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SOLVABILITY OF NEUMANN BOUNDARY-VALUE PROBLEMS WITH CARATHÉODORY NONLINEARITIES

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Abstract. We propose a sufficient condition, on the nonlinear term, for the existence of solutions. This new condition is weaker than the usual sign condition and than the assumption on the existence of constant upper and lower solutions.

1. Introduction

This paper is devoted to the study of the existence of solutions to Neumann boundary-value problems for nonlinear second order differential equations with Carathéodory nonlinearities. More specifically, we consider the problem

$$\begin{gathered} y''(t) = f(t, y(t), y'(t)) \quad 0 < t < 1 \\ y'(0) = y'(1) = 0 \end{gathered} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is an L^1-Carathéodory function, i.e. f satisfies the following conditions:

(i) $f(\cdot, y, z)$ is measurable for all $(y, z) \in \mathbb{R}^2$.
(ii) $f(t, \cdot, \cdot)$ is continuous for almost all $t \in [0,1]$.
(iii) For each $K > 0$ there exists $h_K \in L^1(0,1)$ such that $|y| + |z| \leq K$ implies $|f(t, y, z)| \leq h_K(t)$ for almost all $t \in [0,1]$.

Problem (1.1) has been investigated by several authors under suitable conditions on the nonlinearity. See for instance [1, 2, 4, 5, 6] and the references therein. In most of these works the nonlinearity is assumed to be either continuous or of the Carathéodory class. The techniques involved are based on the upper and lower solution method, the topological degree, or the topological transversality theorem. We should point out that a different class of Neumann problems has been considered in [7, 8, 9, 10]. Our assumptions and techniques of proofs are different and our results cannot be trivially deduced from the previous works. In fact, we generalize the results in [5, 6] and some of the results in [2, 4].

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2. Preliminaries

Let I denote the real interval $[0,1]$. Let $X = AC^1(I)$ denote the Banach space of real-valued functions that are absolutely continuous, together with their first derivatives, on I. This space is endowed with the norm

$$\|y\| = \max\{|y(t)| + |y'(t)|;\ t \in I\}\,.$$

Let $\mathrm{Car}(I \times \mathbb{R}^2)$ be the set of real-valued functions satisfying the Carathéodory conditions (i), (ii), (iii) above.

By a solution to (1.1) we mean a function $y \in X_0 := \{u \in X : u'(0) = u'(1) = 0\}$ satisfying the differential equation in (1.1) almost everywhere on I. Note that the homogeneous problem $y'' = 0, y'(0) = y'(1) = 0$ has nontrivial solutions. So, we shall consider the following problem, for $m > 1$:

$$\begin{gathered} y''(t) = \frac{1}{m}y(t) + f(t, y(t), y'(t)), \quad 0 < t < 1 \\ y'(0) = y'(1) = 0 \end{gathered} \tag{2.1}$$

and consider (1.1) as a limiting case when $m \to +\infty$.

Our aim is to provide sufficient conditions on f that will make (2.1) solvable. First, we show that solutions to (2.1) are uniformly bounded, independently of m. Then, we use the Arzela-Ascoli theorem to obtain the solvability of (1.1).

Since our arguments are based on the topological transversality theorem (see [3, 1] for definitions and properties), we shall consider a one-parameter family of problems related to (2.1). For $0 \le \lambda \le 1$, consider

$$\begin{gathered} y''(t) = \frac{1}{m}y(t) + \lambda f(t, y(t), y'(t)), \quad 0 < t < 1 \\ y'(0) = y'(1) = 0\,. \end{gathered} \tag{2.2}$$

Note that for $\lambda = 1$, the above equation is exactly (2.1). Also note that when $\lambda = 0$, equation (2.2) has only the trivial solution. It is clear that (2.2) is equivalent to

$$y'(t) - y'(0) - \frac{1}{m}\int_0^t y(s)ds = \lambda \int_0^t f(s, y(s), y'(s))ds\,.$$

Define the linear operator $L_m : X_0 \to C_0(I)$ by

$$(L_m y)(t) = y'(t) - y'(0) - \frac{1}{m}\int_0^t y(s)ds, \quad t \in I\,.$$

Here $C_0(I) = \{u \in C(I); u(0) = 0\}$. Also, define $N_f(\lambda, \cdot) : X_0 \to C(I)$ by

$$N_f(\lambda, y)(t) = \lambda \int_0^t f(s, y(s), y'(s))ds, \quad t \in I\,.$$

It follows that (2.2) is equivalent to

$$L_m y = N_f(\lambda, y), \tag{2.3}$$

in the sense that every solution of (2.2) is a solution of (2.3) and vice-versa.

Lemma 2.1. *The operator L_m is invertible.*

Proof. Note that the equation $L_m y = p$ is equivalent to

$$y(t) = y(0) + \frac{1}{m}\int_0^t (t-s)y(s)ds + \int_0^t p(s)ds\,.$$

Since this equation is a linear Volterra integral equation, it has a unique solution (see for instance [11]). This completes the proof. $\square$

Lemma 2.2. *The operator $N_f(\lambda, \cdot)$ is continuous and completely continuous.*

The proof of this lemma can be found in [5].

3. A Priori Estimates

In this section we present a new sufficient condition on $f \in \mathrm{Car}(I \times \mathbb{R}^2)$ in order to obtain a priori bounds, independent of λ and m, on solutions y of (2.2). This condition is an improvement of condition (H1) in [6] and is more general than the assumption on the existence of constant upper and lower solutions in the correct order or in the reverse order in [2].

Proposition 3.1. *Assume $f \in \mathrm{Car}(I \times \mathbb{R}^2)$ satisfies the condition*

(C1) *There exists $M_0 > 0$ such that* $\Big[\int_0^1 f(t, M_0, 0)dt\Big]\Big[\int_0^1 f(t, -M_0, 0)dt\Big] < 0$.

Then any possible solution y of

$$\begin{gathered} y''(t) = \frac{1}{m} y(t) + \lambda f_1(t, y(t), y'(t)) \quad 0 < t < 1 \\ y'(0) = y'(1) = 0 \end{gathered} \tag{3.1}$$

satisfies $|y(t)| \le M_0$ for all $t \in I$.

Proof. The proof is similar to that of Lemma 2.2 in [6], but we reproduce it here for the sake of completeness. Without loss of generality, we prove only the case when $\int_0^1 f(t, M_0, 0)dt > 0$ and $\int_0^1 f(t, -M_0, 0)dt < 0$. The other case is similar.

Consider the modified problem (3.1) with

$$f_1(t, y, z) = \begin{cases} \max\Big\{f(t, y, z), -\frac{M_0}{m} + \int_0^1 f(t, M_0, 0)dt\Big\} & y > M_0 \\ f(t, y, z) & -M_0 \le y \le M_0 \\ \min\Big\{f(t, y, z), \frac{M_0}{m} + \int_0^1 f(t, -M_0, 0)dt\Big\} & y < -M_0 \end{cases}$$

We remark that any solution y of (3.1) that satisfies $|y(t)| \le M_0$ is a solution of (2.2), because in this case $f_1(t, y(t), y'(t) \equiv f(t, y(t), y'(t))$.

Let y be a solution of (3.1), and let $t_0 \in I$ be a value where y achieves its positive maximum. Then $y'(t_0) = 0$.

Assume that $y(t_0) > M_0$ and $t_0 \in (0, 1)$. Then there exists $a > 0$ such that $y(t) > M_0$ for all $t \in [t_0, t_0 + a]$. It follows from the differential equation in (3.1) and the definition of f_1 that for all $t \in [t_0, t_0 + a]$,

$$y''(t) \ge \frac{y(t)}{m} - \frac{M_0}{m} + \int_0^1 f(t, M_0, 0)ds \;= \frac{y(t) - M_0}{m} + \int_0^1 f(t, M_0, 0)ds > 0\,.$$

This implies that $y'(t) = \int_{t_0}^t y''(s)ds > 0$ for all $t \in [t_0, t_0 + a]$, which yields

$$y(t) - y(t_0) = \int_{t_0}^t y'(\tau)d\tau > 0 \quad \text{for all } t \in [t_0, t_0, a].$$

This contradicts that $y(t_0)$ is the maximum of y. Hence $y(t) \le M_0$ for all $t \in (0, 1)$.

If $t_0 = 0$, then assuming $y(0) > M_0$ we shall arrive at a contradiction. Indeed,

$$y''(0) = \frac{y(0)}{m} + \lambda f_1(0, y(0), 0)$$

implies

$$y''(0) \geq \frac{y(0) - M_0}{m} + \int_0^1 f(s, M_0, 0) ds > 0 \,.$$

So y' is strictly increasing to the right of $t = 0$ (but sufficiently near 0). Then $y'(t) > y'(0) = 0$ for t near 0; and so, y is strictly increasing to the right of $t = 0$ and $y(0)$ is not the maximum of y on I. This is the desired contradiction. Hence $y(0) \leq M_0$.

Similarly, we can show that $y(1) \leq M_0$. Thus $y(t) \leq M_0$ for all $t \in I$.

Now, in case y achieves a negative minimum at $t = \tau_0$ such that $y(\tau_0) < -M_0$ and $\tau_0 \in (0, 1)$ then there exists $b > 0$ such that $y(t) < -M_0$ for all $t \in [\tau_0, \tau_0 + b]$. It follows from the differential equation in (3.1) and the definition of f_1 that for all $t \in [\tau_0, \tau_0 + b]$,

$$y''(t) \leq \frac{y(t) + M_0}{m} + \int_0^1 f(s, -M_0, 0) ds \leq 0$$

which leads to $y'(t) = \int_{\tau_0}^t y''(s) ds < 0$ for all $t \in [\tau_0, \tau_0 + b]$ and

$$y(t) - y(\tau_0) = \int_{\tau_0}^t y'(s) ds < 0 \quad \text{for all } t \in [\tau_0, \tau_0 + b] \,.$$

This contradicts that $y(\tau_0)$ is the minimum of y on I.

We can handle the case of a minimum at $\tau_0 = 0$ or $\tau_0 = 1$ in a similar way as above. Hence, we have proved that

$$-M_0 \leq y(t) \leq M_0 \text{ for all } t \in I,$$

which completes the proof. □

Remark 3.2. Hypothesis (H1) in [6] states that there exists $M > 0$ such that $f(t, M, 0) \geq 0$ and $f(t, -M, 0) \leq 0$ almost everywhere in I. Our condition is much weaker than (H1), since we allow the possibility of $f(t, M_0, 0) < 0$ on a subset of I with positive measure as long as $\int_0^1 f(t, M_0, 0) dt$ remains positive; or $f(t, -M_0, 0) > 0$ on a subset of I with positive measure as long as $\int_0^1 f(t, -M_0, 0) dt$ remains negative.

Remark 3.3. From the definition of upper and lower solutions, it follows that (H1) in [6] implies that $-M_0$ is a lower solution and M_0 is an upper solution. Hence our condition is more general than the assumption of the existence of constant upper and lower solutions. Moreover, the case $\int_0^1 f(t, M_0, 0) dt < 0$ and $\int_0^1 f(t, -M_0, 0) dt > 0$ is more general than the condition of existence of constant upper and lower solutions in the reverse order (see [2]).

Our next result gives an a priori bound on the first derivative of any solution y of (3.1) satisfying $|y(t)| \leq M_0$ for all $t \in I$.

Proposition 3.4. *Assume $f \in \mathrm{Car}(I \times \mathbb{R}^2)$ satisfies the condition*

(C2) *There exist $q \in L^1(I)$, $\Phi : [0, +\infty) \to (0, +\infty)$ nondecreasing with $1/\Phi$ integrable over bounded intervals, and*

$$\int_{M_0}^{+\infty} \frac{d\sigma}{\Phi(\sigma)} > \|q\|_{L^1}$$

such that $|f(t, y, z))| \leq q(t)\Phi(|z|)$ for all $(t, y) \in I \times [-M_0, M_0]$ and all $z \in \mathbb{R}$.

Then, there exists $M_1 > 0$ such that $|y'(t)| \leq M_1$ for all $t \in I$ for any solution y of (3.1) *with $|y(t)| \leq M_0$ for all $t \in I$.*

Proof. Let y be a solution of (3.1) such that $|y(t)| \leq M_0$ for all $t \in I$. Condition (C2) implies

$$|y''(t)| \leq \frac{|y(t)|}{m} + q(t)\Phi(|y'(t)|) \quad \text{for all } t \in I$$

Since $m > 1$ and $|y(t)| \leq M_0$, we have

$$|y''(t)| \leq M_0 + q(t)\Phi(|y'(t)|) \quad \text{for all } t \in I\,.$$

On the other hand,

$$|y'(t)| = \Big|\int_0^t y''(s)ds\Big| \leq \int_0^t |y''(s)|ds \quad \text{for all } t \in I.$$

Hence

$$|y'(t)| \leq M_0 t + \int_0^t q(s)\Phi(|y'(s)|)ds \quad \text{for all } t \in I\,.$$

Since $0 \leq t \leq 1$, we infer that

$$|y'(t)| \leq M_0 + \int_0^t q(s)\Phi(|y'(s)|)ds \quad \text{for all } t \in I\,.$$

Let

$$u(t) = M_0 + \int_0^t q(s)\Phi(|y'(s)|)ds \quad \text{for all } t \in I$$

Then $|y'(t)| \leq u(t)$ and $u'(t) = q(t)\Phi(|y'(t)|)$ for all $t \in I$. Since Φ is nondecreasing,

$$u'(t) \leq q(t)\Phi(u(t)) \quad \text{for all } t \in I$$

Therefore,

$$\frac{u'(t)}{\Phi(u(t))} \leq q(t) \quad \text{for all } t \in I.$$

It follows that

$$\int_0^t \frac{u'(s)ds}{\Phi(u(s))} \leq \int_0^t q(s)ds \leq \int_0^1 q(s)ds = \|q\|_{L^1}.$$

This implies

$$\int_{M_0}^{u(t)} \frac{d\sigma}{\Phi(\sigma)} \leq \|q\|_{L^1}.$$

The condition on Φ implies that there exists $M_1 > 0$ such that $u(t) \leq M_1$ for all $t \in I$. Therefore, $|y'(t)| \leq M_1$ for all $t \in I$, which completes the proof. $\square$

4. Existence of Solutions

In this section we state and prove our existence result.

Theorem 4.1. *Assume that $f \in \mathrm{Car}(I \times \mathbb{R}^2)$ satisfies conditions (C1) and (C2). Then problem* (1.1) *has at least one solution.*

Proof. We have seen in the above discussion that any possible solution y of (3.1) satisfies
$$|y(t)| \leq M_0 \quad \text{and} \quad |y'(t)| \leq M_1 \quad \text{for all } t \in I\,.$$
Let $M := M_0 + M_1$. Then $\|y\| \leq M$. It is clear that problem (3.1) is equivalent to
$$y = L_m^{-1} N_{f_1}(\lambda, y) \tag{4.1}$$
Let $U := \{y \in X_0; \|y\| < 1+M\}$. Then we can easily show that for any λ, the operator $L_m^{-1}N_{f_1}(\lambda, \cdot)$ is compact (see [4]) and has no fixed point on ∂U, the boundary of U. Therefore, $L_m^{-1}N_{f_1}(.,.) : [0,1] \times \overline{U} \to X_0$ is a compact homotopy without fixed point on ∂U. Since $L_m^{-1}N_{f_1}(0,\cdot) \equiv 0$ is essential, then by the topological transversality theorem (see [1, 3]) $L_m^{-1}N_{f_1}(1,\cdot)$ is essential. Consequently, there exists $y \in U$ such that $y = L_m^{-1}N_{f_1}(1, y)$, which means that y is a solution of (3.1) for $\lambda = 1$. But, we have seen that any solution of (3.1), satisfying $|y(t)| \leq M_0$ is also a solution of (2.2). Hence (2.2), with $\lambda = 1$, has at least one solution. But (2.2) is exactly (2.1) for $\lambda = 1$. Hence, we have proved that for each $m > 1$, problem (2.1) has at least one solution, which we denote by y_m. Moreover, y_m, satisfies the estimates
$$|y_m(t)| \leq M_0 \quad \text{and} \quad |y_m'(t)| \leq M_1 \quad \text{for all} t \in I.$$
Furthermore, M_0 and M_1 are independent of m. This shows that the sequences $\{y_m(t)\}$ and $\{y_m'(t)\}$ are uniformly bounded.

Now,
$$y_m'(t) = \int_0^t y''(s)ds = \frac{1}{m}\int_0^t y_m(s)ds + \int_0^t f(s, y_m(s), y_m'(s)ds.$$
This implies
$$y_m'(t_2) - y'(t_1) = \frac{1}{m}\int_{t_1}^{t_2} y_m(s)ds + \int_{t_1}^{t_2} f(s, y_m(s), y_m'(s))ds\,.$$
Since $m > 1$ and $f \in \mathrm{Car}(I \times \mathbb{R}^2)$, we have
$$|y_m'(t_2) - y_m'(t_1)| \leq M_0|t_2 - t_1| + \int_{t_1}^{t_2} h_{M_0}(s)ds\,.$$
This shows that $\{y_m'\}$ is equicontinuous. Also, $y_m(t) = y_m(0) + \int_0^t y_m'(s)ds$ implies
$$y_m(\tau_2) - y_m(\tau_1) = \int_{\tau_1}^{\tau_2} y_m'(s)ds$$
By proposition 3.4 we have $|y_m'(t) \leq M_1$ for all t. Thus
$$|y_m(\tau_2) - y_m(\tau_1)| \leq M_1|\tau_2 - \tau_1|$$
So that $\{y_m\}$ is also equicontinuous.

By the Arzela-Ascoli theorem, we can extract from $\{y_m\}$ and $\{y_m'\}$ subsequences, which we label the same, and that are uniformly convergent on I. Let $y(t) = \lim_{m\to+\infty} y_m(t)$ and $z(t) = \lim_{m\to+\infty} y_m'(t)$. Since $y_m(t) = y_m(0) + \int_0^t y_m'(s)ds$, and the convergence of $\{y_m\}$ $\{y_m'\}$ is uniform, we obtain
$$y(t) = y(0) + \int_0^t z(s)ds$$
which implies that $y'(t) = z(t)$; i.e., $y'(t) = \lim_{m\to} y_m'(t)$. Moreover y is a solution of (1.1). This completes the proof $\square$

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