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EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR EIGENVALUE PROBLEMS

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Abstract. In this paper, we discuss the existence, nonexistence, and multiplicity of positive solutions for a class of singular eigenvalue problems. Some of our theorems are new, while others extend earlier results obtained by Zhang and Kong [12]. The interesting point is that the authors obtain the relation between the existence of solutions and the parameter λ. The arguments are based on the fixed point index theory and the upper and lower solutions method.

1. Introduction

The deformations of an elastic beam are described by a fourth-order differential equation

$$u^{(4)} = f(t, u, u'').$$

Most of the available literature on fourth-order boundary value problems, for example [1, 2, 5, 7, 8, 9, 10], discusses the case when f is either continuous or a Caratheodory function and is concerned with the existence and uniqueness of positive solutions for boundary value problems for the above differential equation. However, only a small number of papers have discussed fourth-order singular eigenvalue problems; see for example [11, 14].

In this paper, we study the fourth-order singular differential equation

$$u^{(4)}(t) = \lambda g(t) f(u(t)), \quad 0 < t < 1, \tag{1.1}$$

subject to one of the following boundary conditions:

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

$$u(0) = u'(1) = u''(0) = u'''(1) = 0, \tag{1.3}$$

where $\lambda > 0$. The following assumptions will stand throughout this paper:

- (H1) $f \in C([0, +\infty), (0, +\infty))$ and is nondecreasing on $[0, +\infty)$. Furthermore, there exist $\bar{\delta} > 0, m \geq 2$ such that $f(u) > \bar{\delta} u^m, u \in [0, +\infty)$;
- (H2) $g \in C((0, 1), (0, +\infty))$ and $0 < \int_0^1 s(1-s)g(s)ds < +\infty$.

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It is the purpose of this paper to obtain the existence and the nonexistence of positive solutions, and multiplicity results for the eigenvalue problems (EP) (1.1)-(1.2) and (1.1)-(1.3) by employing new technique (different from the one used in [12]). Very few papers discuss the connection between the existence of solutions and the parameter λ. The work done by others [14] does not cover the general case given in (1.1)-(1.2) and (1.1)-(1.3).

In this paper, we use mainly the following fixed point index theory to obtain multiplicity results for (1.1)-(1.2) and (1.1)-(1.3).

Lemma 1.1 ([6])**.** *Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $\theta \in \Omega$. Suppose $A : P \cap \bar{\Omega} \to P$ is a completely continuous operator, that satisfies*

$$Ax = \mu x,\ x \in P \cap \partial\Omega \Longrightarrow \mu < 1.$$

Then $i(A, P \cap \Omega, P) = 1$.

Lemma 1.2 ([6])**.** *Suppose $A : P \cap \bar{\Omega} \to P$ is a completely continuous operator, and satisfies:*

(1) $\inf_{x \in P \cap \partial\Omega} \|Ax\| > 0$;
(2) $Ax = \mu x, x \in P \cap \partial\Omega \Longrightarrow \mu \notin (0, 1]$.
Then $i(A, P \cap \Omega, P) = 0$.

In Section 2, we provide some necessary background. In particular, we state some properties of the Green's function associated with (1.1)-(1.2) and some Lemmas. In Section 3, we present our main result and discuss an example.

2. Preliminaries

For the convenience of the reader, we present here the necessary definitions and Lemmas. Let $E = C[0, 1]$ be a real Banach space with the norm $\|u\| = \max_{0 \le t \le 1} |u(t)|$. Let $S = \{\lambda > 0$ such that (1.1) has at least one solution$\}$ and $P = \{u \in E : u(t) \ge 0, t \in [0, 1]\}$. It is clear that P is a cone of E.

We deal first with (1.1)-(1.2). Define

$$G_1(t, \xi) = \begin{cases} t(1-\xi), & 0 \le t \le \xi \le 1, \\ \xi(1-t), & 0 \le \xi \le t \le 1. \end{cases}$$

$$\begin{aligned} G(t, s) &= \int_0^1 G_1(t, \xi) G_1(\xi, s) d\xi \\ &= \begin{cases} t(1-s)\frac{2s-s^2-t^2}{6}, & 0 \le t \le s \le 1, \\ s(1-t)\frac{2t-t^2-s^2}{6}, & 0 \le s \le t \le 1. \end{cases} \end{aligned}$$

It is easy to prove that $G_1(t, s)$ and $G(t, s)$ have the following properties.

Proposition 2.1. *For all $t, s \in [0, 1]$, we have*

$$\begin{gathered} G_1(t, s) > 0, \quad \text{for } (t, s) \in (0, 1) \times (0, 1); \\ G_1(t, s) \le G_1(s, s) = s(1-s), \quad \text{for } 0 \le t, s \le 1; \\ 0 \le G_1(t, s) \le \frac{1}{4}, \quad \text{for } 0 \le t, s \le 1; \\ G(t, s) \le \frac{1}{6} G_1(s, s) = \frac{1}{6} s(1-s), \quad \text{for } 0 \le t, s \le 1. \end{gathered} \tag{2.1}$$

Proposition 2.2. *For all $t \in [\theta, 1-\theta]$, we have*

$$G_1(t,s) \geq \theta G_1(s,s), \quad \theta \in (0, \frac{1}{2}),\ s \in [0,1]. \tag{2.2}$$

In fact

$$\frac{G_1(t,s)}{G_1(s,s)} = \begin{cases} \frac{t}{s}, & 0 \leq t \leq s \leq 1, \\ \frac{1-t}{1-s}, & 1 \geq t \geq s \geq 0. \end{cases} \geq \begin{cases} t \geq \theta, & t \leq s, \\ 1-t \geq \theta, & t \geq s. \end{cases}$$

Therefore, for all $t \in [\theta, 1-\theta]$, we have

$$G_1(t,s) \geq \theta G_1(s,s), \quad \theta \in (0, \frac{1}{2}),\ s \in [0,1].$$

Definition 2.3. Let $\alpha(t) \in C^2[0,1] \cap C^4(0,1)$. We say that α is a lower solution of (1.1)-(1.2) if it satisfies

$$\begin{gathered} \alpha^{(4)}(t) \leq \lambda g(t) f(u(t)), \quad 0 < t < 1, \\ \alpha(0) \leq 0, \alpha(1) \leq 0, \alpha''(0) \geq 0, \quad \alpha''(1) \geq 0. \end{gathered}$$

Definition 2.4. Let $\beta(t) \in C^2[0,1] \cap C^4(0,1)$. We say that β is an upper solution of (1.1)-(1.2) if it satisfies

$$\begin{gathered} \beta^{(4)}(t) \geq \lambda g(t) f(u(t)), \quad 0 < t < 1, \\ \beta(0) \geq 0, \beta(1) \geq 0, \beta''(0) \leq 0, \quad \beta''(1) \leq 0. \end{gathered}$$

First, we consider the following eigenvalue problem

$$\begin{gathered} u^{(4)}(t) = \lambda g(t) f(u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = h \geq 0. \end{gathered} \tag{2.3}$$

Define $T_\lambda^h : E \to E$ by

$$T_\lambda^h u(t) = h + \int_0^1 G(t,s) \lambda g(s) f(u(s)) ds - \int_0^1 G_1(t,s) h ds. \tag{2.4}$$

From (2.4), it is easy to obtain the following lemma, which is proved by a direct computation.

Lemma 2.5. *Suppose that (H1) and (H2) are satisfied. Then* (1.1)-(1.2) *has a solution u if and only if u is a fixed point of T_λ^0.*

To prove the following results we define the cone

$$Q = \{u \in C[0,1] | u(t) \geq 0, \min_{\theta \leq t \leq 1-\theta} u(t) \geq M_\theta \|u\|\} \tag{2.5}$$

where $\|u\| = \max_{t \in [0,1]} |u(t)|$, $M_\theta = \theta^2(1 - 6\theta^2 + 4\theta^3)$, $\theta \in (0, \frac{1}{2})$. It is clear that $Q \subset P$.

Lemma 2.6. *Suppose that (H1) and (H2) are satisfied. Then $T_\lambda^0 Q \subset Q$ is completely continuous and nondecreasing.*

Proof. For any $u \in P$, by (2.1) and (2.4), we have

$$\begin{aligned} T_\lambda^0 u(t) &= \int_0^1 G(t,s) \lambda g(s) f(u(s)) ds \\ &\leq \frac{1}{6} \int_0^1 \lambda s(1-s) g(s) f(u(s)) ds. \end{aligned}$$

Therefore,

$$\|T_\lambda^0 u\| \le \frac{1}{6}\int_0^1 \lambda s(1-s)g(s)f(u(s))ds.$$

On the other hand, by (2.2), for any $\theta \le t \le 1-\theta$, we have

$$G(t,s) = \int_0^1 G_1(t,\xi)G_1(\xi,s)d\xi \ge M_\theta \frac{1}{6}s(1-s). \tag{2.6}$$

Therefore,

$$\begin{aligned}
\min_{\theta\le t\le 1-\theta} T_\lambda^0 u(t) &= \min_{\theta\le t\le 1-\theta}\int_0^1 G(t,s)\lambda g(s)f(u(s))ds\\
&\ge M_\theta \frac{1}{6}\int_0^1 \lambda s(1-s)g(s)f(u(s))ds\\
&\ge M_\theta \|T_\lambda^0 u\|.
\end{aligned}$$

Hence $T_\lambda^0 P \subset Q$ and then $T_\lambda^0 Q \subset Q$ by $Q \subset P$. By similar arguments in [2, 9, 12, 14], $T_\lambda^0 : Q \to Q$ is completely continuous. Since f is increasing on $[0,+\infty)$, it is easy to obtain that T_λ^0 is nondecreasing on $[0,+\infty)$. $\square$

Remark 2.7. Reasoning as in the proofs of Lemmas 2.5 and 2.6, we conclude that $T_\lambda^h : Q \to Q$ is completely continuous and that $u(t)$ is a solution of (2.3) if and only if $u(t)$ is a fixed point of T_λ^h.

Lemma 2.8. *Suppose that $\lambda \in S, S_1 = (\lambda,+\infty)\cap S \not\equiv \emptyset$. Then there exists $R(\lambda) > 0$, such that $\|u_{\lambda'}\| \le R(\lambda)$, where $\lambda' \in S_1$, and $u_{\lambda'} \in Q$ is a solution of (1.1)-(1.2) with λ' instead of λ.*

Proof. For any $\lambda' \in S$, let $u_{\lambda'}$ be a solution of (1.1)-(1.2) with λ' instead of λ. Then

$$u_{\lambda'}(t) = T_{\lambda'}^0 u_{\lambda'}(t) = \int_0^1 G(t,s)\lambda' g(s)f(u_{\lambda'}(s))ds.$$

Let $R(\lambda) = \max\{[\frac{1}{6}\lambda' M_\theta^{m+1}\bar\delta\int_\theta^{1-\theta} G_1(s,s)g(s)ds]^{-1}, 1\}$. Next we shall prove that $\|u_{\lambda'}\| \le R(\lambda)$. Indeed, if $\|u_{\lambda'}\| < 1$, the result is easily obtained. On the other hand, if $\|u_{\lambda'}\| \ge 1$, then we have by (H1) and (2.6),

$$\begin{aligned}
\frac{1}{\|u_{\lambda'}\|} &\ge \frac{\min_{\theta\le t\le 1-\theta} u_{\lambda'}(t)}{\|u_{\lambda'}\|^2}\\
&= \frac{1}{\|u_{\lambda'}\|^2}\min_{\theta\le t\le 1-\theta}\int_0^1 G(t,s)\lambda' g(s)f(u_{\lambda'}(s))ds\\
&\ge \frac{1}{\|u_{\lambda'}\|^2} M_\theta \int_\theta^{1-\theta}\frac{1}{6}G_1(s,s)\lambda' g(s)\bar\delta u_{\lambda'}(s)^m ds\\
&\ge \frac{1}{\|u_{\lambda'}\|^2} M_\theta^{m+1}\int_\theta^{1-\theta}\frac{1}{6}G_1(s,s)\lambda' g(s)\bar\delta \|u_{\lambda'}\|^m ds\\
&\ge \frac{1}{6}\lambda' M_\theta^{m+1}\bar\delta\int_\theta^{1-\theta} G_1(s,s)g(s)ds.
\end{aligned}$$

Therefore, $\|u_{\lambda'}\| \le R(\lambda)$ and the conclusion of Lemma 2.8 follows. $\square$

Lemma 2.9 ([4]). *Suppose that $f : [0,+\infty) \to (0,+\infty)$ is continuous and increasing. If s, s_0 and M are such that $0 < s < s_0, M > 0$, then there exist $\bar{s} \in (s, s_0), h_0 \in (0,1)$ such that*

$$sf(u+h) < \bar{s}f(u), u \in [0,M], h \in (0,h_0).$$

3. Main results

In this section, we apply Lemmas 1.1 and 1.2 to establish nonexistence and existence of positive solutions, as well as multiplicity results for (1.1)-(1.2) and (1.1)-(1.3). Our approach depends on the upper and lower solutions method and the fixed point index theory. We deal with (1.1)-(1.2) first.

Theorem 3.1. *Let (H1) and (H2) be satisfied. Then there exists $0 < \lambda^* < +\infty$ such that*

1. *EP (1.1)-(1.2) has no solution for $\lambda > \lambda^*$;*
2. *EP (1.1)-(1.2) has at least one positive solution for $\lambda = \lambda^*$;*
3. *EP (1.1)-(1.2) has at least two positive solutions for $0 < \lambda < \lambda^*$.*

Proof. First, we prove that the conclusion (1) of Theorem 3.1 holds. If $\beta(t)$ is a solution of the boundary-value problem

$$\begin{gathered} u^{(4)}(t) = g(t) \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \tag{3.1}$$

then, by Lemma 2.5, we have $\beta(t) = \int_0^1 G(t,s)g(s)ds$. Let $\beta_0 = \max_{t\in[0,1]} \beta(t)$; therefore, by (H1) and (2.4),

$$T_\lambda^0 \beta(t) \le T_\lambda^0 \beta_0 = \int_0^1 G(t,s)\lambda g(s)f(\beta_0)ds < \beta(t), \quad \forall 0 < \lambda < \frac{1}{f(\beta_0)}.$$

This implies that $\beta(t)$ is an upper solution of T_λ^0. On the other hand, let $\alpha(t) \equiv 0, t \in [0,1]$, then $\alpha(t)$ is a lower solution of T_λ^0, and $\alpha(t) < \beta(t), t \in [0,1]$. Clearly T_λ^0 is completely continuous on $[\alpha,\beta]$. Therefore, T_λ^0 has a fixed point $u_\lambda \in [\alpha,\beta]$, and therefore u_λ is a solution of (1.1)-(1.2). Hence, for any $0 < \lambda < \frac{1}{f(\beta_0)}$, we have $\lambda \in S$, which implies that $S \neq \emptyset$.

On the other hand, if $\lambda_1 \in S$, then we must have $(0,\lambda_1) \subset S$. In fact, let u_{λ_1} be a solution of (1.1)-(1.2). Then, by Lemma 2.5, we have

$$u_{\lambda_1}(t) = T_{\lambda_1}^0 u_{\lambda_1}(t), t \in [0,1].$$

Therefore, for any $\lambda \in (0,\lambda_1)$, by (2.4), we have

$$\begin{aligned} T_\lambda^0 u_{\lambda_1}(t) &= \int_0^1 G(t,s)\lambda g(s)f(u_{\lambda_1}(s)ds \\ &\le \int_0^1 G(t,s)\lambda_1 g(s)f(u_{\lambda_1}(s)ds \\ &= T_{\lambda_1}^0 u_{\lambda_1}(t) \\ &= u_{\lambda_1}(t), \end{aligned}$$

which implies that u_{λ_1} is an upper solution of T_λ^0. Combining this with the fact that $\alpha(t) \equiv 0$ $(t \in [0,1])$ is a lower solution of T_λ^0, then, by Lemma 2.5, EP (1.1)-(1.2) has a solution, therefore $\lambda \in S$. Thus we have $(0,\lambda_1) \subset S$.

Let $\lambda^* = \sup S$, now we prove that $\lambda^* < +\infty$. If this is not true, then we must have $N \subset S$, where N denotes natural number numbers. Therefore, for any $n \in N$, by Lemma 2.5, there exists $u_n \in Q$ satisfying

$$u_n = T_n^0 u_n = \int_0^1 G(t,s) n g(s) f(u_n(s) ds.$$

Let $K = [\frac{\bar{\delta} M_\theta^{m+1}}{6} \int_\theta^{1-\theta} G_1(s,s) g(s) ds]^{-1}$. Suppose $\|u_n\| \geq 1$. Then we have

$$\begin{aligned}
1 &\geq \frac{1}{\|u_n\|} \\
&\geq \frac{\min_{\theta \leq t \leq 1-\theta} u_n(t)}{\|u_n\|^2} \\
&= \frac{1}{\|u_n\|^2} \min_{\theta \leq t \leq 1-\theta} \int_0^1 G(t,s) n g(s) f(u_n(s) ds \\
&\geq \frac{1}{\|u_n\|^2} M_\theta \int_\theta^{1-\theta} \frac{1}{6} G_1(s,s) n g(s) \bar{\delta} u_n(s)^m ds \\
&\geq \frac{1}{\|u_n\|^2} M_\theta^{m+1} \int_\theta^{1-\theta} \frac{1}{6} G_1(s,s) n g(s) \bar{\delta} \|u_n\|^m ds \\
&\geq \frac{1}{6} n M_\theta^{m+1} \bar{\delta} \int_\theta^{1-\theta} G_1(s,s) g(s) ds.
\end{aligned}$$

If $\|u_n\| \leq 1$, then

$$\begin{aligned}
1 \geq \|u_n\| &\geq \min_{\theta \leq t \leq 1-\theta} \int_0^1 G(t,s) n g(s) f(u_n(s) ds \\
&\geq M_\theta \int_\theta^{1-\theta} \frac{1}{6} G_1(s,s) n g(s) f(0) ds.
\end{aligned}$$

Hence $n \leq \{K, (M_\theta \int_\theta^{1-\theta} \frac{1}{6} G_1(s,s) g(s) f(0) ds)^{-1}\}$, this contradicts the fact that N is unbounded; therefore $\lambda^* < +\infty$, and the proof of the conclusion (1) is complete.

Secondly, we verify the conclusion (2) of Theorem 3.1. Let $\{\lambda_n\} \subset [\frac{\lambda^*}{2}, \lambda^*)$, $\lambda_n \to \lambda^* (n \to \infty)$, $\{\lambda_n\}$ be an increasing sequence. Suppose u_n is solution of (1.1) with λ_n instead of λ. By Lemma 2.8, there exists $R(\frac{\lambda^*}{2}) > 0$ such that $\|u_n\| \leq R(\frac{\lambda^*}{2})$, $n = 1, 2, \cdots$. Hence u_n is a bounded set. It is clear that $\{u_n\}$ is an equicontinuous set of $C[0,1]$. Therefore, by the Ascoli-Arzela theorem, it follows that $\{u_n\}$ is compact set, and therefore $\{u_n\}$ has a convergent subsequence. Without loss of generality, we suppose that u_n is convergent: $u_n \to u^* (n \to +\infty)$. Since $u_n = T_{\lambda_n}^0 u_n$, by control convergence theorem (f is bounded), we have $u^* = T_{\lambda^*}^0 u^*$. Therefore, by Lemma 2.5, u^* is a solution of (1.1)-(1.2) with λ^* instead of λ. Hence the conclusion (2) of Theorem 3.1 holds.

Finally, we prove the conclusion (3) of Theorem 3.1. Let $\alpha(t) \equiv h$ ($t \in [0,1]$). Then for any $\lambda \in (0, \lambda^*)$, $\alpha(t)$ is a lower solution of (2.3). On the other hand, by Lemma 2.8, there exists $R(\lambda) > 0$ such that $\|u_{\lambda'}\| \leq R(\lambda)$, $\lambda' \in [\lambda, \lambda^*]$, where $u_{\lambda'}$ is a solution of (1.1) with λ' instead of λ. Also by Lemma 2.9, there exist $\bar{\lambda} \in [\lambda, \lambda^*], h_0 \in (0,1)$ satisfying

$$\lambda f(u+h) < \bar{\lambda} f(u), u \in [0, R(\lambda)], h \in (0, h_0).$$

Let $u_{\bar\lambda}$ be a solution of (1.1)-(1.2) with $\bar\lambda$, and $\bar u_\lambda(t) = u_{\bar\lambda} + h$, $h \in (0, h_0)$. Then

$$\begin{aligned}
\bar u_\lambda(t) &= u_{\bar\lambda} + h\\
&= \int_0^1 G(t,s)\bar\lambda g(s) f(u_{\bar\lambda}(s))ds + h\\
&\geq h + \int_0^1 G(t,s)\lambda g(s) f(u_{\bar\lambda}(s)+h)ds - \int_0^1 G_1(t,s)hds\\
&= T_\lambda^h \bar u_\lambda(t).
\end{aligned}$$

Combining this with $\bar u_\lambda(0) = \bar u_\lambda(1) \geq h$, $\bar u''_\lambda(0) = 0 \leq h$, $\bar u''_\lambda(1) = 0 \leq h$, we have the $\bar u_\lambda(t)$ is an upper solution of (2.3). Therefore (2.3) has solution. Let $v_\lambda(t)$ be a solution of (2.3). Let $\Omega = \{u \in Q | u(t) < v_\lambda(t), t \in [0,1]\}$. It is clear that $\Omega \subset Q$ is a bounded open set. If $u \in \partial\Omega$, then there exists $t_0 \in [0,1]$, such that $u(t_0) = v_\lambda(t_0)$. Therefore, for any $\mu \geq 1$, $h \in (0, h_0)$, $u \in \partial\Omega$, we have

$$\begin{aligned}
T_\lambda^0 u(t_0) &< h + T_\lambda^0 u(t_0) - \int_0^1 G_1(t,s)hds\\
&\leq h + T_\lambda^0 v_\lambda(t_0) - \int_0^1 G_1(t,s)hds\\
&= T_\lambda^h v_\lambda(t_0)\\
&= v_\lambda(t_0)\\
&= u(t_0)\\
&\leq \mu u(t_0).
\end{aligned}$$

Hence for any $\mu \geq 1$, we have $T_\lambda^0 u \neq \mu u$, $u \in \partial\Omega$. Therefore, by Lemma 1.1,

$$i(T_\lambda^0, \Omega, Q) = 1. \tag{3.2}$$

It remains to prove that the conditions of Lemma 1.2 are satisfied Firstly, we check the condition (1) of Lemma 1.2 is fulfilled. In fact, for any $u \in Q$, we have by (H1) and (2.5),

$$\begin{aligned}
T_\lambda^0 u(\frac{1}{2}) &= \int_0^1 G(\frac{1}{2}, s)\lambda g(s) f(u(s))ds\\
&\geq \int_\theta^{1-\theta} G(\frac{1}{2}, s)\lambda g(s)\bar\delta M_\theta^m \|u\|^m ds\\
&= \|u\|^m \int_\theta^{1-\theta} G(\frac{1}{2}, s)\lambda g(s)\bar\delta M_\theta^m ds\\
&= \|u\|^{m-1} \int_\theta^{1-\theta} G(\frac{1}{2}, s)\lambda g(s)\bar\delta M_\theta^m ds \|u\|
\end{aligned} \tag{3.3}$$

Choose $\bar R > 0$ such that $\bar R^{m-1} \int_\theta^{1-\theta} G(\frac{1}{2}, s)\lambda g(s)\bar\delta M_\theta^m ds > 1$. Therefore, for any $R > \bar R$ and $B_R \subset Q$, by (3.3),

$$\|T_\lambda^0 u\| > \|u\| > 0, u \in \partial B_R, \tag{3.4}$$

where $B_R = \{u \in Q | \|u\| < R\}$. Hence the condition (1) of Lemma 1.2 is fulfilled.

Now we prove that the condition (2) of Lemma 1.2 is satisfied. In fact, if the condition (2) of Lemma 1.2 does not hold, then there exist $u_1 \in Q \cap \partial B_R, 0 < \mu_1 \leq 1$, such that $T_\lambda^0 u_1 = \mu_1 u_1$. Therefore, $\|T_\lambda^0 u_1\| \le \|u_1\|$. This conflicts with (3.4). Hence the condition (2) of Lemma 1.2 is satisfied. Therefore by Lemma 1.2, we have

$$i(T_\lambda^0, B_R, Q) = 0. \tag{3.5}$$

Consequently, by the additivity of the fixed point index, we get

$$0 = i(T_\lambda^0, B_R, Q) = i(T_\lambda^0, \Omega, Q) + i(T_\lambda^0, B_R \setminus \bar{\Omega}, Q).$$

Since $i(T_\lambda^0, \Omega, Q) = 1$, $i(T_\lambda^0, B_R \setminus \bar{\Omega}, Q) = -1$. Therefore, by the solution property of the fixed point index, there is a fixed point of T_λ^0 in Ω and a fixed point of T_λ^0 in $B_R \setminus \bar{\Omega}$, respectively. Therefore by Lemma 2.5, EP (1.1)-(1.2) has at least two solutions. Furthermore, (1.1)-(1.2) has at least two positive solutions by (H1) and (H2). The proof of Theorem 3.1 is complete. $\square$

Now we study (1.1)-(1.3). The method is similar to the method above. Define

$$\hat{G}(t,s) = \min\{t,s\} = \begin{cases} t, & t \le s,\ 0 \le t \le s \le 1, \\ s, & s \le t,\ 0 \le s \le t \le 1, \end{cases}$$

$$\begin{aligned} \tilde{G}(t,s) &= \int_0^1 \hat{G}(t,r)\hat{G}(r,s)dr \\ &= \begin{cases} \frac{s^3}{3} + \frac{s(t^2-s^2)}{2} + st(1-t), & 0 \le s \le t \le 1, \\ \frac{t^3}{3} + \frac{t(s^2-t^2)}{2} + ts(1-s), & 0 \le t \le s \le 1. \end{cases} \end{aligned}$$

It is easy to prove that $\hat{G}(t,s)$ and $\tilde{G}(t,s)$ have the following properties.

Proposition 3.2. *For all $t,s \in [0,1]$, $\alpha \in (0,\frac{1}{2})$ we have*

$$\begin{gathered} \hat{G}(t,s) > 0, \quad t,s \in (0,1), \\ \hat{G}(t,s) \le \hat{G}(s,s) = s, \quad t,s \in [0,1], \\ \hat{G}(t,s) \ge \alpha\hat{G}(s,s), \quad t \in [\alpha, 1-\alpha],\ s \in [0,1], \\ \tilde{G}(t,s) \le \frac{1}{2}s, \quad t,s \in [0,1]; \\ \tilde{G}(t,s) \ge \frac{1}{2}M_\alpha s, \quad t \in [\alpha, 1-\alpha],\ s \in [0,1] \end{gathered}$$

where $M_\alpha = \alpha^2(1-2\alpha)$.

Define the cone

$$\hat{Q} = \{u \in C[0,1] | u(t) \ge 0, \min_{\alpha \le t \le 1-\alpha} u(t) \ge M_\alpha \|u\|\}$$

and let

(H3) $g \in C((0,1),(0,+\infty))$ and $0 < \int_0^1 sg(s)ds < +\infty$

Theorem 3.3. *Let (H1) and (H3) be satisfied. Then there exists $0 < \lambda^* < +\infty$ such that:*

(1) *EP (1.1)-(1.3) has no solution for $\lambda > \lambda^*$;*
(2) *EP (1.1)-(1.3) has at least one positive solution for $\lambda = \lambda^*$;*
(3) *EP (1.1)-(1.3) has at least two positive solutions for $0 < \lambda < \lambda^*$.*

As an example we consider the eigenvalue problem

$$\begin{gathered} u^{(4)}(t)=\lambda\frac{1}{t(1-t)}2^{2u},\quad 0<t<1,\\ u(0)=u(1)=u''(0)=u''(1)=0. \end{gathered} \tag{3.6}$$

It is clear that (3.6) is not covered by the results in [1, 2, 5, 7, 8, 9, 10, 11, 12, 13, 14].

Let $g(t)=\frac{1}{t(1-t)}$, $f(u)=2^{2u}$. It is obvious that $g(t)$ is singular at both $t=0$ and at $t=1$. However, hypothesis (H2) is satisfied. In addition, for $\bar{\bar{\delta}}=1>0, m=2$, we have $f(u)=2^{2u}=\bar{\bar{\delta}}2^{2u}>u^2=u^m>0$. So that (H1) is satisfied.

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