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MULTIPLE SOLUTIONS FOR NONHOMOGENEOUS SCHRÖDINGER-POISSON SYSTEM WITH p-LAPLACIAN

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Abstract. This article concerns the existence of solutions to the Schrödinger-Poisson system

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u + \lambda\phi u = |u|^{q-2}u + h(x) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered}$$

where $4/3 < p < 12/5$, $p < q < p^* = 3p/(3-p)$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\lambda > 0$, and $h \neq 0$. The multiplicity results are obtained by using Ekeland's variational principle and the mountain pass theorem.

1. Introduction and statement of main results

This article concerns the existence of solutions to the Schrödinger-Poisson system

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u + \lambda\phi u = |u|^{q-2}u + h(x) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.1}$$

where $4/3 < p < 12/5$, $p < q < p^* = \frac{3p}{3-p}$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\lambda > 0$, and $h \neq 0$. The system (1.1) can be viewed as a perturbation of the system

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u + \lambda\phi u = |u|^{q-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.2}$$

This system was first considered by Du, Su, and Wang in [11] where the variational framework was built and the existence of nontrivial solutions was established via the mountain pass theorem. For $p = 2$, the system (1.2) reduces to the following classical Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + u + \lambda\phi u = |u|^{q-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.3}$$

where $\lambda > 0$ and $q \in (2, 6)$. Such a system, also known as the nonlinear Schrödinger-Maxwell equation, has an interesting physical context. According to a classical model, the interaction of a charged particle with an electromagnetic field can be described by coupling a nonlinear Schrödinger equation and a Poisson equation. For more details on the physical aspects of the system we refer to the pioneering works of Benci and Fortunado [5, 6] and the references therein. In the past decades,

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the existence of solutions to the system (1.3) has been discussed in [4] for $q \in (3, 6)$, in [9, 10] for $q \in [4, 6)$, and in [2, 3, 21, 25, 31] for $q \in (2, 6)$ or general nonlinearity.

For $p = 2$, the system (1.1) reduces to the nonhomogeneous Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + u + \lambda\phi u = |u|^{q-2}u + h(x) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.4}$$

where $\lambda > 0$, $q \in (2, 6)$ and $h(x) \not\equiv 0$. In [22], Salvatore obtained multiple radial solutions to the system (1.4) for $q \in (4, 6)$ and $h \in L^2(\mathbb{R}^3)$ being radial with small L^2-norm. In [16], Jiang, Wang and Zhou considered the system (1.4) with $h \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ being a nonnegative radial function and satisfying $(x, \nabla h) \in L^2(\mathbb{R}^3)$. Applying the Ekeland's variational principle and the mountain pass theorem, it was proved in [16] that the system (1.4) admitted two radial solutions for $q \in (2, 6)$ with small L^2-norm $|h|_{L^2(\mathbb{R}^3)}$ of h and for $q \in (2, 3]$ with $\lambda > 0$ also small. For other works related to the system (1.4) or to similar systems involving certain potentials, we refer to [8, 13, 17, 20, 26, 27, 30, 32, 33] and the references therein.

After an accurate bibliographic review, we see that it is open question the existence of multiple solutions to the quasilinesr system (1.1) with $4/3 < p < 12/5$ and $h \neq 0$. Inspired by this fact, We aim to establish the existence of multiple solutions to system (1.1). We use $\tau' = \frac{\tau}{\tau - 1}$ to denote the Hölder conjugate of $\tau > 1$. We impose on h the following assumption.

- (H1) h is a nonzero radial function and for $(p^*)' \le s \le p'$,
  - (i) $h \in L^s(\mathbb{R}^3)$ with the L^s-norm denoted by $|h|_{L^s(\mathbb{R}^3)}$;
  - (ii) $(x, \nabla h) \in L^s(\mathbb{R}^3)$ where the gradient ∇h is in the weak sense.

We will prove the following theorems.

Theorem 1.1. *Assume that* (H1) *holds and* $\frac{6p}{p+2} < q < p^*$. *Then there exists* $\Lambda > 0$ *such that for* $|h|_{L^s(\mathbb{R}^3)} < \Lambda$ *the system* (1.1) *admits two solutions for any* $\lambda > 0$.

Theorem 1.2. *Assume that* (H1)(i) *holds and* $p < q \le \frac{6p}{p+2}$. *There exist* $\Lambda > 0$ *and* $\lambda_* > 0$ *such that for* $|h|_{L^s(\mathbb{R}^3)} < \Lambda$, *system* (1.1) *admits two solutions for any* $\lambda \in (0, \lambda_*)$.

Remark 1.3. The first attempt of the study on the Schrödinger-Poisson system (1.2) with p-Laplacian were made in [11]. Now the results in Theorems 1.1 and 1.2 extend the results in [16, 22] from $p = 2$ to the quasilinear case $4/3 < p < 12/5$. This range of p was first determined in [11]. We observe a phenomenon that the solvability of the system (1.1) can be considered for a large class of radial functions h satisfying (H1). In this sense the existence results in [16] may be extended to the case that h and $(x, \nabla h)$ belonging to $L^s(\mathbb{R}^3)$ with $6/5 \le s \le 2$.

Notice that for $p \neq 2$, it is difficult to prove the Pohožaev identity which is essential to establish the boundedness of Palais-Smale sequences for $q \in (3, 6)$ in [16]. To overcome this difficulty, for $6p/(p + 2) < q < p^*$ we introduce an auxiliary functional and use an indirect method to do that: see our proof of Lemma 4.3. It also should be pointed out that our method is more applicable. As far as we know, this article is the first attempt to study the nonhomogeneous Schrödinger-Poisson system with p-Laplacian.

The proofs of the main results will be obtained by exploiting suitable variational methods. In Section 2, we give some preliminary results concerning the variational structure for the system (1.1). In Section 3, with the aid of the Ekeland's variational principle [12], we obtain by Theorem 3.3 a solution of (1.1) with negative energy for $p < q < p^*$. In Section 4 we obtain a solution of (1.1) with positive energy and discuss with two cases of $\frac{6p}{p+2} < q < p^*$ and $p < q \leq \frac{6p}{p+2}$. In Subsection 4.1, we use the scaling technique beginning in [14] and developing in [11] to obtain the boundedness of a Palais-Smale sequence for $\frac{6p}{p+2} < q < p^*$ and find a positive energy solution by using the mountain pass theorem [1], see Theorem 4.1. In Subsection 4.2, by using the cut-off technique as in [15] and combining some delicate analysis, we prove a positive energy solution of (1.1) with $p < q \leq \frac{6p}{p+2}$ and $\lambda > 0$ small, see Theorem 4.4. Then Theorems 1.1 and 1.2 will follow from Theorem 3.3, Theorems 4.1 and 4.4.

2. Preliminaries

In this section we give some preliminary results related to the variational structure of system (1.1). We will use the following function spaces.

- $L^s(\Omega)$, the Lebesgue space endowed with the norm $|u|_{L^s(\Omega)} = \big(\int_\Omega |u|^s\,dx\big)^{1/s}$ for $1 \leq s < \infty$.
- $W^{1,p}(\mathbb{R}^3)$, the Sobolev space with the norm $\|u\| = \Big(\int_{\mathbb{R}^3} |\nabla u|^p + |u|^p\,dx\Big)^{1/p}$, and $W^{1,p}_r(\mathbb{R}^3) = \big\{u \in W^{1,p}(\mathbb{R}^3) :\ u(x) = u(|x|)\big\}$.
- $D^{1,2}(\mathbb{R}^3)$, the completion of $C_0^\infty(\mathbb{R}^3)$ with the norm $\|u\|_D = \Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^{1/2}$. It is a Hilbert space with the inner product $\langle v, w\rangle = \int_{\mathbb{R}^3} \nabla v \nabla w\,dx$.

It follows from the classical Sobolev embedding theorems that $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^\ell(\mathbb{R}^3)$ are continuous for all $\ell \in [p, p^*]$ and $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is continuous. Restricted to the radial case, it holds that the embedding $W^{1,p}_r(\mathbb{R}^3) \hookrightarrow L^\ell(\mathbb{R}^3)$ is compact for any $p < \ell < p^*$. See [19, Theorem II.1] or [23, Theorem 1].

We will use C to denote various positive constants. We will use the following elementary inequality (see [24, p240]) in later arguments: There exists $c_p > 0$ such that for all $\xi, \eta \in \mathbb{R}^3$, we have

$$\begin{gathered}
\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\big)_{\mathbb{R}^3} \geq c_p|\xi - \eta|^p \quad \text{for } p \geq 2,\\
\big(|\xi| + |\eta|\big)^{2-p}\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\big)_{\mathbb{R}^3} \geq c_p|\xi - \eta|^2 \quad \text{for } 1 < p < 2.
\end{gathered} \tag{2.1}$$

For each fixed $u \in W^{1,p}(\mathbb{R}^3)$, we define a linear functional $\mathcal{K} : D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ by

$$\mathcal{K}(v) = \int_{\mathbb{R}^3} u^2 v\,dx.$$

By the Hölder and Sobolev inequalities, we have

$$|\mathcal{K}(v)| \leq \Big(\int_{\mathbb{R}^3} |u|^{12/5}\,dx\Big)^{5/6}\Big(\int_{\mathbb{R}^3} |v|^6\,dx\Big)^{1/6} \leq C\|u\|^2\|v\|_D.$$

Therefore $\mathcal{K}$ is continuous on $D^{1,2}(\mathbb{R}^3)$. By the Lax-Milgram theorem, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ satisfying the equation $-\Delta\phi_u = u^2$. According to [18, Theorem 6.21], ϕ_u has the explicit expression $\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy \geq 0$.

It defines a mapping $u \mapsto \phi_u$ from $W^{1,p}(\mathbb{R}^3)$ to $D^{1,2}(\mathbb{R}^3)$ such that $\phi_u \geq 0$ solves uniquely the Poisson equation $-\Delta\phi = u^2$ for $u \in W^{1,p}(\mathbb{R}^3)$.

Proposition 2.1 ([11, Proposition 2.1]). *The mapping $u \mapsto \phi_u$ enjoys the following properties.*

(i) $\|\phi_u\|_D \le A\|u\|^2$ *for all* $u \in W^{1,p}(\mathbb{R}^3)$ *where* $A > 0$ *is a constant;*
(ii) *if* $u_n \rightharpoonup u$ *in* $W^{1,p}(\mathbb{R}^3)$*, then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$*;*
(iii) *if* $u \in W^{1,p}_r(\mathbb{R}^3)$ *then* $\phi_u \in D^{1,2}_r(\mathbb{R}^3) := \{\phi \in D^{1,2}(\mathbb{R}^3) : \phi(x) = \phi(|x|)\}$.

We note here that the third conclusion comes from a fact that the convolution of two radial functions is still radial.

Now we are ready to establish the variational framework of (1.1). For $h \in L^s(\mathbb{R}^3)$ with $(p^*)' \le s \le p'$, arguing as in [5, 6], by Proposition 2.1 and the implicit function theorem, the functional

$$I_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^3}(|\nabla u|^p + |u|^p)\,dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2\,dx - \frac{1}{q}\int_{\mathbb{R}^3}|u|^q\,dx - \int_{\mathbb{R}^3}h(x)u\,dx$$

is a well-defined C^1 functional on $W^{1,p}(\mathbb{R}^3)$ with derivative

$$\begin{aligned}\langle I'_\lambda(u), v\rangle &= \int_{\mathbb{R}^3}(|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv)\,dx + \lambda\int_{\mathbb{R}^3}\phi_u uv\,dx \\ &\quad - \int_{\mathbb{R}^3}|u|^{q-2}uv\,dx - \int_{\mathbb{R}^3}h(x)v\,dx, \quad \forall u, v \in W^{1,p}(\mathbb{R}^3).\end{aligned}$$

Furthermore, $u \in W^{1,p}(\mathbb{R}^3)$ is a critical point of I_λ if and only if the couple $(u, \phi_u) \in W^{1,p}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of the system (1.1). Then we will prove Theorems 1.1 and 1.2 by looking for critical points of I_λ.

The following result is crucial and can be proved by applying some ideas from Boccardo and Murat [7]. We include the proof for completeness.

Lemma 2.2. *Let* $\{u_n\} \subset W^{1,p}(\mathbb{R}^3)$ *be bounded and satisfy* $I'_\lambda(u_n) \to 0$ *as* $n \to \infty$*. Then, up to a subsequence, there exists* $u \in W^{1,p}(\mathbb{R}^3)$ *such that* $\nabla u_n(x) \to \nabla u(x)$ *a.e. in* $\mathbb{R}^3$*.*

Proof. Since $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$, up to a subsequence, there exists $u \in W^{1,p}(\mathbb{R}^3)$ such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } W^{1,p}(\mathbb{R}^3), \\ u_n \to u \quad \text{in } L^\ell_{\rm loc}(\mathbb{R}^3),\ p \le \ell < p^*, \\ u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^3. \end{gathered} \tag{2.2}$$

We will prove that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \mathbb{R}^3. \tag{2.3}$$

Let $\upsilon \in C_0^\infty(\mathbb{R}^3, [0,1])$ satisfy

$$\upsilon\big|_{B_R} = 1 \quad \text{and} \quad \operatorname{supt}\upsilon \subset B_{2R},$$

where $B_R = \{x \in \mathbb{R}^3 : |x| \le R\}$. Since $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^3)$, it follows that

$$(u_n - u)\upsilon \rightharpoonup 0 \quad \text{in } W^{1,p}(\mathbb{R}^3). \tag{2.4}$$

Then, by (2.2) and the Hölder inequality, as $n \to \infty$,

$$\begin{gathered} \int_{\mathbb{R}^3}(|u_n|^{\ell-2}u_n - |u|^{\ell-2}u)\left[(u_n - u)\upsilon\right]\,dx = o(1), \\ \int_{\mathbb{R}^3}\left[(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla\upsilon\right](u_n - u)\,dx = o(1). \end{gathered} \tag{2.5}$$

Using the Hölder and Sobolev inequalities, we deduce by Proposition 2.1(i) and (2.2) that

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)\upsilon \, dx \\
&\le |\phi_{u_n}|_{L^6(B_{2R})} |u_n(u_n-u)\upsilon|_{L^{6/5}(B_{2R})} + |\phi_u|_{L^6(B_{2R})} |u(u_n-u)\upsilon|_{L^{6/5}(B_{2R})} \\
&\le C\|\phi_{u_n}\|_D |u_n(u_n-u)\upsilon|_{L^{6/5}(B_{2R})} + C\|\phi_u\|_D |u(u_n-u)\upsilon|_{L^{6/5}(B_{2R})} \\
&\le C\|u_n\|^2 |u_n(u_n-u)\upsilon|_{L^{6/5}(B_{2R})} + C\|u\|^2 |u(u_n-u)\upsilon|_{L^{6/5}(B_{2R})} \\
&\le C\left(\|u_n\|^2 |u_n|_{L^{12/5}(B_{2R})} + \|u\|^2 |u|_{L^{12/5}(B_{2R})}\right) |u_n - u|_{L^{12/5}(B_{2R})} \\
&= o(1).
\end{aligned} \tag{2.6}$$

By (2.4) and $I'_\lambda(u_n) \to 0$ in $[W^{1,p}(\mathbb{R}^3)]^*$, we have that as $n \to \infty$,

$$\langle I'_\lambda(u_n) - I'_\lambda(u), (u_n - u)\upsilon \rangle = o(1). \tag{2.7}$$

It follows from (2.5)–(2.7) that as $n \to \infty$,

$$\int_{\mathbb{R}^3} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\upsilon \, dx = o(1). \tag{2.8}$$

Set $e_n := (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u)_{\mathbb{R}^3}$. Then, as $n \to \infty$,

$$\int_{B_R} e_n \, dx = o(1). \tag{2.9}$$

By (2.1) and (2.9), for $2 \le p < 12/5$, we have

$$C\int_{B_R} |\nabla u_n - \nabla u|^p \, dx \le \int_{B_R} e_n \, dx = o(1), \tag{2.10}$$

and for $4/3 < p < 2$,

$$\begin{aligned}
C\int_{B_R} |\nabla u_n - \nabla u|^p \, dx &\le \int_{B_R} e_n^{p/2} (|\nabla u_n| + |\nabla u|)^{\frac{p(2-p)}{2}} \, dx \\
&\le \Big(\int_{B_R} e_n \, dx\Big)^{p/2} \Big(\int_{B_R} (|\nabla u_n| + |\nabla u|)^p \, dx\Big)^{\frac{2-p}{2}} \\
&\le C\Big(\int_{B_R} e_n \, dx\Big)^{p/2}.
\end{aligned} \tag{2.11}$$

It follows from (2.9)–(2.11) that

$$\lim_{n\to\infty} \int_{B_R} |\nabla u_n - \nabla u|^p \, dx = 0.$$

Up to a subsequence, we have $\nabla u_n(x) \to \nabla u(x)$ a.e. in B_R. It follows from the arbitrariness of B_R that (2.3) holds. The proof is complete. $\square$

3. A solution with negative energy

In this section we find a solution of (1.1) with negative energy for $p < q < p^*$, and h satisfying (H1)(i) and small $|h|_{L^s(\mathbb{R}^3)}$.

Lemma 3.1. *Assume that $h \in L^s(\mathbb{R}^3)$ with $(p^*)' \le s \le p'$. Then there exist $\rho > 0$, $\Lambda > 0$ and $\alpha > 0$ such that $I_\lambda(u) \ge \alpha$ for $u \in W^{1,p}(\mathbb{R}^3)$ with $\|u\| = \rho$, $\lambda > 0$ and $|h|_{L^s(\mathbb{R}^3)} < \Lambda$.*

Proof. For $u \in W^{1,p}(\mathbb{R}^3)$ and $\lambda > 0$, since $\phi_u \geq 0$, it follows from Hölder and Sobolev inequalities that

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p}\|u\|^p - \frac{1}{q}|u|^q_{L^q(\mathbb{R}^3)} - |h|_{L^s(\mathbb{R}^3)}|u|_{L^{s'}(\mathbb{R}^3)} \\ &\geq \frac{1}{p}\|u\|^p - \frac{S_q^q}{q}\|u\|^q - S_{s'}|h|_{L^s(\mathbb{R}^3)}\|u\| \\ &= \|u\|\Big(\frac{1}{p}\|u\|^{p-1} - \frac{S_q^q}{q}\|u\|^{q-1} - S_{s'}|h|_{L^s(\mathbb{R}^3)}\Big), \end{aligned} \tag{3.1}$$

where S_ℓ denotes the embedding constant of $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^\ell(\mathbb{R}^3)$ for $p \leq \ell \leq p^*$. Since $q > p$, there exists a unique $\rho > 0$ such that the function $f(t) = \frac{1}{p}t^{p-1} - \frac{S_q^q}{q}t^{q-1}$ attains its unique maximum $f(\rho) = \max_{t\geq 0} f(t) > 0$. Take $\Lambda = f(\rho)/S_{s'}$ and $\alpha = \rho\big(f(\rho) - S_{s'}|h|_{L^s(\mathbb{R}^3)}\big)$. Then by (3.1) we have that when $|h|_{L^s(\mathbb{R}^3)} < \Lambda$, $I_\lambda(u) \geq \alpha$ for any $\|u\| = \rho$. □

Next we work on the Sobolev space $W_r^{1,p}(\mathbb{R}^3)$ of radial functions.

Lemma 3.2. *Assume that h satisfies* (H1)(i). *Then each bounded sequence $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^3)$ satisfying $I'_\lambda(u_n) \to 0$ has a strongly convergent subsequence.*

Proof. Let $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^3)$ be bounded. Going if necessary to a subsequence, there exists $u \in W_r^{1,p}(\mathbb{R}^3)$ such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } W_r^{1,p}(\mathbb{R}^3), \\ u_n \to u \quad \text{in } L^q(\mathbb{R}^3), \quad p < q < p^*, \\ u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^3. \end{gathered} \tag{3.2}$$

We will complete the proof by showing $u_n \to u$ in $W_r^{1,p}(\mathbb{R}^3)$. By $I'_\lambda(u_n) \to 0$ and (3.2) we obtain

$$\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \to 0, \quad \text{as } n \to \infty. \tag{3.3}$$

By Proposition 2.1, the boundedness of $\{u_n\}$, the Hölder inequality and (3.2), we obtain that, as $n \to \infty$,

$$\begin{gathered} \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)\,dx = o(1), \\ \int_{\mathbb{R}^3} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)\,dx = o(1). \end{gathered} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\int_{\mathbb{R}^3} e_n + (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,dx = o(1). \tag{3.5}$$

For $2 \leq p < 12/5$, by (2.1) we obtain

$$\begin{gathered} \int_{\mathbb{R}^3} e_n\,dx \geq C\int_{\mathbb{R}^3} |\nabla u_n - \nabla u|^p\,dx, \\ \int_{\mathbb{R}^3} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,dx \geq C\int_{\mathbb{R}^3} |u_n - u|^p\,dx. \end{gathered} \tag{3.6}$$

For $4/3 < p < 2$, from the boundedness of $\{u_n\}$ and the proof of (2.11), we obtain

$$\int_{\mathbb{R}^3} |\nabla(u_n - u)|^p\,dx \leq C\Big(\int_{\mathbb{R}^3} e_n\,dx\Big)^{p/2}, \tag{3.7}$$

$$\int_{\mathbb{R}^3} |u_n - u|^p \, dx \le C \left(\int_{\mathbb{R}^3} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \, dx \right)^{p/2}. \tag{3.8}$$

It follows from (3.5), (3.6)–(3.8) that $\|u_n - u\| \to 0$ as $n \to \infty$. □

Theorem 3.3. *Assume that* (H1)(i) *holds and $p < q < p^*$. Then I_λ has a critical point $u_* \in W_r^{1,p}(\mathbb{R}^3)$ with $I_\lambda(u_*) < 0$ for $\lambda > 0$ provided $|h|_{L^s(\mathbb{R}^3)} < \Lambda$, where Λ was given in Lemma 3.1.*

Proof. We first find a function $w \in W_r^{1,p}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} h(x)w(x)\,dx > 0$. It follows from $h \in L^s(\mathbb{R}^3)$ that $|h|^{s-2}h \in L^{s'}(\mathbb{R}^3)$. Then there exists a radial sequence $\{h_n\} \subset C_0^\infty(\mathbb{R}^3)$ such that $h_n \to |h|^{s-2}h$ strongly in $L^{s'}(\mathbb{R}^3)$ since $C_0^\infty(\mathbb{R}^3)$ is dense in $L^{s'}(\mathbb{R}^3)$ and h is radial. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\left|h_{n_0} - |h|^{s-2}h\right|_{L^{s'}(\mathbb{R}^3)} \le \frac{1}{2}|h|^{s-1}_{L^s(\mathbb{R}^3)}.$$

By Hölder's inequality, we conclude that

$$\int_{\mathbb{R}^3} h(x)h_{n_0}(x)\,dx \ge -|h|_{L^s(\mathbb{R}^3)} \left|h_{n_0} - |h|^{s-2}h\right|_{L^{s'}(\mathbb{R}^3)} + |h|^s_{L^s(\mathbb{R}^3)} > 0.$$

It is clear that $h_{n_0} \in W_r^{1,p}(\mathbb{R}^3)$. Taking $w(x) = h_{n_0}(x)$, we get $\int_{\mathbb{R}^3} h(x)w(x)\,dx > 0$.

Now for $t > 0$ small enough, we have

$$I_\lambda(tw) = \frac{t^p}{p}\|w\|^p + \frac{t^4}{4}\lambda \int_{\mathbb{R}^3} \phi_w w^2\,dx - \frac{t^q}{q}\int_{\mathbb{R}^3} |w|^q\,dx - t\int_{\mathbb{R}^3} hw\,dx < 0.$$

It follows that

$$c_* = \inf_{u \in \bar{B}_\rho} I_\lambda(u) < 0,$$

where $\bar{B}_\rho = \{u \in W_r^{1,p}(\mathbb{R}^3) : \|u\| \le \rho\}$ and ρ is given by Lemma 3.1. Applying the Ekeland variational principle [12], we obtain a sequence $\{u_n\} \subset \bar{B}_\rho$ satisfying

$$c_* \le I_\lambda(u_n) \le c_* + \frac{1}{n}, \tag{3.9}$$

$$I_\lambda(v) \ge I_\lambda(u_n) - \frac{1}{n}\|v - u_n\| \quad \text{for all } v \in \bar{B}_\rho. \tag{3.10}$$

It must be that $\|u_n\| < \rho$ for all $n \in \mathbb{N}$ large. Otherwise, we may assume that $\|u_n\| = \rho$, up to a subsequence. By Lemma 3.1, we see that $I_\lambda(u_n) \ge \alpha > 0$. Then there is a contradiction by taking the limit in (3.9) as $n \to \infty$. We can assume that $\|u_n\| < \rho$ for all $n \in \mathbb{N}$. Now we show that $I'_\lambda(u_n) \to 0$. For any $z \in W_r^{1,p}(\mathbb{R}^3)$ with $\|z\| = 1$, we choose sufficiently small $\delta > 0$ such that $\|u_n + tz\| < \rho$ for all $|t| < \delta$. By (3.10), we have

$$\frac{I_\lambda(u_n + tz) - I_\lambda(u_n)}{t} \ge -\frac{1}{n}.$$

Letting $t \to 0$, we obtain $\langle I'_\lambda(u_n), z\rangle \ge -1/n$. Similarly, replacing z with $-z$ in the above arguments, we obtain $\langle I'_\lambda(u_n), z\rangle \le 1/n$. Then, we deduce that, for any $z \in W_r^{1,p}(\mathbb{R}^3)$ with $\|z\| = 1$, $\langle I'_\lambda(u_n), z\rangle \to 0$ as $n \to \infty$. Thus $\{u_n\}$ is a bounded $(\mathrm{PS})_{c_*}$ sequence of I_λ. Finally, by Lemma 3.2, there exists $u_* \in W_r^{1,p}(\mathbb{R}^3)$ such that $I_\lambda(u_*) = c_* < 0$ and $I'_\lambda(u_*) = 0$. □

4. A solution with positive energy

In this section we find a solution of (1.1) with positive energy. In Subsection 4.1 we consider the case $\frac{6p}{p+2} < q < p^*$ and in Subsection 4.2 we consider the case $p < q \le \frac{6p}{p+2}$. We still work on $W_r^{1,p}(\mathbb{R}^3)$.

4.1. Case $\frac{6p}{p+2} < q < p^*$.

In this subsection we will prove the following theorem.

Theorem 4.1. *Assume that* (H1) *holds and* $\frac{6p}{p+2} < q < p^*$. *Then* I_λ *has a critical point* $u^* \in W_r^{1,p}(\mathbb{R}^3)$ *with* $I_\lambda(u^*) > 0$ *for* $\lambda > 0$ *provided* $|h|_{L^s(\mathbb{R}^3)} < \Lambda$, *where* Λ *was given in Lemma 3.1.*

Lemma 4.2. *Assume that* (H1)(i) *holds and* $\frac{6p}{p+2} < q < p^*$.

(i) *There exist* $\rho > 0$, $\Lambda > 0$ *and* $\alpha > 0$, *such that* $I_\lambda(u) \ge \alpha$ *for* $u \in W_r^{1,p}(\mathbb{R}^3)$ *with* $\|u\| = \rho$, $\lambda > 0$, *and* $|h|_{L^s(\mathbb{R}^3)} < \Lambda$.

(ii) *There exists* $\upsilon \in W_r^{1,p}(\mathbb{R}^3)\backslash\{0\}$ *such that* $\|\upsilon\| > \rho$ *and* $I_\lambda(\upsilon) < 0$.

Proof. Item (i) follows from the argument of the proof of Lemma 3.1.

(ii) Take any fixed $u \in W_r^{1,p}(\mathbb{R}^3)\backslash\{0\}$ and define $u_t(x) = t^{\frac{p+2}{4-p}} u(tx)$. Then we have

$$\begin{aligned} I_\lambda(u_t) = \frac{t^{\beta_1}}{p}\int_{\mathbb{R}^3} |\nabla u|^p\,dx + \frac{t^{\beta_2}}{p}\int_{\mathbb{R}^3} |u|^p\,dx + \frac{t^{\beta_1}}{4}\lambda\int_{\mathbb{R}^3} \phi_u u^2\,dx \\ - \frac{t^{\beta_3}}{q}\int_{\mathbb{R}^3} |u|^q\,dx - t^{\beta_4}\int_{\mathbb{R}^3} h\left(\frac{x}{t}\right) u\,dx, \end{aligned}$$

where

$$\begin{gathered} \beta_1 = \frac{9p-12}{4-p}, \quad \beta_2 = \frac{p^2+5p-12}{4-p}, \\ \beta_3 = \frac{(p+2)q-12+3p}{4-p}, \quad \beta_4 = \frac{4p-10}{4-p}. \end{gathered} \tag{4.1}$$

It follows from $4/3 < p < 12/5$ and $\frac{6p}{p+2} < q$ that $\beta_3 > \beta_1 > \beta_2$, $\beta_3 > 0$ and $\beta_4 < 0$. Therefore there exists $t_0 > 0$ such that $I_\lambda(u_{t_0}) < 0$. The conclusion (ii) follows by taking $\upsilon = u_{t_0}$. $\square$

Since $I_\lambda(0) = 0$, by Lemma 4.2, the functional I_λ satisfies the hypotheses of the mountain pass theorem [1] and a mountain pass level of I_λ can be defined as

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0, \tag{4.2}$$

where $\Gamma = \{\gamma \in C([0,1], W_r^{1,p}(\mathbb{R}^3)) : \gamma(0) = 0 \text{ and } I_\lambda(\gamma(1)) < 0\}$. We define an auxiliary functional $J_\lambda : W_r^{1,p}(\mathbb{R}^3) \to \mathbb{R}$ as follows with the numbers β_i given by (4.1):

$$\begin{aligned} J_\lambda(u) = \frac{\beta_1}{p}\int_{\mathbb{R}^3} |\nabla u|^p\,dx + \frac{\beta_2}{p}\int_{\mathbb{R}^3} |u|^p\,dx + \frac{\lambda\beta_1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \frac{\beta_3}{q}\int_{\mathbb{R}^3} |u|^q\,dx \\ - \beta_4\int_{\mathbb{R}^3} hu\,dx + \int_{\mathbb{R}^3} (x, \nabla h(x))u\,dx. \end{aligned}$$

Lemma 4.3. *Assume that* (H1) *holds and* $\frac{6p}{p+2} < q < p^*$. *There exists a bounded sequence* $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^3)$ *satisfying*

$$I_\lambda(u_n) \to c, \quad I'_\lambda(u_n) \to 0, \quad J_\lambda(u_n) \to 0.$$

Proof. We follow the idea in Jeanjean [14] and modified in [11]. Define the map

$$\Phi(\sigma, v)(x) = e^{\frac{p+2}{4-p}\sigma} v(e^\sigma x), \quad \sigma \in \mathbb{R},\ v \in W_r^{1,p}(\mathbb{R}^3).$$

A simple computation shows that

$$\begin{aligned} I_\lambda(\Phi(\sigma, v)) = \frac{e^{\beta_1\sigma}}{p}\int_{\mathbb{R}^3} |\nabla v|^p\,dx + \frac{e^{\beta_2\sigma}}{p}\int_{\mathbb{R}^3} |v|^p\,dx + \frac{\lambda e^{\beta_1\sigma}}{4}\int_{\mathbb{R}^3}\phi_v v^2\,dx \\ - \frac{e^{\beta_3\sigma}}{q}\int_{\mathbb{R}^3}|v|^q\,dx - e^{\beta_4\sigma}\int_{\mathbb{R}^3} h\left(\frac{x}{e^\sigma}\right) v\,dx, \end{aligned}$$

and $I_\lambda(\Phi(0,0)) = 0$. It is standard to verify that $I_\lambda \circ \Phi$ is continuously Fréchet-differentiable on $\mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)$. We set

$$\begin{gathered} \bar\Gamma = \left\{\bar\gamma \in C([0,1], \mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)) : \bar\gamma(0) = (0,0) \text{ and } (I_\lambda \circ \Phi)(\bar\gamma(1)) < 0\right\}, \\ \bar c = \inf_{\bar\gamma\in\bar\Gamma}\sup_{t\in[0,1]} (I_\lambda \circ \Phi)(\bar\gamma(t)). \end{gathered} \tag{4.3}$$

It can be proved that $\Gamma = \{\Phi \circ \bar\gamma : \bar\gamma \in \bar\Gamma\}$. It follows that $c = \bar c$. Let $\bar\gamma = (0, \gamma)$. For each $\epsilon \in (0, \frac{c}{2})$, there exists $\gamma \in \Gamma$ such that

$$\sup(I_\lambda \circ \Phi)(0, \gamma) \le c + \epsilon.$$

Then, by [28, Theorem 2.8], there exists $(\sigma, v) \in \mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)$ such that

(a) $c - 2\epsilon \le (I_\lambda \circ \Phi)(\sigma, v) \le c + 2\epsilon$,
(b) $\operatorname{dist}\{(\sigma, v), (0, \gamma)\} \le 2\sqrt{\epsilon}$, where $\operatorname{dist}\{(\sigma, v), (\varsigma, w)\} = (|\sigma - \varsigma|^2 + \|v - w\|^2)^{1/2}$,
(c) $(I_\lambda \circ \Phi)'(\sigma, v) \to 0$ in $[\mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)]^*$.

Therefore, there exists a sequence $\{(\sigma_n, v_n)\} \subset \mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)$ such that as $n \to \infty$,

$$\sigma_n \to 0, \quad (I_\lambda \circ \Phi)(\sigma_n, v_n) \to c, \quad (I_\lambda \circ \Phi)'(\sigma_n, v_n) \to 0.$$

For every $(\zeta, w) \in \mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)$, it holds

$$\langle (I_\lambda \circ \Phi)'(\sigma_n, v_n), (\zeta, w)\rangle = \langle I'_\lambda(\Phi(\sigma_n, v_n)), \Phi(\sigma_n, w)\rangle + J_\lambda(\Phi(\sigma_n, v_n))\zeta.$$

Taking $u_n = \Phi(\sigma_n, v_n)$, we have

$$I_\lambda(u_n) \to c, \quad I'_\lambda(u_n) \to 0, \quad J_\lambda(u_n) \to 0. \tag{4.4}$$

Now we prove that $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$. By (4.4), for n large enough,

$$\begin{aligned} c + 1 &\ge I_\lambda(u_n) - \frac{1}{\beta_3} J_\lambda(u_n) \\ &= \frac{\beta_3 - \beta_1}{p\beta_3}\int_{\mathbb{R}^3}|\nabla u_n|^p\,dx + \frac{\beta_3 - \beta_2}{p\beta_3}\int_{\mathbb{R}^3}|u_n|^p\,dx + \frac{\lambda(\beta_3 - \beta_1)}{4\beta_3}\int_{\mathbb{R}^3}\phi_{u_n} u_n^2\,dx \\ &\quad - \frac{\beta_3 - \beta_4}{\beta_3}\int_{\mathbb{R}^3} h u_n\,dx - \frac{1}{\beta_3}\int_{\mathbb{R}^3}(x, \nabla h) u_n\,dx \\ &\ge \frac{\beta_3 - \beta_1}{p\beta_3}\int_{\mathbb{R}^3}|\nabla u_n|^p\,dx + \frac{\beta_3 - \beta_2}{p\beta_3}\int_{\mathbb{R}^3}|u_n|^p\,dx \\ &\quad - \frac{\beta_3 - \beta_4}{\beta_3}\int_{\mathbb{R}^3} h u_n\,dx - \frac{1}{\beta_3}\int_{\mathbb{R}^3}(x, \nabla h) u_n\,dx. \end{aligned}$$

It follows that

$$c + 1 + \frac{\beta_3 - \beta_4}{\beta_3}\int_{\mathbb{R}^3} h u_n\,dx + \frac{1}{\beta_3}\int_{\mathbb{R}^3}(x, \nabla h) u_n\,dx \ge \frac{\beta_3 - \beta_1}{p\beta_3}\|u_n\|^p. \tag{4.5}$$

It is easy to see that $\int_{\mathbb{R}^3} hu_n\,dx \le C\|u_n\|$. We deduce from (H1), the Hölder and Sobolev inequalities that

$$
\Big|\int_{\mathbb{R}^3}(x,\nabla h)u_n\,dx\Big| \le \Big(\int_{\mathbb{R}^3}|(x,\nabla h)|^s\,dx\Big)^{1/s}\Big(\int_{\mathbb{R}^3}|u_n|^{s'}\,dx\Big)^{1/s'} \le C\|u_n\|.
$$

Therefore by (4.5) that $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$. $\square$

Proof of Theorem 4.1. It follows from Lemmas 4.2, 4.3, and 3.2. $\square$

4.2. Case $p < q \le \frac{6p}{p+2}$.

Theorem 4.4. *Assume that* (H1)(i) *holds and $p < q \le \frac{6p}{p+2}$. Then there exists $\lambda_* > 0$ such that I_λ has a critical point $u^* \in W_r^{1,p}(\mathbb{R}^3)$ with $I_\lambda(u^*) > 0$ for each $\lambda \in (0,\lambda_*)$ provided $|h|_{L^s(\mathbb{R}^3)} < \Lambda$, where Λ is given in Lemma 3.1.*

We adopt some techniques from [15] to do the proof. We introduce a smooth function $\chi \in C^\infty(\mathbb{R}_+,[0,1])$ which satisfies

$$
\chi(t) = \begin{cases} 1 & \text{for } t \in [0,\frac{1}{2}],\\ 0 & \text{for } t \ge 1,\\ \in [0,1] & \text{for } t \in (\frac{1}{2},1), \end{cases}
\qquad |\chi'|_\infty \le 4.
$$

We define a penalized functional $I_{\lambda,M} : W_r^{1,p}(\mathbb{R}^3) \to \mathbb{R}$ as

$$
\begin{aligned}
I_{\lambda,M}(u) = \frac{1}{p}\int_{\mathbb{R}^3}(|\nabla u|^p + |u|^p)\,dx + \frac{\lambda}{4}L_M(u)\int_{\mathbb{R}^3}\phi_u u^2\,dx \\
- \frac{1}{q}\int_{\mathbb{R}^3}|u|^q\,dx - \int_{\mathbb{R}^3} hu\,dx,
\end{aligned}
\tag{4.6}
$$

where $M > 0$ and $L_M(u) = \chi\big(\frac{\|u\|^p}{M^p}\big)$. It is standard to prove that $I_{\lambda,M}$ belongs to C^1, and for all $u, v \in W_r^{1,p}(\mathbb{R}^3)$,

$$
\begin{aligned}
\langle I'_{\lambda,M}(u), v\rangle = (1 + a_{\lambda,M}(u))\int_{\mathbb{R}^3}(|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv)\,dx \\
+ \lambda L_M(u)\int_{\mathbb{R}^3}\phi_u uv\,dx - \int_{\mathbb{R}^3}|u|^{q-2}uv\,dx - \int_{\mathbb{R}^3} hv\,dx,
\end{aligned}
\tag{4.7}
$$

where

$$
a_{\lambda,M}(u) = \frac{p\lambda}{4M^p}\chi'\Big(\frac{\|u\|^p}{M^p}\Big)\int_{\mathbb{R}^3}\phi_u u^2\,dx. \tag{4.8}
$$

From the definition one sees that if u is a critical point of $I_{\lambda,M}$ and $\|u\| \le M/2$, then u is a critical point of I_λ. We first verify that the penalized functional $I_{\lambda,M}$ possesses a mountain pass geometry for each $M > 0$.

Lemma 4.5. *Assume that* (H1)(i) *holds and $p < q \le \frac{6p}{p+2}$. For every $M > 0$,*

- (i) *there exist $\rho > 0$, $\Lambda > 0$ and $\alpha > 0$, such that $I_{\lambda,M}(u) \ge \alpha$ for $u \in W_r^{1,p}(\mathbb{R}^3)$ with $\|u\| = \rho$, $|h|_{L^s(\mathbb{R}^3)} < \Lambda$ and $\lambda > 0$.*
- (ii) *there exists $\omega \in W_r^{1,p}(\mathbb{R}^3)\backslash\{0\}$ such that $\|\omega\| > \rho$ and $I_{\lambda,M}(\omega) < 0$.*

Proof. Item (i) follows from an argument similar to the one in the proof of Lemma 3.1.

(ii) Arguing as in the proof of Theorem 3.3, we can choose a function $\omega_1 \in W_r^{1,p}(\mathbb{R}^3)$ such that $\|\omega_1\| = 1$ and $\int_{\mathbb{R}^3} h(x)\omega_1(x)\,dx > 0$. For each $M > 0$ and $t \geq M$, it follows from the definition of χ that $L_M(t\omega_1) = 0$. Thus

$$I_{\lambda,M}(t\omega_1) = \frac{1}{p}t^p - \frac{1}{q}t^q \int_{\mathbb{R}^3} |\omega_1|^q\,dx - t\int_{\mathbb{R}^3} h(x)\omega_1\,dx.$$

Since $p < q$, we take $\omega = t_M\omega_1$ and $t_M > M$ large, so that $\|\omega\| > \rho$ and $I_{\lambda,M}(\omega) < 0$. This completes the proof. $\square$

Lemma 4.6. *For $M > 0$ and $\lambda > 0$ fixed, each bounded sequence $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^3)$ satisfying $I'_{\lambda,M}(u_n) \to 0$ admits a strongly convergent subsequence.*

Proof. Let $\{u_n\}$ be bounded in $W_r^{1,p}(\mathbb{R}^3)$. Up to a subsequence, there exists $u \in W_r^{1,p}(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $W_r^{1,p}(\mathbb{R}^3)$, $u_n \to u$ in $L^q(\mathbb{R}^3)$ for all $p < q < p^*$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$. Therefore

$$\langle I'_{\lambda,M}(u_n) - I'_{\lambda,M}(u), u_n - u\rangle \to 0, \quad \text{as } n \to \infty. \tag{4.9}$$

Similar to (3.4), we conclude that, as $n \to \infty$,

$$\begin{gathered}\int_{\mathbb{R}^3} \phi_{u_n} u_n(u_n - u)\,dx = o(1), \quad \int_{\mathbb{R}^3} \phi_u u(u_n - u)\,dx = o(1), \\ \int_{\mathbb{R}^3} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)\,dx = o(1).\end{gathered} \tag{4.10}$$

We set

$$[u, v] = \int_{\mathbb{R}^3} (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)\,dx.$$

From (4.7), (4.9), and (4.10), a direct computation shows that, as $n \to \infty$,

$$\begin{aligned}&(1 + a_{\lambda,M}(u_n))\,([u_n, u_n - u] - [u, u_n - u]) \\ &+ (a_{\lambda,M}(u_n) - a_{\lambda,M}(u))[u, u_n - u] = o(1).\end{aligned} \tag{4.11}$$

By Proposition 2.1(i), we have that for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx = -\int_{\mathbb{R}^3} \phi_{u_n} \Delta\phi_{u_n}\,dx = \|\phi_{u_n}\|_D^2 \leq A^2\|u_n\|^4. \tag{4.12}$$

Notice that if $\|u_n\| \geq M$ then $\chi'\big(\frac{\|u_n\|^p}{M^p}\big) = 0$. It follows from (4.8) and (4.12) that

$$|a_{\lambda,M}(u_n)| \leq \frac{p\lambda}{4M^p}\big|\chi'\big(\frac{\|u_n\|^p}{M^p}\big)\big|\,\big|\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx\big| \leq p\lambda A^2 M^{4-p}. \tag{4.13}$$

It can be shown in a same way that $|a_{\lambda,M}(u)|$ is bounded. By Lemma 2.2, we have that $\nabla u_n(x) \to \nabla u(x)$ a.e. in $\mathbb{R}^3$. Combing with $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$, we deduce by [29, Proposition 5.4.7] that

$$[u, u_n - u] = o(1). \tag{4.14}$$

It follows from (4.11) and (4.14) that

$$[u_n, u_n - u] - [u, u_n - u] = o(1). \tag{4.15}$$

Arguing as in the proof of Lemma 3.2 we obtain that $\|u_n - u\| \to 0$ as $n \to \infty$. $\square$

By Lemma 4.5, we can define the following mountain pass level of $I_{\lambda,M}$ for each $M>0$,

$$c_M = \inf_{\gamma\in\Gamma_M}\sup_{t\in[0,1]} I_{\lambda,M}(\gamma(t)) > 0,$$

where $\Gamma_M := \big\{\gamma\in C([0,1],W_r^{1,p}(\mathbb{R}^3)) : \gamma(0)=0, I_{\lambda,M}(\gamma(1))<0\big\}$. Then, by the mountain pass theorem [1], there exists $\{u_n\}\subset W_r^{1,p}(\mathbb{R}^3)$ such that

$$I_{\lambda,M}(u_n)\to c_M,\quad I'_{\lambda,M}(u_n)\to 0 \text{ in } [W_r^{1,p}(\mathbb{R}^3)]^*. \tag{4.16}$$

Next we prove that $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$ for large M and small λ.

Lemma 4.7. *There exist $M>0$ and $\lambda_*>0$ such that for all $\lambda\in(0,\lambda_*)$, the sequence $\{u_n\}$ given by* (4.16) *satisfies*

$$\|u_n\|\le \frac{M}{2}. \tag{4.17}$$

Proof. First of all, from (4.6), (4.7) and (4.16) we have

$$\begin{aligned}
c_M+1+\|u_n\| &\ge I_{\lambda,M}(u_n)-\frac{1}{q}\langle I'_{\lambda,M}(u_n),u_n\rangle\\
&= \big(\frac1p-\frac1q\big)\|u_n\|^p+\big(\frac{\lambda}{4}-\frac{\lambda}{q}\big)L_M(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx\\
&\quad -\frac{a_{\lambda,M}(u_n)}{q}\|u_n\|^p-\frac{q-1}{q}\int_{\mathbb{R}^3}h(x)u_n\,dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big(\frac1p-\frac1q\big)\|u_n\|^p &\le c_M+1+\|u_n\|+\big(\frac{\lambda}{q}-\frac{\lambda}{4}\big)L_M(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx\\
&\quad +\frac{a_{\lambda,M}(u_n)}{q}\|u_n\|^p+\frac{q-1}{q}\int_{\mathbb{R}^3}h(x)u_n\,dx.
\end{aligned}\tag{4.18}$$

We claim that $\{u_n\}$ is bounded. Indeed, by definition, when $\|u_n\|\ge M$, $L_M(u_n)=0$, $\chi'\big(\frac{\|u_n\|^p}{M^p}\big)=0$ and so (4.18) reads

$$\big(\frac1p-\frac1q\big)\|u_n\|^p\le c_M+1+\|u_n\|+\frac{q-1}{q}\int_{\mathbb{R}^3}h(x)u_n\,dx.$$

Thus $\{u_n\}$ is bounded. By using (4.12), (4.13), and Hölder's inequality,

$$\big(\frac{\lambda}{q}-\frac{\lambda}{4}\big)L_M(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx\le\big|\frac{\lambda}{q}-\frac{\lambda}{4}\big|L_M(u_n)A^2\|u_n\|^4\le\lambda A^2M^4, \tag{4.19}$$

$$\frac{a_{\lambda,M}(u_n)}{q}\|u_n\|^p\le|a_{\lambda,M}(u_n)|\|u_n\|^p\le p\lambda A^2M^{4-p}M^p=p\lambda A^2M^4, \tag{4.20}$$

$$\begin{aligned}
\frac{q-1}{q}\int_{\mathbb{R}^3}h(x)u_n\,dx &\le\frac{q-1}{q}\int_{\mathbb{R}^3}|h(x)u_n|\,dx\\
&\le|h|_{L^s(\mathbb{R}^3)}|u_n|_{L^{s'}(\mathbb{R}^3)}\le C\Lambda\|u_n\|.
\end{aligned}\tag{4.21}$$

Let ω_1 be the function taken in the proof of Lemma 4.5(ii). By (4.6), we have

$$I_{\lambda,M}(M\omega_1)\le\frac{M^p}{p}-\frac{M^q}{q}|\omega_1|_q^q.$$

Then there exists $M_1 > 0$ such that $I_{\lambda,M}(M\omega_1) < 0$ for all $M \geq M_1$. Thus

$$\begin{aligned} c_M &\leq \max_{t\in[0,1]} I_{\lambda,M}(tM\omega_1) \leq \max_{t\in[0,1]} \Big\{\frac{1}{p}(Mt)^p - \frac{1}{q}(Mt)^q|\omega_1|_q^q\Big\} \\ &\quad + \max_{t\in[0,1]} \frac{\lambda}{4}(tM)^4 L_M(tM\omega_1)\int_{\mathbb{R}^3}\phi_{\omega_1}\omega_1^2\,dx \\ &\leq C + \lambda A^2 M^4. \end{aligned} \tag{4.22}$$

It follows from (4.18)–(4.22) that, for all $M \geq M_1$,

$$\Big(\frac{1}{p} - \frac{1}{q}\Big)\|u_n\|^p \leq C + 1 + (p+2)\lambda A^2 M^4 + (1 + C\Lambda)\|u_n\|. \tag{4.23}$$

Take $\lambda_* = (A^2M^4)^{-1}$. Then it follows from (4.23) that (4.17) holds for any $M \geq M_1$ and $\lambda \in (0, \lambda_*)$. The proof is complete. $\square$

Proof of Theorem 4.4. Combining Lemmas 4.5–4.7 and the mountain pass theorem, for $M > 0$ large enough and $\lambda > 0$ small, we can find a critical point u^* for $I_{\lambda,M}$ at the level $c_M > 0$ with $\|u^*\| \leq \frac{M}{2}$. Thus u^* is a critical point for I_λ with $I_\lambda(u^*) = c_M > 0$. $\square$

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