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Minimax principles for critical-point theory in applications to quasilinear boundary-value problems *

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Abstract

Using the variational method developed in [7], we establish the existence of solutions to the equation $-\Delta_p u = f(x,u)$ with Dirichlet boundary conditions. Here Δ_p denotes the p-Laplacian and $\int_0^s f(x,t)\,dt$ is assumed to lie between the first two eigenvalues of the p-Laplacian.

1 Introduction

Consider the Dirichlet problem for the p-Laplacian ($p > 1$),

$$\begin{aligned} -\Delta_p u &= f(x,u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth; that is,

$$|f(x,s)| \le A|s|^{q-1} + B, \quad \forall s \in \mathbb{R},\ \text{a.e. } x \in \Omega, \tag{F_0}$$

and some positive constants A, B, where $1 \le q < \frac{Np}{N-p}$ if $N \ge p+1$, and $1 \le q < \infty$ if $1 \le N < p$. It is well known that weak solutions $u \in W_0^{1,p}(\Omega)$ of (1) are the critical points of the C^1 functional

$$\Phi(u) = \frac{1}{p}\int |\nabla u|^p\,dx - \int F(x,u)\,dx\,,$$

where $F(x,s) = \int_0^s f(x,t)\,dt$.

We are interested in the situation where Φ is strongly indefinite in the sense that it is neither bounded from above or from below. Let λ_1 and λ_2 be the first and the second eigenvalues of $-\Delta_p$ on $W_0^{1p}(\Omega)$. It is known that $\lambda_1 > 0$ is a simple eigenvalue, and that $\sigma(-\Delta_p)\cap]\lambda_1, \lambda_2[= \emptyset$, where $\sigma(-\Delta_p)$ is the spectrum of $-\Delta_p$, (cf. [1]).

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We shall assume the following conditions

$$\lim_{|s|\to\infty}[f(x,s)s - pF(x,s)] = \pm\infty \quad \text{uniformly for a.e. } x\in\Omega\,, \tag{$F_1^\pm$}$$

$$\limsup_{s\to\infty}\frac{pF(x,s)}{|s|^p} < \lambda_2\,, \tag{F_2}$$

and

$$\Big[\int F(x,t\varphi_1)\,dx - \frac{1}{p}|t|^p\Big] \to \infty, \quad \text{as } |t|\to\infty\,, \tag{F_3}$$

where φ_1 is the normalized λ_1- eigenfunction. We note that φ_1 does not change sign in Ω.

Now, we are ready to state our main result.

Theorem 1.1 *Assume* $(F_0), (F_1^+), (F_2)$ *and* (F_3). *Then (1) has a weak solution in* $W_0^{1,p}(\Omega)$.

Similarly, we have

Theorem 1.2 *Assume* $(F_0), (F_1^-), (F_2)$ *and* (F_3). *Then (1) has a weak solution in* $W_0^{1,p}(\Omega)$.

As an immediate consequence, we obtain the following corollary.

Corollary 1.1 *If* F *satisfies* $(F_0), (F_1^-)$, *and*

$$\lambda_1 \le \liminf_{s\to\infty}\frac{pF(x,s)}{|s|^p} \le \limsup_{s\to\infty}\frac{pF(x,s)}{|s|^p} < \lambda_2, \tag{F_3'}$$

then (1) has a solution.

The nonlinear case ($p\neq 2$) when the nonlinearity $pF(x,s)/|s|^p$ stays asymptotically between λ_1 and λ_2 has been studied by just a few authors. A contribution in this direction is [8], where the authors use a topological method to study the case $N=1$. Another contribution was made by D. G. Costa and C.A.-Magalhães [5] who studied the case when $pF(x,s)/|s|^p$ interacts asymptotically with the first eigenvalue λ_1.

We point out, that the variational method used in the linear case ($p=2$) can not be extended to the nonlinear case. To overcome this difficulty, we introduce the idea of linking and proving an abstract min-max theorem.

2 Preliminaries. An abstract theorem

In this section we prove a critical-point theorem for the real functional Φ on a real Banach space X. Let X^* denote the dual of X, and $\|.\|$ denote the norm in X and in X^*. For Φ a continuously Fréchet differentiable map from X to $\mathbb{R}$, let $\Phi'(u)$ denote its Fréchet derivative. For $\Phi\in C^1(X,\mathbb{R})$ and $c\in\mathbb{R}$, let

$$K_c = \{x\in E : \Phi(x)=c, \Phi'(x)=0\},$$
$$\Phi^c = \{x\in X : \Phi(u)\ge c\}.$$

Thus K_c is the set of critical points of Φ, and Φ has value c.

Definition Given $c \in \mathbb{R}$, we shall say that $\Phi \in C^1(X, \mathbb{R})$ satisfies the condition (C_c), if

i) any bounded sequence $(u_n) \subset E$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ possesses a convergent subsequence;

ii) there exist constants $\delta, R, \alpha > 0$ such that

$$\|\Phi'(u)\|\|u\| \geq \alpha \text{ for any } u \in \Phi^{-1}([c-\delta, c+\delta]) \text{ with } \|u\| \geq R.$$

Definition If $\Phi \in C^1(X, \mathbb{R})$ satisfies the condition (C_c) for every $c \in \mathbb{R}$, we say that Φ satisfies (C).

This condition was introduced by Cerami [3], and recently was generalized by the first author in [7]. It was shown in [2] that condition (C) suffices to get a deformation lemma.

Lemma 2.1 (Deformation Lemma) *Let X be a real Banach space and let $\Phi \in C^1(X, \mathbb{R})$ satisfy (C_c). Then there exists $\bar{\varepsilon} > 0$, $\varepsilon \in]0, \bar{\varepsilon}[$ and an homeomorphism $\eta : X \to X$ such that:*

1. *$\eta(x) = x$ if $x \notin \Phi^{-1}[c - \bar{\varepsilon}, c + \bar{\varepsilon}[$;*
2. *If $K_c = \emptyset$, $\eta(\Phi^{c-\varepsilon}) \subset \Phi^{c+\varepsilon}$.*

Now, we define the class of closed symmetric subsets of X as

$$\Sigma = \{A \subset X : A closed, A = -A\}.$$

Definition For a non-empty set A in Σ, following Coffman [4], we define the Krasnoselskii genus as

$$\gamma(A) = \begin{cases} \inf\{m : \exists h \in C(A, \mathbb{R}^m \setminus \{0\}); h(-x) = -h(x)\} \\ \infty \quad \text{if } \{...\} \text{ is empty, in particular if 0 is in } A. \end{cases}$$

For A empty we define $\gamma(A) = 0$.

Next we state the existence of critical points for a class of perturbations of p-homogeneous real valued C^1 functionals defined on a real Banach space.

Theorem 2.1 *Let Φ be a C^1 functional on X satisfying condition (C), and let Q be a closed connected subset such that $\partial Q \cap (-\partial Q) \neq \emptyset$. Assume that*

i) *$\forall K \in A_2$ there exists $v_K \in K$ and there exists $\beta \in \mathbb{R}$ such that $\Phi(v_K) \geq \beta$ and $\Phi(-v_K) \geq \beta$*

ii) *$a = \sup_{\partial Q} \Phi < \beta$.*

iii) *$\sup_Q \Phi(x) < \infty$.*

Then Φ has a critical value $c \geq \beta$.

For the proof of this theorem, we will use lemma 1.1 and the following lemma.

Lemma 2.2 *Under the hypothesis of Theorem 2.1, we have*

$$h(Q) \cap \Phi^{\delta} \neq \emptyset; \quad \forall \delta, \delta < \beta, \forall h \in \Gamma, \qquad (H_1),$$

where $\Gamma = \{h \in C(X, X) : h(x) = x \text{ in } \partial Q\}$.

Proof : First we claim that *If A is nonempty connected symmetric then* $\gamma(A) > 1$.

Indeed, if $\gamma(A) = 1$, then there exists a map h continuous and even such that $h(A) \subset \mathbb{R} \setminus \{0\}$. Since h is even continuous, $h(A)$ is a symmetric interval. Therefore, $0 \in h(A)$ which is a contradiction and the claim is proved.

Let $h \in \Gamma$ and put $K = \overline{h(Q) \cup -h(Q)}$. Clearly we have

$$\partial Q \cap -\partial Q \subset h(Q) \cap -h(Q).$$

Therefore, K is a closed, connected, symmetric subset, and by the claim above $\gamma(K) \geq 2$.

On the other hand, by **i)** of Theorem 2.1 there exists $v_K \in K$ such that

$$\Phi(v_K) \geq \beta \quad \text{and} \quad \Phi(-v_K) \geq \beta\,.$$

Let $\delta < \beta$, then there exists $v_1 \in h(Q) \cup -h(Q)$ such that

$$\Phi(v_1) \geq \delta \quad \text{and} \quad \Phi(-v_1) \geq \delta\,.$$

Indeed, if this is not the case, then for every $v \in h(Q) \cup -h(Q)$ we have $\Phi(v) < \delta$ or $\Phi(-v) < \delta$. Then, since Φ is continuous, for every $v \in K$ $\Phi(v) \leq \delta$ or $\Phi(-v) \leq \delta$. Which is a contradiction. Moreover, $h(Q) \cap \Phi^{\delta} \neq \emptyset$, and the conclusion easily follows. ♢

Proof of Theorem 2.1. Suppose that $c = \inf_{h\in\Gamma} \sup_{x\in Q} \Phi(h(x))$ is not a critical value (i.e. $K_c = \emptyset$). Let $\bar\varepsilon < \beta - a$, then by lemma 2.1 there exists $\eta : X \to X$ an homeomorphism such that

$$\begin{gathered}\eta(x) = x \quad \text{if } x \notin \Phi^{-1}[c-\bar\varepsilon, c+\bar\varepsilon[, \text{ with } \bar\varepsilon < \gamma - a;\\ \eta(\Phi^{c-\varepsilon}) \subset \Phi^{c+\varepsilon}\,.\end{gathered} \tag{2}$$

By (H_1) there exists a sequence $(x_n)_n \subset Q$ such that

$$\gamma \leq \sup_n \Phi(h(x_n)), \quad \forall h \in \Gamma\,.$$

This implies $\beta \leq c$. Then by **iii**) we have $\beta \leq c < \infty$.

On the other hand, since $\bar\varepsilon < \beta - a$ and $\beta \leq c$, it results from *ii*) that

$$\Phi(x) < c - \bar\varepsilon, \quad \forall x \in \partial Q\,.$$

This leads to

$$\eta(x) = x \quad \text{for } x \text{ in } \partial Q. \tag{3}$$

Hence, we have $\eta^{-1} \circ h \in \Gamma$, and by the definition of c there exists $\tilde x \in Q$ such that

$$\Phi\left(\eta^{-1} \circ h(\tilde x)\right) \geq c - \varepsilon\,.$$

Hence, by (2) we obtain

$$c+\varepsilon \leq \Phi\left(\eta\left[\eta^{-1}\circ h(\tilde{x})\right]\right)=\Phi(h(\tilde{x})).$$

Therefore, we get the contradiction

$$c+\varepsilon \leq \inf_{h\in\Gamma}\sup_{x\in Q}\Phi(h(x))=c\,.$$

Which completes the present proof. ♢

3 Proof of Theorem 1.1

In this section we shall use Theorem 2.1 for proving Theorem 1.1. The Sobolev space $W_0^{1,p}(\Omega)$ will be the Banach space X, endowed with the norm $\|u\|=(\int_\Omega |\nabla u|^p\,dx)^{\frac{1}{p}}$ and the C^1 functional Φ will be

$$\Phi(u)=\frac{1}{p}\int_\Omega |\nabla u|^p\,dx-\int_\Omega F(x,u)\,dx\,.$$

To apply Theorem 2.1, we shall do separate studies of the "compactness" of Φ and its "geometry". First, we prove that Φ satisfies the condition (C).

Lemma 3.1 *Assume F satisfies $(F_0),(F_2)$ and (F_1^+). Then for every $c\in\mathbb{R}$, Φ satisfies the condition (C_c) on $W_0^{1,p}(\Omega)$.*

Proof: We first verify the condition $(C_c)(i)$. Let $(u_n)_n\subset W_0^{1,p}(\Omega)$, be bounded and such that $\Phi'(u_n)\to 0$ in $W^{-1,p'}(\Omega)$. We have

$$-\Delta_p u_n - f(x,u_n)\to 0 \quad \text{in } W^{-1,p'}(\Omega).$$

And as $-\Delta_p$ is an homeomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ (cf [9]), we have

$$u_n-(-\Delta)_p^{-1}[f(x,u_n)]\to 0 \quad \text{in } W_0^{1,p}(\Omega)\,. \tag{4}$$

Since (u_n) is bounded, there is a subsequence (u_n') weakly converging to some $u_0\in W_0^{1,p}(\Omega)$. On the other hand, as the map $u\mapsto f(x,u)$ is completely continuous from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ then

$$(-\Delta_p)^{-1}[f(x,u_n')]\to(-\Delta_p)^{-1}[f(x,u_0)] \quad \text{in } W_0^{1,p}(\Omega). \tag{5}$$

By (4), (5) we deduce that (u_n') converges in $W_0^{1,p}(\Omega)$.

Let us now prove that the condition $(C_c)(ii)$ is satisfied for every $c\in\mathbb{R}$. Assume that F satisfies $(F_0),(F_2),(F_1^+)$ and again, by contradiction, let $c\in\mathbb{R}$ and $(u_n)_n\subset W_0^{1,p}(\Omega)$ such that:

$$\Phi(u_n)\to c \tag{6}$$

$$\|u_n\||\langle\Phi'(u_n),v\rangle|\leq\varepsilon_n\|v\| \quad \forall v\in W_0^{1,p}(\Omega) \tag{7}$$

$$\|u_n\|\to\infty,\varepsilon_n=\|u_n\|\|\Phi'(u_n)\|\to 0,\quad \text{as } n\to\infty,$$

where $\langle .,. \rangle$ is the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$. It follows that

$$\lim_{n\to\infty} |\langle \Phi'(u_n), u_n \rangle - p\Phi(u_n)| = pc\,.$$

More precisely, we have

$$\lim_{n\to\infty} \int_\Omega [f(x,u_n)u_n(x) - pF(x,u_n)]\;dx = pc\,. \tag{8}$$

Put $z_n = u_n/\|u_n\|$, we have $\|z_n\| = 1$ and, passing if necessary to a subsequence, we may assume that: $z_n \rightharpoonup z$ weakly in $W_0^{1,p}(\Omega)$, $z_n \to z$ strongly in $L^p(\Omega)$ and $z_n(x) \to z(x)$ a.e. in Ω.

On the other hand, note that $\limsup_{s\to\infty} \frac{pF(x,s)}{|s|^p} < \lambda_2$ and (F_0) implies

$$F(x,s) \le \frac{\lambda_2}{p}|s|^p + b(x), \quad \forall s \in \mathbb{R}, b \in L^p(\Omega)\,. \tag{9}$$

Therefore, passing to the limit in the equality

$$\frac{1}{\|u_n\|^p}\Phi(u_n) = \frac{1}{p} - \frac{1}{\|u_n\|^p}\int F(x,u_n)\,dx$$

and, using (9), it results

$$\frac{1}{p}(1 - \lambda_2\|z\|^p_{L^p}) \le 0$$

which shows that $z \not\equiv 0$. Now, by (F_1^+) and (F_0) there exist $M > 0$, such that

$$f(x,s)s - pF(x,s) \ge -M + b_1(x), \forall s \in \mathbb{R}, \quad a.e. x \in \Omega\,;$$

hence,

$$\begin{aligned}\int_\Omega [f(x,u_n)u_n(x) - pF(x,u_n)]\;dx \quad &\ge \quad \int_{\{x:z(x)\neq 0\}} f(x,u_n)u_n(x) - pF(x,u_n)\,dx \\ &\quad -M|\{x\in\Omega : z(x) = 0\}| - \|b_1\|_{L^1}.\end{aligned}$$

An application of Fatou's lemma yields

$$\int_\Omega [f(x,u_n)u_n(x) - pF(x,u_n)]\;dx \to \infty, \quad \text{as } n \to \infty,$$

which is a contradiction to (8). Thus the proof of lemma 3.1 is complete. $\diamondsuit$

Now, we will show that Φ satisfies the geometric conditions $i), ii), iii)$ of Theorem 2.1.

Lemma 3.2 *Assume that F satisfies the hypothesis of Theorem 1.1. Then we have*

i) $\Phi(v) \to -\infty$, *as* $\|v\| \to \infty$ *with* $v \in X_1$

ii) $\forall K \in A_2$, *there exists* $v_K \in K$, *and* $\beta \in \mathbb{R}$ *such that* $\Phi(v_k) \ge \beta$ *and* $\Phi(-v_K) \ge \beta$.

Proof: i) Let X_1 denote the eigenspace associated to the eigenvalue λ_1. Since dim $X_1 = 1$, we set $X_1 = \{t\varphi_1 : t \in \mathbb{R}\}$. Thus for every $v \in X_1, v = t\varphi_1, t \in \mathbb{R}$, we obtain

$$\begin{aligned}\Phi(v) &= \frac{1}{p}\int |\nabla t\varphi_1|^p - \int F(x,t\varphi_1)\,dx \\ &= \frac{1}{p}|t|^p\int |\nabla\varphi_1|^p - \int F(x,t\varphi_1)\,dx.\end{aligned}$$

Since $\int |\nabla\varphi_1|^p = 1$, by (F_3), we obtain

$$\Phi(v) = -\left[\int F(x,t\varphi_1)\,dx - \frac{1}{p}|t|^p\right] \to -\infty, \quad \text{as } |t| \to \infty\,.$$

ii) Let us recall that the Lusternik-Schnirelaman theory gives

$$\lambda_2 = \inf_{K\in A_2}\sup\Big\{\int |\nabla u|^p, \int |u|^p = 1, u \in K\Big\}.$$

However, for every $K \in A_2$ and $\epsilon > 0$ there exists $v_K \in K$ such that

$$(\lambda_2 - \epsilon)\int |v_K|^p\,dx \le \int |\nabla v_K|^p\,dx\,. \tag{10}$$

Indeed, we shall treat the following two possible cases:
Case 1. $0 \in K$, (10) is proved by setting $v_K = 0$.
Case 2. $0 \notin K$, we consider

$$\Pi : K \to \tilde{K}, v \mapsto \frac{v}{\|v\|_{L^p}}.$$

Note that Π is an odd map. By the genus properties we have $\gamma(\Pi(K)) \ge 2$ and by the definition of λ_2 there exists $\tilde{v_K} \in \tilde{K}$ such that

$$\int |\tilde{v_K}|^p\,dx = 1 \quad \text{and} \quad (\lambda_2 - \epsilon) \le \int |\nabla\tilde{v_K}|^p\,dx\,.$$

Thus (10) is satisfied by setting $v_K = \Pi^{-1}(\tilde{v}_K)$.

On the other hand, we note that $\limsup_{s\to\infty}\frac{pF(x,s)}{|s|^p} < \lambda_2$ and (F_0) implies

$$F(x,s) \le (\lambda_2 - 2\epsilon)\frac{|s|^p}{p} + D, \forall s \in \mathbb{R} \tag{11}$$

for some constant $D > 0$. Therefore, by using (10) and (11), we obtain the estimate

$$\begin{aligned}\Phi(v_K) &\ge \frac{1}{p}\int |\nabla v_K|^p\,dx - \frac{(\lambda_2 - 2\epsilon)}{p}\int |v_K|^p\,dx - D|\Omega| \\ &\ge \frac{1}{p}\left[1 - \frac{(\lambda_2 - 2\epsilon)}{(\lambda_2 - \epsilon)}\right]\int |\nabla v_K|^p\,dx - D|\Omega|\,. \end{aligned}\tag{12}$$

The argument is similar for

$$\Phi(-v_K) \geq \frac{1}{p}\left[1 - \frac{(\lambda_2 - 2\epsilon)}{(\lambda_2 - \epsilon)}\right] \int |\nabla v_K|^p \, dx - D|\Omega| \,. \tag{13}$$

It is clear from (12) and (13) that for every $K \in A_2$ we have

$$\Phi(\pm v_K) \geq -D|\Omega| = \beta.$$

Which completes the proof. $\diamondsuit$

Proof of theorem 1.1: In view of Lemmas 3.1 and 3.2, we may apply Theorem 2.1 letting $Q = B_R \cap X_1$, where, $B_R = \{u \in W_0^{1,p} : \|u\| \leq R\}$ with $R > 0$ being such that $\sup_{v \in \partial Q} \Phi(v) < \beta$. It follows that the functional Φ has a critical value $c \geq \beta$ and, hence, the problem (1) has a weak solution $u \in W_0^{1,p}(\Omega)$, the theorem is proved.

Proof of Corollary 1.1: The proof of this corollary follows closely the arguments in [5]. It suffices to prove that (F_1^-) and (F_3') implies (F_3). Let us suppose that $g(x,s) = f(x,s) - \lambda_1 |s|^{p-1}s$ and $G(x,s) = F(x,s) - \frac{1}{p}\lambda_1|s|^p$. Then, by (F_1^-), for every $M > 0$ there exists $s_M > 0$ such that

$$g(x,s)s - pG(x,s) \leq -M, \forall |s| \geq s_M, \text{ a.e. } x \in \Omega \,. \tag{14}$$

Using (14) and integrating the relation

$$\frac{d}{ds}\left[\frac{G(x,s)}{|s|^p}\right] = \frac{g(x,s)s - pG(x,s)}{|s|^{p+1}}$$

over an interval $[t, T] \subset [s_M, \infty[$ which was also explored in [6], we get

$$\frac{G(x,T)}{T^p} - \frac{G(x,t)}{t^p} \leq -\frac{M}{p}\left[\frac{1}{T^p} - \frac{1}{t^p}\right].$$

Therefore, since $\liminf_{T\to\infty} \frac{G(x,T)}{T^p} \geq 0$ by (F_3'), we obtain

$$G(x,t) \geq \frac{M}{p}, \forall t \geq s_M, \text{ a.e. } x \in \Omega$$

In the same way we show that $G(x,t) \geq \frac{M}{p}$, for every $t \leq -s_M$, and almost every $x \in \Omega$. By (F_3') and $M > 0$ being arbitrary, we have (F_3) which completes the proof. $\diamondsuit$

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