Electronic Journal of Differential Equations, Vol. 2006(2006), No. 64, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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A BOUNDARY BLOW-UP FOR SUB-LINEAR ELLIPTIC PROBLEMS WITH A NONLINEAR GRADIENT TERM

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Abstract. By a perturbation method and constructing comparison functions, we show the exact asymptotic behaviour of solutions to the semilinear elliptic problem

$$\Delta u - |\nabla u|^q = b(x)g(u), \quad u > 0 \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = +\infty,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $q \in (1,2]$, $g \in C[0,\infty) \cap C^1(0,\infty)$, $g(0) = 0$, g is increasing on $[0,\infty)$, and b is non-negative non-trivial in Ω, which may be singular or vanishing on the boundary.

1. Introduction and statement of main results

The purpose of this paper is to investigate the exact asymptotic behaviour of solutions near the boundary for the problem

$$\Delta u - |\nabla u|^q = b(x)g(u), \quad u > 0 \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = +\infty, \tag{1.1}$$

where the last condition means that $u(x) \to +\infty$ as $d(x) = \operatorname{dist}(x, \partial\Omega) \to 0$, and the solution is called "a large solution" or "an explosive solution", Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 1)$, $q \in (1,2]$. The functions g and b satisfy

- (G1) $g \in C^1(0,\infty) \cap C[0,\infty)$, $g(0) = 0$, g is increasing on $[0,\infty)$.
- (G2) $\int_t^\infty \frac{ds}{\sqrt{2G(s)}} = \infty$, for all $t > 0$, $G(s) = \int_0^s g(z)dz$.
- (B1) $b \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$, is non-negative and non-trivial in Ω.

The main feature of this paper is the presence of the three terms: The nonlinear term $g(u)$ which is sub-linear at infinity, the nonlinear gradient term $|\nabla u|^q$, and the weight $b(x)$ which may be singular or vanishing on the boundary.

First, we review the model

$$\Delta u = b(x)g(u) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = +\infty. \tag{1.2}$$

For g satisfying (G1) and the Keller-Osserman condition

- (G3) $\int_t^\infty \frac{ds}{\sqrt{2G(s)}} < \infty$,

2000 *Mathematics Subject Classification.* 35J60, 35B25, 35B50, 35R05.
Key words and phrases. Semilinear elliptic equations; large solutions; asymptotic behaviour.

Submitted February 5, 2006. Published May 20, 2006.
Supported by grant 10071066 from the National Natural Science Foundation of China.

problem (1.2) arises in many branches of applied mathematics and has been discussed by many authors; see for instance [2, 4, 5, 6, 7, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

For $g(s) = s^p$, $p \in (0, 1]$, little is known. Lair and Wood [15] showed that if $b \in C(\bar{\Omega})$ then (1.2) has no solution. Then Lair [14] showed that if g satisfies (G1), $b \in C(\bar{\Omega})$ is non-negative in Ω and is positive near the boundary then (1.2) has no solution if and only if (G2) holds. Bachar and Zeddini [1, Theorem 3] showed that if $b \in C(\bar{\Omega})$ and there exist positive constants c_1, c_2 such that $g(s) \le c_1 s + c_2$, for all $s \ge 0$, then (1.2) has no solution. Chuaqui et al. [4] showed that when $\Omega = B$, $g(s) = s^p$, $p \in (0, 1)$, and $b(|x|) = b(r)$ satisfies

(B2) $\lim_{r\to 0^+}(1-r)^\gamma b(r) = c_0 > 0$ for some $\gamma > 0$,

then (1.2) has at least one solution if and only if $\gamma \ge 2$. Moreover, if $\gamma > 2$, then, for any solution u, to problem (1.2),

$$\lim_{r\to 0^+}(1-r)^\beta u(r) = \Big(\frac{c_0}{\beta(\beta+1)}\Big)^{1/(1-p)},$$

where $\beta = (\gamma - 2)/(1-p)$. If $\gamma = 2$, then, for any solution u to problem (1.2),

$$\lim_{r\to 0^+}\frac{u(r)}{(-\ln(1-r))^{1/(1-p)}} = \big(c_0(1-p)\big)^{1/(1-p)}.$$

Yang [26] showed that if $b \in C[0, 1)$ is non-negative non-trivial in $[0, 1)$, g satisfies (G1) and

$$\int_1^\infty \frac{ds}{g(s)} = \infty, \tag{1.3}$$

then (1.2) has one solution if and only if

$$\int_0^1 (1-r)b(r)dr = \infty. \tag{1.4}$$

Moreover, if $b(r) \sim (1-r)^{-\gamma}$ as $r \to 1$, $\gamma \ge 2$, and $p \in (0, 1)$, $g(s) \sim s(\ln s)^p$ as $s \to \infty$, then, for any solution u to problem (1.2),

$$u(r) \sim \begin{cases} (1-r)^{-(\gamma-2)/(2-p)} & \text{if } \gamma > 2; \\ (-\ln(1-r))^{2/(2-p)} & \text{if } \gamma = 2. \end{cases}$$

He also showed that (1.2) has no solution provided that Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ ($N \ge 1$), g satisfies (G1) and (1.3), b satisfies (B1) and

$$b(x) \le C(d(x))^{-2}(-\ln(d(x)))^{-p}, \tag{1.5}$$

where $p > 1$ and $C > 0$.

Let's return to problem (1.1). When $b \equiv 1$ on Ω: for $g(u) = u$, Lasry and Lions [15] established the model (1.1) which arises from the description of the basic stochastic control problem, and showed by a perturbation method and a sub-supersolutions method that if $q \in (1, 2]$ then problem (1.1) has a unique solution $u \in C^2(\Omega)$. Moreover,

(i) when $1 < q < 2$,

$$\lim_{d(x)\to 0} u(x)(d(x))^{(2-q)/(q-1)} = (2-q)^{-1}(q-1)^{-(2-q)/(q-1)}; \tag{1.6}$$

(ii) when $q = 2$,

$$\lim_{d(x)\to 0} u(x)/(-\ln(d(x))) = 1. \tag{1.7}$$

For $g(u) = u^p$, $p > 0$, by the theory of ordinary differential equation and the comparison principle, Bandle and Giarrusso [3] showed that
(iii) if $1 < q \le 2$, then problem (1.1) has one solution in $C^2(\Omega)$;
(iv) if $\max\{2p/(p+1), 1\} < q < 2$, then every solution u to problem (1.1) satisfies (1.6);
(v) if $q = 2$, then every solution u to problem (1.1) satisfies (1.7).

For the other results of large solutions to elliptic problems with nonlinear gradient terms, see [8, 9, 29, 30, 31, 32] and the references therein. In this note, by a perturbation method and constructing comparison functions, we show how the weight b affects the exact asymptotic behaviour of solutions near the boundary, to problems (1.1).

Our main results are state in the following theorems.

Theorem 1.1. *Let $1 < q < 2$, and assume (G1) and (B1).*

(I) *If the following convergence is uniform for $\xi \in [a, b]$ with $0 < a < b$,*
$$\lim_{d(x)\to 0} b(x)(d(x))^{\frac{q}{q-1}} g\big(\xi (d(x))^{-\frac{2-q}{q-1}}\big) = 0, \tag{1.8}$$
then every solution to problem (1.1) *satisfies* (1.6)*;*

(II) *if $g(u) = u^p$, $p \in (0, 1]$ and*
$$\lim_{d(x)\to 0} b(x)(d(x))^{\frac{q-p(2-q)}{q-1}} = C_0 > 0, \tag{1.9}$$
then every solution to problem (1.1) *satisfies*
$$\lim_{d(x)\to 0} u(x)(d(x))^{(2-q)/(q-1)} = \xi_0, \tag{1.10}$$
provided that
(i) *$p = 1$, $C_0 \in \big(0, \frac{2-q}{(q-1)^2}\big)$. In this case,*
$$\xi_0 = \big(\frac{q-1}{2-q}\big)^{q/(q-1)} \big(\frac{2-q}{(q-1)^2} - C_0\big)^{1/(q-1)};$$
(ii) *$p \in (0, 1)$, $C_0 \in (0, \bar C)$ where*
$$\bar C = \Big(\frac{q(1-p)}{p(q-p)(q-1)}\Big)^{\frac{q-p}{q-1}} \Big(\frac{p(1-p)}{q(q-1)}\Big)^{\frac{1-p}{q-1}} \Big(\frac{2-q}{q-1}\Big)^{\frac{q(1-p)}{q-1}}.$$
In this case, $\xi_0 = \xi_2$, where the equation
$$\frac{2-q}{q-1} = C_0 \xi^{p-1} + \big(\frac{2-q}{q-1}\big)^q \xi^{q-1}, \tag{1.11}$$
has just two positive solutions ξ_1 and ξ_2 with
$$0 < \xi_1 < \Big(\frac{C_0(1-p)}{q-1}\Big)^{1/(q-p)} \Big(\frac{2-q}{q-1}\Big)^{q/(q-p)} < \xi_2.$$

Theorem 1.2. *Let $q = 2$, and assume (G1) and (B1).*

(I) *If the following convergence is uniform for $\xi \in [a, b]$ with $0 < a < b$,*
$$\lim_{d(x)\to 0} b(x)(d(x))^2 g(-\xi \ln(d(x))) = 0, \tag{1.12}$$
then every solution to problem (1.1) *satisfies* (1.7)*;*

(II) *if* $g(u) = u^p$, $p \in (0,1]$ *and*

$$\lim_{d(x)\to 0} b(x)(d(x))^2(-\ln(d(x)))^p = C_0 > 0, \tag{1.13}$$

then every solution u *to problem* (1.1) *satisfies*

$$\lim_{d(x)\to 0} u(x)/(-\ln(d(x))) = \xi_0, \tag{1.14}$$

provided that
(i) $p = 1$, $C_0 \in (0,1)$, $\xi_0 = 1 - C_0$;
(ii) $p \in (0,1)$, $C_0 = 2^p/4$, $\xi_0 = 1/2$;
(iii) $p \in (0,1)$, $C_0 \in (0, 2^p/4)$, $\xi_0 = \xi_2$, *where the equation*

$$\xi - \xi^2 = C_0\xi^p,$$

has just two positive solutions ξ_1 *and* ξ_2 *with* $0 < \xi_1 < 1/2 < \xi_2 < 1$.

2. Proof of theorems

Lemma 2.1 (The comparison principle, [10, Theorem 10.1])**.** *Let* $\Psi(x,s,\xi)$ *satisfy the following two conditions*

(D1) Ψ *is non-increasing in* s *for each* $(x,\xi) \in (\Omega \times \mathbb{R}^N)$;
(D2) Ψ *is continuously differentiable with respect to the variable* ξ *in* $\Omega \times (0,\infty) \times \mathbb{R}^N$.

If $u, v \in C(\bar{\Omega}) \cap C^2(\Omega)$ *satisfy* $\Delta u + \Psi(x,u,\nabla u) \geq \Delta v + \Psi(x,v,\nabla v)$ *in* Ω *and* $u \leq v$ *on* $\partial\Omega$, *then* $u \leq v$ *in* Ω.

Lemma 2.2 (Taylor's formula)**.** *Let* $\alpha \in \mathbb{R}$, $x \in [-x_0, x_0]$ *with* $x_0 \in (0,1)$. *Then there exists* $\varepsilon_1 > 0$ *small enough such that for* $\varepsilon \in (0, \varepsilon_1)$

$$(1 + \varepsilon x)^\alpha = 1 + \alpha\varepsilon x + o(\varepsilon^2). \tag{2.1}$$

Proof of Theorem 1.1. Given an arbitrary $\varepsilon \in (0, \xi_0/2)$, let $\xi_{2\varepsilon} = \xi_0 + \varepsilon$, $\xi_{1\varepsilon} = \xi_0 - \varepsilon$. It follows that

$$\frac{1}{2}\xi_0 < \xi_{1\varepsilon} < \xi_{2\varepsilon} < 2\xi_0.$$

For $\delta > 0$, we define

$$\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}.$$

Since $\partial\Omega \in C^2$, there exists a constant $\delta > 0$ which only depends on Ω such that

$$d(x) \in C^2(\bar{\Omega}_{2\delta}) \quad \text{and} \quad |\nabla d| \equiv 1 \quad \text{on } \Omega_{2\delta}. \tag{2.2}$$

(I) When (1.8) holds, $\xi_0 = (2-q)^{-1}(q-1)^{-(2-q)/(q-1)}$. It follows from Lemma 2.2 that there exists $\varepsilon_1 > 0$ small enough such that for $\varepsilon \in (0, \varepsilon_1)$

$$\begin{aligned}
\frac{2-q}{(q-1)^2}\xi_{2\varepsilon} - \Big(\frac{2-q}{q-1}\Big)^q \xi_{2\varepsilon}^q &= \frac{2-q}{(q-1)^2}(\xi_0 + \varepsilon) - \Big(\frac{2-q}{q-1}\Big)^q (\xi_0 + \varepsilon)^q \\
&= \frac{2-q}{(q-1)^2}\varepsilon - \Big(\frac{2-q}{q-1}\Big)^q \xi_0^q \Big((1 + \frac{\varepsilon}{\xi_0})^q - 1\Big) \\
&= -\frac{(q-1)(2-q)}{(q-1)^2}\varepsilon + o(\varepsilon^2);
\end{aligned}$$

and

$$\frac{2-q}{(q-1)^2}\xi_{1\varepsilon} - (\frac{2-q}{q-1})^q \xi_{1\varepsilon}^q = \frac{2-q}{(q-1)^2}(\xi_0 - \varepsilon) - (\frac{2-q}{q-1})^q (\xi_0 - \varepsilon)^q$$
$$= \frac{(q-1)(2-q)}{(q-1)^2}\varepsilon + o(\varepsilon^2).$$

Denote

$$c_1 = \frac{(q-1)(2-q)}{(q-1)^2}.$$

It follows by (2.2) and (1.8) that corresponding to $\varepsilon \in (0, \varepsilon_1)$, there is $\delta_\varepsilon \in (0, \delta)$ sufficiently small such that

$$\frac{2-q}{q-1}|\xi_{i\varepsilon} d(x)\Delta d(x)| + |b(x)(d(x))^2 g\left(-\xi_{i\varepsilon}\ln(d(x))\right)| < \frac{c_1}{2}\varepsilon, \tag{2.3}$$

for all $x \in \Omega_{2\delta_\varepsilon}$, $i = 1, 2$.
(II) (i) When $p = 1$, $C_0 \in \left(0, \frac{2-q}{(q-1)^2}\right)$. As the result of (I), we see that for $\varepsilon \in (0, \varepsilon_1)$,

$$\left(\frac{2-q}{(q-1)^2} - C_0\right)\xi_{2\varepsilon} - \left(\frac{2-q}{q-1}\right)^q \xi_{2\varepsilon}^q = -(q-1)\left(\frac{2-q}{(q-1)^2} - C_0\right)\varepsilon + o(\varepsilon^2);$$

and

$$\left(\frac{2-q}{(q-1)^2} - C_0\right)\xi_{1\varepsilon} - \left(\frac{2-q}{q-1}\right)^q \xi_{1\varepsilon}^q = (q-1)\left(\frac{2-q}{(q-1)^2} - C_0\right)\varepsilon + o(\varepsilon^2).$$

(ii) When $p \in (0, 1)$, $C_0 \in (0, \bar{C})$. Since

$$\left(\frac{C_0(1-p)}{q-1}(\frac{q-1}{2-q})^q\right)^{1/(q-p)} < \xi_0,$$

it follows that

$$(q-1)(\frac{2-q}{q-1})^q \xi_0^{q-p} - C_0(1-p) > 0.$$

Then by Lemma 2.2, there exists $\varepsilon_1 > 0$ small enough such that for $\varepsilon \in (0, \varepsilon_1)$

$$\begin{aligned}
&\frac{2-q}{(q-1)^2}\xi_{2\varepsilon} - \left(\frac{2-q}{q-1}\right)^q \xi_{2\varepsilon}^q - C_0\xi_{2\varepsilon}^p \\
&= \frac{2-q}{(q-1)^2}(\xi_0 + \varepsilon) - \left(\frac{2-q}{q-1}\right)^q (\xi_0 + \varepsilon)^q - C_0(\xi_0 + \varepsilon)^p \\
&= \frac{2-q}{(q-1)^2}\varepsilon - C_0\xi_0^p\left((1 + \frac{\varepsilon}{\xi_0})^p - 1\right) - \left(\frac{2-q}{q-1}\right)^q \xi_0^q\left((1 + \frac{\varepsilon}{\xi_0})^q - 1\right) \\
&= -\xi_0^{-1}\left(q\left(\frac{2-q}{q-1}\right)^q \xi_0^q + pC_0\xi_0^p - \frac{2-q}{(q-1)^2}\xi_0\right)\varepsilon + o(\varepsilon^2) \\
&= -\xi_0^{-(2-p)}\left((q-1)\left(\frac{2-q}{q-1}\right)^q \xi_0^{(q-p)} - C_0(1-p)\right)\varepsilon + o(\varepsilon^2);
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2-q}{(q-1)^2}\xi_{1\varepsilon} - \left(\frac{2-q}{q-1}\right)^q \xi_{1\varepsilon}^q - C_0\xi_{1\varepsilon}^p \\
&= \frac{2-q}{(q-1)^2}(\xi_0-\varepsilon) - \left(\frac{2-q}{q-1}\right)^q (\xi_0-\varepsilon)^q - C_0(\xi_0-\varepsilon)^p \\
&= \xi_0^{-(2-p)}\left((q-1)\left(\frac{2-q}{q-1}\right)^q \xi_0^{(q-p)} - C_0(1-p)\right)\varepsilon + o(\varepsilon^2).
\end{aligned}$$

Denote

$$c_2 = \xi_0^{-(2-p)}\left((q-1)\left(\frac{2-q}{q-1}\right)^q \xi_0^{(q-p)} - C_0(1-p)\right).$$

We see by (2.2) and (1.9) that corresponding to $\varepsilon \in (0,\varepsilon_1)$, there is $\delta_\varepsilon \in (0,\delta)$ sufficiently small such that

$$\frac{2-q}{q-1}|\xi_{i\varepsilon}d(x)\Delta d(x)| + |b(x)(d(x))^2 g(-\xi_{i\varepsilon}\ln(d(x)))| < C_0\xi_{i\varepsilon}^p + \frac{c_2}{2}\varepsilon, \tag{2.4}$$

for all $x \in \Omega_{2\delta_\varepsilon}$, $i=1,2$. Let $\beta \in (0,\delta_\varepsilon)$ be arbitrary, we define

$$\begin{gathered}
\overline{u}_\beta = \xi_{2\varepsilon}(d(x)-\beta)^{-(2-q)/(q-1)},\ x \in D_\beta^- = \Omega_{2\delta_\varepsilon}/\bar{\Omega}_\beta; \\
\underline{u}_\beta = \xi_{1\varepsilon}(d(x)+\beta)^{-(2-q)/(q-1)},\ x \in D_\beta^+ = \Omega_{2\delta_\varepsilon-\beta}.
\end{gathered}$$

It follows that for $(x,\beta) \in D_\beta^- \times (0,\delta_\varepsilon)$,

$$\begin{aligned}
&\Delta\overline{u}_\beta(x) - |\nabla\overline{u}_\beta(x)|^q - b(x)g(\overline{u}_\beta(x)) \\
&= (d(x)-\beta)^{-q/(q-1)}\Big(\frac{\xi_{2\varepsilon}(2-q)}{(q-1)^2} - \frac{\xi_{2\varepsilon}(2-q)}{q-1}(d(x)-\beta)\Delta d(x) - \xi_{2\varepsilon}^q(\frac{2-q}{q-1})^q \\
&\quad - b(x)g\left(\xi_{2\varepsilon}(d(x)-\beta)^{-(2-q)/(q-1)}\right)(d(x)-\beta)^{q/(q-1)}\Big) \\
&\le 0;
\end{aligned}$$

and for $(x,\beta) \in D_\beta^+ \times (0,\delta_\varepsilon)$

$$\begin{aligned}
&\Delta\underline{u}_\beta(x) - |\nabla\underline{u}_\beta(x)|^q - b(x)g(\underline{u}_\beta(x)) \\
&= (d(x)+\beta)^{-q/(q-1)}\Big(\frac{\xi_{1\varepsilon}(2-q)}{(q-1)^2} - \frac{\xi_{1\varepsilon}(2-q)}{q-1}(d(x)+\beta)\Delta d(x) - \xi_{1\varepsilon}^q(\frac{2-q}{q-1})^q \\
&\quad - b(x)g\left(\xi_{2\varepsilon}(d(x)+\beta)^{-(2-q)/(q-1)}\right)(d(x)+\beta)^{q/(q-1)}\Big) \\
&\ge 0.
\end{aligned}$$

Now let u be an arbitrary solution of problem (1.1) and $M_u(2\delta_\varepsilon) = \max_{d(x)\ge 2\delta_\varepsilon} u(x)$. We see that

$$u \le M_u(2\delta_\varepsilon) + \overline{u}_\beta \quad \text{on } \partial D_\beta^-.$$

Since $\underline{u}_\beta = \xi_{1\varepsilon}(2\delta_\varepsilon)^{-(2-q)/(q-1)} = M_{\underline{u}}(2\delta_\varepsilon)$ whenever $d(x) = 2\delta_{2\varepsilon} - \beta$, we see that

$$\underline{u}_\beta \le u + M_{\underline{u}}(2\delta_\varepsilon) \quad \text{on } \partial D_\beta^+.$$

It follows by (G1) and Lemma 2.1 that

$$\begin{gathered}
u \le M_u(2\delta_\varepsilon) + \overline{u}_\beta, \quad x \in D_\beta^-; \\
\underline{u}_\beta \le u + M_{\underline{u}}(2\delta_\varepsilon), \quad x \in D_\beta^+.
\end{gathered}$$

Hence, for $x \in D_\beta^- \cap D_\beta^+$, and letting $\beta \to 0$, we have

$$\xi_{1\varepsilon} - \frac{M_{\underline{u}}(2\delta_\varepsilon)}{(d(x))^{-(2-q)/(q-1)}} \le \frac{u(x)}{(d(x))^{-(2-q)/(q-1)}} \le \xi_{2\varepsilon} + \frac{M_u(2\delta_\varepsilon)}{(d(x))^{-(2-q)/(q-1)}};$$

i.e.,

$$\xi_{1\varepsilon} \le \lim_{d(x)\to 0} \inf \frac{u(x)}{(d(x))^{-(2-q)/(q-1)}} \le \lim_{d(x)\to 0} \sup \frac{u(x)}{(d(x))^{-(2-q)/(q-1)}} \le \xi_{2\varepsilon}.$$

Letting $\epsilon \to 0$, and using the definitions of $\xi_{1\varepsilon}$ and $\xi_{2\varepsilon}$, we have

$$\lim_{d(x)\to 0} \frac{u(x)}{(d(x))^{-(2-q)/(q-1)}} = \xi_0.$$

□

Proof of Theorem 1.2. We proceed as in the proof of Theorem 1.1. Given an arbitrary $\varepsilon \in (0, \xi_0/2)$, let

$$\xi_{2\varepsilon} = \xi_0 + \varepsilon, \quad \xi_{1\varepsilon} = \xi_0 - \varepsilon.$$

Note that

$$\frac{1}{2}\xi_0 < \xi_{1\varepsilon} < \xi_{2\varepsilon} < 2\xi_0.$$

When $p = 1$, $C_0 \in (0,1)$, $\xi_0 = 1 - C_0$, we see that

$$(1 - C_0)\xi_{2\varepsilon} - \xi_{2\varepsilon}^2 = -\varepsilon\xi_0 - o(\varepsilon^2) \quad \text{and} \quad (1 - C_0)\xi_{1\varepsilon} - \xi_{1\varepsilon}^2 = \varepsilon\xi_0 - o(\varepsilon^2).$$

It follows by (2.2) and (1.12) that there is $\delta_\varepsilon \in (0, \delta)$ sufficiently small such that

$$|\xi_{i\varepsilon} d(x)\Delta d(x)| + |b(x)(d(x))^2 g(-\xi_{i\varepsilon} \ln(d(x)))| < \frac{\xi_0}{2}\varepsilon, \tag{2.5}$$

for all $x \in \Omega_{2\delta_\varepsilon}$, $i = 1, 2$. When $p \in (0,1)$ and $\xi_0 \ge \frac{1}{2}$, we see that $\xi_0 > \frac{1-p}{2-p}$ and for $\varepsilon \in (0, \varepsilon_1)$,

$$\begin{aligned}\xi_{2\varepsilon} - \xi_{2\varepsilon}^2 - C_0\xi_{2\varepsilon}^p &= \xi_0 + \varepsilon - (\xi_0 + \varepsilon)^2 - C_0(\xi_0 + \varepsilon)^p \\ &= -(2-p)\left(\xi_0 - \frac{1-p}{2-p}\right)\varepsilon + o(\varepsilon^2);\end{aligned}$$

and

$$\begin{aligned}\xi_{1\varepsilon} - \xi_{1\varepsilon}^2 - C_0\xi_{1\varepsilon}^p &= \xi_0 - \varepsilon - (\xi_0 - \varepsilon)^2 - C_0(\xi_0 - \varepsilon)^p \\ &= (2-p)\left(\xi_0 - \frac{1-p}{2-p}\right)\varepsilon + o(\varepsilon^2).\end{aligned}$$

Denote

$$c_3 = (2-p)\Big(\xi_0 - \frac{1-p}{2-p}\Big).$$

It follows by (2.2) and (1.13) that there is $\delta_\varepsilon \in (0, \delta)$ sufficiently small such that

$$|\xi_{i\varepsilon} d(x)\Delta d(x)| + |b(x)(d(x))^2 g(-\xi_{i\varepsilon} \ln(d(x)))| < C_0\xi_{i\varepsilon}^p + \frac{c_3}{2}\varepsilon, \tag{2.6}$$

for all $x \in \Omega_{2\delta_\varepsilon}$, $i = 1, 2$. Let $\beta \in (0, \delta_\varepsilon)$ be arbitrary, we define

$$\begin{aligned}\overline{u}_\beta &= -\xi_{2\varepsilon} \ln(d(x) - \beta), \quad x \in D_\beta^- = \Omega_{2\delta_\varepsilon}/\bar{\Omega}_\beta; \\ \underline{u}_\beta &= -\xi_{1\varepsilon} \ln(d(x) + \beta), \quad x \in D_\beta^+ = \Omega_{2\delta_\varepsilon - \beta}.\end{aligned}$$

It follows that for $(x,\beta)\in D_\beta^-\times(0,\delta_\varepsilon)$,

$$\begin{aligned}\Delta\overline{u}_\beta(x)-|\nabla\overline{u}_\beta(x)|^2-b(x)g(\overline{u}_\beta(x)) &= (d(x)-\beta)^2\Big(\xi_{2\varepsilon}-\xi_{2\varepsilon}(d(x)-\beta)\Delta d(x)-\xi_{2\varepsilon}^2\\ &\quad -b(x)(d(x)-\beta)^2 g(\xi_{2\varepsilon}\ln(d(x)-\beta))\Big)\\ &\le 0;\end{aligned}$$

and for $(x,\beta)\in D_\beta^+\times(0,\delta_\varepsilon)$,

$$\begin{aligned}\Delta\underline{u}_\beta(x)-|\nabla\underline{u}_\beta(x)|^2-b(x)g(\underline{u}_\beta(x)) &= (d(x)+\beta)^2\Big(\xi_{2\varepsilon}-\xi_{2\varepsilon}(d(x)+\beta)\Delta d(x)-\xi_{2\varepsilon}^2\\ &\quad -b(x)(d(x)+\beta)^2 g(\xi_{2\varepsilon}\ln(d(x)+\beta))\Big)\\ &\ge 0.\end{aligned}$$

The rest of the proof is the same as in Theorem 1.1, we omit it. $\square$

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