Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 20, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NONEXISTENCE OF SOLUTIONS TO CAUCHY PROBLEMS FOR FRACTIONAL TIME SEMI-LINEAR PSEUDO-HYPERBOLIC SYSTEMS

SALEM ABDELMALEK, MAHA BAJNEED, KHALED SIOUD

Abstract. We study Cauchy problems time fractional semi-linear pseudo-hyperbolic equations and systems. Using the method of nonlinear capacity, we show that there are no solutions for certain nonlinearities and initial data. Our work complements the work by Aliev and col. [1, 6, 7].

1. Introduction

In this article, we study Cauchy problems for time fractional pseudo-hyperbolic equations and systems. We start by considering the time fractional equation

$$u_{tt} + \eta(-\Delta)^k u_{tt} + (-\Delta)^\ell u + \xi(-\Delta)^r D^\alpha_{0|t} u + \gamma D^\beta_{0|t} u = f(u), \tag{1.1}$$

for $x \in \mathbb{R}^N$, $t > 0$, supplemented with the initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

and with

$$\eta, \xi, \gamma \geq 0 \quad \text{for } r, k \in \mathbb{N} \cup \{0\},\ \ell \in \mathbb{N},\ 0 < \beta \leq \alpha \leq 1, \tag{1.3}$$

Δ is the Laplacian and $D^\alpha_{0|t}$ is the left-sided Riemann-Liouville fractional derivative of order α.

The aim of this paper is to show, using the method of nonlinear capacity proposed by Pokhozhaev in 1997 [18] and developed successfully and jointly with Mitidieri [15, 16, 17], that under certain conditions, there are no solutions to (1.1)-(1.2).

For the non fractional case $\alpha = \beta = 1$, Lions' monograph [13] considered equation (1.1) in the case where $\eta = 0$ and $f(u) = -|u|^p u$. A step forward was achieved by [10, 15, 20] where they considered the absence of global solutions for the case where $\eta = \xi = 0$ and $f(u) = |u|^{p-1}u$ or $f(u) = \pm|u|^p$. Kato [10] showed that for $\ell = 1$, $\xi = 0$, and $1 < p < 1 + \frac{2}{N}$, problem (1.1)-(1.2) admits no global solution under a certain condition on the initial data. A further study of John [9] considered the case $\ell = 1$, $\xi = 0$, $\eta = 0, \gamma = 0$ and $f(u) = |u|^p$ for u close to zero. This was generalized to $\ell \in \mathbb{N}$, $\xi = \eta = 0$, $\gamma > 0$, $\alpha = 1$, by Zhang [20] and Kirane and

2010 *Mathematics Subject Classification.* 80A23, 65N21, 26A33, 45J05, 34K37, 42A16.
Key words and phrases. Semi-linear pseudo-hyperbolic equation; non-existence of solutions; non-linear capacity method.

Submitted June 25, 2015. Published January 12, 2016.

Qafsaoui [12]. The two studies proved that the critical exponent for this case is in fact $p = 1 + \frac{2}{N}$.

The existence of global solutions of problem (1.1)-(1.3) for the non-fractional case $\alpha = \beta = 1$, $\eta, \xi, \gamma \geq 0$, $r, k \in \mathbb{N} \cup \{0\}$ and $\ell \in \mathbb{N}$ was achieved by Aliev and Kazymov [5].

Recently, by using the method of the test function Aliev and col. [1, 6, 7] established sufficient conditions for the nonexistence of global solutions of problem (1.1)-(1.3) for the non-fractional case $\alpha = \beta = 1$: Aliev and Lichaei [6] considered the case $\alpha = \beta = 1$ and $\eta, \xi, \gamma > 0$ for $r = k \in \mathbb{N} \cup \{0\}$, $\ell \in \mathbb{N}$, and $f(u) \geq C|u|^p$. Aliev and Kazymov [1] examined the case
$\alpha = \beta = 1$, $k = 0$, $r = 0$, $\ell \in \mathbb{N}$ and $f(u) = \frac{1}{(1+|x|^2)^s}|u|^p$. Aliev and Mamedov [7] treated the non existence of global solutions of a semilinear hyperbolic equation with an anisotopic elliptic part ($\alpha = \beta = 1$, $k = r = 0$),

$$u_{tt} + \varepsilon u_t + \sum_{k=1}^{N} (-1)^{\ell_k} D_{x_k}^{2\ell_k} u = f(u), \quad f(u) \geq c|u|^p.$$

Our work will complement the results of [6] for $r, k \in \mathbb{N} \cup \{0\}$, $\eta, \xi, \gamma \geq 0$ and $\ell \in \mathbb{N}$ and extend it to the time-fractional case $0 < \beta \leq \alpha < 1$, using the test function method.

In the second part of this paper, we study the Cauchy problem for the time-fractional pseudo-hyperbolic system

$$\begin{aligned} u_{tt} + \eta_1(-\Delta)^{k_1} u_{tt} + (-\Delta)^{\ell_1} u + \xi_1(-\Delta)^{r_1} D_{0|t}^{\alpha_1} u + \gamma_1 D_{0|t}^{\beta_1} u = f(v) = |v|^p \\ v_{tt} + \eta_2(-\Delta)^{k_2} v_{tt} + (-\Delta)^{\ell_2} v + \xi_2(-\Delta)^{r_2} D_{0|t}^{\alpha_2} v + \gamma_2 D_{0|t}^{\beta_2} v = g(u) = |u|^q \end{aligned} \tag{1.4}$$

posed in $Q_\infty := \mathbb{R}^N \times (0, \infty)$, subject to the initial conditions

$$\begin{gathered} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x) x \in \mathbb{R}^N \end{gathered} \tag{1.5}$$

with $p, q > 1$, $r_i, k_i \in \mathbb{N} \cup \{0\}$, $\ell_i \in \mathbb{N}$, $\eta_i, \xi_i, \gamma_i \geq 0$ and $0 < \beta_i \leq \alpha_i \leq 1$ for $i = 1, 2$.

The non-existence of global solutions in the case of a non fractional system of two (or more) equations with $\alpha_i = 0$ or 1 and $\beta_i = 0$ or 1, is investigated in numerous studies of Aliev and colleagues: Aliev, Mammadzada, and Lichaei [8] considered the case $\beta_i = 1$, $\gamma_i = \eta_i = 1$, $\xi_i = 0$, $\ell_1 = 1$, $\ell_2 = 2$, $p = \frac{7}{2}$ and $q = \frac{5}{2}$; Aliev and Kazymov [4] examined the case $\beta_i = 1$, $\gamma_i = \eta_i = 1$, $\xi_i = 0$, $\ell_i \in \mathbb{N}$, and $f_i(u,v) \geq C_{i,1}|u|^{p_i} + C_{i,2}|v|^{q_i}$; Aliev and Kazymov [2] considered the case $\beta_i = 1$, $\gamma_i = \eta_i = 1$, $\xi_i = 0$, $\ell_i \in \mathbb{N}$, and $f(v) \geq C|v|^p$ and $g(u) \geq C|u|^q$; Aliev and Kazymov [3] dealt with a system of three equations that is similar to the case presented in [4].

Our work will complement these papers for the system of two equations in the cases $\gamma_i, \eta_i, \xi_i > 0$, $r_i, k_i \in \mathbb{N} \cup \{0\}$, $\ell_i \in \mathbb{N}$ and extend it to the time-fractional case, using again the test function method.

2. Preliminaries

For the convenience of the reader, we start by recalling some basic definitions and properties which will be useful throughout this paper.

Definition 2.1. The left- and right-sided Riemann-Liouville integrals of order $0 < \alpha < 1$ for an integrable function are defined as

$$\big(I^{\alpha}_{0|t} f\big)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2.1}$$

$$\big(I^{\alpha}_{t|T} f\big)(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s) ds, \tag{2.2}$$

where Γ is the Euler gamma function.

Definition 2.2. Let $AC[0,T]$ be the space of functions f which are absolutely continuous on $[0,T]$. The left and right-handed Riemann-Liouville fractional derivatives of order $n-1 < \gamma < n$ for a function

$$f \in AC^n[0,T] := \{f : [0,T] \to \mathbb{R}, D^{n-1} f \in AC[0,T]\}, \quad n \in \mathbb{N}$$

is defined as (see [11])

$$D^{\gamma}_{0|t} f(t) := D^n (I^{n-\gamma}_{0|t} f)(t), \quad t > 0, \tag{2.3}$$

$$D^{\gamma}_{t|T} f(t) := (-1)^n D^n (I^{n-\gamma}_{t|T} f)(t), \tag{2.4}$$

where D is the usual time derivative.

Furthermore, for every $f, g \in C([0,T])$ such that $D^{\alpha}_{0|t} f(t), D^{\alpha}_{0|t} g(t)$ exist and are continuous for all $t \in [0,T]$, $0 < \alpha < 1$, the formula of integration by parts can be given according to Love and Young [14] by

$$\int_0^T g(t)(D^{\alpha}_{0|t} f)(t) dt = \int_0^T f(t)(D^{\alpha}_{t|T} g)(t) dt. \tag{2.5}$$

In addition, [19, Lemma 2.2] provides us with the formula

$$D^{\alpha}_{t|T} f(t) := \frac{1}{\Gamma(1-\alpha)} \Big[\frac{f(T)}{(T-t)^{\alpha}} - \int_t^T (t-s)^{-\alpha} f'(s) ds \Big] \tag{2.6}$$

or

$$D^{\alpha}_{t|T} f(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (t-s)^{-\alpha} f(s) ds. \tag{2.7}$$

3. Non-existence of global solutions of one equation

In this section, we study the non-existence of global solutions for the time-fractional semi-linear pseudo-hyperbolic equation (1.1) for certain initial data with $f(u) = |u|^p$. Before we state our result. let us define the weak solution of problem (1.1)-(1.3).

In this article, Q_T denotes the set $Q_T := \mathbb{R}^N \times (0,T)$, $0 < T \le +\infty$. We set

$$\int_{Q_T} f := \int_{\mathbb{R}^N} \int_0^T f(x,t)\, dx\, dt, \quad \int_{Q_\infty} f := \int_{\mathbb{R}^N} \int_0^\infty f(x,t)\, dx\, dt,$$

$$\int_{\mathbb{R}^N} f := \int_{\mathbb{R}^N} f(x,0) dx.$$

Definition 3.1. The function $u \in L^p_{\rm loc}(Q_\infty)$ is a weak solution of problem (1.1)-(1.3) on Q_T with initial data $u_0(x), u_1(x) \in L^1_{\rm loc}(\mathbb{R}^N)$ if it satisfies

$$\begin{aligned}
&\int_{Q_T} |u|^p\varphi + \int_{\mathbb{R}^N} u_1(x)\varphi(x,0) + \eta\int_{\mathbb{R}^N} u_1(x)(-\Delta)^k\varphi(x,0)\\
&= \int_{\mathbb{R}^N} u_0(x)\varphi_t(x,0) + \eta\int_{\mathbb{R}^N} u_0(x)(-\Delta)^k\varphi_t(x,0) + \int_{Q_T} u\varphi_{tt}\\
&\quad + \eta\int_{Q_T} u(-\Delta)^k\varphi_{tt} - \xi\int_{Q_T} u(-\Delta)^r D^\alpha_{t|T}\varphi + \gamma\int_{Q_T} uD^\beta_{t|T}\varphi\\
&\quad + \int_{Q_T} u(-\Delta)^\ell\varphi,
\end{aligned}\tag{3.1}$$

for any test-function $\varphi \in C^{2d}_x{}_t{}^2(Q_T)$ with $d = \max\{\ell, k, r\}$ such that φ is positive, $\varphi \equiv 0$ outside a compact $K \subset \mathbb{R}^n$, $\varphi(x,T) = \varphi_t(x,T) = 0$ and $D^\beta_{t|T}\varphi, D^\alpha_{t|T}\varphi \in C(Q_T)$.

As for the result on the non-existence of a global solution, the constants η, ξ and γ will not play a role, and thus will be taken equal to one.

Theorem 3.2. *Assume that*

1. $r, k \in \mathbb{N}\cup\{0\}$, $\ell \in \mathbb{N}$ *and* $0 < \beta \le \alpha < 1$*;*
2. $u_0, u_1, \in L^1(\mathbb{R}^N)$ *such that* $\int_{\mathbb{R}^N} u_0(x)dx > 0$, $\int_{\mathbb{R}^N} u_1(x)dx > 0$
3. $1 < p \le 1 + \dfrac{2\ell}{N + 2\ell(\frac{1}{\beta} - 1)} =: p_c$.

Then problem (1.1)–(1.3) *does not admit any global in time nontrivial solution.*

Proof. The proof is by contraction. Let u be a global weak solution of problem (1.1)-(1.3) and φ be a non-negative function (satisfying the conditions of Definition 3.1) that will be specified later.

Using ϵ-Young's inequality

$$ab \le \epsilon a^p + c(\epsilon)b^{\widetilde{p}}, \quad p > 1,\ a \ge 0,\ b \ge 0,\ p + \widetilde{p} = p\widetilde{p},\ \epsilon > 0,$$

we can write

$$\begin{gathered}
\int_{Q_T} u\varphi_{tt} \le \epsilon\int_{Q_T} |u|^p\varphi + c(\epsilon)\int_{Q_T} |\varphi_{tt}|^{\widetilde{p}}\varphi^{-\widetilde{p}/p},\\
\int_{Q_T} u(-\Delta)^k\varphi_{tt} \le \epsilon\int_{Q_T} |u|^p\varphi + c(\epsilon)\int_{Q_T} |(-\Delta)^{k_1}\varphi_{tt}|^{\widetilde{p}}\varphi^{-\widetilde{p}/p},\\
\int_{Q_T} u(-\Delta)^r D^\alpha_{t|T}\varphi \le \epsilon\int_{Q_T} |u|^p\varphi + c(\epsilon)\int_{Q_T} |(-\Delta)^r D^\alpha_{t|T}\varphi|^{\widetilde{p}}\varphi^{-\widetilde{p}/p},\\
\int_{Q_T} uD^\beta_{t|T}\varphi \le \epsilon\int_{Q_T} |u|^p\varphi + c(\epsilon)\int_{Q_T} |D^\beta_{t|T}\varphi|^{\widetilde{p}}\varphi^{-\widetilde{p}/p},\\
\int_{Q_T} u(-\Delta)^\ell\varphi \le \epsilon\int_{Q_T} |u|^p\varphi + c(\epsilon)\int_{Q_T} |(-\Delta)^\ell\varphi|^{\widetilde{p}}\varphi^{-\widetilde{p}/p}.
\end{gathered}\tag{3.2}$$

Using inequalities (3.2) in (3.1), we obtain the inequality

$$
\begin{aligned}
&\int_{Q_T} |u|^p\varphi + \int_{\mathbb{R}^N} u_1(x)\varphi(x,0) + \int_{\mathbb{R}^N} u_1(x)(-\Delta)^k\varphi(x,0)\\
&- \int_{\mathbb{R}^N} u_0(x)\varphi_t(x,0) - \int_{\mathbb{R}^N} u_0(x)(-\Delta)^k\varphi(x,0)\\
&\le \epsilon_1 \int_{Q_T} |u|^p\varphi + C_1\Big\{ \int_{Q_T} |\varphi_{tt}|^{\widetilde{p}}\varphi^{-\widetilde{p}/p} + \int_{Q_T} |(-\Delta)^k\varphi_{tt}|^{\widetilde{p}}\varphi^{-\widetilde{p}/p}\\
&+ \int_{Q_T} |(-\Delta)^r D^{\alpha}_{t|T}\varphi|^{\widetilde{p}}\varphi^{\frac{-\widetilde{p}}{p}} + \int_{Q_T} |D^{\beta}_{t|T}\varphi|^{\widetilde{p}}\varphi^{-\widetilde{p}/p} + \int_{Q_T} |(-\Delta)^{\ell}\varphi|^{\widetilde{p}}\varphi^{-\widetilde{p}/p}\Big\}.
\end{aligned}
\tag{3.3}
$$

Setting

$$
\begin{gathered}
A_1 = \int_{Q_T} |\varphi_{tt}|^{\widetilde{p}}\varphi^{-\widetilde{p}/p}, \quad A_2 = \int_{Q_T} |(-\Delta)^k\varphi_{tt}|^{\widetilde{p}}\varphi^{-\widetilde{p}/p},\\
A_3 = \int_{Q_T} |(-\Delta)^r D^{\alpha}_{t|T}\varphi|^{\widetilde{p}}\varphi^{-\widetilde{p}/p}, \quad A_4 = \int_{Q_T} |D^{\beta}_{t|T}\varphi|^{\widetilde{p}}\varphi^{-\widetilde{p}/p},\\
A_5 = \int_{Q_T} |(-\Delta)^{\ell}\varphi|^{\widetilde{p}}\varphi^{-\widetilde{p}/p},
\end{gathered}
$$

and taking $\epsilon = 1/2$, inequality (3.3) becomes

$$
\begin{aligned}
&\int_{Q_T} |u|^p\varphi + \int_{\mathbb{R}^N} u_1(x)\varphi(x,0) + \int_{\mathbb{R}^N} u_1(x)(-\Delta)^k\varphi(x,0)\\
&- \int_{\mathbb{R}^N} u_0(x)\varphi_t(x,0) - \int_{\mathbb{R}^N} u_0(x)(-\Delta)^k\varphi_t(x,0)\\
&\le C\{A_1 + A_2 + A_3 + A_4 + A_5\}.
\end{aligned}
\tag{3.4}
$$

At this stage, we set

$$
\varphi(x,t) = \Psi^{\nu}\Big(\frac{t^2 + |x|^{4\rho}}{R^4}\Big), \quad R > 0,\ \nu \gg 1,\ \rho > 0, \tag{3.5}
$$

where $\Psi \in C_c^{\infty}(\mathbb{R}^+)$ is a decreasing function defined as

$$
\Psi(r) = \begin{cases} 1 & \text{if } r \le 1\\ 0 & \text{if } r \ge 2, \end{cases}
$$

with $0 \le \Psi \le 1$ and $r|\Psi'(r)| < C$.

Note that with this choice of φ, we have

$$
\varphi_t(x,t) = 2\nu t R^{-4}\Psi^{\nu-1}((t^2 + |x|^{4\rho})/R^4)\Psi'((t^2 + |x|^{4\rho})/R^4),
$$

leading to

$$
\varphi_t(x,0) = 0. \tag{3.6}
$$

We also assume that φ satisfies

$$
\int_{Q_T} \varphi^{-p/\widetilde{p}}(|\varphi_{tt}|^{\widetilde{p}} + |(-\Delta)^k\varphi_{tt}|^{\widetilde{p}} + |(-\Delta)^r D^{\alpha}_{t|T}\varphi|^{\widetilde{p}} + |D^{\beta}_{t|T}\varphi|^{\widetilde{p}} + |(-\Delta)^{\ell}\varphi|^{\widetilde{p}}) < \infty,
$$

for that, we will choose $\nu \gg 1$. Therefore, the inequality (3.4) becomes

$$
\begin{aligned}
&\int_{Q_T} |u|^p\varphi + \int_{\mathbb{R}^N} u_1(x)\varphi(x,0) + \int_{\mathbb{R}^N} u_1(x)(-\Delta)^k\varphi(x,0)\\
&\le C\{A_1 + A_2 + A_3 + A_4 + A_5\}.
\end{aligned}
\tag{3.7}
$$

Let us pass to the scaled variables $y = R^{-1/\rho}x$, $\tau = R^{-2}t$ and the function $\widetilde{\varphi}$ given by $\varphi(x,t) = \widetilde{\varphi}(y,\tau)$. In doing so, it follows that

$$\varphi_t = R^{-2}\widetilde{\varphi}_\tau, \quad \varphi_{tt} = R^{-4}\widetilde{\varphi}_{\tau\tau}, \quad D^\beta_{t|T}\varphi = R^{-2\beta}D^\beta_{\tau|T^*}\widetilde{\varphi}, \quad (-\Delta)^m\varphi = R^{\frac{-2m}{\rho}}\Delta\widetilde{\varphi},$$

where $T = R^2T^*$ and $T^*$is a positive constant.

Now, let us set

$$Q = \{(y,\tau),\ 0 \le \tau^2 + y^{4\rho} \le 2\}.$$

Using these definitions, $A_1,\dots,A_5$ can be rewritten as

$$\begin{gathered}
A_1 \le R^{-4\widetilde{p}+\frac{N}{\rho}+2}\int_Q |\widetilde{\varphi}_{\tau\tau}|^{\widetilde{p}}\widetilde{\varphi}^{-\widetilde{p}/p},\\
A_2 \le R^{-(\frac{2k}{\rho}+4)\widetilde{p}+\frac{N}{\rho}+2}\int_Q |(-\Delta)^k\widetilde{\varphi}_{\tau\tau}|^{\widetilde{p}}\widetilde{\varphi}^{\frac{-\widetilde{p}}{p}},\\
A_3 \le R^{-(\frac{2r}{\rho}+2\alpha)\widetilde{p}+\frac{N}{\rho}+2}\int_Q |(-\Delta)^r D^\alpha_{\tau|T^*}\widetilde{\varphi}|^{\widetilde{p}}\varphi^{-\widetilde{p}/p},\\
A_4 \le R^{-2\beta\widetilde{p}+\frac{N}{\rho}+2}\int_Q |D^\beta_{\tau|T^*}\widetilde{\varphi}|^{\widetilde{p}}\widetilde{\varphi}^{\frac{-\widetilde{p}}{p}},\\
A_5 \le R^{\frac{-2\ell}{\rho}\widetilde{p}+\frac{N}{\rho}+2}\int_Q |(-\Delta)^\ell\widetilde{\varphi}|^{\widetilde{p}}\widetilde{\varphi}^{-\widetilde{p}/p}.
\end{gathered}\tag{3.8}$$

In a short form, we can write

$$A_i = C_iR^{\theta_i} \quad \text{for } i = 1,2,\dots,5,$$

where

$$\begin{gathered}
\theta_1 = -4\widetilde{p} + \frac{N}{\rho} + 2, \quad \theta_2 = -\big(\frac{2k}{\rho} + 4\big)\widetilde{p} + \frac{N}{\rho} + 2,\\
\theta_3 = -\big(\frac{2r}{\rho} + 2\alpha\big)\widetilde{p} + \frac{N}{\rho} + 2, \quad \theta_4 = -2\beta\widetilde{p} + \frac{N}{\rho} + 2,\\
\theta_5 = \frac{-2\ell}{\rho}\widetilde{p} + \frac{N}{\rho} + 2.
\end{gathered}\tag{3.9}$$

As $\beta \le \alpha < 1$, we observe that

$$\theta_2 \le \theta_1 \le \theta_4 \quad \text{and} \quad \theta_3 \le \theta_4.$$

For $R \ge 1$, we have $R^{\theta_i} \le R^{\theta_4} + R^{\theta_5}$ for $i = 1,2,\dots,5$ and then inequality (3.7) becomes

$$\int_{Q_T} |u|^p\varphi + \int_{\mathbb{R}^N} u_1(x)\varphi(x,0) + \int_{\mathbb{R}^N} u_1(x)(-\Delta)^k\varphi(x,0) \le K(R^{\theta_4} + R^{\theta_5}), \tag{3.10}$$

where K is a positive constant. We have

$$\begin{aligned}
\int_{\mathbb{R}^N} u_1(x)(-\Delta)^k\varphi(x,0) &= \int_{\overline{Q_R}} u_1(x)(-\Delta)^k\varphi(x,0)\\
&= R^{\frac{N}{\rho}-\frac{2k}{\rho}}\int_{\overline{Q}} u_1(R^{1/\rho}y)(-\Delta^k)\widetilde{\varphi}(y,0),
\end{aligned}$$

where

$$\overline{Q_R} = \{(x,0); R^{1/\rho} \le |x| \le 2^{1/(4\rho)}R^{1/\rho}\}, \quad \overline{Q} = \{(y,0); 1 \le |y| \le 2\}.$$

We assume that φ satisfies

$$\big\|(-\Delta^k)\widetilde{\varphi}(\cdot,0)\big\|_\infty < \infty,$$

for that, we will choose $\nu \gg 1$. It follows that

$$\int_{\mathbb{R}^N} u_1(x)(-\Delta)^k\varphi(x,0) \le CR^{\frac{N}{\rho}-\frac{2k}{\rho}} \int_{\overline{Q}} |u_1(R^{1/\rho}y)| \le CR^{-\frac{2k}{\rho}} \int_{\overline{Q_R}} |u_1(x)|,$$

then, passing to the limit as $R \to +\infty$, we have

$$\int_{\mathbb{R}^N} u_1(x)(-\Delta)^k\varphi(x,0) \to 0. \tag{3.11}$$

Now, if $\theta_4 < 0$ and $\theta_5 < 0$, (i.e. $\max(\theta_4,\theta_5) < 0$), that means

$$p < p_1(\rho) = 1 + \frac{2\beta\rho}{N + 2\rho(1-\beta)},$$
$$p < p_2(\rho) = 1 + \frac{2\ell}{N + 2(\rho-\ell)}$$

which is equivalent to

$$p < \overline{p}(\rho) = \min(p_1(\rho), p_2(\rho)).$$

As p_1 is a decreasing function and p_2 is an increasing function, the maximum value of the function $\min(p_1,p_2)$ will be at the point $\rho = \frac{\ell}{\beta}$, when the two functions p_1 and p_2 are equal

$$p_1(\frac{\ell}{\beta}) = p_2(\frac{\ell}{\beta}) = 1 + \frac{2\ell}{N + 2\ell(\frac{1}{\beta}-1)} =: p_c, \quad \text{(i.e. } \theta_4 = \theta_5).$$

Therefore, for

$$1 < p < p_c, \tag{3.12}$$

we have $R^{\theta_4} + R^{\theta_5}$ tends to zero when $R \to \infty$ and the inequalities (3.10) and (3.11) yield

$$\int_{Q_\infty} |u|^p + \int_{\mathbb{R}^N} u_1(x) \le 0.$$

As

$$\int_{\mathbb{R}^N} u_1(x) > 0,$$

we get a contradiction. This proves the theorem in the case (3.12).

For the border case where $p = p_c$ which corresponds to $\theta_4 = \theta_5 = 0$ and $\rho = \frac{\ell}{\beta}$, let

$$Q_{T,R} = \{(x,t), R^4 \le t^2 + |x|^{4\rho} \le 2R^4\},$$

if we use the Hölder inequality in the estimate of $\int_{Q_T} u\varphi_{tt}$ instead of the ϵ-Young inequality, we obtain

$$\begin{aligned}\int_{Q_T} u\varphi_{tt} = \int_{Q_{T,R}} u\varphi_{tt} &\le \Big(\int_{Q_{T,R}} |u|^p\varphi\Big)^{1/p}\Big(\int_{Q_{T,R}} \varphi^{-\widetilde{p}/p}|\varphi_{tt}|^{\widetilde{p}}\Big)^{1/\widetilde{p}}\\ &\le (A_1)^{1/\widetilde{p}}(\int_{Q_{T,R}} |u|^p\varphi)^{1/p},\end{aligned}$$

and similarly

$$\int_{Q_T} u(-\Delta)^k\varphi_{tt} = \int_{Q_{T,R}} u(-\Delta)^k\varphi_{tt} \le (A_2)^{1/\widetilde{p}} \int_{Q_{T,R}} |u|^p\varphi,$$
$$\int_{Q_T} u(-\Delta)^r D^\alpha_{t|T}\varphi = \int_{Q_{T,R}} u(-\Delta)^r D^\alpha_{t|T}\varphi \le (A_3)^{1/\widetilde{p}}\Big(\int_{Q_{T,R}} |u|^p\varphi\Big)^{1/p},$$

$$
\begin{gathered}
\int_{Q_T} u D^{\beta}_{t|T}\varphi = \int_{Q_{T,R}} u D^{\beta}_{t|T}\varphi \le (A_4)^{1/\widetilde{p}}\Big(\int_{Q_{T,R}} |u|^p\varphi\Big)^{1/p},\\
\int_{Q_T} u(-\Delta)^{\ell}\varphi = \int_{Q_{T,R}} u(-\Delta)^{\ell}\varphi \le (A_5)^{1/\widetilde{p}}\Big(\int_{Q_{T,R}} |u|^p\varphi\Big)^{1/p}.
\end{gathered}
$$

Thus, we obtain

$$
\begin{aligned}
\int_{Q_T} |u|^p\varphi + \int_{\mathbb{R}^N} u_1(x) &\le \big(A_1^{1/\widetilde{p}} + A_2^{1/\widetilde{p}} + A_3^{1/\widetilde{p}} + A_4^{\frac{1}{\widetilde{p}}} + A_5^{1/\widetilde{p}}\big)\Big(\int_{Q_{T,R}} |u|^p\varphi\Big)^{1/p}\\
&\le C\Big(\int_{Q_{T,R}} |u|^p\varphi\Big)^{1/p}.
\end{aligned}
$$

As $\int_{Q_T} |u|^q < +\infty$, we have

$$
\lim_{R\to+\infty}\int_{Q_{T,R}} |u|^q\varphi \le \lim_{R\to+\infty}\int_{Q_{T,R}} |u|^q = 0.
$$

Passing to the limit as $R \to +\infty$, we find that $\int_{Q_\infty} |u|^q + \int_{\mathbb{R}^N} u_1(x) = 0$, which contradicts $\int_{\mathbb{R}^N} u_1 > 0$. This prove the theorem in the case $p = p_c$. □

4. A pseudo-hyperbolic system

This section is concerned with the fractional time pseudo-hyperbolic system (1.4))-(1.5).

Definition 4.1. The couple of functions (u, v), $u \in L^q_{\rm loc}(Q_\infty)$ and $v \in L^p_{\rm loc}(Q_\infty)$ is a weak solution of (1.4)-(1.5) on Q_T with initial data $u_0(x), u_1(x), v_0(x)$ and $v_1(x) \in L^1_{\rm loc}(\mathbb{R}^N)$, if it satisfies

$$
\begin{aligned}
&\int_{Q_T} |v|^p\varphi + \int_{\mathbb{R}^N} u_1(x)\varphi(x,0) + \eta_1\int_{\mathbb{R}^N} u_1(x)(-\Delta)^{k_1}\varphi(x,0)\\
&= \int_{\mathbb{R}^N} u_0(x)\varphi_t(x,0) + \eta_1\int_{\mathbb{R}^N} u_0(x)(-\Delta)^{k_1}\varphi_t(x,0) + \int_{Q_T} u\varphi_{tt}\\
&\quad + \int_{Q_T} u(-\Delta)^{\ell_1}\varphi + \eta_1\int_{Q_T} u(-\Delta)^{k_1}\varphi_{tt} - \xi_1\int_{Q_T} u(-\Delta)^{r_1}D^{\alpha_1}_{t|T}\varphi\\
&\quad + \gamma_1\int_{Q_T} u D^{\beta_1}_{t|T}\varphi,
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
&\int_{Q_T} |u|^q\varphi + \int_{\mathbb{R}^N} v_1(x)\varphi(x,0) + \eta_2\int_{\mathbb{R}^N} v_1(x)(-\Delta)^{k_2}\varphi(x,0)\\
&= \int_{\mathbb{R}^N} v_0(x)\varphi_t(x,0) + \eta_2\int_{\mathbb{R}^N} v_0(x)(-\Delta)^{k_2}\varphi_t(x,0) + \int_{Q_T} v\varphi_{tt}\\
&\quad + \int_{Q_T} v(-\Delta)^{\ell_2}\varphi + \eta_2\int_{Q_T} v(-\Delta)^{k_2}\varphi_{tt} - \xi_2\int_{Q_T} v(-\Delta)^{r_1}D^{\alpha_2}_{t|T}\varphi\\
&\quad + \gamma_2\int_{Q_T} v D^{\beta_2}_{t|T}\varphi,
\end{aligned}
\tag{4.2}
$$

for any test-function $\varphi \in C^{2\ell 2}_{x\ \ t}(Q_T)$, $\ell = \max\{\ell_1, \ell_2\}$ being positive, $\varphi \equiv 0$ outside a compact $K \subset \mathbb{R}^n$, $\varphi(x,T) = \varphi_t(x,T) = 0$ and $D^{\alpha_1}_{t|T}\varphi, D^{\beta_1}_{t|T}\varphi, D^{\alpha_2}_{t|T}\varphi, D^{\beta_2}_{t|T}\varphi \in C(Q_T)$.

Theorem 4.2. *Assume that*

(1) $r_i, k_i \in \mathbb{N} \cup \{0\}$, $\ell_i \in \mathbb{N}$ *and* $0 < \beta_i \le \alpha_i < 1$, $i = 1, 2$;
(2) $u_0, u_1, v_0, v_1 \in L^1(\mathbb{R}^N)$ *such that* $\int_{\mathbb{R}^N} u_0(x) > 0$, $\int_{\mathbb{R}^N} u_1(x) > 0$, $\int_{\mathbb{R}^N} v_0(x) > 0$ *and* $\int_{\mathbb{R}^N} v_1(x) > 0$;
(3) $p > 1$, $q > 1$,
$$pq \le \min\Big(1 + \frac{2(p\beta_2 + \beta_1)\overline{\rho}}{N + 2(1-\beta_1)\overline{\rho}}, 1 + \frac{2(q\beta_1 + \beta_2)\overline{\rho}}{N + 2(1-\beta_2)\overline{\rho}}\Big)$$
where $\overline{\rho} = \min(\frac{\ell_1}{\beta_1}, \frac{\ell_2}{\beta_2})$.

Then problem (1.4)-(1.5) *does not admit any global non trivial solution.*

Proof. The proof is by contraction. Let (u, v) be a global weak solution of (1.4)-(1.5) and φ be a non-negative function (satisfying the conditions of Definition 4.1).

Applying Hölder inequality to $\int_{Q_T} u\varphi_{tt}$, we obtain
$$\int_{Q_T} u\varphi_{tt} \le \Big(\int_{Q_T} |u|^q\varphi\Big)^{1/q}\Big(\int_{Q_T} \varphi^{\frac{-\widetilde{q}}{q}}|\varphi_{tt}|^{\widetilde{q}}\Big)^{1/\widetilde{q}} \le (A_1)^{1/\widetilde{q}}\Big(\int_{Q_T} |u|^q\varphi\Big)^{1/q}$$
and similarly
$$\begin{gathered}
\int_{Q_T} u(-\Delta)^{k_1}\varphi_{tt} \le (A_2)^{1/\widetilde{q}}\Big(\int_{Q_T} |u|^q\varphi\Big)^{1/q},\\
\int_{Q_T} u(-\Delta)^{r_1} D^{\alpha_1}_{t|T}\varphi \le (A_3)^{\frac{1}{\widetilde{q}}}\Big(\int_{Q_T} |u|^q\varphi\Big)^{1/q},\\
\int_{Q_T} uD^{\beta_1}_{t|T}\varphi \le (A_4)^{1/\widetilde{q}}\Big(\int_{Q_T} |u|^q\varphi\Big)^{1/q},\\
\int_{Q_T} u(-\Delta)^{\ell_1}\varphi \le (A_5)^{1/\widetilde{q}}\Big(\int_{Q_T} |u|^q\varphi\Big)^{1/q},
\end{gathered} \tag{4.3}$$
where
$$\begin{gathered}
A_1 = \int_{Q_T} |\varphi_{tt}|^{\widetilde{q}}\varphi^{\frac{-\widetilde{q}}{q}}\,dx\,dt,\\
A_2 = \int_{Q_T} |(-\Delta)^{k_1}\varphi_{tt}|^{\widetilde{q}}\varphi^{-\widetilde{q}/q}\,dx\,dt,\\
A_3 = \int_{Q_T} |(-\Delta)^{r_1}D^{\alpha_1}_{t|T}\varphi|^{\widetilde{q}}\varphi^{-\widetilde{q}/q}\,dx\,dt,\\
A_4 = \int_{Q_T} |D^{\beta_1}_{t|T}\varphi|^{\widetilde{q}}\varphi^{-\widetilde{q}/q}\,dx\,dt,\\
A_5 = \int_{Q_T} |(-\Delta)^{\ell_1}\varphi|^{\widetilde{q}}\varphi^{-\widetilde{q}/q}\,dx\,dt.
\end{gathered}$$
Now, let φ be the test function defined by the expression (3.5). Using the previous estimates (4.3) and the properties (3.5) and (3.6) of the function φ in equation (4.1), we obtain the inequality
$$\begin{aligned}
&\int_{Q_T} |v|^p\varphi + \int_{\mathbb{R}^N} u_1\varphi(x,0)\\
&\le \Big(\int_{\mathbb{R}^N} |u|^q\varphi\Big)^{1/q}\Big[A_1^{1/\widetilde{q}} + A_2^{1/\widetilde{q}} + A_3^{1/\widetilde{q}} + A_4^{1/\widetilde{q}} + A_5^{1/\widetilde{q}}\Big].
\end{aligned} \tag{4.4}$$

Similarly, for the equation (4.2), we have

$$\begin{aligned}&\int_{Q_T}|u|^p\varphi+\int_{\mathbb{R}^N}v_1\varphi(x,0)\\&\leq\Big(\int_{\mathbb{R}^N}|v|^q\varphi\Big)^{1/q}\big[B_1^{1/\widetilde{q}}+B_2^{1/\widetilde{q}}+B_3^{\frac{1}{\widetilde{q}}}+B_4^{1/\widetilde{q}}+B_5^{\frac{1}{\widetilde{q}}}\big],\end{aligned}\tag{4.5}$$

where

$$\begin{gathered}B_1=\int_{Q_T}|\varphi_{tt}|^{\widetilde{q}}\varphi^{\frac{-\widetilde{q}}{q}}\,dx\,dt,\\B_2=\int_{Q_T}|u(-\Delta)^{k_2}\varphi_{tt}|^{\widetilde{q}}\varphi^{\frac{-\widetilde{q}}{q}}\,dx\,dt,\\B_3=\int_{Q_T}|(-\Delta)^{r_2}D^{\alpha_2}_{t|T}\varphi|^{\widetilde{q}}\varphi^{\frac{-\widetilde{q}}{q}}\,dx\,dt,\\B_4=\int_{Q_T}|D^{\beta_2}_{t|T}\varphi|^{\widetilde{q}}\varphi^{\frac{-\widetilde{q}}{q}}\,dx\,dt,\\B_5=\int_{Q_T}|(-\Delta)^{\ell_2}\varphi|^{\widetilde{q}}\varphi^{\frac{-\widetilde{q}}{q}}\,dx\,dt.\end{gathered}$$

Now, we estimate $A_1,\dots,A_5$ and $B_1,\dots,B_5$ in the same way as in Section 3, we obtain inequalities similar to those given in (3.7) and (3.8)

$$A_i=C_iR^{\theta_i},\quad B_i=D_iR^{\delta_i}\quad\text{for } i=1,2,\dots,5,\tag{4.6}$$

where

$$\begin{gathered}\theta_1=-4\widetilde{q}+\frac{N}{\rho}+2,\quad\delta_1=-4\widetilde{p}+\frac{N}{\rho}+2,\\\theta_2=-\big(\frac{2k_1}{\rho}+4\big)\widetilde{q}+\frac{N}{\rho}+2,\quad\delta_2=-\big(\frac{2k_2}{\rho}+4\big)\widetilde{p}+\frac{N}{\rho}+2,\\\theta_3=-\big(\frac{2r_1}{\rho}+2\alpha_1\big)\widetilde{q}+\frac{N}{\rho}+2,\quad\delta_3=-\big(\frac{2r_2}{\rho}+2\alpha_2\big)\widetilde{p}+\frac{N}{\rho}+2,\\\theta_4=-2\beta_1\widetilde{q}+\frac{N}{\rho}+2,\quad\delta_4=-2\beta_2\widetilde{p}+\frac{N}{\rho}+2,\\\theta_5=\frac{-2\ell_1}{\rho}\widetilde{q}+\frac{N}{\rho}+2,\quad\delta_5=\frac{-2\ell_2}{\rho}\widetilde{p}+\frac{N}{\rho}+2.\end{gathered}\tag{4.7}$$

If we set

$$I=\Big(\int_{Q_T}|v|^p\varphi\Big)^{1/p}\quad\text{and}\quad J=(\int_{Q_T}|u|^q\varphi)^{1/q},$$

inequalities (4.4) and (4.5) become

$$I^P+\int_{\mathbb{R}^N}u_1\varphi(x,0)\leq J(C_1R^{\theta_1/\widetilde{q}}+C_2R^{\theta_2/\widetilde{q}}+C_3R^{\theta_3/\widetilde{q}}+C_4R^{\theta_4/\widetilde{q}}+C_5R^{\theta_5/\widetilde{q}}),\tag{4.8}$$

$$J^q+\int_{\mathbb{R}^N}v_1\varphi(x,0)\leq I(D_1R^{\frac{\delta_1}{\widetilde{q}}}+D_2R^{\delta_2/\widetilde{q}}+D_3R^{\delta_3/\widetilde{q}}+D_4R^{\delta_4/\widetilde{q}}+D_5R^{\delta_5/\widetilde{q}}).\tag{4.9}$$

We can observe that under the conditions on α_i and β_i, we have

$$\theta_2\leq\theta_1\leq\theta_4,\quad\theta_3\leq\theta_4,\quad\delta_2\leq\delta_1\leq\delta_4,\quad\delta_3\leq\delta_4.$$

Hence, for $R \geq 1$, we have $R^{\theta i} \leq R^{\theta_4} + R^{\theta_5}$ and $R^{\delta i} \leq R^{\delta_4} + R^{\delta_5}$, and consequently, inequalities (4.8) and (4.9) can be rewritten as

$$I^p + \int_{\mathbb{R}^N} u_1 \varphi(x,0) \leq CJ\big(R^{\frac{\theta_4}{\widetilde{q}}} + R^{\theta_5/\widetilde{q}}\big), \tag{4.10}$$

$$J^P + \int_{\mathbb{R}^N} v_1 \varphi(x,0) \leq DI(R^{\frac{\delta_4}{\widetilde{q}}} + R^{\frac{\delta_5}{\widetilde{q}}}), \tag{4.11}$$

where

$$C = \sum_{i=1}^{5} C_i, \quad \text{and} \quad D = \sum_{i=1}^{5} D_i.$$

Since $\int_{\mathbb{R}^N} u_1 \varphi(x,0) \geq 0$ and $\int_{\mathbb{R}^N} v_1 \varphi(x,0) \geq 0$, inequalities (4.10) and (4.11) yield

$$I^p \leq CJ\big(R^{\theta_4/\widetilde{q}} + R^{\theta_5//\widetilde{q}}\big), \tag{4.12}$$

$$J^P \leq DI\big(R^{\delta_4/\widetilde{q}} + R^{\delta_5/\widetilde{q}}\big). \tag{4.13}$$

The constants C and D will be updated at each step of the calculation and will not play a role. This implies that

$$I^{pq} \leq CI(R^{\delta_4/\widetilde{p}} + R^{\delta_5/\widetilde{p}})(R^{\theta_4/\widetilde{q}} + R^{\frac{\theta_5}{\widetilde{q}}})^q, \ J^{pq} \leq CJ(R^{\frac{\theta_4}{\widetilde{q}}} + R^{\frac{\theta_5}{\widetilde{q}}})(R^{\delta_4/\widetilde{p}} + R^{\delta_5/\widetilde{p}})^p,$$

leading to

$$I^{pq-1} \leq C(R^{\delta_4/\widetilde{p}} + R^{\delta_5/\widetilde{p}})(R^{\theta_4/\widetilde{q}} + R^{\frac{\theta_5}{\widetilde{q}}})^q, \ J^{pq-1} \leq C(R^{\frac{\theta_4}{\widetilde{q}}} + R^{\frac{\theta_5}{\widetilde{q}}})(R^{\delta_4/\widetilde{p}} + R^{\delta_5/\widetilde{p}})^p. \tag{4.14}$$

Now, let

$$S_1 = \frac{1}{\widetilde{p}} \max(\delta_4, \delta_5) + \frac{q}{\widetilde{q}} \max(\theta_4, \theta_5), \quad S_2 = \frac{1}{\widetilde{q}} \max(\theta_4, \theta_5) + \frac{p}{\widetilde{p}} \max(\delta_4, \delta_5).$$

If

$$S_1 < 0, \quad \text{and} \quad S_2 < 0, \tag{4.15}$$

we have $(R^{\delta_4/\widetilde{p}}+R^{\delta_5/\widetilde{p}})(R^{\theta_4/\widetilde{q}}+R^{\theta_5/\widetilde{q}})^q \to 0$ and $(R^{\frac{\theta_4}{\widetilde{q}}}+R^{\theta_5/\widetilde{q}})(R^{\delta_4/\widetilde{p}}+R^{\delta_5/\widetilde{p}})^p \to 0$ as $R \to \infty$. Hence, by (4.14), both I and J vanish as $R \to \infty$. This implies that $J^q = \int_{Q_T} |u|^q \varphi$ converges to $\int_{Q_\infty} |u|^q \varphi = 0$ and $I^p = \int_{Q_T} |v|^p \varphi$ converges to $\int_{Q_\infty} |v|^p \varphi = 0$. Consequently, $u \equiv 0$ and $v \equiv 0$.

As $S_1 < 0$, then we have $\max(\theta_4, \theta_5) < 0$ or $\max(\delta_4, \delta_5) < 0$. Suppose that $\max(\theta_4, \theta_5) < 0$ and let again $R \to \infty$. By (4.10), we obtain $\int_{\mathbb{R}^N} u_1 = 0$, which contradicts $\int_{\mathbb{R}^N} u_1 > 0$.

Now, we return to the condition (4.15) that lead to the contradiction. Inequalities $S_1 < 0$ and $S_2 < 0$ are equivalent to

$$\begin{aligned} S_1 &= -2\Big(q \min(\beta_1, \frac{\ell_1}{\rho}) + \min(\beta_2, \frac{\ell_2}{\rho})\Big) + (\frac{N}{\rho} + 2)\frac{pq-1}{p} < 0 \\ S_2 &= -2\Big(p \min(\beta_2, \frac{\ell_2}{\rho}) + \min(\beta_1, \frac{\ell_1}{\rho})\Big) + (\frac{N}{\rho} + 2)\frac{pq-1}{q} < 0. \end{aligned} \tag{4.16}$$

Let us take $\rho = \overline{\rho} = \min(\frac{\ell_1}{\beta_1}, \frac{\ell_2}{\beta_2})$. We have

$$\min\big(\beta_1, \frac{\ell_1}{\rho}\big) = \beta_1 \quad \text{and} \quad \min\big(\beta_2, \frac{\ell_2}{\rho}\big) = \beta_2.$$

The inequalities in (4.16) can now be written as

$$\begin{aligned} S_1 &= -2(q\beta_1+\beta_2)\overline{\rho}+(N+2\overline{\rho})\frac{pq-1}{p}<0 \\ S_2 &= -2(p\beta_2+\beta_1)\overline{\rho}+(N+2\overline{\rho})\frac{pq-1}{p}<0, \end{aligned} \tag{4.17}$$

which are equivalent to

$$1<pq<\min\Big(1+\frac{2(p\beta_2+\beta_1)\overline{\rho}}{N+2(1-\beta_1)\overline{\rho}},1+\frac{2(q\beta_1+\beta_2)\overline{\rho}}{N+2(1-\beta_2)\overline{\rho}}\Big).$$

Let us now consider the border case where

$$pq=\min\Big(1+\frac{2(p\beta_2+\beta_1)\overline{\rho}}{N+2(1-\beta_1)\overline{\rho}},1+\frac{2(q\beta_1+\beta_2)\overline{\rho}}{N+2(1-\beta_2)\overline{\rho}}\Big),$$

which corresponds to

$$S_1=0,\ S_2\le 0 \quad\text{or}\quad S_1\le 0,\ S_2=0. \tag{4.18}$$

Let us take the case $S_1=0$, $S_2\le 0$ (the second case: $S_1\le 0$, $S_2=0$ is similar). We have

$$\widetilde{p}\widetilde{q}S_1=\widetilde{q}\max(\delta_4,\delta_5)+q\widetilde{p}\max(\theta_4,\theta_5)=0, \tag{4.19}$$

$$\widetilde{p}\widetilde{q}S_2=\widetilde{p}\max(\theta_4,\theta_5)+p\widetilde{q}\max(\delta_4,\delta_5)\le 0. \tag{4.20}$$

From (4.19) and (4.20), we have

$$\widetilde{p}\max(\theta_4,\theta_5)=-\frac{\widetilde{q}\max(\delta_4,\delta_5)}{q}\quad\text{and}\quad \frac{\widetilde{q}}{q}(pq-1)\max(\delta_4,\delta_5)\le 0,$$

which implies

$$\max(\delta_4,\delta_5)=\max(\delta_1,\delta_2,\dots,\delta_5)\le 0.$$

Moreover, using Young's inequality $(a+b)^r\le 2^{r-1}(a^r+b^r)$ for $r\ge 1$, the inequalities in (4.14) lead to

$$I^{pq-1}\le 2R^{1/\widetilde{p}\max(\delta_4,\delta_5)}2^{q-1}R^{\frac{q}{\widetilde{q}}\max(\theta_4,\theta_5)}=2^qR^{S_1}=2^q,$$

and similarly

$$J^{pq-1}\le 2^pR^{S_2}=2^p,$$

for every $T\in(0,+\infty)$. Hence, $I<+\infty$ and $J<+\infty$ for every $T\in(0,+\infty)$, and thus $\int_{Q_\infty}|u|^q<+\infty$ and $\int_{Q_\infty}|v|^q<+\infty$.

Now, let $Q_{T,R}=\big\{(x,t),\ R^4\le t^2+|x|^{4\overline{\rho}}\le 2R^4\big\}$. For the first inequality of (4.3), we obtain

$$\begin{aligned} \int_{Q_T}u\varphi_{tt}&=\int_{Q_{T,R}}u\varphi_{tt}\le\Big(\int_{Q_{T,R}}|u|^q\varphi\Big)^{1/q}\Big(\int_{Q_{T,R}}\varphi^{\frac{-\widetilde{q}}{q}}|\varphi_{tt}|^{\widetilde{q}}\Big)^{1/\widetilde{q}} \\ &\le (A_1)^{1/\widetilde{q}}\Big(\int_{Q_{T,R}}|u|^q\varphi\Big)^{1/q}\le C_1R^{\theta_1/\widetilde{q}}. \end{aligned}$$

Doing the same for the remaining inequalities of (4.3), we obtain a new estimate of (4.1),

$$\begin{aligned} 0&<\int_{\mathbb{R}^N}u_1(x) \\ &\le\int_{Q_{T,R}}|v|^p\varphi+\int_{\mathbb{R}^N}\{u_1(x)+u_0(x)\}\varphi(x,0) \end{aligned}$$

$$
\begin{aligned}
&\le \Big(\int_{Q_{T,R}} |u|^q \varphi\Big)^{1/q} \big[C_1 R^{\theta_1/\widetilde{q}} + C_2 R^{\theta_2/\widetilde{q}} + C_3 R^{\theta_3/\widetilde{q}} + C_4 R^{\theta_4/\widetilde{q}} + C_5 R^{\theta_5/\widetilde{q}}\big] \\
&\le C\Big(\int_{Q_{T,R}} |u|^q \varphi\Big)^{1/q} \Big(R^{\theta_4/\widetilde{q}} + R^{\theta_5/\widetilde{q}}\Big) \\
&\le C\Big(\int_{Q_{T,R}} |u|^q \varphi\Big)^{1/q}.
\end{aligned}
$$

Let $R \to \infty$. Since $\int_{Q_\infty} |u|^q < +\infty$, the right-hand side of the above inequality approaches zero when $R \to \infty$, while the left-hand side $\int_{\mathbb{R}^N} u_1(x)$ is assumed to be positive, this is a contradiction.

Similarly, the second case $S_1 \le 0$, $S_2 = 0$ leads to a contradiction. □

Acknowledgements. The authors would like to thank Taibah University for grant No 4113 which made this work possible. Special thanks also go to Prof. M. Kirane for his continuous support and guidance, which have helped improve the quality of this paper.

References

[1] A. B. Aliev, A. A. Kazymov; *Effect of Weight Function in Nonlinear Part on Global Solvability of Cauchy Problem for Semi-Linear Hyperbolic Equations*, Inter. J. of Modern Nonlinear Theory and Appl. 2013, 2, pp. 102-106.

[2] A. B. Aliev, A. A. Kazymov; *Existence non existence and asymptotic behavior of global solutions of the Cauchy problem for of semi-linear hyperbolic equations with damping terms*, Nonlinear Analysis. 2012, 75, pp. 91-102.

[3] A. B. Aliev, A. A. Kazymov; *Global Solvability and Behavior of Solutions of the Cauchy Problem for a System of Three Semilinear Hyperbolic Equations with Dissipation*, in: Transactions of National Academy of Sciences of Azerbaijan, Series of Physical-Technical and Mathematical Sciences, Vol. XXX, Mathematics , No 4, 2011, pp. 3–18.

[4] A. B. Aliev, A. A. Kazymov; *Global Solvability and Behavior of Solutions of the Cauchy Problem for a System of Two Semilinear Hyperbolic Equations with Dissipation*, Differ. Equations, 2013, Vol. 49, No 4, pp. 476-486.

[5] A. B. Aliev, A. A. Kazymov; *Global weak solutions of the Cauchy problem for semilinear pseudo-hyperbolic equations*, Differential Equations 2009, Vol 45, pp 175-185.

[6] A. B. Aliev, B. H. Lichaei; *Existence and non-existence of global solutions of the Cauchy problem for higher order semilinear pseudo-hyperbolic equations*, Nonlinear Analysis: Theory, Methods & Applications, Vol 72, Issues 7–8, 1 April 2010, Pages 3275–3288.

[7] A. B. Aliev, F. V. Mamedov; *Existence and nonexistence of global solutions of the Cauchy problem for semi-linear hyperbolic equations with dissipation and anisotropic elliptic part*, Differ. Equations, 2010, Vol. 46, No 3, pp. 307-317.

[8] A. B. Aliev, K. S. Mammadzada, B. H. Lichaei; *Existence of a global weak solution of the Cauchy problem for systems of semilinear hyperbolic equations*, in: Transactions of National Academy of Sciences of Azerbaijan, Series of Physical-Technical and Mathematical Sciences, Vol. XXX, Mathematics and Mechanics, No1, 2010, pp. 13–24.

[9] F. John; Blow-up of solutions of nonlinear wave equations in three space dimensions, manuscripta mathematica, Vol. 28, Issue 1-3,(1979), pp 235-268.

[10] T. Kato; *Blow-up of solutions of some nonlinear hyperbolic equations*, Comm. Pure Appl.Math.,1980, Vol. 33, pp. 501-505.

[11] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*. Elsevier (2006).

[12] M. Kirane, M. Qafsaoui; *Fujita's exponent for a semilinear wave equation with linear damping*, Advanced Nonlinear Studies 2 (2002), pp. 41-49.

[13] J. L. Lions; *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Paris, Dunod 1969.

[14] E. R. Love, L. C. Young; *In fractional integration by parts.* Proc. London Math. Soc., Ser. 2, 44, (1938) pp. 1-35 .

[15] E. Mitidieri, S.I. Pokhozhaev; *A Priori Estimates and the Absence of Solutions of Nonlinear Partial Differential Equations and Inequalities*, Tr. Mat. Inst. Steklova, 2001, Vol. 234, pp. 1–384.

[16] E. Mitidieri, S. I. Pohozaev; *Liouville Theorems for Some Classes of Nonlinear Nonlocal Problems*, Proc. Steklov Math. Inst., Vol. 248 (2005), pp. 164-184.

[17] E. Mitidieri, S. I. Pohozaev; *Towards a unified approach to nonexistence of solutions for a class of differential inequalities*, Milan J. Math., Vol. 72 (2004), 129–162.

[18] S. I. Pokhozhaev; *essentially nonlinear capacities induced by differential operators*, Dokl Akad Nauk 1997, Vol. 357, No5, pp.592-594.

[19] S. G. Samko, A. A. Kilbas, O.I. Marichev; *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Beach, Yverdon and New York (1993).

[20] Q. S. Zhang; *A blow-up result for a nonlinear wave equation with damping. The critical case*, C.R. Acad. Sci. Paris. Sér. I Math., 2001, Vol 333, pp. 109-114.

Salem Abdelmalek
Department of Mathematics, College of Sciences, Yanbu, Taibah University, Saudi Arabia
E-mail address: `sallllm@gmail.com`

Maha Bajneed
Department of Mathematics, Faculty of Sciences at Yanbu, Taibah University, Saudi Arabia
E-mail address: `m_bajneed@hotmail.com`

Khaled Sioud
Department of Mathematics, Faculty of Sciences at Yanbu, Taibah University, Saudi Arabia
E-mail address: `sioudkha@aol.com`