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EXISTENCE OF SOLUTIONS FOR ONE-DIMENSIONAL WAVE EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this article we study an initial and boundary-value problem with a nonlocal integral condition for a one-dimensional wave equation. We prove existence and uniqueness of classical solution and find its Fourier representation. The basis used consists of a system of eigenfunctions and adjoint functions.

1. Introduction

Certain problems of modern physics and technology can be effectively described in terms of nonlocal problems for partial differential equations. These nonlocal conditions arise mainly when the data on the boundary cannot be measured directly.

The first paper, devoted to second-order partial differential equations with nonlocal integral conditions goes back to Cannon [4]. Later, the problems with nonlocal integral conditions for parabolic equations were investigated by Kamynin [10], Ionkin [9], Yurchuk [18], Bouziani [2]; problems for elliptic equations with operator nonlocal conditions were considered by Mikhailov and Guschin [7], Scubachevski [17], Paneiah [13].

Then, Gordeziani and Avalishvili [5], Bouziani [3] devoted a few papers to nonlocal problems for hyperbolic equations. Pulkina [14, 15] studied the nonlocal analogue to classical Goursat problem.

In this paper we investigate the nonlocal analogue to classical mixed problem, which involves initial, boundary and nonlocal integral conditions. In the rectangular domain $D = \{(x,t) : 0 < x < l,\ 0 < t < T\}$, we consider the equation

$$\mathcal{L}U \equiv U_{tt} - U_{xx} = F(x,t) \tag{1.1}$$

with initial data

$$U(x,0) = \Phi(x), \quad U_t(x,0) = \Psi(x), \tag{1.2}$$

Dirichlet boundary condition

$$U(0,t) = 0 \tag{1.3}$$

and the nonlocal condition

$$\int_0^l U(x,t)\,dx = 0, \tag{1.4}$$

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where $\Phi(x)$, $\Psi(x)$ are given, $\Phi(x) \in C[0,l] \cap C^2(0,l)$, $\Psi(x) \in C[0,l] \cap C^1(0,l)$ and satisfy the compatibility conditions

$$\Phi(0) = 0, \quad \Psi(0) = 0, \quad \int_0^l \Phi(x)\,dx = \int_0^l \Psi(x)\,dx = 0.$$

Note that we do not lose generality by assuming that (1.3) and (1.4) are homogeneous. Indeed, if $U(0,t) = m(t)$ and $\int_0^l U(x,t)\,dx = n(t)$, we introduce a new unknown function $v(x,t) = U(x,t) - W(x,t)$, where

$$W(x,t) = (1 - \frac{2x}{l})m(t) + \frac{2x}{l^2}n(t).$$

Then (1.1) is converted into the similar equation

$$v_{tt} - v_{xx} = g(x,t), \quad g(x,t) = F(x,t) - \mathcal{L}W,$$

while the Dirichlet and integral conditions are now homogeneous.

The presence of integral conditions complicates the application of standard techniques. Therefore, we first reduce (1.1)-(1.4) to an equivalent problem.

Lemma 1.1. *Problem (1.1)-(1.4) is equivalent to (1.1)-(1.3) and*

$$U_x(0,t) - U_x(l,t) = \int_0^l F(x,t)\,dx. \tag{1.5}$$

Proof. Let $U(x,t)$ is a solution of (1.1)-(1.4). Integrating (1.1) with respect to x over $(0,l)$, and taking in account (1.4), we obtain

$$U_x(0,t) - U_x(l,t) = \int_0^l F(x,t)\,dx.$$

Let now $U(x,t)$ be a solution of (1.1)-(1.3), (1.5). We need only to show that $\int_0^l U(x,t)\,dx = 0$. For this end we integrate again (1.1) and obtain

$$\frac{d^2}{dt^2}\int_0^l U(x,t)\,dx = 0.$$

By virtue of the compatibility conditions,

$$\int_0^l U(x,0)\,dx = 0, \quad \int_0^l U_t(x,0)\,dx = 0\,.$$

Then $\int_0^l U(x,t)\,dx = 0$ is a unique solution to homogeneous Cauchy problem. $\square$

Introduce a new unknown function $u(x,t) = U(x,t) - w(x,t)$, where $w(x,t) = -\frac{x^2}{2l}\int_0^l F(x,t)\,dx$. Then (1.1)-(1.3), (1.5) is transformed now into

$$u_{tt} - u_{xx} = g(x,t), \tag{1.6}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \tag{1.7}$$

$$u(0,t) = 0, \tag{1.8}$$

$$u_x(0,t) = u_x(l,t), \tag{1.9}$$

where

$$\begin{aligned} g(x,t) &= F(x,t) + \frac{x^2}{2l}\int_0^l F_{tt}(x,t)\,dx - \frac{1}{l}\int_0^l F(x,t)\,dx, \\ \varphi(x) &= \Phi(x) + \frac{x^2}{2l}\int_0^l F(x,0)\,dx, \\ \psi(x) &= \Psi(x) + \frac{x^2}{2l}\int_0^l F_t(x,0)\,dx. \end{aligned}$$

2. Uniqueness

Theorem 2.1. *There exists at most one solution to (1.6)-(1.9).*

Proof. Let $u_1(x,t)$, $u_2(x,t)$ be two different solutions of (1.6)-(1.9). Then $u(x,t) = u_1(x,t) - u_2(x,t)$ is a nontrivial solution to the homogeneous problem

$$\begin{gathered} u_{tt} - u_{xx} = 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \\ u(0,t) = 0, \quad u_x(0,t) = u_x(l,t). \end{gathered}$$

As $u \in C^1(\bar{D}) \cap C^2(D)$, then $u(x,t)$ takes on certain value for $x = l$. Let $u(l,t) = \mu(t)$. Consider mixed problem for the equation $u_{tt} - u_{xx} = 0$ with homogeneous initial data and the boundary conditions

$$u(0,t) = 0, \quad u(l,t) = \mu(t).$$

Note, that $\mu(t)$ is required to satisfy the compatibility conditions $\mu(0) = 0$ and $\mu'(0) = 0$.

For all conditions to be homogeneous, we let $\tilde{u} = u - \frac{x}{l}\mu(t)$. Then, taking in account the compatibility conditions for $\mu(t)$, we obtain

$$\begin{gathered} \tilde{u}_{tt} - \tilde{u}_{xx} = \frac{x}{l}\mu''(t), \\ \tilde{u}(x,0) = 0, \quad \tilde{u}_t(x,0) = 0, \\ \tilde{u}(0,t) = 0, \quad \tilde{u}(l,t) = 0. \end{gathered}$$

It is well known that there exists unique solution $\tilde{u}(x,t)$ to this problem [1], hence $u(x,t)$ assumes the form

$$u(x,t) = \frac{2l}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\left(\int_0^t \mu''(\tau)\sin\frac{k\pi(t-\tau)}{l}\,d\tau\right)\sin\frac{k\pi x}{l} + \frac{x}{l}\mu(t).$$

Now we find that

$$\begin{aligned} u_x(0,t) &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^t \mu''(\tau)\sin\frac{k\pi(t-\tau)}{l}\,d\tau + \frac{1}{l}\mu(t), \\ u_x(l,t) &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^t \mu''(\tau)\sin\frac{k\pi(t-\tau)}{l}\,d\tau + \frac{1}{l}\mu(t) \end{aligned}$$

and consider

$$u_x(l,t) - u_x(0,t) = \frac{4}{\pi}\int_0^t \mu''(\tau)\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin\frac{(2m-1)\pi(t-\tau)}{l}\,d\tau.$$

As in [6]

$$\sum_{m=1}^{\infty} \frac{\sin{(2m-1)x}}{2m-1} = \begin{cases} \pi/4, & \text{if } 0 < x < \pi \\ -\pi/4, & \text{if } \pi < x < 2\pi\,. \end{cases}$$

Then by (1.9) we can write

$$0 = |u_x(l,t) - u_x(0,t)| = \int_0^t \mu''(\tau)\; d\tau.$$

Taking into account the compatibility conditions $\mu(0) = \mu'(0) = 0$, we easily obtain $\mu(t) \equiv 0$. Now from the uniqueness theorem [1], we obtain $u(x,t) \equiv 0$. □

3. Existence

Obviously, the solution to the problem (1.6)-(1.9), if it exists, is a sum of solutions to the following two problems:

Problem $\mathcal{H}$

$$\begin{gathered} u_{tt} - u_{xx} = 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \\ u(0,t) = 0, \quad u_x(0,t) = u_x(l,t) \end{gathered}$$

Problem $\mathcal{NH}$

$$\begin{gathered} u_{tt} - u_{xx} = g(x,t), \\ u(x,0) = u_t(x,0) = 0, \quad u(0,t) = 0, \quad u_x(0,t) = u_x(l,t). \end{gathered}$$

First consider problem $\mathcal{H}$ and use separation of variables. Let $u(x,t) = X(x)T(t)$. Substituting in the equation $u_{tt} - u_{xx} = 0$ and taking into account (1.8), (1.9), we obtain

$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X'(0) = X'(l). \tag{3.1}$$

Note that problem (3.1) is not self-adjoint: The adjoint problem is

$$Y''(x) + \bar{\lambda} Y(x) = 0, \quad Y'(l) = 0, \quad Y(l) = Y(0). \tag{3.2}$$

The eigenvalues and eigenfunctions of problem (3.1) are

$$\lambda_k = (\frac{2\pi k}{l})^2, \quad k = 1, 2, \dots \tag{3.3}$$

$$X_0 = x, \quad X_k = \sin\frac{2\pi k x}{l} \tag{3.4}$$

respectively. Note, that for $k > 0$ the functions (3.4) are not orthonormal with X_0. To construct a basis in L_2, we complete (3.4) by using adjoint functions.

Following M. Keldysh [11], we define an adjoint function $\tilde{X}_k$, corresponding eigenvalue λ_k from (3.3), as a solution to the boundary-valued problem

$$\tilde{X}_k''(x) + \lambda_k \tilde{X}_k(x) = -2\sqrt{\lambda_k} X_k(x), \quad \tilde{X}_k(0) = 0, \quad \tilde{X}_k'(0) = \tilde{X}_k'(l). \tag{3.5}$$

We obtain

$$\tilde{X}_k(x) = x\cos\frac{2\pi k x}{l}, \quad k = 1, 2, \dots$$

Rewrite now a system of eigenvalue and adjoint functions of (3.1) as

$$X_0 = x, \quad X_{2k-1}(x) = x\cos\frac{2\pi k x}{l}, \quad X_{2k}(x) = \sin\frac{2\pi k x}{l}. \tag{3.6}$$

In a similar way we find the system of eigenvalue and adjoint functions (3.2):

$$Y_0(x)=\frac{2}{l^2},\quad Y_{2k-1}(x)=\frac{4}{l^2}\cos\frac{2\pi kx}{l},\quad Y_{2k}(x)=\frac{4(l-x)}{l^2}\sin\frac{2\pi kx}{l}\,, \tag{3.7}$$

where for every λ_k with $k>0$, $X_{2k}(x)$, $Y_{2k}(x)$ are eigenvalue functions, $X_{2k-1}(x)$, $Y_{2k-1}(x)$ are adjoint functions of the problems (3.1) and (3.2) respectively. Direct calculations show that (3.6) and (3.7) form a biorthogonal system for $x\in(0,l)$:

$$(X_i,Y_j)=\int_0^l X_i(x)Y_j(x)\,dx=\delta_{ij}.$$

As it was shown in [8] the system (3.6) is complete and forms a basis in $L_2(0,l)$. Hence, an arbitrary function $f(x)\in L_2(0,l)$ may be expanded as

$$f(x)=A_0X_0(x)+\sum_{k=1}^{\infty}(A_{2k}X_{2k}(x)+A_{2k-1}X_{2k-1}(x)),$$

where

$$A_i=\int_0^l f(x)Y_i(x)\,dx. \tag{3.8}$$

Returning to the separation variables technique, for $T(t)$ we obtain

$$T_k(t)=a_k\sin\frac{2\pi kt}{l}+b_k\cos\frac{2\pi kt}{l}.$$

We assume now that a solution to $\mathcal{H}$ is of the form

$$u(x,t)=A_0X_0+\sum_{k=1}^{\infty}\left((A_{2k}X_{2k}+A_{2k-1}X_{2k-1})T_k-\frac{lt}{2\pi k}A_{2k-1}X_{2k}T_k'\right). \tag{3.9}$$

Substitute $T_k(t)$ and rewrite the coefficients. Then

$$\begin{aligned}u(x,t)=&C_0X_0+\sum_{k=1}^{\infty}(X_{2k}(C_{2k}\sin\frac{2\pi kt}{l}+D_{2k}\cos\frac{2\pi kt}{l})\\&+X_{2k-1}(C_{2k-1}\sin\frac{2\pi kt}{l}+D_{2k-1}\cos\frac{2\pi kt}{l})\\&-tX_{2k}(C_{2k-1}\cos\frac{2\pi kt}{l}-D_{2k-1}\sin\frac{2\pi kt}{l})).\end{aligned} \tag{3.10}$$

The initial data (1.7) give us the following two equalities

$$\varphi(x)=C_0X_0+\sum_{k=1}^{\infty}(D_{2k}X_{2k}+D_{2k-1}X_{2k-1}),$$
$$\psi(x)=\sum_{k=1}^{\infty}\left((\frac{2\pi k}{l}C_{2k}-C_{2k-1})X_{2k}+\frac{2\pi k}{l}C_{2k-1}X_{2k-1}\right),$$

and the coefficients can be found via formula (3.8).

Assume a solution to the problem $\mathcal{NH}$ is of the form

$$u(x,t)=V_0(t)X_0(x)+\sum_{k=1}^{\infty}\left(V_{2k}(t)X_{2k}(x)+V_{2k-1}(t)X_{2k-1}(x)\right), \tag{3.11}$$

where $V_i(t)$ are unknown coefficients satisfying the initial conditions $V_i(0) = V_i'(0) = 0$. Substitute (3.11) into the equation $u_{tt} - u_{xx} = g(x,t)$, where $g(x,t)$ has been expanded as a biorthogonal series:

$$g(x,t) = g_0(t)X_0(x) + \sum_{k=1}^{\infty}(g_{2k}(t)X_{2k}(x) + g_{2k-1}(t)X_{2k-1}(x)),$$

with coefficients

$$g_i(t) = \int_0^l g(x,t)Y_i(x)\,dx, \quad i = 0,1,\dots$$

We obtain

$$\begin{aligned}
&V_0''(t)x + \sum_{k=1}^{\infty}\Big(V_{2k}''(t) + \frac{4\pi^2k^2}{l^2}V_{2k}(t)\Big)\sin\frac{2\pi kx}{l}\\
&+\sum_{k=1}^{\infty}\Big(V_{2k-1}''(t) + \frac{4\pi^2k^2}{l^2}V_{2k-1}(t)\Big)x\cos\frac{2\pi kx}{l}\\
&+\sum_{k=1}^{\infty}V_{2k-1}(t)\frac{4\pi k}{l}\sin\frac{2\pi kx}{l}\\
&= \quad g_0(t)X_0(x) + \sum_{k=1}^{\infty}(g_{2k}(t)X_{2k}(x) + g_{2k-1}(t)X_{2k-1}(x)).
\end{aligned}$$

Thus we have a Cauchy problem for the system of ordinary differential equations

$$\begin{gathered}
V_0''(t) = g_0(t)\\
V_{2k}'' + \frac{4\pi k}{l}(\frac{\pi k}{l}V_{2k}(t) + V_{2k-1}(t)) = g_{2k}(t)\\
V_{2k-1}''(t) + \frac{4\pi^2k^2}{l^2}V_{2k-1}(t) = g_{2k-1}(t)
\end{gathered}$$

with initial data

$$V_0(0) = V_0'(0) = 0, \quad V_{2k}(0) = V_{2k}'(0) = 0, \quad V_{2k-1}(0) = V_{2k-1}'(0) = 0,$$

which has a unique solution

$$\begin{gathered}
V_0(t) = \int_0^t (t-\tau)g_0(\tau)\,d\tau,\\
V_{2k-1}(t) = \frac{1}{k\pi}\int_0^t g_{2k-1}(\tau)\sin\frac{k\pi(t-\tau)}{l}\,d\tau,\\
V_{2k}(t) = \frac{1}{k\pi}\int_0^t (g_{2k}(\tau) - 4\pi kV_{2k-1}(\tau))\sin\frac{k\pi(t-\tau)}{l}\,d\tau.
\end{gathered}$$

Theorem 3.1. *Let:*

(1) $g(x,t) \in C^2(D)$, $g_x(x,t) \in C[0,l]$ *for all* $t \in (0,T)$, $|g(x,t)| \le P, (x,t) \in D$
(2) $\varphi \in C[0,l] \cap C^2(0,l)$, $\psi \in C[0,l]$, $\varphi(0) = 0$, $\varphi'(0) = \varphi'(l)$, $\psi(0) = 0$.

Then there exists the solution to (1.6)–(1.9),

$$u(x,t) \in C(\bar{D}) \cap C^1(\bar{D} \setminus \{t = T\}) \cap C^2(D)$$

which has the form of a sum of (3.9) and (3.11).

Series Proof. It is sufficient to prove uniform convergence of the series (3.9) and (3.11) and the series, obtained with formal differentiation. Let $|\varphi'(x)| \leq M_1$, $|\varphi''(x)| \leq M_2$, $|\psi(x)| \leq N$, $|\psi'(x)| \leq N_1$, $|g_x| \leq P_1$, $|g_{xx}| \leq P_2$.

Integrating C_i, D_i, V_i by parts and taking in account the abovementioned assumptions, we obtain:

$$
\begin{gathered}
|D_{2k}| \leq \frac{1}{k^2}\frac{l(lM_2+2M_1)}{\pi^2}, \quad |D_{2k-1}| \leq \frac{1}{k^2}\frac{M_2 l}{\pi^2}, \\
|C_{2k}| \leq \frac{1}{k^2}\frac{l(N_1+2N)}{2\pi^2}, \quad |C_{2k-1}| \leq \frac{1}{k^2}\frac{N_1 l}{\pi^2}, \\
|V_{2k}| \leq \frac{1}{k^2}\frac{4T^2(2p_1+P_2 l)}{\pi^2}, \quad |V_{2k-1}| \leq \frac{1}{k^2}\frac{2TP_1}{\pi^2},
\end{gathered}
$$

and hence the series (3.9) and (3.11) and the series, obtained with formal differentiation, converge uniformly. $\square$

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