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EXISTENCE OF NONTRIVIAL SOLUTIONS FOR A PERTURBATION OF CHOQUARD EQUATION WITH HARDY-LITTLEWOOD-SOBOLEV UPPER CRITICAL EXPONENT

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Abstract. In this article, we consider the problem

$$-\Delta u = \Big(\int_{\mathbb{R}^N} \frac{|u|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u|^{2_\mu^*-2}u + f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $N \geqslant 3$, $\mu \in (0,N)$ and $2_\mu^* = \frac{2N-\mu}{N-2}$. Under suitable assumptions on $f(x,u)$, we establish the existence and non-existence of nontrivial solutions via the variational method.

1. Introduction

In this article, we consider the problem

$$-\Delta u = \Big(\int_{\mathbb{R}^N} \frac{|u|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u|^{2_\mu^*-2}u + f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geqslant 3$, $\mu \in (0,N)$, $2_\mu^* = \frac{2N-\mu}{N-2}$ and $f(x,u)$ is a sign-changing nonlinearity satisfying certain assumptions. Equation (1.1) is closely related to the nonlinear Choquard equation as follows:

$$-\Delta u + V(x)u = (|x|^\mu * |u|^p)\,|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $\frac{2N-\mu}{N} \leqslant p \leqslant \frac{2N-\mu}{N-2}$. For $p=2$ and $\mu=1$, the equation (1.2) goes back to the description of the quantum theory of a polaron at rest by Pekar in 1954 [20] and the modeling of an electron trapped in its own hole in 1976 in the work of Choquard, as a certain approximation to Hartree-Fock theory of one-component plasma [21]. For $p = \frac{2N-1}{N-2}$ and $\mu = 1$, by using the Green function, it is obvious that equation (1.2) can be regarded as a generalized version of Schrödinger-Newton equation

$$\begin{gathered} -\Delta u + V(x)u = |u|^{\frac{N+1}{N-2}}\phi \quad \text{in } \mathbb{R}^N,\\ -\Delta\phi = |u|^{\frac{N+1}{N-2}} \quad \text{in } \mathbb{R}^N. \end{gathered}$$

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The existence and qualitative properties of solutions of Choquard type equations (1.2) have been widely studied in the previous decades (see [18]). Moroz and Van Schaftingen [17] considered (1.2) with lower critical exponent $\frac{2N-\mu}{N}$ if the potential $1-V(x)$ should not decay to zero at infinity faster than the inverse of $|x|^2$. In [1], the authors studied (1.2) with critical growth in the sense of Trudinger-Moser inequality and studied the existence and concentration of the ground states.

In 2016, Gao and Yang [9] firstly investigated the critical Choquard equation

$$-\Delta u=\Big(\int_{\mathbb{R}^N}\frac{|u|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u|^{2_\mu^*-2}u+\lambda u\quad\text{in }\Omega,\tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^N$, with lipschitz boundary, $N\geqslant 3$, $\mu\in(0,N)$ and $\lambda>0$. By using variational methods, they established the existence, multiplicity and nonexistence of nontrivial solutions to equation (1.3). In equation (1.3), λu is a linear perturbed term.

In [16], the authors studied the following critical Choquard equation

$$\begin{gathered}-\Delta u=\Big(\int_{\mathbb{R}^N}\frac{|u|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u|^{2_\mu^*-2}u+\lambda u^{-q}\text{ in }\Omega,\\ u>0\quad\text{in }\Omega,\quad u=0\text{ on }\partial\Omega,\end{gathered}\tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N\geqslant 3)$, $\mu\in(0,N)$, $0<q<1$ and $\lambda>0$. By using variational methods and the Nehari manifold, they established the existence and multiplicity of nontrivial solutions to (1.4). In equation (1.4), λu^{-q} is a singular perturbed term.

In [11], the authors studied the critical Choquard equation

$$-\Delta u=\Big(\int_{\mathbb{R}^N}\frac{|u|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u|^{2_\mu^*-2}u+\lambda f(u)\quad\text{in }\Omega,\tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N\geqslant 3$ and $\mu\in(0,N)$. By using variational methods, they established the nonexistence, existence and multiplicity of nontrivial solutions to equation (1.5) with different kinds of perturbed terms.

Very recently, Alves, Gao, Squassina and Yang [2] studied the singularly perturbed critical Choquard equation

$$-\varepsilon^2\Delta u+V(x)u=\varepsilon^{\mu-3}\Big(\int_{\mathbb{R}^N}\frac{Q(y)G(u(y))}{|x-y|^\mu}\,\mathrm{d}y\Big)Q(x)g(u)\quad\text{in }\mathbb{R}^3,$$

where $0<\mu<3$, ε is a positive parameter, V,Q are two continuous real function on $\mathbb{R}^3$ and G is the primitive of g which is of critical growth due to the Hardy-Littlewood-Sobolev inequality. Under suitable assumptions on g, they first establish the existence of ground states for the critical Choquard equation with constant coefficient. They also establish existence and multiplicity of semi-classical solutions and characterize the concentration behavior by variational methods.

Inspired by [9, 11, 16], we are interested in the problem that how the sign-changing Hardy term or the sign-changing superlinear nonlocal term will effect the existence and nonexistence of solutions for the equation (1.1). The main difference between equation (1.1) and equations (1.3), (1.4) and (1.5) are not only the working domain $\mathbb{R}^N$ but also the sign-changing perturbed term $f(x,u)$.

In this paper, we study problem (1.1) with two kinds of perturbation. Our results are divided into two classes:

Perturbation with sign-changing Hardy term. For problem (1.1) with a sign-changing Hardy term $f(x,u) = g(x)\frac{u}{|x|^2}$,

$$-\Delta u = \Big(\int_{\mathbb{R}^N} \frac{|u|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u|^{2^*_\mu-2}u + g(x)\frac{u}{|x|^2} \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

We suppose that g satisfies the following hypotheses:

(A7) $g \in C(\mathbb{R}^N)$, $g_{\max}$ and $g_{\min}$ are well-defined, where $g_{\max} := \max_{x\in\mathbb{R}^N} g(x)$ and $g_{\min} := \min_{x\in\mathbb{R}^N} g(x)$;
(A8) the sets $\Omega_1 := \{x \in \mathbb{R}^N | g(x) > 0\}$ and $\Omega_2 := \{x \in \mathbb{R}^N | g(x) < 0\}$ have finite positive Lebesgue measure;
(A9) there exist r_ε and r_g such that $\Omega_1 \cup \Omega_2 \subset \overline{B(0,r_g)} \setminus B(0,r_\varepsilon)$, and $g(x) = 0$ in $\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)$, where $0 < r_\varepsilon < r_g < \infty$;
(A10) $\int_{\Omega_1} g(x)\,\mathrm{d}x > 2\big(\frac{r_g}{r_\varepsilon}\big)^{4N} \int_{\Omega_2}(-g(x))\,\mathrm{d}x$;
(A11) $g_{\max} \in (0, \frac{(N-2)^2}{4})$;
(A12) $|x_1 - x_2| \geqslant 2r_\varepsilon$ for any $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$.

Firstly, we firstly present a nontrivial example. Let $\widetilde{19} := (19,0,0)$ and $-\widetilde{19} := (-19,0,0)$, and $\widetilde{19}, -\widetilde{19} \in \mathbb{R}^3$. Then

$$g(x) = \begin{cases} \frac{1}{10}e^{-|x-\widetilde{19}|^2} - \frac{1}{10}e^{-1} & \text{in } B(\widetilde{19},1), \\ \frac{1}{10}e^{-1} - \frac{1}{10}e^{-10^4|x-\widetilde{19}|^2} & \text{in } B(-\widetilde{19},0.01), \\ 0 & \text{otherwise}. \end{cases}$$

The function $g(x)$ satisfies hypotheses (A7)–(A12).

Theorem 1.1. *Let $N \geq 3$ and $\mu \in (0,N)$, then* (1.6) *has no weak solution when $g(x)$ is a differential functional and $(x \cdot \nabla g(x))$ has a fixed sign.*

Theorem 1.2. *Assume that* (A7)–(A11) *hold. Let $N \geqslant 3$ and $\mu \in (0,N)$, then* (1.6) *has a ground state solution.*

Perturbation with a sign-changing superlinear nonlocal term. We are interested in problem (1.1) with a sign-changing superlinear nonlocal term

$$\begin{aligned} -\Delta u &= \Big(\int_{\mathbb{R}^N} \frac{|u|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u|^{2^*_\mu-2}u \\ &\quad + \Big(\int_{\mathbb{R}^N} \frac{g(y)|u|^p}{|x-y|^\mu}\,\mathrm{d}y\Big)g(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{1.7}$$

Theorem 1.3. *Assume that* (A7)–(A10), (A12) *hold. Let $N \geqslant 3$, $\mu \in (0,N)$ and $p \in (\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2})$, then problem* (1.7) *has a nontrivial solution.*

We need to point out the main features of problem (1.1) are three-fold: (1) Because of the Hardy-Littlewood-Sobolev upper critical term, it is difficult to establish the Pohozaev type of identity on entire space; (2) Since the sign-changing perturbed term, it is difficult to estimate the Mountain-Pass level c; (3) The loss of compactness due to the Hardy-Littlewood-Sobolev upper critical exponent which makes it difficult to verify the (PS) condition.

We refer the readers to [5, 7, 8, 19, 22] for equations involving different kinds of sign-changing perturbed term, the difference between the present paper and previous papers not only the assumptions on perturbed term but also the method of estimate the Mountain-Pass level c.

The extremal function of best constant plays a key role in estimating the Mountain-Pass level c. In previous papers, they estimate the Mountain-Pass level c by σ small enough or large enough (where σ defined in (2.2)). In present paper, we estimate the Mountain-Pass level c by $\sigma \in [r_\varepsilon, r_g]$ (where r_ε and r_g defined in (A9)).

This article is organized as follows: In Section 2, we present notation and useful preliminary lemmas. In Section 3, we investigate the critical Choquard equation perturbed by a sign-changing Hardy term; In Section 4, we investigate the critical Choquard equation perturbed by a sign-changing superlinear nonlocal term.

2. Preliminaries

$D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_D^2 = \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x.$$

It is well known that $\frac{(N-2)^2}{4}$ is the best constant in the Hardy inequality

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x, \quad \text{for any } u \in D^{1,2}(\mathbb{R}^N).$$

By (A7)–(A11), we derive that

$$\|u\|_g^2 = \int_{\mathbb{R}^N} \Big(|\nabla u|^2 - g(x)\frac{u^2}{|x|^2}\Big) \,\mathrm{d}x,$$

is an equivalent norm in $D^{1,2}(\mathbb{R}^N)$, since the following inequalities hold:

$$\Big(1 - \frac{4g_{\max}}{(N-2)^2}\Big) \|u\|_D^2 \leqslant \|u\|_g^2 \leqslant \Big(1 - \frac{4g_{\min}}{(N-2)^2}\Big)\|u\|_D^2.$$

We recall the Sobolev inequality

$$S\Big(\int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x\Big)^{2/2^*} \leqslant \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x, \quad \text{for any } u \in D^{1,2}(\mathbb{R}^N),$$

where $S > 0$ is the Sobolev constant (see [23]).

Lemma 2.1 (Hardy-Littlewood-Sobolev inequality [14])**.** *Let $t, r > 1$ and $\mu \in (0, N)$ with $\frac{1}{t} + \frac{1}{r} + \frac{\mu}{N} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(N, \mu, r, t)$, independent of f, h such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)||h(y)|}{|x-y|^\mu} \,\mathrm{d}x \,\mathrm{d}y \leqslant C(N, \mu, r, t)\|f\|_t \|h\|_r.$$

If $t = r = \frac{2N}{2N-\mu}$, then

$$C(N, \mu, r, t) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \Big(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\Big)^{-1+\frac{\mu}{N}}.$$

For $\mu \in (0, N)$, we define the best constant

$$S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x}{(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} \,\mathrm{d}x \,\mathrm{d}y)^{1/2^*_\mu}}. \tag{2.1}$$

The authors in [9, Lemma 1.2] proved that $S_{H,L}$ is attained in $\mathbb{R}^N$ by the extremal function:

$$w_\sigma(x) = \sigma^{\frac{2-N}{2}} w(\frac{x}{\sigma}), \quad w(x) = \frac{\mathfrak{C}}{(1+|x|^2)^{\frac{N-2}{2}}}, \tag{2.2}$$

where $\mathfrak{C} > 0$ is a fixed constant. By the definition of convolution, we set

$$\begin{gathered}
|x|^{-\mu} * (|u_n|^{2^*_\mu}) := \int_{\mathbb{R}^N} \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu} \,\mathrm{d}y \\
|x|^{-\mu} * (g|u_n|^p)(\mathbb{R}^N) := \int_{\mathbb{R}^N} \frac{g(y)|u_n(y)|^p}{|x-y|^\mu} \,\mathrm{d}y, \\
|x|^{-\mu} * (g|u_n|^p)(\Omega_i) := \int_{\Omega_i} \frac{g(y)|u_n(y)|^p}{|x-y|^\mu} \,\mathrm{d}y, \quad (i = 1, 2).
\end{gathered}$$

Lemma 2.2 ([9, Lemma 2.3])**.** *Let $N \geqslant 3$ and $0 < \mu < N$. If $\{u_n\}$ is a bounded sequence in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, then the following hold,*

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|x|^{-\mu} * (|u_n|^{2^*_\mu}))|u_n|^{2^*_\mu} \,\mathrm{d}x - \int_{\mathbb{R}^N} (|x|^{-\mu} * (|u_n - u|^{2^*_\mu}))|u_n - u|^{2^*_\mu} \,\mathrm{d}x \\
&\to \int_{\mathbb{R}^N} (|x|^{-\mu} * (|u|^{2^*_\mu}))|u|^{2^*_\mu} \,\mathrm{d}x.
\end{aligned}$$

3. Perturbation with a sign-changing Hardy term

In this section we study the existence and nonexistence of solutions for the critical Choquard equation with a sign-changing Hardy term, i.e.

$$-\Delta u = \Big(\int_{\mathbb{R}^N} \frac{|u|^{2^*_\mu}}{|x-y|^\mu} \,\mathrm{d}y\Big)|u|^{2^*_\mu - 2}u + g(x)\frac{u}{|x|^2}, \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

We introduce the energy functional associated with (1.6) as

$$I_1(u) = \frac{1}{2}\|u\|_D^2 - \frac{1}{2}\int_{\mathbb{R}^N} g(x)\frac{|u|^2}{|x|^2} \,\mathrm{d}x - \frac{1}{2 \cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|^\mu} \,\mathrm{d}x \,\mathrm{d}y.$$

3.1. **Non-existence result.**

In this subsection, if $N \geqslant 3$, $\mu \in (0, N)$ and $(x \cdot \nabla g(x))$ has a fixed sign, we prove that problem (1.6) does not have any solution by Pohozaev type of identity.

Proof of Theorem 1.1. We use the same cut-off function which was used in [6]. More precisely, for $\epsilon > 0$ and $\epsilon_1 > 0$, we define $\tilde{\psi}_{\epsilon,\epsilon_1}(x) = \psi_\epsilon(x)\bar{\psi}_{\epsilon_1}(x)$, where $\psi_\epsilon(x) = \psi(\frac{|x|}{\epsilon})$ and $\bar{\psi}_{\epsilon_1}(x) = \psi(\frac{|x|}{\epsilon_1})$, ψ and $\bar{\psi}$ are smooth functions in $\mathbb{R}$ with the properties $0 \leqslant \psi, \bar{\psi} \leqslant 1$, with supports of ψ and $\bar{\psi}$ in $(1, \infty)$ and $(-\infty, 2)$ respectively and $\psi(t) = 1$ for $t \geqslant 2$, and $\bar{\psi}(t) = 1$ for $t \leqslant 1$.

Let u be a weak solution of problem (1.6). Then u is smooth away from origin and hence $(x \cdot \nabla u)\tilde{\psi}_{\epsilon,\epsilon_1} \in C_c^2(\mathbb{R}^N)$ (see [6]). Multiplying problem (1.6) by $(x \cdot \nabla u)\tilde{\psi}_{\epsilon,\epsilon_1}$

and integrating by parts we obtain

$$
\begin{aligned}
&-\int_{\mathbb{R}^N} \Delta u(x\cdot\nabla u)\tilde{\psi}_{\epsilon,\epsilon_1}\,\mathrm{d}x \\
&=\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u|^{2^*_\mu-2}u(x\cdot\nabla u)\tilde{\psi}_{\epsilon,\epsilon_1}\,\mathrm{d}x \\
&\quad+\int_{\mathbb{R}^N} g(x)\frac{u}{|x|^2}(x\cdot\nabla u)\tilde{\psi}_{\epsilon,\epsilon_1}\,\mathrm{d}x,
\end{aligned}\tag{3.2}
$$

We can show that

$$
\lim_{\epsilon_1\to\infty}\lim_{\epsilon\to 0}-\int_{\mathbb{R}^N}\Delta u(x\cdot\nabla u)\tilde{\psi}_{\epsilon,\epsilon_1}\,\mathrm{d}x=-\Big(\frac{N-2}{2}\Big)\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x,\tag{3.3}
$$

$$
\begin{aligned}
&\lim_{\epsilon_1\to\infty}\lim_{\epsilon\to 0}\int_{\mathbb{R}^N}\frac{g(x)}{|x|^2}u(x\cdot\nabla u)\tilde{\psi}_{\epsilon,\epsilon_1}\,\mathrm{d}x \\
&=-\Big(\frac{N-2}{2}\Big)\int_{\mathbb{R}^N}\frac{g(x)}{|x|^2}u^2\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}\frac{(x\cdot\nabla g(x))}{|x|^2}u^2\,\mathrm{d}x.
\end{aligned}\tag{3.4}
$$

We just show the critical term,

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u(x)|^{2^*_\mu-2}u(x)(x\cdot\nabla u(x))\tilde{\psi}_{\epsilon,\epsilon_1}(x)\,\mathrm{d}x \\
&=-\int_{\mathbb{R}^N}u(x)\nabla\Big(x\tilde{\psi}_{\epsilon,\epsilon_1}(x)\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y|u(x)|^{2^*_\mu-2}u(x)\Big)\,\mathrm{d}x \\
&=-N\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\tilde{\psi}_{\epsilon,\epsilon_1}(x)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad-(2^*_\mu-1)\int_{\mathbb{R}^N}(x\cdot\nabla u(x))\tilde{\psi}_{\epsilon,\epsilon_1}(x)\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y|u(x)|^{2^*_\mu-2}\Big)\,\mathrm{d}x \\
&\quad+\mu\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}x\cdot(x-y)\tilde{\psi}_{\epsilon,\epsilon_1}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}x\cdot(\psi_\epsilon(x)\nabla\bar{\psi}_{\epsilon_1}(x)+\bar{\psi}_{\epsilon_1}(x)\nabla\psi_\epsilon(x))\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}
$$

Note that $\nabla\bar{\psi}_{\epsilon_1}(x)$ and $\nabla\psi_\epsilon(x)$ have supports in $\{\epsilon_1<|x|<2\epsilon_1\}$ and $\{\epsilon<|x|<2\epsilon\}$, respectively. Since $|x\cdot(\psi_\epsilon(x)\nabla\bar{\psi}_{\epsilon_1}(x)+\bar{\psi}_{\epsilon_1}(x)\nabla\psi_\epsilon(x))|\leqslant C$, applying the dominated convergence theorem, we have

$$
\begin{aligned}
&\lim_{\epsilon_1\to\infty}\lim_{\epsilon\to 0}\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u(x)|^{2^*_\mu-2}u(x)(x\cdot\nabla u(x))\tilde{\psi}_{\epsilon,\epsilon_1}(x)\,\mathrm{d}x \\
&=-N\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad-\lim_{\epsilon_1\to\infty}\lim_{\epsilon\to 0}(2^*_\mu-1)\int_{\mathbb{R}^N}(x\cdot\nabla u(x))\tilde{\psi}_{\epsilon,\epsilon_1}(x)\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y|u(x)|^{2^*_\mu-2}\Big)\,\mathrm{d}x \\
&\quad+\mu\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}x\cdot(x-y)\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}
$$

which implies

$$
\lim_{\epsilon_1\to\infty}\lim_{\epsilon\to 0}2^*_\mu\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u(x)|^{2^*_\mu-2}u(x)(x\cdot\nabla u(x))\tilde{\psi}_{\epsilon,\epsilon_1}(x)\,\mathrm{d}x
$$

$$
\begin{aligned}
&= -N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \mu \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x\cdot(x-y)\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\lim_{\epsilon_1\to\infty}\lim_{\epsilon\to 0} 2^*_\mu \int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\Big)|u(y)|^{2^*_\mu-2}u(y)(y\cdot\nabla u(y))\tilde{\psi}_{\epsilon,\epsilon_1}(y)\,\mathrm{d}y \\
&= -N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \mu \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x\cdot(x-y)\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

Hence, we know that

$$
\begin{aligned}
&\lim_{\epsilon_1\to\infty}\lim_{\epsilon\to 0} 2^*_\mu \int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u(x)|^{2^*_\mu-2}u(x)(x\cdot\nabla u(x))\tilde{\psi}_{\epsilon,\epsilon_1}(x)\,\mathrm{d}x \\
&= \frac{\mu-2N}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x \\
&= -\Big(\frac{N-2}{2}\Big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
\tag{3.5}
$$

Therefore, putting (3.3)–(3.5) into (3.2), we obtain

$$
\begin{aligned}
&-\Big(\frac{N-2}{2}\Big)\Big(\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x - \int_{\mathbb{R}^N}\frac{g(x)}{|x|^2}u^2\,\mathrm{d}x\Big) \\
&= -\frac{1}{2}\int_{\mathbb{R}^N}\frac{(x\cdot\nabla g(x))}{|x|^2}u^2\,\mathrm{d}x - \Big(\frac{N-2}{2}\Big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

Also from (1.6), we have

$$
\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x - \int_{\mathbb{R}^N}\frac{g(x)}{|x|^2}u^2\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x.
$$

Then we obtain

$$
\int_{\mathbb{R}^N}\frac{(x\cdot\nabla g(x))}{|x|^2}u^2\,\mathrm{d}x = 0,
$$

which is not possible if $(x\cdot\nabla g(x))$ has a fixed sign and $u\not\equiv 0$. □

3.2. **Existence of a ground state solution.** In this subsection, we study the existence of ground state solution for problem (1.6) on$\mathbb{R}^N$. The following Lemma plays an important role in estimating the Mountain-Pass levels.

Lemma 3.1. *Assume that* (A7)–(A10) *hold. Then for all* $\sigma\in[r_\varepsilon, r_g]$, *we have*

$$
\int_{\mathbb{R}^N} g(x)\frac{|w_\sigma(x)|^2}{|x|^2}\,\mathrm{d}x > 0.
$$

Proof. By using (2.2), we have

$$
w_\sigma(x) = \frac{\mathfrak{C}\sigma^{\frac{2-N}{2}}}{(1+|\frac{x}{\sigma}|^2)^{\frac{N-2}{2}}}. \tag{3.6}
$$

According to (A8), (A9) and (3.6), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} g(x)\frac{|w_\sigma(x)|^2}{|x|^2}\,\mathrm{d}x &= \int_{\mathbb{R}^N} g(x)\frac{\mathfrak{C}^2\sigma^{2-N}}{(1+|\frac{x}{\sigma}|^2)^{N-2}|x|^2}\,\mathrm{d}x\\ &= \int_{\Omega_1\cup\Omega_2} g(x)\frac{\mathfrak{C}^2\sigma^{N-2}}{(\sigma^2+|x|^2)^{N-2}|x|^2}\,\mathrm{d}x.\end{aligned}$$

Since $\Omega_1\cup\Omega_2\subset\overline{B(0,r_g)}\setminus B(0,r_\varepsilon)$, we have $|x|\in[r_\varepsilon,r_g]$ in $\Omega_1\cup\Omega_2$. From the fact that $\int_{\Omega_1} g(x)\,\mathrm{d}x>0$ and $\int_{\Omega_2} g(x)\,\mathrm{d}x<0$, we obtain

$$\int_{\mathbb{R}^N} g(x)\frac{|w_\sigma(x)|^2}{|x|^2}\,\mathrm{d}x \geqslant \frac{\mathfrak{C}^2\sigma^{N-)}}{(\sigma^2+r_g^2)^{N-2}r_g^2}\int_{\Omega_1} g(x)\,\mathrm{d}x + \frac{\mathfrak{C}^2\sigma^{N-2}}{(\sigma^2+r_\varepsilon^2)^{N-2}r_\varepsilon^2}\int_{\Omega_2} g(x)\,\mathrm{d}x.$$

Keeping in mind that $\int_{\Omega_2} g(x)\,\mathrm{d}x<0$ and $\sigma\in[r_\varepsilon,r_g]$, we know that

$$\begin{aligned}\int_{\mathbb{R}^N} g(x)\frac{|w_\sigma(x)|^2}{|x|^2}\,\mathrm{d}x \geqslant& \frac{\mathfrak{C}^2\sigma^{N-2}}{(2r_g^2)^{N-2}r_g^2}\int_{\Omega_1} g(x)\,\mathrm{d}x + \frac{\mathfrak{C}^2\sigma^{N-2}}{(2r_\varepsilon^2)^{N-2}r_\varepsilon^2}\int_{\Omega_2} g(x)\,\mathrm{d}x\\ =& \frac{\mathfrak{C}^2\sigma^{N-2}}{2^{N-2}r_g^{2N-2}}\int_{\Omega_1} g(x)\,\mathrm{d}x + \frac{\mathfrak{C}^2\sigma^{N-2}}{2^{N-2}r_\varepsilon^{2N-2}}\int_{\Omega_2} g(x)\,\mathrm{d}x.\end{aligned} \tag{3.7}$$

By (A10), we have

$$\int_{\Omega_1} g(x)\,\mathrm{d}x > 2\big(\frac{r_g}{r_\varepsilon}\big)^{4N}\int_{\Omega_2}(-g(x))\,\mathrm{d}x > \big(\frac{r_g}{r_\varepsilon}\big)^{2N-2}\int_{\Omega_2}(-g(x))\,\mathrm{d}x. \tag{3.8}$$

Inserting (3.8) into (3.7), we deduce that $\int_{\mathbb{R}^N} g(x)\frac{|w_\sigma(x)|^2}{|x|^2}\,\mathrm{d}x>0$. $\square$

We show that the functional I_1 satisfies the Mountain-Pass geometry, and estimate the Mountain-Pass levels.

Lemma 3.2. *Assume that the hypotheses of Theorem 1.2 hold, there exists a $(PS)_c$ sequence of I_1 at a level c, where $0<c<c^*=\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$.*

Proof. **Step 1.** We prove that I_1 satisfies all the conditions in Mountain-pass theorem.

(i) $I_1(0)=0$;

(ii) For any $u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}$, we have

$$\begin{aligned}I_1(u) \geqslant& \frac{1}{2}\|u\|_g^2 - \frac{1}{2\cdot 2_\mu^* S_{H,L}^{2_\mu^*}}\|u\|_D^{2\cdot 2_\mu^*}\\ \geqslant& \frac{1}{2}\Big(1-\frac{4g_{\max}}{(N-2)^2}\Big)\|u\|_D^2 - \frac{1}{2\cdot 2_\mu^* S_{H,L}^{2_\mu^*}}\|u\|_D^{2\cdot 2_\mu^*}.\end{aligned}$$

Because of $2<2\cdot 2_\mu^*$, there exists a sufficiently small positive number ρ such that

$$\vartheta := \inf_{\|u\|_D=\rho} I_1(u) > 0 = I_1(0).$$

(iii) Given $u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}$ such that $\lim_{t\to\infty} I_1(tu)=-\infty$. We could choose $t_u>0$ corresponding to u such that $I_1(tu)<0$ for all $t>t_u$ and $\|t_u u\|_D>\rho$. Set

$$c = \inf_{\Upsilon\in\Gamma_u}\max_{t\in[0,1]} I_1(\Upsilon(t)),$$

where $\Gamma_u=\{\Upsilon\in C([0,1],D^{1,2}(\mathbb{R}^N)) : \Upsilon(0)=0, \Upsilon(1)=t_u u\}$.

Step 2. Here we show $0 < c < c^*$. Using Lemma 3.1, there exists $\sigma \in [r_\varepsilon, r_g]$ such that $\int_{\mathbb{R}^N} g(x)\frac{|w_\sigma(x)|^2}{|x|^2}\,\mathrm{d}x > 0$. For all $t \geqslant 0$, we obtain

$$
\begin{aligned}
0 < c &\leqslant \sup_{t\geqslant 0} I_1(tw_\sigma)\\
&\leqslant \frac{N+2-\mu}{4N-2\mu}\Big(\frac{\int_{\mathbb{R}^N}|\nabla w_\sigma(x)|^2\,\mathrm{d}x - \int_{\mathbb{R}^N} g(x)\frac{|w_\sigma(x)|^2}{|x|^2}\,\mathrm{d}x}{(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_\sigma(x)|^{2^*_\mu}|w_\sigma(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y)^{1/2^*_\mu}}\Big)^{\frac{2N-\mu}{N+2-\mu}}\\
&< \frac{N+2-\mu}{4N-2\mu}\Big(\frac{\int_{\mathbb{R}^N}|\nabla w_\sigma(x)|^2\,\mathrm{d}x}{(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_\sigma(x)|^{2^*_\mu}|w_\sigma(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y)^{1/2^*_\mu}}\Big)^{\frac{2N-\mu}{N+2-\mu}}\\
&= \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.
\end{aligned}
$$

which means $0 < c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. □

Lemma 3.3. *Assume that the hypotheses of Theorem 1.2 hold. If $\{u_n\}$ is a $(PS)_c$ sequence of I_1, then $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$.*

Proof. The $(PS)_c$ sequence $\{u_n\}$ defined in Lemma 3.2. From the definition of $(PS)_c$ sequence, we have

$$
\begin{aligned}
c^* + \|u_n\|_D \geqslant c^* + o(1)\|u_n\|_D &\geqslant I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\\
&= \frac{2^*_\mu - 1}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Applying above inequality and (A11), we know

$$
c^* \geqslant I(u_n) \geqslant \frac{1}{2}\Big(1 - \frac{4g_{\max}}{(N-2)^2}\Big)\|u_n\|_D^2 - \frac{1}{2^*_\mu - 1}(c^* + \|u_n\|_D).
$$

Set

$$
f_1(t) = \frac{1}{2}\Big(1 - \frac{4g_{\max}}{(N-2)^2}\Big)t^2 - \frac{1}{2^*_\mu - 1}t - \frac{2^*_\mu c^*}{2^*_\mu - 1}.
$$

We have two solutions of $f_1(\cdot)$ as follows:

$$
\begin{aligned}
t' &= \frac{\frac{1}{2^*_\mu-1} + \sqrt{(\frac{1}{2^*_\mu-1})^2 + \frac{2\cdot 2^*_\mu\cdot c^*}{2^*_\mu-1}(1-\frac{4g_{\max}}{(N-2)^2})}}{1-\frac{4g_{\max}}{(N-2)^2}} > 0,\\
t'' &= \frac{\frac{1}{2^*_\mu-1} - \sqrt{(\frac{1}{2^*_\mu-1})^2 + \frac{2\cdot 2^*_\mu\cdot c^*}{2^*_\mu-1}(1-\frac{4g_{\max}}{(N-2)^2})}}{1-\frac{4g_{\max}}{(N-2)^2}} < 0.
\end{aligned}
$$

Therefore, $0 \leqslant \|u_n\|_D \leqslant t'$, this implies that $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$. □

To check that functional I_1 satisfies the $(PS)_c$ condition, we give the following Lemma.

Lemma 3.4. *Assume that the hypotheses of Theorem 1.2 hold. If $\{u_n\}$ is a bounded sequence in $D^{1,2}(\mathbb{R}^N)$, up to a subsequence, $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$ and*

$u_n \to u$a.e. in $\mathbb{R}^N$ as $n \to \infty$, then

$$\int_{\mathbb{R}^N} g(x)\frac{|u_n|^2}{|x|^2}\,\mathrm{d}x \to \int_{\mathbb{R}^N} g(x)\frac{|u|^2}{|x|^2}\,\mathrm{d}x.$$

In addition, for any $\varphi \in D^{1,2}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} g(x)\frac{u_n\varphi}{|x|^2}\,\mathrm{d}x \to \int_{\mathbb{R}^N} g(x)\frac{u\varphi}{|x|^2}\,\mathrm{d}x.$$

as $n \to \infty$.

Proof. **Step 1.** Define $v_n := u_n - u$. According to (A8), (A9) and Brézis-Lieb lemma in [3], we have

$$\int_{\Omega_1} g(x)\frac{|u_n|^2}{|x|^2}\,\mathrm{d}x = \int_{\Omega_1} g(x)\frac{|v_n|^2}{|x|^2}\,\mathrm{d}x + \int_{\Omega_1} g(x)\frac{|u|^2}{|x|^2}\,\mathrm{d}x + o(1), \quad \text{as } n \to \infty, \tag{3.9}$$

and

$$\int_{\Omega_2} g(x)\frac{|u_n|^2}{|x|^2}\,\mathrm{d}x = \int_{\Omega_2} g(x)\frac{|v_n|^2}{|x|^2}\,\mathrm{d}x + \int_{\Omega_2} g(x)\frac{|u|^2}{|x|^2}\,\mathrm{d}x + o(1), \quad \text{as } n \to \infty. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$\int_{\Omega_1\cup\Omega_2} g(x)\frac{|u_n|^2}{|x|^2}\,\mathrm{d}x = \int_{\Omega_1\cup\Omega_2} g(x)\frac{|v_n|^2}{|x|^2}\,\mathrm{d}x + \int_{\Omega_1\cup\Omega_2} g(x)\frac{|u|^2}{|x|^2}\,\mathrm{d}x + o(1), \tag{3.11}$$

as $n \to \infty$.

Step 2. Furthermore, we estimate the term involving v_n in (3.11). Since $v_n \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^N)$, we have $v_n \to 0$ in $L^2(B(0,r_g) \setminus B(0,r_\varepsilon))$. According to (A8) and (A9), we obtain

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega_1\cup\Omega_2} g(x)\frac{|v_n|^2}{|x|^2}\,\mathrm{d}x &= \lim_{n\to\infty} \int_{B(0,r_g)\setminus B(0,r_\varepsilon)} g(x)\frac{|v_n|^2}{|x|^2}\,\mathrm{d}x \\ &\leqslant \frac{g_{\max}}{r_\varepsilon^2} \lim_{n\to\infty} \int_{B(0,r_g)\setminus B(0,r_\varepsilon)} |v_n|^2\,\mathrm{d}x = 0. \end{aligned} \tag{3.12}$$

Keeping in mind that $g_{\min} < 0$, similar to (3.12), we obtain

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega_1\cup\Omega_2} g(x)\frac{|v_n|^2}{|x|^2}\,\mathrm{d}x &= \lim_{n\to\infty} \int_{B(0,r_g)\setminus B(0,r_\varepsilon)} g(x)\frac{|v_n|^2}{|x|^2}\,\mathrm{d}x \\ &\geqslant \frac{g_{\min}}{r_\varepsilon^2} \lim_{n\to\infty} \int_{B(0,r_g)\setminus B(0,r_\varepsilon)} |v_n|^2\,\mathrm{d}x \to 0. \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$0 \leqslant \lim_{n\to\infty} \int_{\Omega_1\cup\Omega_2} g(x)\frac{|v_n|^2}{|x|^2}\,\mathrm{d}x \leqslant 0,$$

then

$$\lim_{n\to\infty} \int_{\Omega_1\cup\Omega_2} g(x)\frac{|v_n|^2}{|x|^2}\,\mathrm{d}x = 0. \tag{3.14}$$

Step 3. Putting (3.14) into (3.11), we know

$$\int_{\Omega_1\cup\Omega_2} g(x)\frac{|u_n|^2}{|x|^2}\,\mathrm{d}x = \int_{\Omega_1\cup\Omega_2} g(x)\frac{|u|^2}{|x|^2}\,\mathrm{d}x + o(1) \quad \text{as } n \to \infty. \tag{3.15}$$

Since $g \equiv 0$ in $\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)$, by (3.15), we have
$$\int_{\mathbb{R}^N} g(x)\frac{|u_n|^2}{|x|^2}\,\mathrm{d}x = \int_{\mathbb{R}^N} g(x)\frac{|u|^2}{|x|^2}\,\mathrm{d}x + o(1), \quad \text{as } n \to \infty.$$

Step 4. In addition, the boundedness of u_n in $D^{1,2}(\mathbb{R}^N)$ yields that u_n are bounded in $L^2(\Omega_1, |x|^{-2})$ and $L^2(\Omega_2, |x|^{-2})$, respectively. Therefore, up to a subsequence, we have the following weak convergence
$$g(x)u_n \rightharpoonup g(x)u \quad \text{in } L^2(\Omega_1, |x|^{-2}),$$
$$g(x)u_n \rightharpoonup g(x)u \quad \text{in } L^2(\Omega_2, |x|^{-2}).$$
Then
$$g(x)u_n \rightharpoonup g(x)u \quad \text{in } L^2(\Omega_1 \cup \Omega_2, |x|^{-2}).$$
Since $g \equiv 0$ in $\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)$, we know that
$$g(x)u_n \rightharpoonup g(x)u \quad \text{in } L^2(\mathbb{R}^N, |x|^{-2}).$$
For any $\varphi \in D^{1,2}(\mathbb{R}^N)$, we have
$$\int_{\mathbb{R}^N} g(x)\frac{u_n\varphi}{|x|^{-2}}\,\mathrm{d}x \to \int_{\mathbb{R}^N} g(x)\frac{u\varphi}{|x|^2}\,\mathrm{d}x.$$
$\square$

Now we check functional I_1 satisfies the $(PS)_c$ condition.

Lemma 3.5. *Assume that the hypotheses of Theorem 1.2 hold. If $\{u_n\}$ is a $(PS)_c$ sequence of I_1 with $0 < c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$, then $\{u_n\}$ has a convergent subsequence.*

Proof. **Step 1.** Since $D^{1,2}(\mathbb{R}^N)$ is a reflexive space. And $\{u_n\}$ is a bounded sequence in $D^{1,2}(\mathbb{R}^N)$, up to a subsequence, we can assume that
$$u_n \rightharpoonup u \quad \text{in } D^{1,2}(\mathbb{R}^N), \quad u_n \to u \quad \text{a.e. in } \mathbb{R}^N,$$
$$u_n \to u \quad \text{in } L^r_{loc}(\mathbb{R}^N) \quad \text{for all } r \in [1, 2^*).$$
Then
$$|u_n|^{2^*_\mu} \rightharpoonup |u|^{2^*_\mu} \quad \text{in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N), \quad \text{as } n \to \infty.$$
By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\mu}}(\mathbb{R}^N)$, we know
$$|x|^{-\mu} * |u_n|^{2^*_\mu} \rightharpoonup |x|^{-\mu} * |u|^{2^*_\mu} \quad \text{in } L^{\frac{2N}{\mu}}(\mathbb{R}^N), \quad \text{as } n \to \infty.$$
Combining with the fact that
$$|u_n|^{2^*_\mu-2}u_n \rightharpoonup |u|^{2^*_\mu-2}u \quad \text{in } L^{\frac{2N}{N+2-\mu}}(\mathbb{R}^N), \quad \text{as n } \to \infty,$$
we obtain
$$(|x|^{-\mu} * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu-2}u_n \rightharpoonup (|x|^{-\mu} * |u|^{2^*_\mu})|u|^{2^*_\mu-2}u \quad \text{in } L^{\frac{2N}{N+2}}(\mathbb{R}^N) \quad \text{as } n \to \infty.$$
For any $\varphi \in D^{1,2}(\mathbb{R}^N)$, we obtain
$$\begin{aligned} 0 \leftarrow \langle I_1'(u_n), \varphi\rangle &= \int_{\mathbb{R}^N} \nabla u_n \nabla\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N} g(x)\frac{u_n\varphi}{|x|^2}\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu-2}u_n(y)\varphi(y)}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

Passing to the limit as $n \to \infty$, by using Lemma 3.4, we obtain

$$\begin{aligned} 0 = &\int_{\mathbb{R}^N} \nabla u \nabla \varphi \, \mathrm{d}x - \int_{\mathbb{R}^N} g(x) \frac{u\varphi}{|x|^2} \, \mathrm{d}x \\ &- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu - 2} u(y) \varphi(y)}{|x-y|^\mu} \, \mathrm{d}y \, \mathrm{d}x, \end{aligned}$$

for any $\varphi \in D^{1,2}(\mathbb{R}^N)$, which means that u is a weak solution of problem (1.6). Taking $\varphi = u \in D^{1,2}(\mathbb{R}^N)$ as a test function in (1.6), we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} g(x) \frac{|u|^2}{|x|^2} \, \mathrm{d}x = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu} u(y)}{|x-y|^\mu} \, \mathrm{d}y \, \mathrm{d}x,$$

which implies that $\langle I_1'(u), u \rangle = 0$.

Step 2. From $\langle I_1'(u), u \rangle = 0$, we obtain

$$\begin{aligned} I_1(u) = &I_1(u) - \frac{1}{2} \langle I_1'(u), u \rangle \\ = &\Big(\frac{1}{2} - \frac{1}{2 \cdot 2^*_\mu} \Big) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \geqslant 0. \end{aligned}$$

Define $v_n := u_n - u$, then we know $v_n \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^N)$. According to the Brézis-Lieb lemma, Lemma 2.2 and Lemma 3.4, we have

$$\begin{aligned} c \leftarrow I_1(u_n) = &\frac{1}{2} \|u_n\|_D^2 - \frac{1}{2} \int_{\mathbb{R}^N} g(x) \frac{|u_n|^2}{|x|^2} \, \mathrm{d}x \\ &- \frac{1}{2 \cdot 2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \\ = &I_1(u) + \frac{1}{2} \|v_n\|_D^2 - \frac{1}{2 \cdot 2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \\ \geqslant &\frac{1}{2} \|v_n\|_D^2 - \frac{1}{2 \cdot 2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y, \end{aligned} \tag{3.16}$$

since $I_1(u) \geqslant 0$. Similarly, since $\langle I_1'(u), u \rangle = 0$, we obtain

$$\begin{aligned} o(1) = &\langle I_1'(u_n), u_n \rangle \\ = &\|u_n\|_D^2 - \int_{\mathbb{R}^N} g(x) \frac{|u_n|^2}{|x|^2} \, \mathrm{d}x - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \\ = &\langle I_1'(u), u \rangle + \|v_n\|_D^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \\ = &\|v_n\|_D^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{3.17}$$

From this equality, there exists a nonnegative constant b such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 \, \mathrm{d}x \to b \quad \text{and} \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \to b,$$

as $n \to \infty$. From (3.16) and (3.17), we obtain

$$c \geqslant \frac{N + 2 - \mu}{4N - 2\mu} b. \tag{3.18}$$

By the definition of the best constant $S_{H,L}$ in (2.1), we have

$$S_{H,L}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)|^{2^*_\mu}|v_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/2^*_\mu}\leqslant\int_{\mathbb{R}^N}|\nabla v_n|^2\,\mathrm{d}x,$$

which gives $S_{H,L}b^{1/2^*_\mu}\leqslant b$. Thus we have that either $b=0$ or $b\geqslant S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$.

If $b\geqslant S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$, then from (3.18), we obtain

$$\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}\leqslant\frac{N+2-\mu}{4N-2\mu}b\leqslant c.$$

This is in contradiction to $c<\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. Thus $b=0$, and $\|u_n-u\|_D\to0$, as $n\to\infty$. □

Proof of Theorem 1.2. **Step 1.** Applying Lemma 3.2, we obtain that I_1 possesses a mountain pass geometry. Then from the Maintain Pass Theorem, there is a sequence $\{u_n\}\subset D^{1,2}(\mathbb{R}^N)$ satisfying $I_1(u_n)\to c$ and $I_1'(u_n)\to0$, where $0<\vartheta\leqslant c<\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. Moreover, according to Lemma 3.3 and Lemma 3.5, $\{u_n\}$ satisfying $(PS)_c$ condition. We have a nontrivial solution u_0 to problem (1.6). In following text, we show the existence of ground state solution to problem (1.6).

Step 2. Define

$$K_1=\{u\in D^{1,2}(\mathbb{R}^N)|\langle I_1'(u),u\rangle=0,\ u\neq0\},$$
$$E_1=\{I_1(u)|u\in K_1\}.$$

In Step 1, we have $u_0\neq0$ and $\langle I_1'(u_0),u_0\rangle=0$. Hence, we know $K_1\neq\emptyset$.

Now, we claim that any limit point of a sequence in K_1 is different from zero. For any $u\in K_1$, according to $\langle I_1'(u),u\rangle=0$ and (2.1), it follows that

$$\begin{aligned}0=\langle I_1'(u),u\rangle&=\|u\|_g^2-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\&\geqslant\Big(1-\frac{4g_{\max}}{(N-2)^2}\Big)\|u\|_D^2-\frac{1}{S_{H,L}^{2^*_\mu}}\|u\|_D^{2\cdot2^*_\mu}.\end{aligned}$$

From the above expression, we obtain

$$\Big(1-\frac{4g_{\max}}{(N-2)^2}\Big)\|u\|_D^2\leqslant\frac{1}{S_{H,L}^{2^*_\mu}}\|u\|_D^{2\cdot2^*_\mu},$$

which gives

$$0<\Big(\Big(1-\frac{4g_{\max}}{(N-2)^2}\Big)S_{H,L}^{2^*_\mu}\Big)^{\frac{1}{2\cdot2^*_\mu-2}}\leqslant\|u\|_D,\quad\text{for any }u\in K_1.$$

Hence, any limit point of a sequence in K_1 is different from zero. Now, we claim that E_1 has an infimum. In fact, for any $u\in K_1$, we have

$$0=\langle I_1'(u),u\rangle=\|u\|_g^2-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.$$

Then

$$I_1(u)=\frac12\|u\|_g^2-\frac{1}{2\cdot2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y$$

$$
\begin{aligned}
&=\Big(\frac{1}{2}-\frac{1}{2\cdot 2_\mu^*}\Big)\|u\|_g^2\\
&\geqslant\Big(\frac{1}{2}-\frac{1}{2\cdot 2_\mu^*}\Big)\Big(1-\frac{4g_{\max}}{(N-2)^2}\Big)\|u\|_D^2\\
&\geqslant\Big(\frac{1}{2}-\frac{1}{2\cdot 2_\mu^*}\Big)\Big(\Big(1-\frac{4g_{\max}}{(N-2)^2}\Big)S_{H,L}\Big)^{\frac{2\cdot 2_\mu^*}{2\cdot 2_\mu^*-2}}>0.
\end{aligned}
$$

Therefore, we obtain

$$
0<\Big(\frac{1}{2}-\frac{1}{2\cdot 2_\mu^*}\Big)\Big(\Big(1-\frac{4g_{\max}}{(N-2)^2}\Big)S_{H,L}\Big)^{\frac{2\cdot 2_\mu^*}{2\cdot 2_\mu^*-2}}\leqslant \overline{E}_1:=\inf\{I_1(u)|u\in K_1\}.
$$

Step 3. (i) For each $u\in D^{1,2}(\mathbb{R}^N)$ with $u\not\equiv 0$, and $t\in(0,\infty)$, we set

$$
\begin{gathered}
f_2(t)=I_1(tu)=\frac{t^2}{2}\|u\|_g^2-\frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y,\\
f_2'(t)=t\|u\|_g^2-t^{2\cdot 2_\mu^*-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.
\end{gathered}
$$

This implies that $f_2'(\cdot)=0$ if and only if $\|u\|_g^2=t^{2\cdot 2_\mu^*-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y$. Set

$$
f_3(t)=t^{2\cdot 2_\mu^*-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.
$$

We know that $\lim_{t\to 0}f_3(t)\to 0$, $\lim_{t\to\infty}f_3(t)\to\infty$ and $f_3(\cdot)$ is strictly increasing on $(0,\infty)$. This shows that $f_2(\cdot)$ admits a unique critical point t_u on $(0,\infty)$ such that $f_2(\cdot)$ takes the maximum at t_u. This is showing that $t_uu\in K_1$.

To prove the uniqueness of t_u, let us assume that $0<\bar{t}<\bar{\bar{t}}$ satisfy $f_2'(\bar{t})=f_2'(\bar{\bar{t}})=0$. We obtain

$$
\begin{aligned}
\|u\|_g^2&=\bar{t}^{2\cdot 2_\mu^*-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&=\bar{\bar{t}}^{2\cdot 2_\mu^*-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Since $\bar{t}^{2\cdot 2_\mu^*-2}<\bar{\bar{t}}^{2\cdot 2_\mu^*-2}$, the above equality leads to the contradiction: $u=0$.

Hence, for each $u\in D^{1,2}(\mathbb{R}^N)$ with $u\not\equiv 0$, there exists a unique $t_u>0$ such that $t_uu\in K_1$.

(ii) Set $\Phi(u)=\langle I_1'(u),u\rangle$, for any $u\in K_1$, then

$$
\begin{aligned}
\langle\Phi'(u),u\rangle&=\langle\Phi'(u),u\rangle-q\Phi(u)\\
&\leqslant(2-q)\|u\|_g^2-(2\cdot 2_\mu^*-q)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant(2-q)\|u\|_g^2<0,
\end{aligned}
$$

where $2<q<2\cdot 2_\mu^*$. Thus, for any $u\in K_1$, we obtain $\Phi'(u)\neq 0$.

(iii) If $u\in K_1$ and $I_1(u)=\overline{E}_1$ then since $\overline{E}_1$ is the minimum of I_1 on K_1, Lagrange multiplier theorem implies that there exists $\lambda\in\mathbb{R}$ such that $I_1'(u)=\lambda\Phi'(u)$. Thus

$$
\langle\lambda\Phi'(u),u\rangle=\langle I_1'(u),u\rangle=\Phi(u)=0.
$$

According to the previous result, $\lambda = 0$, and so, $I_1'(u) = 0$, Then u is ground state solution for problem (1.6).

Step 4. By the Ekeland variational principle, there exists $\{\bar{u}_n\} \subset K_1$ and $\lambda_n \in \mathbb{R}$ such that

$$I_1(\bar{u}_n) \to \overline{E}_1 \quad \text{and} \quad I_1'(\bar{u}_n) - \lambda_n \Phi'(\bar{u}_n) \to 0 \quad \text{in } (D^{1,2}(\mathbb{R}^N))^{-1},$$

we can show that $\{\bar{u}_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$. Hence, taking into account that

$$|\langle I_1'(\bar{u}_n), \bar{u}_n\rangle - \langle \lambda_n \Phi'(\bar{u}_n), \bar{u}_n\rangle| \leqslant \|I_1'(\bar{u}_n) - \lambda_n \Phi'(\bar{u}_n)\|_{D^{-1}} \|\bar{u}_n\|_D \to 0,$$

we have

$$\langle I_1'(\bar{u}_n), \bar{u}_n\rangle - \lambda_n \langle \Phi'(\bar{u}_n), \bar{u}_n\rangle \to 0,$$

Using that $\langle I_1'(\bar{u}_n), \bar{u}_n\rangle = 0$ and $\langle \Phi'(\bar{u}_n), \bar{u}_n\rangle \neq 0$, we conclude that $\lambda_n \to 0$. Consequently, $I_1'(\bar{u}_n) \to 0$ in $(D^{1,2}(\mathbb{R}^N))^{-1}$. Hence $\{\bar{u}_n\}$ is a $(PS)_{\overline{E}_1}$ sequence of I_1.

By Lemma 3.5, we obtain that $\{\bar{u}_n\}$ has a strongly convergent subsequence (still denoted by $\{\bar{u}_n\}$). Hence, there exists $\bar{u}_0 \in D^{1,2}(\mathbb{R}^N)$ such that $\bar{u}_n \to \bar{u}_0$ in $D^{1,2}(\mathbb{R}^N)$. By using Step 2, we know $\bar{u}_0 \neq 0$. By weak lower semicontinuity of $\|\cdot\|_g$, we have

$$\begin{aligned}
\overline{E}_1 \leqslant I_1(\bar{u}_0) &= \Big(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\Big)\|\bar{u}_0\|_g^2 \\
&\leqslant \liminf_{n\to\infty} \Big(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\Big)\|\bar{u}_n\|_g^2 \\
&= \liminf_{n\to\infty} I_1(\bar{u}_n) = \lim_{n\to\infty} I_1(\bar{u}_n) = \overline{E}_1,
\end{aligned}$$

which implies that $I_1(\bar{u}_0) = \overline{E}_1$. Therefore, $\bar{u}_0$ is a ground state solution of problem (1.6). $\square$

4. Perturbation with a sign-changing superlinear nonlocal term

In this section, we study the existence of nontrivial solutions for the critical Choquard equation with a sign-changing superlinear nonlocal term, i.e.

$$-\Delta u = \Big(\int_{\mathbb{R}^N} \frac{|u|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\Big)|u|^{2_\mu^*-2}u + \Big(\int_{\mathbb{R}^N} \frac{g(y)|u|^p}{|x-y|^\mu}\,\mathrm{d}y\Big)g(x)|u|^{p-2}u. \tag{4.1}$$

We introduce the energy functional associated with (1.7) as

$$\begin{aligned}
I_2(u) =& \frac{1}{2}\|u\|_D^2 - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)g(y)|u_n(x)|^p|u_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\
&- \frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

The following Lemma plays an important role in estimating the Mountain-Pass levels.

Lemma 4.1. *Assume that* (A7)–(A10), (A12) *hold. Let* $N \geqslant 3$, $\mu \in (0, N)$ *and* $p \in (\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2})$. *Then for all* $\sigma \in [r_\varepsilon, r_g]$, *we have*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)g(y)|w_\sigma(x)|^p|w_\sigma(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y > 0.$$

Proof. According to (A8), (A9) and (3.6), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(x)g(y)|w_\sigma(x)|^p|w_\sigma(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2}\frac{\mathfrak{C}^{2p}\sigma^{p(N-2)}g(x)g(y)}{(\sigma^2+|x|^2)^{\frac{p}{2}(N-2)}|x-y|^\mu(\sigma^2+|y|^2)^{\frac{p}{2}(N-2)}}\,\mathrm{d}x\,\mathrm{d}y\\
&=\Big(\int_{\Omega_1}\int_{\Omega_1}\frac{\mathfrak{C}^{2p}\sigma^{p(N-2)}g(x)g(y)}{(\sigma^2+|x|^2)^{\frac{p}{2}(N-2)}|x-y|^\mu(\sigma^2+|y|^2)^{\frac{p}{2}(N-2)}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\int_{\Omega_2}\int_{\Omega_1}\frac{\mathfrak{C}^{2p}\sigma^{p(N-2)}g(x)g(y)}{(\sigma^2+|x|^2)^{\frac{p}{2}(N-2)}|x-y|^\mu(\sigma^2+|y|^2)^{\frac{p}{2}(N-2)}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\int_{\Omega_1}\int_{\Omega_2}\frac{\mathfrak{C}^{2p}\sigma^{p(N-2)}g(x)g(y)}{(\sigma^2+|x|^2)^{\frac{p}{2}(N-2)}|x-y|^\mu(\sigma^2+|y|^2)^{\frac{p}{2}(N-2)}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\int_{\Omega_2}\int_{\Omega_2}\frac{\mathfrak{C}^{2p}\sigma^{p(N-2)}g(x)g(y)}{(\sigma^2+|x|^2)^{\frac{p}{2}(N-2)}|x-y|^\mu(\sigma^2+|y|^2)^{\frac{p}{2}(N-2)}}\,\mathrm{d}x\,\mathrm{d}y\Big)\\
&=\mathfrak{C}^{2p}\sigma^{p(N-2)}\Big(A_1+A_2+A_3+A_4\Big).
\end{aligned}\tag{4.2}$$

Since $\Omega_1\cup\Omega_2\subset\overline{B(0,r_g)}\setminus B(0,r_\varepsilon)$, we have $|x|,|y|\in[r_\varepsilon,r_g]$ in Ω_1. By using $g(x),g(y)>0$ on Ω_1, $\sigma\in[r_\varepsilon,r_g]$ and Fubini's theorem, we obtain

$$\begin{aligned}
A_1&\geqslant\int_{\Omega_1}\int_{\Omega_1}\frac{g(x)g(y)}{(\sigma^2+r_g^2)^{p(N-2)}|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\geqslant\int_{\Omega_1}\int_{\Omega_1}\frac{g(x)g(y)}{(\sigma^2+r_g^2)^{p(N-2)}(|x|+|y|)^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\geqslant\frac{1}{(\sigma^2+r_g^2)^{p(N-2)}|2r_g|^\mu}\int_{\Omega_1}\int_{\Omega_1}g(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\\
&\geqslant\frac{1}{2^{p(N-2)+\mu}\cdot r_g^{2p(N-2)+\mu}}\int_{\Omega_1}g(x)\,\mathrm{d}x\int_{\Omega_1}g(y)\,\mathrm{d}y.
\end{aligned}\tag{4.3}$$

Similar to (4.3), we have

$$A_4\geqslant\frac{1}{2^{p(N-2)+\mu}\cdot r_g^{2p(N-2)+\mu}}\int_{\Omega_2}g(x)\,\mathrm{d}x\int_{\Omega_2}g(y)\,\mathrm{d}y\geqslant 0.\tag{4.4}$$

Keeping in mind that $g(x)>0$ on Ω_1 and $g(y)<0$ on Ω_2. Since $x\in\Omega_1$, $y\in\Omega_2$, $\Omega_1\cap\Omega_2=\emptyset$ and (A12), we have $|x-y|\geqslant 2r_\varepsilon$. Then

$$\begin{aligned}
A_2&\geqslant\int_{\Omega_2}\int_{\Omega_1}\frac{g(x)g(y)}{(\sigma^2+r_\varepsilon^2)^{p(N-2)}|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\geqslant\int_{\Omega_2}\int_{\Omega_1}\frac{g(x)g(y)}{(\sigma^2+r_\varepsilon^2)^{p(N-2)}|2r_\varepsilon|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\geqslant\frac{1}{2^{p(N-2)+\mu}\cdot r_\varepsilon^{2p(N-2)+\mu}}\int_{\Omega_1}g(x)\,\mathrm{d}x\int_{\Omega_2}g(y)\,\mathrm{d}y.
\end{aligned}\tag{4.5}$$

Similar to (4.5), we have

$$\begin{aligned} A_3 \geqslant & \frac{1}{2^{p(N-2)+\mu} \cdot r_\varepsilon^{2p(N-2)+\mu}} \int_{\Omega_2} g(x)\,\mathrm{d}x \int_{\Omega_1} g(y)\,\mathrm{d}y \\ = & \frac{1}{2^{p(N-2)+\mu} \cdot r_\varepsilon^{2p(N-2)+\mu}} \int_{\Omega_1} g(x)\,\mathrm{d}x \int_{\Omega_2} g(y)\,\mathrm{d}y. \end{aligned} \tag{4.6}$$

Combining (4.3), (4.5) and (4.6), we deduce that

$$\begin{aligned} A_1 + A_2 + A_3 \geqslant \frac{1}{2^{p(N-2)+\mu}} \int_{\Omega_1} g(x)\,\mathrm{d}x \Big(& \frac{1}{r_g^{2p(N-2)+\mu}} \int_{\Omega_1} g(y)\,\mathrm{d}y \\ & + \frac{2}{r_\varepsilon^{2p(N-2)+\mu}} \int_{\Omega_2} g(y)\,\mathrm{d}y \Big). \end{aligned} \tag{4.7}$$

By (A10), we have

$$\begin{aligned} \int_{\Omega_1} g(y)\,\mathrm{d}y > & 2\Big(\frac{r_g}{r_\varepsilon}\Big)^{4N} \int_{\Omega_2} (-g(y))\,\mathrm{d}y \\ > & 2\Big(\frac{r_g}{r_\varepsilon}\Big)^{4N-\mu} \int_{\Omega_2} (-g(y))\,\mathrm{d}y \\ > & 2\Big(\frac{r_g}{r_\varepsilon}\Big)^{2p(N-2)+\mu} \int_{\Omega_2} (-g(y))\,\mathrm{d}y. \end{aligned} \tag{4.8}$$

Inserting (4.8) into (4.7), we deduce that

$$A_1 + A_2 + A_3 > 0. \tag{4.9}$$

Inserting (4.4) and (4.9) into (4.2), we obtain

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)g(y)|w_\sigma(x)|^p |w_\sigma(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \geqslant A_1 + A_2 + A_3 + A_4 > 0.$$

□

We show that the functional I_2 satisfies the Mountain-Pass geometry, and estimate the Mountain-Pass levels.

Lemma 4.2. *Assume that the hypotheses of Theorem 1.3 hold. Then there exists a $(PS)_c$ sequence of I_2 at a level c, where $0 < c < c^* = \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$.*

Proof. **Step 1.** For any $u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}$, we have

$$I_2(u) \geqslant \frac{1}{2}\|u\|_D^2 - \frac{C_1}{2p}\|u\|_D^{2p} - \frac{1}{2 \cdot 2_\mu^* S_{H,L}^{2_\mu^*}} \|u\|_D^{2\cdot 2_\mu^*}.$$

We just prove that I_2 satisfies the above condition in Mountain-pass theorem, the others similar to Lemma 3.2.

Step 2. Using Lemma 4.1, there exists $\sigma \in [r_\varepsilon, r_g]$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)g(y)|w_\sigma(x)|^p |w_\sigma(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y > 0.$$

Let

$$
\begin{gathered}
B_1 := \int_{\mathbb{R}^N} |\nabla w_\sigma|^2 \,\mathrm{d}x > 0, \\
B_2 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)g(y)|w_\sigma(x)|^p |w_\sigma(y)|^p}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y > 0, \\
B_3 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_\sigma(x)|^{2_\mu^*} |w_\sigma(y)|^{2_\mu^*}}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y > 0.
\end{gathered} \tag{4.10}
$$

Set

$$
h_1(t) = I_2(tw_\sigma) = \frac{t^2}{2} B_1 - \frac{t^{2p}}{2p} B_2 - \frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*} B_3.
$$

We know that $h_1(0) = 0$, $\lim_{t\to\infty} h_1(t) = -\infty$. On the interval $[0,\infty)$, we can see that $h_1'(t) = 0$ if and only if

$$
h_1'(t) = tB_1 - t^{2p-1} B_2 - t^{2\cdot 2_\mu^* -1} B_3 = 0. \tag{4.11}
$$

From (4.11), we have

$$
B_1 = t^{2p-2} B_2 + t^{2\cdot 2_\mu^* -2} B_3. \tag{4.12}
$$

Define

$$
k(t) = t^{2p-2} B_2 + t^{2\cdot 2_\mu^* -2} B_3. \tag{4.13}
$$

By Lemma 4.1 and (4.10), we know that $B_2 > 0$. Observe $k(\cdot)$ is strictly increasing on $[0,\infty)$, $k(t) = 0$ if and only if $t = 0$, and $\lim_{t\to\infty} k(t) = \infty$. From the fact that $h_1'(t) = t(B_1 - k(t)) = 0$, we have two solutions t_1 and t_2 such that $t_1 = 0$ and t_2 satisfies

$$
B_1 = t_2^{2p-2} B_2 + t_2^{2\cdot 2_\mu^* -2} B_3. \tag{4.14}
$$

By $B_1, B_2, B_3 > 0$ and (4.14), we have $t_2 > 0$ and

$$
\begin{aligned}
h_1(t_2) &= \frac{t_2^2}{2} B_1 - \frac{t_2^{2p}}{2p} B_2 - \frac{t_2^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*} B_3 \\
&= \frac{t_2^2}{2}\Big(t_2^{2p-2} B_2 + t_2^{2\cdot 2_\mu^* -2} B_3\Big) - \frac{t_2^{2p}}{2p} B_2 - \frac{t_2^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*} B_3 \\
&= \Big(\frac{1}{2} - \frac{1}{2p}\Big) t_2^{2p} B_2 + \Big(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\Big) t_2^{2\cdot 2_\mu^*} B_3 > 0.
\end{aligned} \tag{4.15}
$$

So h_1 does not achieve its maximum at $t_1 = 0$.

Next, we prove that h_1 achieves its maximum at t_2. Applying (4.14), we know

$$
\begin{aligned}
h_1'(t) &= t(B_1 - k(t)) \\
&= t[(t_2^{2p-2} - t^{2p-2}) B_2 + (t_2^{2\cdot 2_\mu^* -2} - t^{2\cdot 2_\mu^* -2}) B_3] > 0, \quad \text{for } t \in (0, t_2),
\end{aligned} \tag{4.16}
$$

and

$$
h_1'(t) = t[(t_2^{2p-2} - t^{2p-2}) B_2 + (t_2^{2\cdot 2_\mu^* -2} - t^{2\cdot 2_\mu^* -2}) B_3] < 0, \quad \text{for } t \in (t_2, \infty). \tag{4.17}
$$

Let $t_3 = \big(2_\mu^* \cdot \frac{B_1}{B_3}\big)^{\frac{1}{2\cdot 2_\mu^* -2}}$. Since $t_3 > 0$, we have

$$
h_1(t_3) = \frac{t_3^2}{2} B_1 - \frac{t_3^{2p}}{2p} B_2 - \frac{t_3^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*} B_3 = -\frac{t_3^{2p}}{2p} B_2 < 0. \tag{4.18}
$$

Now, we claim that $t_2 < t_3$.

Suppose on the contrary that $t_2 = t_3$, we obtain

$$h_1(t_3) < 0 < h_1(t_2),$$

which contradicts with $t_2 = t_3$.

Suppose on the contrary that $t_2 > t_3$. Applying $h_1(t_1) = 0$, (4.18), (4.16) and $t_3 \in (t_1, t_2)$, we obtain

$$0 = h_1(t_1) < h_1(t_3) < 0,$$

which is a contradiction. Hence, $t_2 < t_3$.

According to Extreme value theorem, we know that h_1 achieves its maximum on compact set $[0, t_3]$. Applying $h_1'(t_2) = 0$, (4.16) and (4.17), we obtain that $h_1(t_2)$ is the maximum of h_1 on $[0, t_3]$.

By using (4.17) and $h_1(t_3) < 0$, we obtain $h_1(t) < 0$ for $t \in (t_3, \infty)$. Hence, we deduce that $h_1(t_2)$ is the maximum of h_1 on $[0, \infty)$.

Step 3. Set $h_2(t) = \frac{t^2}{2}B_1 - \frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}B_3$. Similar to Step 2, we obtain that the maximum of h_2 attained at $t_4 = \big(\frac{B_1}{B_3}\big)^{\frac{1}{2\cdot 2_\mu^* - 2}} > 0$.

Next, we prove that $t_4 > t_2$. By (4.11), we have

$$h_1'(t_4) = t_4(B_1 - t_4^{2\cdot 2_\mu^* - 1}B_3) - t_4^{2p-1}B_2 = -t_4^{2p-1}B_2 < 0.$$

Similar to the proof of $t_3 > t_2$ in Step 2, we know that $t_4 > t_2$.

Furthermore, we show that $\max_{t\geqslant 0} h_1(t) < \max_{t\geqslant 0} h_2(t)$. We have

$$\begin{aligned}\max_{t\geqslant 0} h_1(t) = h_1(t_2) =&\Big(\frac{1}{2} - \frac{1}{2p}\Big)t_2^{2p}B_2 + \Big(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\Big)t_2^{2\cdot 2_\mu^*}B_3\\ <&t_2^2\Big(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\Big)\Big(t_2^{2p-2}B_2 + t_2^{2\cdot 2_\mu^* - 2}B_3\Big)\\ =&t_2^2\Big(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\Big)B_1,\end{aligned} \tag{4.19}$$

and

$$\max_{t\geqslant 0} h_2(t) = h_2(t_4) = t_4^2\Big(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\Big)B_1. \tag{4.20}$$

According to (4.19), (4.20) and $t_2 < t_4$, we know

$$\begin{aligned}\max_{t\geqslant 0} h_1(t) <&t_2^2\Big(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\Big)B_1\\ <&t_4^2\Big(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\Big)B_1 = \max_{t\geqslant 0} h_2(t) = h_2(t_4).\end{aligned}$$

Step 4. From the above argument, we have

$$0 < c \leqslant \sup_{t\geqslant 0} I_2(tw_\sigma) = \max_{t\geqslant 0} h_1(t) < h_2(t_4) = \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}},$$

which means that $0 < c < \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. □

Lemma 4.3. *Assume that the hypotheses of Theorem 1.3 hold. If $\{u_n\}$ is a $(PS)_c$ sequence of I_2, then $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$.*

Proof. Similar to the proof of Lemma 3.3, we have Lemma 4.3. We omit it. □

To check that the functional I_2 satisfies $(PS)_c$ condition, we give the following Lemmas.

Lemma 4.4. *Assume that the hypotheses of Theorem 1.3 hold. If* $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ *is a sequence converging weakly to* $u \in D^{1,2}(\mathbb{R}^N)$ *as* $n \to \infty$*, then*

$$\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|u_n(x)-u(x)|^p|u_n(y)-u(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Proof. Set $v_n := u_n - u$, then we know $v_n \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^N)$. According to (A8), (A9) and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant g_{\max}^2 \lim_{n\to\infty}\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant g_{\max}^2 C\|v_n\|^{2p}_{L^{\frac{2Np}{2N-\mu}}(\Omega_1\cup\Omega_2)} \to 0 \quad (\text{since } \frac{2Np}{2N-\mu} \in (2,2^*)).
\end{aligned} \tag{4.21}$$

On the other hand,

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&= \lim_{n\to\infty}\Big(\int_{\Omega_1}\int_{\Omega_1} \frac{g(x)g(y)|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \int_{\Omega_1}\int_{\Omega_2} \frac{g(x)g(y)|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \int_{\Omega_2}\int_{\Omega_1} \frac{g(x)g(y)|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \int_{\Omega_2}\int_{\Omega_2} \frac{g(x)g(y)|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\Big)\\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{4.22}$$

Applying (A8) and (A9), we have

$$J_1 \geqslant 0 \quad \text{and} \quad J_4 \geqslant 0. \tag{4.23}$$

Keeping in mind that $g(x) > 0$ on Ω_1 and $g(y) < 0$ on Ω_2. According to $g_{\min} < 0$, $p \in (\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2}) \subset [1, 2^*)$ and Fubini's theorem, we obtain

$$\begin{aligned}
J_2 &\geqslant g_{\max}g_{\min} \lim_{n\to\infty}\int_{\Omega_1}\int_{\Omega_2} \frac{|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\geqslant \frac{g_{\max}g_{\min}}{(2r_\varepsilon)^\mu} \lim_{n\to\infty}\int_{\Omega_1}\int_{\Omega_2} |v_n(x)|^p|v_n(y)|^p\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{g_{\max}g_{\min}}{(2r_\varepsilon)^\mu}\Big(\lim_{n\to\infty}\int_{\Omega_1}|v_n(x)|^p\,\mathrm{d}x\Big)\Big(\lim_{n\to\infty}\int_{\Omega_2}|v_n(y)|^p\,\mathrm{d}y\Big) \to 0.
\end{aligned} \tag{4.24}$$

Similar to (4.24), we have

$$J_3 \geqslant 0. \tag{4.25}$$

Putting (4.23)–(4.25) into (4.22), we obtain

$$\lim_{n\to\infty}\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \geqslant 0. \tag{4.26}$$

Combining (4.21) and (4.26), we have

$$0 \leqslant \lim_{n\to\infty} \int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \leqslant 0.$$

Thus,

$$\lim_{n\to\infty} \int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y = 0.$$

□

Lemma 4.5. *Assume that the hypotheses of Theorem 1.3 hold. If $\{u_n\}$ is a bounded sequence in $L^{\frac{2Np}{2N-\mu}}(\Omega_1\cup\Omega_2)$ such that $u_n \to u$ a.e. on $\Omega_1\cup\Omega_2$ as $n\to\infty$, then*

$$\begin{aligned}
&\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|u_n(x)|^p|u_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&-\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|u(y)|^p|u(x)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|u_n(x)-u(x)|^p|u_n(y)-u(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y + o(1),
\end{aligned}$$

Proof. **Step 1.** Since $\{u_n\}$ is a bounded sequence in $L^{\frac{2Np}{2N-\mu}}(\Omega_1\cup\Omega_2)$ and $u_n \to u$ a.e. on $\Omega_1\cup\Omega_2$ as $n\to\infty$. Set $v_n := u_n - u$. By using Hardy-Littlewood-Sobolev inequality and Brézis-Lieb lemma in [3], we have

$$\int_{\Omega_i} g(x)|u_n|^p\,\mathrm{d}x = \int_{\Omega_i} g(x)|v_n|^p\,\mathrm{d}x + \int_{\Omega_i} g(x)|u|^p\,\mathrm{d}x + o(1), \tag{4.27}$$

for $= 1,2$, as $n\to\infty$, and

$$|x|^{-\mu} * (g|u_n|^p)(\Omega_i) = |x|^{-\mu} * (g|v_n|^p)(\Omega_i) + |x|^{-\mu} * [g|u|^p](\Omega_i) + o(1), \tag{4.28}$$

for $i = 1,2$, as $n\to\infty$.

By using [11, Lemma 2.3], we know that

$$\begin{aligned}
&\int_{\Omega_i} \big(|x|^{-\mu} * (g|u_n|^p)(\Omega_i)\big) g|u_n|^p\,\mathrm{d}x\\
&= \int_{\Omega_i} \big(|x|^{-\mu} * (g|v_n|^p)(\Omega_i)\big) g|v_n|^p\,\mathrm{d}x\\
&\quad + \int_{\Omega_i} \big(|x|^{-\mu} * (g|u|^p)(\Omega_i)\big) g|u|^p\,\mathrm{d}x + o(1), \quad (i=1,2),\ \text{as } n\to\infty.
\end{aligned} \tag{4.29}$$

Step 2. Since $y \in \Omega_1$, $x \in \Omega_2$ and (A12), we have $2r_\varepsilon \leqslant |x-y| \leqslant 2r_g$. According to the properties of convolution and Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\Omega_2} \big(|x|^{-\mu} * (g|v_n|^p)(\Omega_1)\big)\, g(|u_n|^p - |v_n|^p)\,\mathrm{d}x \\
&= \int_{\Omega_2}\int_{\Omega_1} \frac{g(y)|u_n(y)|^p}{|x-y|^\mu} g(x)(|u_n(x)|^p - |v_n(x)|^p)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\Omega_1}\int_{\Omega_2} \frac{g(x)|u_n(x)|^p}{|y-x|^\mu} g(y)|u_n(y)|^p\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\Omega_1}\int_{\Omega_2} \frac{g(x)|v_n(x)|^p}{|y-x|^\mu} g(y)|u_n(y)|^p\,\mathrm{d}x\,\mathrm{d}y \quad \text{(Fubini's theorem)} \\
&= \int_{\Omega_1} \big(|y|^{-\mu} * g(|u_n|^p)(\Omega_2)\big)\,(g|u_n|^p)\,\mathrm{d}y \\
&- \int_{\Omega_1} \big(|y|^{-\mu} * g(|v_n|^p)(\Omega_2)\big)\,(g|u_n|^p)\,\mathrm{d}y \quad \text{(convolution)} \\
&= \int_{\Omega_1} \big(|y|^{-\mu} * g(|u_n|^p)(\Omega_2) - |y|^{-\mu} * g(|v_n|^p)(\Omega_2)\big)\,(g|u_n|^p)\,\mathrm{d}y \\
&= \int_{\Omega_1} \big(|y|^{-\mu} * g(|u_n|^p - |v_n|^p)(\Omega_2)\big)\,(g|u_n|^p)\,\mathrm{d}y \quad \text{(distributivity).}
\end{aligned} \tag{4.30}$$

Since $\{u_n\}$ is a bounded sequence in $L^{\frac{2Np}{2N-\mu}}(\Omega_1)$ and $u_n \to u$ a.e. on Ω_1 as $n \to \infty$, we have

$$g(y)|v_n(y)|^p \rightharpoonup 0 \text{ in } L^{\frac{2Np}{(2N-\mu)(p-1)}}(\Omega_1), \quad \text{as } n \to \infty. \tag{4.31}$$

Similar to (4.31), we obtain

$$g(x)|v_n(x)|^p \rightharpoonup 0 \quad \text{in } L^{\frac{2Np}{(2N-\mu)(p-1)}}(\Omega_2), \quad \text{as } n \to \infty. \tag{4.32}$$

Step 3. Taking the limit $n \to \infty$, by (4.27), (4.28) and (4.30)–(4.32), we deduce that

$$\begin{aligned}
&\int_{\Omega_2} \big(|x|^{-\mu} * (g|u_n|^p)(\Omega_1)\big)\, g|u_n|^p\,\mathrm{d}x - \int_{\Omega_2} \big(|x|^{-\mu} * (g|v_n|^p)(\Omega_1)\big)\, g|v_n|^p\,\mathrm{d}x \\
&= \int_{\Omega_2} \big(|x|^{-\mu} * (g(|u_n|^p - |v_n|^p))(\Omega_1)\big)\, g(|u_n|^p - |v_n|^p)\,\mathrm{d}x \\
&\quad + \int_{\Omega_2} \big(|x|^{-\mu} * (g(|u_n|^p - |v_n|^p))(\Omega_1)\big)\, g|v_n|^p\,\mathrm{d}x \\
&\quad + \int_{\Omega_1} \big(|y|^{-\mu} * g(|u_n|^p - |v_n|^p)(\Omega_2)\big)\, g|v_n|^p\,\mathrm{d}y \quad \text{(by (4.30))} \\
&= \int_{\Omega_2} \big(|x|^{-\mu} * (g|u|^p))(\Omega_1)\big)\, g|u|^p\,\mathrm{d}x + o(1).
\end{aligned} \tag{4.33}$$

Similar to (4.33), we obtain

$$\begin{aligned} &\int_{\Omega_1} \big(|x|^{-\mu} * (g|u_n|^p)(\Omega_2)\big)\, g|u_n|^p \,\mathrm{d}x \\ &= \int_{\Omega_1} \big(|x|^{-\mu} * (g|v_n|^p)(\Omega_2)\big)\, g|v_n|^p \,\mathrm{d}x \\ &\quad + \int_{\Omega_1} \big(|x|^{-\mu} * (g|u|^p)(\Omega_2)\big)\, g|u|^p \,\mathrm{d}x + o(1), \quad \text{as } n \to \infty. \end{aligned} \tag{4.34}$$

Combining (4.29), (4.33) and (4.34), we obtain

$$\begin{aligned} &\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|u_n(x)|^p|u_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|v_n(x)|^p|v_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|u(y)|^p|u(x)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y + o(1) \quad \text{as } n \to \infty. \end{aligned}$$

□

Lemma 4.6. *Assume that the hypotheses of Theorem 1.3 hold. If $\{u_n\}$ is a bounded sequence in $D^{1,2}(\mathbb{R}^N)$, up to a subsequence, $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)g(y)|u_n(x)|^p|u_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)g(y)|u(x)|^p|u(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.$$

In addition, for any $\varphi \in D^{1,2}(\mathbb{R}^N)$,

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)g(y)|u_n(y)|^p|u_n(x)|^{p-2}u_n(x)\varphi(x)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &\to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)g(y)|u(y)|^p|u(x)|^{p-2}u(x)\varphi(x)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

as $n \to \infty$.

Proof. **Step 1.** By Lemmas 4.4 and 4.5, we have

$$\begin{aligned} &\int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|u_n(x)|^p|u_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\Omega_1\cup\Omega_2}\int_{\Omega_1\cup\Omega_2} \frac{g(x)g(y)|u(y)|^p|u(x)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y + o(1), \quad \text{as } n \to \infty. \end{aligned} \tag{4.35}$$

Since $g \equiv 0$ in $\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)$, by (4.35), we have

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)g(y)|u_n(x)|^p|u_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &\to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)g(y)|u(x)|^p|u(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \quad \text{as } n \to \infty. \end{aligned}$$

Step 2. By the Hardy-Littlewood-Sobolev inequality,

$$\int_{\Omega_i} \frac{g(y)|u_n|^p}{|x-y|^\mu}\,\mathrm{d}y \rightharpoonup \int_{\Omega_i} \frac{g(y)|u|^p}{|x-y|^\mu}\,\mathrm{d}y \quad \text{in } L^{\frac{2N}{\mu}}(\Omega_i), (i=1,2), \text{ as } n \to \infty. \tag{4.36}$$

From (4.36), we obtain

$$\int_{\Omega_1\cup\Omega_2} \frac{g(y)|u_n|^p}{|x-y|^\mu}\,\mathrm{d}y \rightharpoonup \int_{\Omega_1\cup\Omega_2} \frac{g(y)|u|^p}{|x-y|^\mu}\,\mathrm{d}y, \quad \text{in } L^{\frac{2N}{\mu}}(\Omega_1\cup\Omega_2) \quad \text{as } n\to\infty. \tag{4.37}$$

Since $g \equiv 0$ in $\mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2)$, by (4.37), we have

$$\int_{\mathbb{R}^N} \frac{g(y)|u_n|^p}{|x-y|^\mu}\,\mathrm{d}y \rightharpoonup \int_{\mathbb{R}^N} \frac{g(y)|u|^p}{|x-y|^\mu}\,\mathrm{d}y \quad \text{in } L^{\frac{2N}{\mu}}(\mathbb{R}^N) \text{ as } n\to\infty. \tag{4.38}$$

Similar to (4.38), we obtain

$$g(x)|u_n|^{p-2}u_n \rightharpoonup g(x)|u|^{p-2}u \quad \text{in } L^{\frac{2Np}{(2N-\mu)(p-1)}}(\mathbb{R}^N) \text{ as } n\to\infty. \tag{4.39}$$

Combining (4.38) and (4.39), as n $\to\infty$, we find that

$$\Big(\int_{\mathbb{R}^N} \frac{g(y)|u_n|^p}{|x-y|^\mu}\,\mathrm{d}y\Big) g(x)|u_n|^{p-2}u_n \rightharpoonup \Big(\int_{\mathbb{R}^N} \frac{g(y)|u|^p}{|x-y|^\mu}\,\mathrm{d}y\Big) g(x)|u|^{p-2}u$$

in $L^{\frac{2Np}{2Np-2N+\mu}}(\mathbb{R}^N)$. Thus, for any $\varphi \in D^{1,2}(\mathbb{R}^N)$,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)g(y)|u_n(y)|^p|u_n(x)|^{p-2}u_n(x)\varphi(x)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\
&\to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)g(y)|u(y)|^p|u(x)|^{p-2}u(x)\varphi(x)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

$\square$

Lemma 4.7. *Assume that the hypotheses of Theorem 1.3 hold. If $\{u_n\}$ is a $(PS)_c$ sequence of I_2 with $0 < c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$, then $\{u_n\}$ has a convergent subsequence.*

The proof of the above lemma is similar to that of Lemma 3.5. We omit it.

Proof of Theorem 1.3. Applying Lemma 4.2, we obtain that I_2 possesses a mountain pass geometry. Then from the Maintain Pass Theorem, there is a sequence $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ satisfying $I_2(u_n) \to c$ and $I_2'(u_n) \to 0$, where

$$0 < \vartheta \leqslant c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

Moreover, according to Lemma 4.3 and Lemma 4.7, $\{u_n\}$ satisfying $(PS)_c$ condition. Hence, we have a nontrivial solution $\tilde{u}_0$ to problem (1.7). $\square$

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