(a, \bar{a})$ to the set of parameters (a, b) setting in (2.4) y_2 instead of $\bar{y}_1$. Then instead of (2.5) we obtain the function

$$T(r, a, b) = 2\pi\Big(1 + \sum_{k=1}^{\infty} p_{2k}(a, b) r^{2k}\Big), \tag{2.6}$$

which coincides with the period function (2.5) when $b = \bar{a}$.

We call the polynomial $p_{2k}(a,b)$ in (2.6) the *k-th isochronicity quantity*. Using (2.5) and the formula for the inversion of series the first three polynomials p_{2k} are computed as:

$$
\begin{aligned}
p_2 &= -\widetilde{H}_3 = \frac{i}{2}\big(Y_1^{(2,1)} - Y_2^{(1,2)}\big)\\
p_4 &= -\widetilde{H}_5 + (\widetilde{H}_3)^2 = \frac{i}{2}\big(Y_1^{(3,2)} - Y_2^{(2,3)}\big) - \frac{1}{4}\big(Y_1^{(2,1)} - Y_2^{(1,2)}\big)^2,\\
p_6 &= -\widetilde{H}_7 + 2\widetilde{H}_3\widetilde{H}_5 - (\widetilde{H}_3)^3\\
&= \frac{i}{2}\big(Y_1^{(4,3)} - Y_2^{(3,4)}\big) - \frac{1}{2}\big(Y_1^{(2,1)} - Y_2^{(1,2)}\big)\big(Y_1^{(3,2)} - Y_2^{(2,3)}\big)\\
&\quad - \frac{i}{8}\big(Y_1^{(2,1)} - Y_2^{(1,2)}\big)^3.
\end{aligned} \tag{2.7}
$$

Since values of the isochronicity quantity p_{2k} are of interest only on the center variety, we should work with the equivalence class $[p_{2k}]$ of p_{2k} in the coordinate ring $\mathbb{C}[V_{\mathscr{C}}]$ of the center variety, which can be viewed as the set of equivalence classes of polynomials $\mathbb{C}[a,b]$ by $V_{\mathscr{C}}$. That is, for polynomials $f,g \in \mathbb{C}[a,b]$,

$$[f] = [g] \quad \text{in } \mathbb{C}[V_{\mathscr{C}}]$$

if and only if

$$f - g \equiv 0 \quad \text{on} \quad V_{\mathscr{C}}.$$

We denote

$$P = \langle p_{2k} : k \in \mathbb{N}\rangle \subset \mathbb{C}[a,b] \quad \text{and} \quad \widetilde{P} = \langle [p_{2k}] : k \in \mathbb{N}\rangle \subset \mathbb{C}[V_{\mathscr{C}}],$$

and for $K \in \mathbb{N}$,

$$P_K = \langle p_2, \dots, p_{2K}\rangle \quad \text{and} \quad \widetilde{P}_K = \langle [p_2], \dots, [p_{2K}]\rangle\,.$$

The ideal P is called the *isochronicity ideal*.

Finally, we remind that given a Noetherian ring R and an ordered set

$$B = \{b_1, b_2, \dots\} \subset R,$$

we construct a basis M_I of the ideal $I = \langle b_1, b_2, \dots\rangle$ as follows:

(a) initially set $M_I = \{b_p\}$, where b_p is the first non-zero element of B;
(b) sequentially check successive elements b_j, starting with $j = p+1$, adding b_j to M_I if and only if $b_j \notin \langle M_I\rangle$

The cardinality of M_I is called the *Bautin depth* of I.

3. An upper bound for critical periods bifurcating from centers of system (1.6)

Along with system (1.6) we consider its complexification

$$
\begin{aligned}
\dot{x} &= ix(1 - a_{31}x^3y - a_{22}x^2y^2 - a_{13}xy^3 - a_{04}y^4),\\
\dot{y} &= -iy(1 - b_{40}x^4 - b_{31}x^3y - b_{22}x^2y^2 - b_{13}xy^3).
\end{aligned} \tag{3.1}
$$

Our study is based on the following theorem which is an immediate corollary of [5, Theorem 5.2 and Remark 5.3].

Theorem 3.1. *Suppose that for the complexification* (1.3) *of the family* (1.2)*:*

(a) $V_{\mathscr{L}} = \mathbf{V}(P_K) \cap V_{\mathscr{C}}$,

(b) *the Bautin depth (i.e., the cardinality of the minimal basis) of* $\widetilde{P}_K$ *in* $\mathbb{C}[V_{\mathscr{C}}]$ *is* m, *and*
(c) *a primary decomposition of* $P_K + \sqrt{\mathscr{B}}$ *can be written* $R \cap N$ *where* R *is the intersection of the ideals in the decomposition that are prime and* N *is the intersection of the remaining ideals in the decomposition.*

Then for any system of family (1.2) *corresponding to* $(a^*, \bar{a}^*) \in V_{\mathscr{C}} \setminus \mathbf{V}(N)$, *at most* $m-1$ *critical periods bifurcate from a center at the origin.*

Thus, to estimate the number of bifurcating critical periods of system (3.1) we have to know the center and linearizability varieties of the system.

First we note that it follows from Corollary 3.4.6 in [14] that for system (3.1) the focus quantities $g_{2k+1,2k+1}$ are zero polynomials. Using the results of [9] we can easily prove the following statement.

Proposition 3.2. *The center variety of system* (3.1) *is defined by the seven first non-zero focus quantities,*

$$\mathbf{V}(\mathscr{B}) = \mathbf{V}(\mathscr{B}_{14}), \tag{3.2}$$

where $\mathscr{B}_{14} = \langle g_{2,2}, g_{4,4}, g_{6,6}, g_{8,8}, g_{10,10}, g_{12,12}, g_{14,14} \rangle$, *and it consists of four components defined by the following prime ideals:*

$$\begin{gathered}
I_1 = \langle a_{22} - b_{22}, a_{31}a_{13} - b_{13}b_{31}, b_{31}^2 a_{04} - a_{13}^2 b_{40}, \\
a_{31}b_{31}a_{04} - b_{13}a_{13}b_{40}, a_{31}^2 a_{04} - b_{13}^2 b_{40} \rangle, \\
I_2 = \langle b_{40}, b_{31}, a_{13}, b_{22}, a_{22}, b_{13} \rangle, \\
I_3 = \langle a_{04}, b_{31}, a_{13}, b_{22}, a_{22}, a_{31} \rangle, \\
I_4 = \langle a_{22} - b_{22}, 3b_{13} - a_{13}, 3a_{31} - b_{31} \rangle.
\end{gathered}$$

Proof. Using the algorithm in [14, Chapter 3] and a *Mathematica* code similar to the one given in [14, Fig. 6.1 of Appendix] we computed the focus quantities $g_{2,2}, g_{4,4}, \dots, g_{14,14}$ (since the expressions are long, we do not present them here, but one can easily compute them using any available computer algebra system). Then, using the routine `minAssGTZ`, which is based on the algorithm of [8], of the computer algebra system Singular [3] we found that the minimal associate primes of $\mathscr{B}_{14}$ are the prime ideals $I_1, \dots, I_4$ in the statement of the theorem.

By the results of [9] if the parameters (a, b) of system (3.1) are from one of the varieties $\mathbf{V}(I_1), \dots, \mathbf{V}(I_4)$, then the corresponding systems have a center. This means that (3.2) holds. $\square$

Note, that taking into account that $\mathbf{V}(\mathscr{B})$ is a complex variety, from (3.2) we obtain that the radical of $\mathscr{B}$ coincides with the radical of $\mathscr{B}_{14}$, that is,

$$\sqrt{\mathscr{B}} = \sqrt{\mathscr{B}_{14}}.$$

To find the linearizability variety of system (3.1) and the isochronicity quantities p_{2k} we have computed the normal form of system (3.1) up to the order 17 and found four first non-zero pairs of the resonant coefficients $Y_1^{(2k+1,2k)}, Y_2^{(2k,2k+1)}$ as follows:

$$\begin{gathered}
Y_1^{(3,2)} = -ia_{22}; \quad Y_2^{(2,3)} = ib_{22}; \\
Y_1^{(5,4)} = i(-2a_{13}a_{31} + a_{31}b_{13} - 3a_{13}b_{31} - 2a_{04}b_{40})/2; \\
Y_2^{(4,5)} = -i(a_{31}b_{13})/2 + (3a_{13}b_{31})/2 + b_{13}b_{31} + a_{04}b_{40};
\end{gathered}$$

$$
\begin{aligned}
Y_1^{(7,6)} &= (4a_{13}a_{22}a_{31} + a_{22}a_{31}b_{13} - 6a_{13}a_{31}b_{22} + a_{31}b_{13}b_{22} - 11a_{13}a_{22}b_{31} \\
&\quad - 6a_{04}a_{31}b_{31} + 2a_{22}b_{13}b_{31} - 3a_{13}b_{22}b_{31} - 10a_{04}b_{31}^2 - 12a_{13}^2b_{40} \\
&\quad - 5a_{04}a_{22}b_{40} - 2a_{04}b_{22}b_{40})/4; \\
Y_2^{(6,7)} &= i(-a_{22}a_{31}b_{13} - 2a_{13}a_{31}b_{22} - a_{31}b_{13}b_{22} + 3a_{13}a_{22}b_{31} + 6a_{22}b_{13}b_{31} \\
&\quad + 11a_{13}b_{22}b_{31} - 4b_{13}b_{22}b_{31} + 12a_{04}b_{31}^2 + 10a_{13}^2b_{40} + 2a_{04}a_{22}b_{40} \\
&\quad + 6a_{13}b_{13}b_{40} + 5a_{04}b_{22}b_{40})/4;
\end{aligned}
$$

$$
\begin{aligned}
Y_1^{(9,8)} &= i(132a_{13}a_{22}^2a_{31} + 36a_{13}^2a_{31}^2 + 108a_{04}a_{22}a_{31}^2 - 150a_{22}^2a_{31}b_{13} - 6a_{13}a_{31}^2b_{13} \\
&\quad + 6a_{31}^2b_{13}^2 + 192a_{13}a_{22}a_{31}b_{22} - 72a_{04}a_{31}^2b_{22} + 204a_{22}a_{31}b_{13}b_{22} \\
&\quad - 324a_{13}a_{31}b_{22}^2 + 18a_{31}b_{13}b_{22}^2 - 198a_{13}a_{22}^2b_{31} - 126a_{13}^2a_{31}b_{31} \\
&\quad + 192a_{04}a_{22}a_{31}b_{31} + 24a_{22}^2b_{13}b_{31} - 24a_{13}a_{31}b_{13}b_{31} + 6a_{31}b_{13}^2b_{31} \\
&\quad - 132a_{13}a_{22}b_{22}b_{31} - 267a_{04}a_{31}b_{22}b_{31} + 24a_{22}b_{13}b_{22}b_{31} + 18a_{13}b_{22}^2b_{31} \\
&\quad - 162a_{13}^2b_{31}^2 - 384a_{04}a_{22}b_{31}^2 - 18a_{13}b_{13}b_{31}^2 - 132a_{04}b_{22}b_{31}^2 - 396a_{13}^2a_{22}b_{40} \\
&\quad - 78a_{04}a_{22}^2b_{40} - 200a_{04}a_{13}a_{31}b_{40} + 27a_{13}a_{22}b_{13}b_{40} - 180a_{13}^2b_{22}b_{40} \\
&\quad - 48a_{04}a_{22}b_{22}b_{40} + 90a_{13}b_{13}b_{22}b_{40} - 800a_{04}a_{13}b_{31}b_{40} - 56a_{04}b_{13}b_{31}b_{40} \\
&\quad - 48a_{04}^2b_{40}^2)/48; \\
Y_2^{(8,9)} &= i(-18a_{22}^2a_{31}b_{13} - 6a_{13}a_{31}^2b_{13} - 6a_{31}^2b_{13}^2 - 24a_{13}a_{22}a_{31}b_{22} \\
&\quad - 204a_{22}a_{31}b_{13}b_{22} - 24a_{13}a_{31}b_{22}^2 + 150a_{31}b_{13}b_{22}^2 - 18a_{13}a_{22}^2b_{31} \\
&\quad + 18a_{13}^2a_{31}b_{31} - 90a_{04}a_{22}a_{31}b_{31} + 324a_{22}^2b_{13}b_{31} + 24a_{13}a_{31}b_{13}b_{31} \\
&\quad + 6a_{31}b_{13}^2b_{31} + 132a_{13}a_{22}b_{22}b_{31} - 27a_{04}a_{31}b_{22}b_{31} - 192a_{22}b_{13}b_{22}b_{31} \\
&\quad + 198a_{13}b_{22}^2b_{31} - 132b_{13}b_{22}^2b_{31} + 162a_{13}^2b_{31}^2 + 180a_{04}a_{22}b_{31}^2 + 126a_{13}b_{13}b_{31}^2 \\
&\quad - 36b_{13}^2b_{31}^2 + 396a_{04}b_{22}b_{31}^2 + 132a_{13}^2a_{22}b_{40} + 56a_{04}a_{13}a_{31}b_{40} \\
&\quad + 267a_{13}a_{22}b_{13}b_{40} + 72a_{22}b_{13}^2b_{40} + 384a_{13}^2b_{22}b_{40} + 48a_{04}a_{22}b_{22}b_{40} \\
&\quad - 192a_{13}b_{13}b_{22}b_{40} - 108b_{13}^2b_{22}b_{40} + 78a_{04}b_{22}^2b_{40} + 800a_{04}a_{13}b_{31}b_{40} \\
&\quad + 200a_{04}b_{13}b_{31}b_{40} + 48a_{04}^2b_{40}^2)/48.
\end{aligned}
$$

Then, using (2.7) for the calculation of p_4 and computing the series expansions (2.6) in order to find p_8, p_{12} and p_{16} we obtain the first four non-zero reduced isochronicity quantities (by the reduced quantities we mean the polynomials obtained in such way that in formulas (2.7) and their extensions to any p_{2k} only terms containing the highest order coefficients of the normal form are taking into account; it is sufficient to work with the reduced quantities since the other terms of p_{2k} are in the ideal $\langle p_2, \dots, p_{2k-2}\rangle$) of system (3.1) as follows:

$$
\begin{aligned}
p_4 &= \frac{1}{2}(a_{22} + b_{22}); \quad p_8 = -\frac{1}{2}(a_{31}b_{13} - a_{31}a_{13} - b_{13}b_{31} - 3a_{13}b_{31} - 2a_{04}b_{40}); \\
p_{12} &= \frac{1}{8}(-4a_{13}a_{22}a_{31} - 2a_{22}a_{31}b_{13} + 4a_{13}a_{31}b_{22} - 2a_{31}b_{13}b_{22} + 14a_{13}a_{22}b_{31} \\
&\quad + 6a_{04}a_{31}b_{31} + 4a_{22}b_{13}b_{31} + 14a_{13}b_{22}b_{31} - 4b_{13}b_{22}b_{31} + 22a_{04}b_{31}^2 + 22a_{13}^2b_{40} \\
&\quad + 7a_{04}a_{22}b_{40} + 6a_{13}b_{13}b_{40} + 7a_{04}b_{22}b_{40});
\end{aligned}
$$

$$\begin{aligned}
p_{16} = \frac{1}{48}(&-66a_{13}a_{22}^2a_{31} - 18a_{13}^2a_{31}^2 - 54a_{04}a_{22}a_{31}^2 + 66a_{22}^2a_{31}b_{13} - 6a_{31}^2b_{13}^2 \\
&- 108a_{13}a_{22}a_{31}b_{22} + 36a_{04}a_{31}^2b_{22} - 204a_{22}a_{31}b_{13}b_{22} + 150a_{13}a_{31}b_{22}^2 \\
&+ 66a_{31}b_{13}b_{22}^2 + 90a_{13}a_{22}^2b_{31} + 72a_{13}^2a_{31}b_{31} - 141a_{04}a_{22}a_{31}b_{31} \\
&+ 150a_{22}^2b_{13}b_{31} + 24a_{13}a_{31}b_{13}b_{31} + 132a_{13}a_{22}b_{22}b_{31} + 120a_{04}a_{31}b_{22}b_{31} \\
&- 108a_{22}b_{13}b_{22}b_{31} + 90a_{13}b_{22}^2b_{31} - 66b_{13}b_{22}^2b_{31} + 162a_{13}^2b_{31}^2 + 282a_{04}a_{22}b_{31}^2 \\
&+ 72a_{13}b_{13}b_{31}^2 - 18b_{13}^2b_{31}^2 + 264a_{04}b_{22}b_{31}^2 + 264a_{13}^2a_{22}b_{40} + 39a_{04}a_{22}^2b_{40} \\
&+ 128a_{04}a_{13}a_{31}b_{40} + 120a_{13}a_{22}b_{13}b_{40} + 36a_{22}b_{13}^2b_{40} + 282a_{13}^2b_{22}b_{40} \\
&+ 48a_{04}a_{22}b_{22}b_{40} - 141a_{13}b_{13}b_{22}b_{40} - 54b_{13}^2b_{22}b_{40} + 39a_{04}b_{22}^2b_{40} \\
&+ 800a_{04}a_{13}b_{31}b_{40} + 128a_{04}b_{13}b_{31}b_{40} + 48a_{04}^2b_{40}^2).
\end{aligned}$$

We now look for the linearizability variety of system (3.1).

Proposition 3.3. *For system* (3.1),

$$V_{\mathscr{L}} = \mathbf{V}(\mathcal{Y}_8) = V_{\mathscr{C}} \cap \mathbf{V}(P_8). \tag{3.3}$$

Proof. Using the routine `minAssGTZ` of Singular we found that the minimal associate primes of ideals $\langle \mathcal{Y}_8 \rangle$ and $\langle \mathscr{B}_{14}, P_8 \rangle$ are the same. Namely, they are the ideals:

$$\begin{gathered}
Q_1 = \langle b_{40}, b_{31}, a_{13}, b_{22}, a_{22}, b_{13} \rangle, \quad Q_2 = \langle b_{40}, b_{31}, b_{22}, a_{22}, a_{31} \rangle, \\
Q_3 = \langle b_{40}, a_{04}, b_{22}, a_{22}, b_{13} - 3a_{13}, a_{31} - 3b_{31} \rangle, \\
Q_4 = \langle b_{40}, a_{04}, b_{22}, a_{22}, b_{13} + a_{13}, a_{31} + b_{31}, \rangle \\
Q_5 = \langle a_{04}, a_{13}, b_{22}, a_{22}, b_{13} \rangle, \quad Q_6 = \langle a_{04}, b_{31}, a_{13}, b_{22}, a_{22}, a_{31} \rangle.
\end{gathered}$$

By the results of [13] systems with the coefficients from the varieties of these ideals are linearizable. This proves (3.3). □

We now can estimate the number of critical periods near a center at the origin of system (1.6).

Theorem 3.4. *At most 3 critical periods bifurcate from nonlinear centers of system* (1.6).

Proof. By Proposition 3.3, part (a) of Theorem 3.1 holds with $K = 8$. We then check that in $\mathbb{C}[V_{\mathcal{C}}]$:

$$[p_8] \notin \langle [p_4] \rangle, \quad [p_{12}] \notin \langle [p_4], [p_8] \rangle, \quad [p_{16}] \notin \langle [p_4], [p_8], [p_{12}] \rangle. \tag{3.4}$$

To this end, with the routine `radical` of the computer algebra system Singular we compute the radical of the Bautin ideal $\mathscr{B} = \mathscr{B}_{14}$ denoted $\mathcal{R}_{14}$, that is,

$$\mathcal{R}_{14} = \sqrt{\mathscr{B}_{14}}$$

(one can also compute $\mathcal{R}_{14}$ using the routine `intersect` of Singular and the ideals $I_1 - I_4$ given in the statement of Proposition 3.2 since it is follows from the proof of Proposition 3.2 that $\mathcal{R}_{14} = \cap_{k=1}^4 I_k$). Then with the `reduce` of Singular we check that for $k = 2, 3, 4$ the remainder of the division of the polynomial p_{4k} by a Groebner basis of the ideal

$$\langle p_4, \dots, p_{4(k-1)}, \mathcal{R}_{14} \rangle$$

is nonzero. That means, that (3.4) holds, which, in turn, yields that the Bautin depth of $\widetilde{P}_8$ in $\mathbb{C}[V_{\mathcal{C}}]$ is 4.

Then, with the routine `primdecGTZ` [4, 8] of Singular we have computed the primary decomposition of the ideal

$$Q = \langle P_8, \mathcal{R}_{14} \rangle$$

and found that

$$Q = \cap_{k=1}^{13} Q_k,$$

where $Q_1, \dots, Q_6$ are prime ideals given in the statement of Proposition 3.2, $Q_7, \dots, Q_{13}$ are some ideals defined by many polynomials (for these reason we do not present them here, however the interested reader can easily compute Q and the primary decomposition $Q = \cap_{k=1}^{13} Q_k$ with an appropriate computer algebra system using the ideals P_8 and $I_1 - I_4$ presented above) whose associate primes are:

$$\begin{aligned}
\sqrt{Q_7} &= \langle b_{40}, b_{31}, b_{22}, a_{22}, b_{13} - 3a_{13}, a_{31} \rangle, \\
\sqrt{Q_8} &= \langle b_{40}, b_{31}, b_{22}, a_{22}, b_{13} + a_{13}, a_{31} \rangle, \\
\sqrt{Q_9} &= \langle a_{04}, a_{13}, b_{22}, a_{22}, b_{13}, a_{31} - 3b_{31} \rangle, \\
\sqrt{Q_{10}} &= \langle a_{04}, a_{13}, b_{22}, a_{22}, b_{13}, a_{31} + b_{31} \rangle, \\
\sqrt{Q_{11}} &= \langle b_{40}, b_{31}, a_{13}, b_{22}, a_{22}, b_{13}, a_{31} \rangle, \\
\sqrt{Q_{12}} &= \langle a_{04}, b_{31}, a_{13}, b_{22}, a_{22}, b_{13}, a_{31} \rangle, \\
\sqrt{Q_{13}} &= \langle b_{40}, a_{04}, b_{31}, a_{13}, b_{22}, a_{22}, b_{13}, a_{31} \rangle.
\end{aligned}$$

Thus, $\sqrt{Q_k} = Q_k$ for $k = 1, \dots, 6$ and $\sqrt{Q_k} \neq Q_k$ for $k = 7, \dots, 13$; that is, the ideals R and N from the statement of Theorem 3.1 are

$$R = \cap_{k=1}^{6} Q_k \quad \text{and} \quad N = \cap_{k=7}^{13} Q_k.$$

To find systems (1.6) whose coefficients are in the variety of the ideal N we perform as follows. Let $T_s = \sqrt{Q_{s+6}}$ for $s = 1, \dots, 7$. Using the `intersect` of Singular we compute the ideal $T = \cap_{k=1}^{7} T_k$ and find that

$$\begin{aligned}
T = \langle & a_{22}, b_{22}, a_{04}b_{40}, a_{13}b_{40}, b_{13}b_{40}, a_{04}b_{31}, a_{04}a_{31}, a_{13}b_{31}, \\
& b_{13}b_{31}, a_{13}a_{31}, -3a_{13}^2 - 2a_{13}b_{13} + b_{13}^2, a_{31}b_{13}, a_{31}^2 - 2a_{31}b_{31} - 3b_{31}^2 \rangle.
\end{aligned}$$

Clearly, $\mathbf{V}(N) = \mathbf{V}(T)$ in $\mathbb{C}^8$.

Since in the case when (3.1) is a complexification of the real system the parameters a_{ks} and b_{sk} are complex conjugate we perform the change of variables

$$\begin{aligned}
a_{31} &= A_{31} + iB_{31}, \quad & b_{13} &= A_{31} - iB_{31}, \\
a_{22} &= A_{22} + iB_{22}, \quad & b_{22} &= A_{22} - iB_{22}, \\
a_{13} &= A_{13} + iB_{13}, \quad & b_{31} &= A_{13} - iB_{13}, \\
a_{04} &= A_{04} + iB_{04}, \quad & b_{40} &= A_{04} - iB_{04},
\end{aligned}$$

where A_{ks}, B_{ks} are real parameters. Substituting these values into the ideal T and computing a Groebner bases of the obtained ideal in the ring

$$\mathbb{Q}[A_{04}, A_{13}, A_{22}, A_{31}, B_{04}, B_{13}, B_{22}, B_{31}]$$

we find the ideal

$$\begin{aligned}T_{\mathbb{R}} = \langle &A_{22}, B_{22}, (B_{13}-B_{31})(3B_{13}+B_{31}), A_{31}^2+B_{31}^2,\\ &A_{31}B_{13}+A_{13}B_{31}, 3A_{13}B_{13}+2A_{31}B_{13}+A_{31}B_{31},\\ &A_{13}A_{31}-B_{13}B_{31}, 3A_{13}^2+2B_{13}B_{31}+B_{31}^2, A_{31}B_{04}+A_{04}B_{31},\\ &-A_{13}B_{04}+A_{04}B_{13}, A_{04}A_{31}-B_{04}B_{31}, A_{04}A_{13}+B_{04}B_{13}, A_{04}^2+B_{04}^2\rangle.\end{aligned} \tag{3.5}$$

The basis of $T_{\mathbb{R}}$ contains the polynomials

$$A_{22}, B_{22}, A_{31}^2+B_{31}^2, A_{04}^2+B_{04}^2.$$

Since $A_{31}, B_{31}, A_{04}, B_{04}$ are real parameters we conclude that

$$A_{22}=B_{22}=A_{31}=B_{31}=A_{04}=B_{04}=0. \tag{3.6}$$

Substituting the values from (3.6) into polynomials of the ideal $T_{\mathbb{R}}$ given in (3.5) we find that also

$$A_{13}=B_{13}=0.$$

It means that the only system of the form (1.6) whose parameters are in the variety of the ideal N is the linear system (1.2), that is, the system $\dot{x}=ix$. Thus, by Theorem 3.1 at most 3 critical periods bifurcate from non-linear isochronous centers of system (1.6). □

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