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HADAMARD TYPE INEQUALITIES VIA FRACTIONAL CALCULUS IN THE SPACE OF EXP-CONVEX FUNCTIONS AND APPLICATIONS

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Abstract. In this article, we study basic properties of exp-convex functions and establish the corresponding Hadamard type integral inequalities along with fractional operators. A comparative analysis between the exp-convexity and classic convexity is discussed. Furthermore, several related integral identities and estimation of upper bounds of inequalities involved with fractional operators are proved. In addition, some indispensable propositions associated with special means are allocated to illustrate the usefulness of our main results. Besides, Mittag-Leffler type convex functions with weaker convexity than exp-convexity are also presented.

1. Introduction

In the past several decades, the role of elementary mathematical inequalities have been rediscovered owing to their applications to different realms of mathematics and applied science. As a matter of fact, the development of mathematical inequalities is very closely related to the advances in the theory of convex function. As we know, the origin of the theory of convex function could be traced back to the literatures from many famous mathematicians, such as Jensen, Hardy, Hadamard. Interesting discussions regarding to convex function have occupied researches in recent decades. One of the most celebrated and sparkled results on convex function, in some sense, is the Hermite-Hadamard integral inequality (or Hadamard type integral inequality). Because of its geometrical significance, there exist an abundance of related studies from a number of mathematicians who provide new proofs, generalizations, extensions and refinements of Hadamard type integral inequality [6, 12, 21, 25]. In addition, various generalized convex functions have sprung up recently, such as quasi-convex function [1, 4, 24], log-convex function [3, 5, 15, 23], s-convex function [2, 14, 28], m-convex function [22], h-convex function [33], (h, m)-convex function[29], co-ordinated convex [16].

On the other hand, the theory of fractional calculus is nearly as old as the classical calculus [8, 13, 37]. During the last few decades, both in mathematics and applied sciences, fractional calculus is recognized as an excellent tool for describing complex dynamic processes incorporating both long range memory effects and hereditary,

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such as physics [19, 31], mechanics [17, 38], engineering [30], biology [9], economy [18, 20] and other branches of technical fields. Nevertheless, it has to be emphasized that only sporadic works have been reported on generalized Hadamard type integral inequalities in the framework of fractional calculus. In [26], the authors establish Hadamard-type inequality via fractional integral operator in the sense of classic convexity. In [35, 36], they present Hadamard-type inequalities associated with Hadamard fractional settings in the presence of classic convexity. In [27], the author identifies some new inequalities of Hermite-Hadamard-type for co-ordinated convex functions on a rectangle of the real plane via Riemann-Liouville fractional integral operator. In addition, more related publications could be found in [10, 11, 32, 34, 39].

It is not unexpected to find that there exist quantities of continuous functions which do not satisfy the strict definition of conventional convex (or, concave) function. To reveal their fundamental properties better, such as geometric characteristics and differentiability, it is reasonable to extend the original notion of convex function to a broader one. It should be noted that Dragomir has proposed the concept of exp-convex function in his letter [7]. In [40], the authors investigate novel Hermite-Hadamard type inequalities for K-conformable fractional integral operator for exponentially convex functions in the classical sense. However, there are no reports on Hadamard type integral inequalities in terms of Riemann-Liouville fractional operators with exp-convexity. Naturally, we put forth two basic questions: What is the essential difference between exp-convexity and ordinary convexity? How to establish the corresponding (Hadamard type) integral inequalities via fractional operators in the sense of exp-convexity? In this paper, we will supply definite answers.

The rest of this paper is organized as follows: In Section 2, some preliminaries on exp-convex function are introduced. Hadamard type integral inequalities/equalities and their generalizations via fractional operators with regard to exp-convex function are proved and discussed in Sections 3 and 4, respectively. Some applications of exp-convex function dealing with special means are provided in Section 5, and the standard definition of Mittag-Leffler type convex function is also posed as the generalization of exp-convexity in the last section.

2. Preliminaries

To prove our main results, some mathematics preliminaries should be provided. First we introduce the definition of exp-convex function as follows.

Definition 2.1 ([7]). A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be exp-convex function, if

$$e^{f(tx+(1-t)y)} \leq te^{f(x)} + (1-t)e^{f(y)}, \tag{2.1}$$

for all $t \in [0, 1]$ and all $x, y \in [a, b]$.

Remark 2.2. The definition of exp-convex indicates that f may not be convex but $\exp(f)$ is convex. Furthermore, if $f(x)$ is convex, then it must be exp-convex. However, the converse is not true. For example, $\ln(x^2)$ is non-convex on $[1, 2]$ but it is exp-convex. Accordingly, compared to the conventional convex function, the exp-convex function has a lower requirement for the properties of function itself and can characterize the geometric properties of a function better. Given that, it has greater potential value and a broader application prospect.

Definition 2.3. The logarithmic-exponential mean of a given function $f(x)$ on $[a,b]$ is defined as

$$\mathrm{LE}(x)=\ln\Big[\frac{e^{f(x)}+e^{f(a+b-x)}}{2}\Big],\quad x\in[a,b]. \tag{2.2}$$

Definition 2.4. The logarithmic-type mean of a given positive function $f(x)$ on $[a,b]$ is defined as

$$\widetilde{\mathrm{LE}}(x)=\ln\big[\frac{f(x)+f(a+b-x)}{2}\big],\quad x\in[a,b]. \tag{2.3}$$

Definition 2.5 ([13])**.** The Gauss hypergeometric function is defined as

$${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \tag{2.4}$$

where $|x|<1$, and $(q)_0=1$, $(q)_n=q(q+1)(q+2)\cdots(q+n-1)$, $(n\in N^*)$.

Definition 2.6 ([13])**.** The left and right Riemann-Liouville fractional integrals with order $\alpha>0$ of a given continuous function $f(x)$, $x\in[a,b]$ are defined as, respectively,

$$\mathrm{D}_{a^+}^{-\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\mathrm{d}t,\quad x>a, \tag{2.5}$$

$$\mathrm{D}_{b^-}^{-\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\mathrm{d}t,\quad x<b, \tag{2.6}$$

where $\Gamma(\cdot)$ is the Gamma function.

3. Hadamard type integral inequalities for exp-convex functions via fractional operators

In this section, we propose several interesting interpolation inequalities as follows. First, the Hadamard type integral inequality with fractional setting in the space of exp-convex functions is established.

Theorem 3.1. *For $\alpha>0$, if $f(x)$ is an exp-convex function and continuous on $[a,b]$, then the following inequalities via fractional integral hold*

$$f\big(\frac{a+b}{2}\big)\le\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b)\le\ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big], \tag{3.1}$$

where $\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b)=\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(x)|_{x=b}$.

Proof. According to the definition of exp-convex function, we have

$$\exp\big(f(\frac{x+y}{2})\big)\le\frac{\exp(f(x))+\exp(f(y))}{2}. \tag{3.2}$$

In this inequality let $x=ta+(1-t)b$, $y=tb+(1-t)a$. Then

$$\exp\big(f(\frac{a+b}{2})\big)\le\frac{\exp(f(ta+(1-t)b))+\exp(f(tb+(1-t)a))}{2}. \tag{3.3}$$

After taking the logarithm on both sides, we obtain

$$f(\frac{a+b}{2})\le\ln\Big[\frac{\exp(f(ta+(1-t)b))+\exp(f(tb+(1-t)a))}{2}\Big]. \tag{3.4}$$

By multiplying the factor $t^{\alpha-1}$ on both sides, and then integrating with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f(\frac{a+b}{2})\mathrm{d}t \\
&\leq \int_0^1 t^{\alpha-1} \ln\Big[\frac{\exp(f(ta+(1-t)b))+\exp(f(tb+(1-t)a))}{2}\Big]\mathrm{d}t;
\end{aligned}\tag{3.5}$$

that is,

$$\frac{f(\frac{a+b}{2})}{\alpha} \leq \int_0^1 t^{\alpha-1} \ln\Big[\frac{\exp(f(ta+(1-t)b))+\exp(f(tb+(1-t)a))}{2}\Big]\mathrm{d}t. \tag{3.6}$$

Combining the inequality (3.6) and making the substitution $t=\frac{b-x}{b-a}$, we obtain

$$\frac{f(\frac{a+b}{2})}{\alpha} \leq \int_b^a \Big(\frac{b-x}{b-a}\Big)^{\alpha-1} \ln\Big[\frac{\exp(f(x))+\exp(f(a+b-x))}{2}\Big]\mathrm{d}\Big(\frac{x}{a-b}\Big). \tag{3.7}$$

Thus we have

$$\begin{aligned}
f(\frac{a+b}{2}) &\leq \alpha\int_a^b \frac{(b-x)^{\alpha-1}}{(b-a)^\alpha} \ln\Big[\frac{\exp(f(x))+\exp(f(a+b-x))}{2}\Big]\mathrm{d}x \\
&= \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(x)|_{x=b}.
\end{aligned}\tag{3.8}$$

For the right side of (3.1), note that

$$\ln\Big[\frac{\exp(f(ta+(1-t)b))+\exp(f(tb+(1-t)a))}{2}\Big] \leq \ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big], \tag{3.9}$$

where the exp-convexity of $f(x)$ has been used.

On the other hand, by multiplying the factor $t^{\alpha-1}$ on both sides of (3.9) and then integrating with respect to t over $[0,1]$, we have

$$\int_0^1 t^{\alpha-1} \ln\Big[\frac{\exp(f(ta+(1-t)b))+\exp(f(tb+(1-t)a))}{2}\Big]\mathrm{d}t \leq \frac{1}{\alpha}\ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big],$$

which implies

$$\frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(x)|_{x=b} \leq \ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big], \tag{3.10}$$

where $t=\frac{b-x}{b-a}$ has been utilized. The proof is complete. $\square$

Now we have established the Hadamard type fractional integral inequalities associated with exp-convexity described by (3.1). In the sequel, we consider special cases of Theorem 3.1.

Corollary 3.2. *Under the conditions of Theorem 3.1, we have*

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b \ln\Big[\frac{e^{f(x)}+e^{f(a+b-x)}}{2}\Big]\mathrm{d}x \leq \ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big]. \tag{3.11}$$

The corollary mentioned above follows from Theorem 3.1 with $\alpha=1$. It is worth noting that (3.11) could be called Hadamard type integral inequality in the space of exp-convex functions (or, Hermite-Hadamard integral inequality for exp-convex functions).

Remark 3.3. *We consider several special cases for such Hermite-Hadamard integral inequality described by* (3.11)*, where* $b>a$.

(1) *If $f(x)=x$, then we have*
$$\frac{a+b}{2}\le\frac{1}{b-a}\int_a^b\ln\Big[\frac{e^x+e^{a+b-x}}{2}\Big]\mathrm{d}x\le\ln\Big[\frac{e^a+e^b}{2}\Big]. \tag{3.12}$$
(2) *If $f(x)=\ln(x)$, then above inequalities* (3.11) *degenerate into the same value $\ln(\frac{a+b}{2})$, where $b>a>0$.*
(3) *If $f(x)=\ln(x^2)$, then*
$$\Big(\frac{a+b}{2}\Big)^2\le\exp\Big\{\frac{1}{b-a}\int_a^b\ln\frac{x^2+(a+b-x)^2}{2}\Big]\mathrm{d}x\Big\}\le\frac{a^2+b^2}{2}. \tag{3.13}$$

In addition, for a monotonic, continuous, exp-convex function, we have the following corollary.

Corollary 3.4. *Under the conditions of Theorem 3.1, if $f(x)$ is a non-increasing function, then*
$$f\Big(\ln\Big[\frac{e^a+e^b}{2}\Big]\Big)\le\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a+}^{-\alpha}\,\mathrm{LE}(b)\le\ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big]. \tag{3.14}$$

Proof. Combining Theorem 3.1 and the inequality $\frac{a+b}{2}\le\ln\big[\frac{e^a+e^b}{2}\big]$ from (3.12) completes the proof. □

The following conclusion can be drawn for a positive and convex function.

Corollary 3.5. *Under the conditions of Theorem 3.1, if $f(x)$ is a positive convex function, we have*
$$\begin{aligned} f(\frac{a+b}{2})&\le\exp\Big[\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\widetilde{\widetilde{\mathrm{LE}}}(b)\Big]\\ &\le\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[\mathrm{D}_{a^+}^{-\alpha}f(b)+\mathrm{D}_{b^-}^{-\alpha}f(a)]\le\frac{f(a)+f(b)}{2}, \end{aligned} \tag{3.15}$$
where $\mathrm{D}_{a^+}^{-\alpha}\widetilde{\widetilde{\mathrm{LE}}}(b)=\mathrm{D}_{a^+}^{-\alpha}\widetilde{\widetilde{\mathrm{LE}}}(x)|_{x=b}$, $\mathrm{D}_{a^+}^{-\alpha}f(b)=\mathrm{D}_{a^+}^{-\alpha}f(x)|_{x=b}$ *and* $\mathrm{D}_{b^-}^{-\alpha}f(a)=\mathrm{D}_{b^-}^{-\alpha}f(x)|_{x=a}$.

Proof. Consider the positive definite and convex function $f(x)$. It is logical to conclude that $g(x)=\ln f(x)$ is an exp-convex function. Utilizing Theorem 3.1, then we have
$$\ln f(\frac{a+b}{2})\le\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a+}^{-\alpha}\widetilde{\widetilde{\mathrm{LE}}}(b)\le\ln\frac{f(a)+f(b)}{2}. \tag{3.16}$$
Taking exponential function on both sides, we obtain
$$f(\frac{a+b}{2})\le\exp\Big[\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\widetilde{\widetilde{\mathrm{LE}}}(b)\Big]\le\frac{f(a)+f(b)}{2}. \tag{3.17}$$
In fact,
$$\begin{aligned} \exp\Big[\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\widetilde{\widetilde{\mathrm{LE}}}(b)\Big]&=\exp\Big[\int_0^1\alpha t^{\alpha-1}\widetilde{\widetilde{\mathrm{LE}}}(ta+(1-t)b)\mathrm{d}t\Big]\\ &=\exp\Big[\frac{\int_0^1\alpha t^{\alpha-1}\widetilde{\widetilde{\mathrm{LE}}}(ta+(1-t)b)\mathrm{d}t}{\int_0^1\alpha t^{\alpha-1}\mathrm{d}t}\Big]\\ &\le\frac{\int_0^1\alpha t^{\alpha-1}\exp(\widetilde{\widetilde{\mathrm{LE}}}(ta+(1-t)b))\mathrm{d}t}{\int_0^1\alpha t^{\alpha-1}\mathrm{d}t} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \alpha t^{\alpha-1}\Big[\frac{f(ta+(1-t)b)+f(tb+(1-t)a)}{2}\Big]\mathrm{d}t \\
&= \frac{1}{2}\int_b^a \alpha\left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\frac{\mathrm{d}u}{a-b} \\
&\quad + \frac{1}{2}\int_a^b \alpha\left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)\frac{\mathrm{d}v}{b-a} \\
&= \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[\mathrm{D}_{a^+}^{-\alpha} f(b)+\mathrm{D}_{b^-}^{-\alpha} f(a)],
\end{aligned}
$$

where Jensen's integral inequality [34] has been used.

On the other hand, due to the convexity of f, we have

$$
\begin{aligned}
\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[\mathrm{D}_{a^+}^{-\alpha} f(b)+\mathrm{D}_{b^-}^{-\alpha} f(a)] &= \int_0^1 \alpha t^{\alpha-1}\Big[\frac{f(ta+(1-t)b)+f(tb+(1-t)a)}{2}\Big]\mathrm{d}t \\
&\le \int_0^1 \alpha t^{\alpha-1}\Big[\frac{tf(a)+(1-t)f(b)+tf(b)+(1-t)f(a)}{2}\Big]\mathrm{d}t \\
&= \int_0^1 \alpha t^{\alpha-1}\frac{f(a)+f(b)}{2}\mathrm{d}t \\
&= \frac{f(a)+f(b)}{2}.
\end{aligned}
$$

Consequently, we complete the proof. □

For a positive, convex and symmetric f, we have the following result.

Remark 3.6. If $f(x)$ is symmetric with respect to the axis $x=\frac{a+b}{2}$ in Corollary 3.5, then we can conclude that

$$
f(\frac{a+b}{2}) \le \exp\Big[\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\widetilde{\mathrm{LE}}(b)\Big] \le \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha} f(b) \le f(b). \tag{3.18}
$$

4. Estimation on bounds of fractional integral via Hadamard type integral inequalities for exp-convex functions

In this section, we prove several explicit bounds in terms of Hadamard type integral inequalities (3.1). First, the estimation on the bound of the right side of (3.1) is proved based on the following integral identity.

Lemma 4.1. *For $\alpha>0$, let $f(x)$ be differentiable on $[a,b]$, then the following relation with fractional setting holds*

$$
\ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big]-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b)=(b-a)\int_0^1 t^\alpha\,\mathrm{LE}'(tb+(1-t)a)\mathrm{d}t. \tag{4.1}
$$

Proof. Let I be the right side of (4.1). Then we have

$$
\begin{aligned}
I &= \int_0^1 t^\alpha \mathrm{d}[\mathrm{LE}(tb+(1-t)a)] \\
&= t^\alpha\,\mathrm{LE}(tb+(1-t)a)|_0^1-\int_0^1 \alpha t^{\alpha-1}\,\mathrm{LE}(tb+(1-t)a)\mathrm{d}t \qquad (4.2)\\
&= \ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big]-\int_0^1 \alpha t^{\alpha-1}\,\mathrm{LE}(tb+(1-t)a)\mathrm{d}t.
\end{aligned}
$$

Noticing that
$$\mathrm{LE}(tb+(1-t)a)=\mathrm{LE}(ta+(1-t)b), \tag{4.3}$$
we obtain
$$\begin{aligned} I &= \ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big]-\int_0^1 \alpha t^{\alpha-1}\,\mathrm{LE}(ta+(1-t)b)\mathrm{d}t \\ &= \ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big]-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b), \end{aligned} \tag{4.4}$$
where the substitution $x=ta+(1-t)b$ has been used. The proof is complete. □

In the light of Lemma above, we can further derive the following conclusion.

Theorem 4.2. *For $\alpha>0$, let $f(x)$ be differentiable on $[a,b]$ and $|\,\mathrm{LE}'(x)|$ be convex, then*
$$\Big|\ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big]-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b)\Big|\le\frac{(b-a)\big|f'(b)e^{f(b)}-f'(a)e^{f(a)}\big|}{(\alpha+1)(e^{f(a)}+e^{f(b)})}.$$

Proof. It suffices to note that
$$|\,\mathrm{LE}'(x)|=\frac{|f'(x)e^{f(x)}-f'(a+b-x)e^{f(a+b-x)}|}{e^{f(x)}+e^{f(a+b-x)}}, \tag{4.5}$$
and according to the convexity and symmetry of $|\,\mathrm{LE}'(x)|$, we obtain
$$\begin{aligned} |\,\mathrm{LE}'(tb+(1-t)a)| &= |\,\mathrm{LE}'(ta+(1-t)b)| \\ &\le t|\,\mathrm{LE}'(a)|+(1-t)|\,\mathrm{LE}'(b)| \\ &= t\frac{|f'(a)e^{f(a)}-f'(b)e^{f(b)}|}{e^{f(a)}+e^{f(b)}}+(1-t)\frac{|f'(b)e^{f(b)}-f'(a)e^{f(a)}|}{e^{f(a)}+e^{f(b)}} \\ &= \frac{|f'(b)e^{f(b)}-f'(a)e^{f(a)}|}{e^{f(a)}+e^{f(b)}}, \end{aligned} \tag{4.6}$$
where $t\in[0,1]$ and substitution $x=tb+(1-t)a$ has been used.

Hence, on account of Lemma 4.1 and (4.6), we obtain
$$\begin{aligned} &\Big|\ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big]-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b)\Big| \\ &= (b-a)\Big|\int_0^1 t^{\alpha}\,\mathrm{LE}'(tb+(1-t)a)\mathrm{d}t\Big| \\ &= (b-a)\Big|\int_0^1 t^{\alpha}\,\mathrm{LE}'(ta+(1-t)b)\mathrm{d}t\Big| \\ &\le (b-a)\int_0^1 |t^{\alpha}\,\mathrm{LE}'(ta+(1-t)b)|\mathrm{d}t \\ &\le (b-a)\int_0^1 t^{\alpha}\mathrm{d}t\frac{|f'(b)e^{f(b)}-f'(a)e^{f(a)}|}{e^{f(a)}+e^{f(b)}} \\ &= \frac{(b-a)|f'(b)e^{f(b)}-f'(a)e^{f(a)}|}{(\alpha+1)(e^{f(a)}+e^{f(b)})}. \end{aligned} \tag{4.7}$$
The proof is complete. □

Next, we present special cases of Theorem 4.2.

Corollary 4.3. *Under the assumptions of Theorem 4.2, we obtain*

$$\Big|\ln\Big[\frac{e^{f(a)}+e^{f(b)}}{2}\Big]-\frac{1}{b-a}\int_a^b \mathrm{LE}(x)\mathrm{d}x\Big|\le\frac{(b-a)|f'(b)e^{f(b)}-f'(a)e^{f(a)}|}{2(e^{f(a)}+e^{f(b)})}. \quad (4.8)$$

The corollary above follows from Theorem 4.2 with $\alpha=1$.

Corollary 4.4. *Under the assumptions of Theorem 4.2, if $f(x)$ is symmetric with respect to the axis $x=\frac{a+b}{2}$, then*

$$\big|f(b)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b)\big|\le\frac{(b-a)|f'(b)|}{\alpha+1}. \quad (4.9)$$

Proof. Obviously, $f(x)=f(a+b-x)$ implies

$$\mathrm{LE}(x)=\ln\Big[\frac{e^{f(x)}+e^{f(a+b-x)}}{2}\Big]=f(x). \quad (4.10)$$

Hence, $f'(x)=-f'(a+b-x)$, $f(a)=f(b)$ and $f'(a)=-f'(b)$. Now by Theorem 4.2, we obtain

$$\begin{aligned}\big|f(b)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b)\big|&\le\frac{(b-a)|f'(b)e^{f(b)}-f'(a)e^{f(a)}|}{2(\alpha+1)e^{f(b)}}\\&\le\frac{(b-a)|f'(b)e^{f(b)}+f'(b)e^{f(b)}|}{2(\alpha+1)e^{f(b)}}\\&=\frac{(b-a)|f'(b)|}{\alpha+1}.\end{aligned} \quad (4.11)$$

Now we complete the proof. □

Next, we provide the estimation on the bound of the left side of (3.1). Before that, we need the following integral identity.

Lemma 4.5. *For $\alpha>0$, let $f(x)$ be differentiable on $[a,b]$. Then the following relation described by fractional setting holds*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b)-f(\frac{a+b}{2})\\&=\frac{b-a}{2}\int_0^1[\xi+(1-t)^\alpha-t^\alpha]\,\mathrm{LE}'(tb+(1-t)a)\mathrm{d}t,\end{aligned} \quad (4.12)$$

where

$$\xi=\begin{cases}-1, & 0\le t\le 1/2,\\ 1, & 1/2<t\le 1.\end{cases}$$

Proof. We start our proof from the right side of (4.12),

$$
\begin{aligned}
I &= \frac{b-a}{2}\int_0^1 [\xi + (1-t)^\alpha - t^\alpha]\,\mathrm{LE}'(tb+(1-t)a)\mathrm{d}t \\
&= \frac{1}{2}\int_0^1 [\xi + (1-t)^\alpha - t^\alpha]\mathrm{d}[\mathrm{LE}(tb+(1-t)a)] \\
&= \frac{1}{2}\int_0^{1/2} [-1 + (1-t)^\alpha - t^\alpha]\mathrm{d}[\mathrm{LE}(tb+(1-t)a)] \\
&\quad + \frac{1}{2}\int_{1/2}^1 [1 + (1-t)^\alpha - t^\alpha]\mathrm{d}[\mathrm{LE}(tb+(1-t)a)] \\
&= \frac{1}{2}[-1 + (1-t)^\alpha - t^\alpha]\,\mathrm{LE}(tb+(1-t)a)|_0^{1/2} \\
&\quad - \frac{1}{2}\int_0^{1/2} [-\alpha(1-t)^{\alpha-1} - \alpha t^{\alpha-1}]\,\mathrm{LE}(tb+(1-t)a)\mathrm{d}t \\
&\quad + \frac{1}{2}[1 + (1-t)^\alpha - t^\alpha]\,\mathrm{LE}(tb+(1-t)a)|_{1/2}^1 \\
&\quad - \frac{1}{2}\int_{1/2}^1 [-\alpha(1-t)^{\alpha-1} - \alpha t^{\alpha-1}]\,\mathrm{LE}(tb+(1-t)a)\mathrm{d}t \\
&= -f\Big(\frac{a+b}{2}\Big) + \frac{1}{2}\int_0^1 [\alpha(1-t)^{\alpha-1} + \alpha t^{\alpha-1}]\,\mathrm{LE}(tb+(1-t)a)\mathrm{d}t.
\end{aligned}
\tag{4.13}
$$

Using the inequalities

$$
\int_0^1 \alpha(1-t)^{\alpha-1}\,\mathrm{LE}(tb+(1-t)a)\mathrm{d}t = \int_0^1 \alpha t^{\alpha-1}\,\mathrm{LE}(ta+(1-t)b)\mathrm{d}t, \tag{4.14}
$$

$$
\int_0^1 \alpha t^{\alpha-1}\,\mathrm{LE}(tb+(1-t)a)\mathrm{d}t = \int_0^1 \alpha t^{\alpha-1}\,\mathrm{LE}(ta+(1-t)b)\mathrm{d}t, \tag{4.15}
$$

and (4.13), we have

$$
\begin{aligned}
I &= -f(\frac{a+b}{2}) + \frac{1}{2}\int_0^1 [\alpha(1-t)^{\alpha-1} + \alpha t^{\alpha-1}]\,\mathrm{LE}(tb+(1-t)a)\mathrm{d}t \\
&= \int_0^1 \alpha t^{\alpha-1}\,\mathrm{LE}(ta+(1-t)b)\mathrm{d}t - f(\frac{a+b}{2}) \\
&= \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b) - f(\frac{a+b}{2}).
\end{aligned}
\tag{4.16}
$$

The proof is complete. □

Theorem 4.6. *For $\alpha > 0$, suppose that $f(x)$ is differentiable on $[a,b]$ and $\left|\mathrm{LE}'(x)\right|$ is a convex function. Then*

$$
\begin{aligned}
&\big|f(\frac{a+b}{2}) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b)\big| \\
&\le \Big(\frac{1}{2} + \frac{1}{2^\alpha(\alpha+1)} - \frac{1}{\alpha+1}\Big)\frac{(b-a)|f'(b)e^{f(b)} - f'(a)e^{f(a)}|}{e^{f(a)}+e^{f(b)}}.
\end{aligned}
\tag{4.17}
$$

Proof. Based on the proof of Theorem 4.2, we know that if $\left|\mathrm{LE}'(x)\right|$ is a convex function, then

$$\left|\mathrm{LE}'(tb+(1-t)a)\right| \le \frac{\left|f'(b)e^{f(b)}-f'(a)e^{f(a)}\right|}{e^{f(a)}+e^{f(b)}}. \tag{4.18}$$

In view of Lemma 4.5, we have

$$\begin{aligned}
&\left|f(\frac{a+b}{2})-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}\,\mathrm{LE}(b)\right| \\
&= \frac{b-a}{2}\Big|\int_0^1 \Big[\xi+(1-t)^\alpha-t^\alpha\Big]\,\mathrm{LE}'(tb+(1-t)a)\mathrm{d}t\Big| \\
&= \frac{b-a}{2}\Big|\int_0^{1/2}[-1+(1-t)^\alpha-t^\alpha]\,\mathrm{LE}'(tb+(1-t)a)\mathrm{d}t \\
&\quad+\int_{1/2}^1[1+(1-t)^\alpha-t^\alpha]\,\mathrm{LE}'(tb+(1-t)a)\mathrm{d}t\Big| \\
&\le \frac{(b-a)\left|f'(b)e^{f(b)}-f'(a)e^{f(a)}\right|}{2(e^{f(a)}+e^{f(b)})}\Big[\int_0^{1/2}\Big[1-(1-t)^\alpha+t^\alpha\Big]\mathrm{d}t \\
&\quad+\int_{1/2}^1\Big[1+(1-t)^\alpha-t^\alpha\Big]\mathrm{d}t\Big] \\
&= \frac{(b-a)\left|f'(b)e^{f(b)}-f'(a)e^{f(a)}\right|}{2(e^{f(a)}+e^{f(b)})}\Big[1+\frac{2}{2^\alpha(\alpha+1)}-\frac{2}{\alpha+1}\Big].
\end{aligned} \tag{4.19}$$

So we complete the proof. □

Corollary 4.7. *Under the conditions of Theorem 4.6, we conclude that*

$$\Big|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b \mathrm{LE}(x)\mathrm{d}x\Big| \le \frac{(b-a)|f'(b)e^{f(b)}-f'(a)e^{f(a)}|}{4(e^{f(a)}+e^{f(b)})}. \tag{4.20}$$

The above corollary follows from Theorem 4.6 with $\alpha=1$. If $f(x)$ has a symmetry, we have the following corollary.

Corollary 4.8. *If $f(x)$ is symmetric with respect to the axis $x=\frac{a+b}{2}$ and other conditions of Theorem 4.6 hold, then*

$$\Big|f(\frac{a+b}{2})-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{a^+}^{-\alpha}f(b)\Big| \le \Big(\frac{1}{2}+\frac{1}{2^\alpha(\alpha+1)}-\frac{1}{\alpha+1}\Big)(b-a)|f'(b)|. \tag{4.21}$$

The proof of the corollary above is almost identical to that of Corollary 4.4, so we omit it. As a by-product, we estimate the bound of fractional integral for given exp-convex function.

Theorem 4.9. *If $f(x)$ is continuous and exp-convex on $[a,b]$, then*

$$\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{b^-}^{-\alpha}f(a) \le f(b)+\frac{1}{\alpha}\Big[{}_2F_1(1,\alpha;\alpha+1;1-\frac{e^{f(b)}}{e^{f(a)}})-1\Big], \tag{4.22}$$

where $f(b)-f(a)<\ln 2$ and $\alpha>0$.

Proof. (i) For $f(a)\neq f(b)$, based on the definition of exp-convex function, we obtain

$$\begin{aligned}
f(tb+(1-t)a) &\le \ln\Big[t(e^{f(b)}-e^{f(a)})+e^{f(a)}\Big] \\
&= \ln\left|e^{f(b)}-e^{f(a)}\right|+\ln(\eta t+m),
\end{aligned} \tag{4.23}$$

where $m = e^{f(a)}/|e^{f(b)} - e^{f(a)}| \in (1, +\infty)$, and

$$\eta = \frac{e^{f(b)} - e^{f(a)}}{|e^{f(b)} - e^{f(a)}|} = \begin{cases} -1, & f(a) > f(b), \\ 1, & f(a) < f(b). \end{cases}$$

By multiplying the factor $\alpha t^{\alpha-1}$ on both sides, and integrating with respect to t over $[0,1]$, it follows that

$$\begin{aligned} &\int_0^1 \alpha t^{\alpha-1} f(tb + (1-t)a)\mathrm{d}t \\ &\leq \int_0^1 \alpha t^{\alpha-1} \ln(\eta t + m)\mathrm{d}t + \int_0^1 \alpha t^{\alpha-1} \ln\big|e^{f(b)} - e^{f(a)}\big|\mathrm{d}t \\ &= \int_0^1 \alpha t^{\alpha-1} \ln(\eta t + m)\mathrm{d}t + \ln\big|e^{f(b)} - e^{f(a)}\big|. \end{aligned} \tag{4.24}$$

For the integral on the right side, we have

$$\begin{aligned} \int_0^1 \alpha t^{\alpha-1} \ln(\eta t + m)\mathrm{d}t &= \int_0^1 \alpha t^{\alpha-1}\big[\ln m + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\eta t)^n}{nm^n}\big]\mathrm{d}t \\ &= \ln m + \alpha \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\eta^n}{nm^n(\alpha+n)} \\ &= \ln m - \sum_{n=1}^{\infty} \frac{\alpha}{n(\alpha+n)}\Big(\frac{-\eta}{m}\Big)^n. \end{aligned} \tag{4.25}$$

Furthermore, the series in (4.25) can be formulated as

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\alpha}{n(\alpha+n)}\Big(\frac{-\eta}{m}\Big)^n &= \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{\alpha+n})\Big(\frac{-\eta}{m}\Big)^n \\ &= -\ln(1 + \frac{\eta}{m}) - \sum_{n=1}^{\infty} \frac{1}{(\alpha+n)}\Big(\frac{-\eta}{m}\Big)^n \\ &= -\ln(1 + \frac{\eta}{m}) - \frac{1}{\alpha}\big[{}_2F_1(1,\, \alpha;\, \alpha+1;\, -\frac{\eta}{m}) - 1\big]. \end{aligned} \tag{4.26}$$

Combining (4.24), (4.25) and (4.26), we obtain

$$\begin{aligned} \int_0^1 \alpha t^{\alpha-1} \ln(\eta t + m)\mathrm{d}t &= \ln m + \ln(1 + \frac{\eta}{m}) + \frac{1}{\alpha}\big[{}_2F_1(1,\, \alpha;\, \alpha+1;\, -\frac{\eta}{m}) - 1\big] \\ &= \ln(m + \eta) + \frac{1}{\alpha}\big[{}_2F_1(1,\, \alpha;\, \alpha+1;\, -\frac{\eta}{m}) - 1\big]. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\mathrm{D}_{b^-}^{-\alpha} f(a) &= \int_0^1 \alpha t^{\alpha-1} f(tb + (1-t)a)\mathrm{d}t \\ &\leq \int_0^1 \alpha t^{\alpha-1} \ln(\eta t + m)\mathrm{d}t + \ln|e^{f(b)} - e^{f(a)}| \\ &= f(b) + \frac{1}{\alpha}\big[{}_2F_1(1, \alpha; \alpha+1; 1 - \frac{e^{f(b)}}{e^{f(a)}}) - 1\big], \end{aligned} \tag{4.27}$$

where $-\eta/m = 1 - e^f(b)/e^f(a)$ has been used.

(ii) For $f(a) = f(b)$, in view of the definition of exp-convex function, we have

$$\begin{aligned} f(tb + (1-t)a) &\le \ln\big[t(e^{f(b)} - e^{f(a)}) + e^{f(a)}\big] \\ &= f(a) = f(b). \end{aligned} \tag{4.28}$$

So, we get

$$\begin{aligned} \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \mathrm{D}_{b^-}^{-\alpha} f(a) &= \int_0^1 \alpha t^{\alpha-1} f(tb + (1-t)a)\mathrm{d}t \\ &\le \int_0^1 \alpha t^{\alpha-1} f(b)\mathrm{d}t \\ &= f(b) + \frac{1}{\alpha}\big[{}_2F_1(1, \alpha; \alpha+1; 0) - 1\big]. \end{aligned} \tag{4.29}$$

Therefore, we have completed the proof. □

Remark 4.10. The constraint condition $f(b) - f(a) < \ln 2$ is required in Theorem 4.9 which is depended greatly both on the series expansion approach and the region of convergence of Gauss hypergeometric function. However, we conjecture that (4.22) will also be valid on a more wider real region in view of other sophisticated techniques/algorithms associated with some special functions.

With $\alpha = 1$ in the theorem above, we can establish another interesting estimating value theorem for exp-convex functions.

Corollary 4.11. *For an exp-convex $f(x)$ defined on $[a, b]$, we have*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le f(b) + \frac{f(b) - f(a)}{e^{f(b)-f(a)} - 1} - 1. \tag{4.30}$$

Proof. As a matter of fact, (4.30) can be obtained after integrating and derivation directly without utilizing series expansion technique. Because of the exp-convexity of f, we have

$$f(tb + (1-t)a) \le \ln\big[te^{f(b)} + (1-t)e^{f(a)}\big]. \tag{4.31}$$

Integrating t over $[0, 1]$, we have

$$\begin{aligned} \int_0^1 f(tb + (1-t)a)\mathrm{d}t &\le \int_0^1 \ln\big[te^{f(b)} + (1-t)e^{f(a)}\big]\mathrm{d}t \\ &= \int_0^1 \ln\big[t(e^{f(b)} - e^{f(a)}) + e^{f(a)}\big]\mathrm{d}t \\ &= f(b) - 1 + \int_0^1 \frac{e^{f(a)}}{t(e^{f(b)} - e^{f(a)}) + e^{f(a)}}\mathrm{d}t \\ &= f(b) + \frac{f(b) - f(a)}{e^{f(b)-f(a)} - 1} - 1. \end{aligned} \tag{4.32}$$

Let $x = tb + (1-t)a$, then we conclude that

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \int_0^1 f(tb + (1-t)a)\mathrm{d}t. \tag{4.33}$$

Combining (4.32) and (4.33), we complete the proof. □

Moreover, as $\alpha = 2$ in Theorem 4.9, we have the following estimation.

Corollary 4.12. *For an exp-convex $f(x)$ defined on $[a,b]$, we have*

$$\frac{2}{(b-a)^2}\int_a^b (x-a)f(x)\mathrm{d}x \le f(b)+m-m^2(f(b)-f(a))-\frac{1}{2}, \tag{4.34}$$

where $m=e^{f(a)}/(e^{f(b)}-e^{f(a)})$.

Proof. The idea is almost identical with Corollary 4.11, so we only present an outline of the proof. For an exp-convex $f(x)$, we have

$$\begin{aligned}\frac{2}{(b-a)^2}\int_a^b (x-a)f(x)\mathrm{d}x &= 2\int_0^1 tf(tb+(1-t)a)\mathrm{d}t\\ &\le 2\int_0^1 t\ln\Big[te^{f(b)}+(1-t)e^{f(a)}\Big]\mathrm{d}t\\ &= f(b)-\int_0^1 \frac{t^2}{t+m}\mathrm{d}t,\end{aligned} \tag{4.35}$$

where $m=e^{f(a)}/(e^{f(b)}-e^{f(a)})$.
Therefore, the result will be obtained immediately after integration by parts. □

5. Applications to special means

We consider the following special means for $b>a>0$:

$$A(a,b)=\frac{a+b}{2},\quad G(a,b)=\sqrt{ab},\quad H(a,b)=\frac{2}{1/a+1/b},$$
$$L(a,b)=\frac{b-a}{\ln b-\ln a},\quad LP(a,b)=\frac{b\ln b-a\ln a}{b-a}.$$

Now according to the results obtained in previous sections, we can obtain some interesting assertions with these special means.

Proposition 5.1. *Let $a,b\in R^+$, $b>a$ and $p\ge 1$. Then*

$$\frac{pa}{L(a,b)}-\frac{a^p}{L(a^p,\,b^p)}\le p-1. \tag{5.1}$$

Proof. Let $f(x)=p\ln x$, $(p\ge 1)$. Obviously, $f(x)$ is exp-convex, in terms of Corollary 4.11, then we obtain

$$\frac{1}{b-a}\int_a^b p\ln x\mathrm{d}x \le p\ln b+\frac{p\ln b-p\ln a}{e^{p\ln b-p\ln a}-1}-1. \tag{5.2}$$

Hence this suffices to show that

$$\frac{p(b\ln b-a\ln a-b+a)}{b-a}\le p\ln b+\Big(\frac{p\ln b-p\ln a}{b^p-a^p}\Big)a^p-1, \tag{5.3}$$

which can be rewritten as

$$\frac{p(b\ln b-a\ln a-b\ln b+a\ln b-b+a)}{b-a}\le \frac{a^p}{L(a^p,\,b^p)}-1. \tag{5.4}$$

We immediately obtain

$$\frac{pa(\ln b-\ln a)}{b-a}-p\le \frac{a^p}{L(a^p,\,b^p)}-1. \tag{5.5}$$

The proof is complete. □

Proposition 5.2. *Let $a, b \in R^+$ with $b > a$. Then*

$$|\ln G(a,b) - LP(a,b) + 1| \le \frac{(b-a)^2}{4ab}, \tag{5.6}$$

$$|\ln A(a,b) - LP(a,b) + 1| \le \frac{(b-a)^2}{8ab}. \tag{5.7}$$

Proof. Let $f(x) = -\ln x = \ln(1/x)$, from Corollary 3.2, we have

$$-\ln\frac{a+b}{2} \le \frac{1}{b-a}\int_a^b \ln\Big(\frac{\frac{1}{x}+\frac{1}{a+b-x}}{2}\Big)\mathrm{d}x \le \ln\Big(\frac{\frac{1}{a}+\frac{1}{b}}{2}\Big). \tag{5.8}$$

Noticing that

$$\begin{aligned}\frac{1}{b-a}\int_a^b \ln\Big(\frac{\frac{1}{x}+\frac{1}{a+b-x}}{2}\Big)\mathrm{d}x &= \frac{1}{b-a}\int_a^b \ln\frac{a+b}{2} - 2\ln x\mathrm{d}x\\ &= \ln\frac{a+b}{2} - \frac{2\,(b\ln b - a\ln a - b + a)}{b-a}\\ &= \ln\frac{a+b}{2} - \frac{2\,(b\ln b - a\ln a)}{b-a} + 2,\end{aligned} \tag{5.9}$$

we obtain

$$-\ln\frac{a+b}{2} \le \ln\frac{a+b}{2} - \frac{2\,(b\ln b - a\ln a)}{b-a} + 2 \le \ln\Big(\frac{\frac{1}{a}+\frac{1}{b}}{2}\Big). \tag{5.10}$$

Equivalently,

$$\ln\frac{1}{A(a,b)} \le \ln A(a,b) - 2LP(a,b) + 2 \le \ln\frac{1}{H(a,b)}. \tag{5.11}$$

In the sequel, we should prove that $|\,\mathrm{LE}'(x)|$ is convex. For $f(x) = \ln(1/x)$, we obtain

$$\begin{aligned}\mathrm{LE}(x) &= \ln\Big[\frac{e^{f(x)} + e^{f(a+b-x)}}{2}\Big]\\ &= \ln\Big[\frac{\frac{1}{x}+\frac{1}{a+b-x}}{2}\Big]\\ &= \ln\frac{a+b}{2} - \ln x - \ln(a+b-x), \quad x \in [a,b].\end{aligned} \tag{5.12}$$

So,

$$\mathrm{LE}'(x) = -\frac{1}{x} + \frac{1}{a+b-x}. \tag{5.13}$$

Hence, we have

$$|\,\mathrm{LE}'(x)| = \begin{cases} \frac{1}{x} - \frac{1}{a+b-x}, & a \le x \le \frac{a+b}{2},\\ -\frac{1}{x} + \frac{1}{a+b-x}, & \frac{a+b}{2} \le x \le b. \end{cases} \tag{5.14}$$

As $a \le x \le \frac{a+b}{2}$, we define $g(x) = \frac{1}{x} - \frac{1}{a+b-x}$, so $g''(x) = \frac{2}{[x-(a+b)]^3} + \frac{2}{x^3}$ is nonincreasing. Thus, $g''(x) \ge g''(\frac{a+b}{2}) = 0$, so $g(x)$ is convex. On the other hand, when $\frac{a+b}{2} \le x \le b$, we set $h(x) = -\frac{1}{x} + \frac{1}{a+b-x}$, so $h''(x) = -\frac{2}{[x-(a+b)]^3} - \frac{2}{x^3}$ is nondecreasing. That is, $h''(x) \ge h''(\frac{a+b}{2}) = 0$, so $h(x)$ is convex. Now from

Corollary 4.3, we have

$$\begin{aligned}\Big|\ln A(a,b)-\ln\frac{1}{H(a,b)}-2LP(a,b)+2\Big| &= |2\ln G(a,b)-2LP(a,b)+2|\\ &\le \frac{(b-a)(\frac{1}{-b^2}+\frac{1}{a^2})}{2(\frac{1}{a}+\frac{1}{b})} \\ &= \frac{(b-a)^2}{2ab}.\end{aligned} \tag{5.15}$$

So

$$|\ln G(a,b)-LP(a,b)+1|\le\frac{(b-a)^2}{4ab}. \tag{5.16}$$

On the other hand, from Corollary 4.7, we obtain

$$\begin{aligned}&\Big|\ln A(a,b)-\ln\frac{1}{A(a,b)}-2LP(a,b)+2\Big|\\ &=2|\ln A(a,b)-LP(a,b)+1|\le\frac{(b-a)^2}{4ab}.\end{aligned} \tag{5.17}$$

Combining (5.16) and (5.17), we obtain the conclusion. ☐

Proposition 5.3. *If $a,b\in R^+$, $p\ge 1$ and $b>a$, then*

$$\frac{pa}{b-a}\Big(1-\frac{a}{L(a,b)}\Big)-\frac{a^p}{b^p-a^p}\Big(1-\frac{a^p}{L(a^p,b^p)}\Big)\le\frac{p-1}{2}. \tag{5.18}$$

Proof. Let $f(x)=p\ln x$, $(p\ge 1)$ be exp-convex. By Corollary 4.12, we have

$$\frac{2p}{(b-a)^2}\int_a^b(x-a)\ln x\mathrm{d}x\le p\ln b+m-m^2(p\ln b-p\ln a)-\frac{1}{2}, \tag{5.19}$$

where $m=e^{f(a)}/(e^{f(b)}-e^{f(a)})=\frac{a^p}{b^p-a^p}$. Denoting

$$I=\frac{2p}{(b-a)^2}\int_a^b(x-a)\ln x\mathrm{d}x,$$

we have

$$\begin{aligned}I&=\frac{2p}{(b-a)^2}\Big[\frac{(x-a)^2}{2}\ln x\Big|_a^b-\frac{1}{2}\int_a^b\frac{(x-a)^2}{x}\mathrm{d}x\Big]\\ &=p\Big[\ln b+\frac{3a-b}{2(b-a)}-\frac{a^2}{(b-a)^2}(\ln b-\ln a)\Big].\end{aligned}$$

So, we obtain

$$\begin{aligned}&p\Big[\ln b+\frac{3a-b}{2(b-a)}-\frac{a^2}{(b-a)^2}(\ln b-\ln a)\Big]\\ &\le p\ln b+m-m^2(p\ln b-p\ln a)-\frac{1}{2}\\ &=p\ln b+\frac{a^p}{b^p-a^p}-\frac{a^{2p}}{(b^p-a^p)^2}(p\ln b-p\ln a)-\frac{1}{2}\\ &=p\ln b+\frac{3a^p-b^p}{2(b^p-a^p)}-\frac{a^{2p}}{(b^p-a^p)^2}(p\ln b-p\ln a).\end{aligned}$$

Further simplification indicates that

$$\frac{p(3a-b)}{2(b-a)} - \frac{3a^p - b^p}{2(b^p - a^p)} \leq \frac{pa^2}{(b-a)^2}(\ln b - \ln a) - \frac{a^{2p}}{(b^p - a^p)^2}(p\ln b - p\ln a), \quad (5.20)$$

or, it can be rewritten as

$$\frac{pa}{b-a} - \frac{pa^2}{(b-a)^2}(\ln b - \ln a) + \frac{a^{2p}}{(b^p - a^p)^2}(p\ln b - p\ln a) - \frac{a^p}{b^p - a^p} \leq \frac{1}{2}p - \frac{1}{2}. \quad (5.21)$$

Thus, we finish this proof. □

6. Conclusions and future work

To enrich geometric properties of common continuous functions, the exp-convexity is studied in this article. Furthermore, some significant integral identities, Hadamard type integral inequalities in the framework of fractional operators including their estimation of the upper bounds are established and clarified in the presence of exp-convexity criterion. Besides, a conjecture about validation of Theorem 4.9 is also posed.

As we know, Mittag-Leffler function is the eigenfunction for fractional order system and plays a leading role in the basic theory of fractional calculus. The standard definition of Mittag-Leffler function is given as follows.

Definition 6.1 ([13])**.** The single-parameter Mittag-Leffler function and the two-parameter Mittag-Leffler function are defined as

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \quad (6.1)$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta > 0, \quad (6.2)$$

respectively.

Compared to exp-convexity, we propose standard Mittag-Leffler type convexity for its better compatibility with fractional order system as well as its weaker convexity than exp-convexity.

Definition 6.2. A function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$, is said to be single-parameter Mittag-Leffler type convex function, if the following inequality holds

$$E_\alpha(f(tx + (1-t)y)) \leq tE_\alpha(f(x)) + (1-t)E_\alpha(f(y)), \quad (6.3)$$

for all $t \in [0,1]$ and $x, y \in [a,b]$.

Definition 6.3. A function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$, is said to be two-parameter Mittag-Leffler type convex function, if the following inequality holds

$$E_{\alpha,\beta}(f(tx + (1-t)y)) \leq tE_{\alpha,\beta}(f(x)) + (1-t)E_{\alpha,\beta}(f(y)), \quad (6.4)$$

for all $t \in [0,1]$ and $x, y \in [a,b]$.

Obviously, as $\alpha = \beta = 1$, the Mittag-Leffler type convex functions above will degenerate into the classic exp-convex function consistently. Such novel Mittag-Leffler type convexity owns special geometric significance and the corresponding researches will be reported in our subsequent works.

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