BROWNIAN MOTION APPLIED

TO HUMAN INTERSECTIONS

THESIS

Presented to the Graduate Council of
Texas State University-San Marcos
in Partial Fulfillment
of the Requirements

for the Degree

Master of SCIENCE

by

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San Marcos, Texas
August 2012

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TO HUMAN INTERSECTIONS

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ACKNOWLEDGEMENTS

I would like to thank my committee members, Doctors Singh, Passty, and White from the Department of Mathematics at Texas State University-San Marcos, for their support and assistance in creating this thesis.

This manuscript was submitted on 2 May, 2012.

TABLE OF CONTENTS

LIST OF FIGURES

Figure **Page**

ABSTRACT

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In this paper, we will adapt Einstein's work on Brownian Motion to pedestrian movement and then use that information to prove three hypotheses:

1) *In general, pedestrian movement at an intersection is not a Brownian Motion.*

2) *Pedestrian movement at intersections in high density cities is a Brownian Motion.*

3) *Pedestrian movement at intersections on a university campus during normal business hours on normal business days is a Brownian Motion.*

We will attempt this by examining the concept of Brownian Motion as presented by three of its main founders, Brown, Weiner, and Einstein, as well as many applications, and then summarizing Einstein's work on developing a diffusivity coefficient. We will then adapt Einstein's Brownian Coefficient of Diffusivity from the molecular case to pedestrian movement. It is during this process that we will prove or disprove our three hypotheses. Finally, we will analyze video logs to determine if the theory holds, and if not, then why it failed.

CHAPTER I

AUTHOR'S INTEREST

Brownian Motion is formally defined as the movement of microscopic particles suspended in a liquid or gas, caused by collisions between these particles and the molecules of the liquid or gas, though it is commonly used in mapping many macroscopic random motions that exist in our world. My first introduction to Brownian Motion was in a Thermodynamics class one year ago, though then it was called the "Drunk Walk". It was called such because, much like walking drunk, there is roughly equal probability of stumbling forward as there is stumbling in any other direction. Always being a math nerd, I fervently believed that everything could be described through math, especially considering that most rigorous sciences use mathematical foundations; but what about human movements? We aren't mindless particles that travel randomly or a liquid that follows the geographical path of least resistance, but instead we make intelligent decisions for the directions we choose. How could math describe something such as this? My key to this problem was probability.

We have many **Decomposition Theorems** throughout the scientific community, ranging from the **Helmholtz Decomposition Theorem**, which states that any smooth vector space disappearing at infinity can be decomposed into rotational and non-rotational pieces, to the basis of **Ito Stochastic Differential Equations.** This last concept is probably the most closely related to what I have come to believe. According to Kiyoshi Ito (1915-2008) any process can be defined as[1]:

$$X_t = X_0 + \int_0^T \alpha(t, X_t)dt + \int_0^T \sigma(t, X_t)dW_t$$

where X_t is a given process, X_0 is an arbitrary function, α is the drift (the change in the average value of X_t), and σ dW_t is the white noise (a random process with mean 0 as $t \rightarrow \infty$). This formula can essentially be used as a rough definition for what a process is. W_t is also known as the **Wiener Process**, named after its creator Norbert Wiener, a man we will provide some background on in the next chapter.

The **Wiener Process** is a continuous-time stochastic process that is often called the standard Brownian Motion. This is one of the best known stochastic processes that utilizes stationary and independent increments. To summarize it briefly, the Wiener Process, W_t, is characterized by:

1) *$W_0=0$;*
2) *W_t, $t \geq 0$, is almost surely continuous; i.e. the probability of this event occurring is one.*
3) *W_t has independent increments with $W_t - W_s \approx N(0,t-s)$ where $N(\mu,\sigma)$ is the normal distribution with expected value μ and variance σ.*

Ito's formula provides a concrete statement for a thought that I can only describe with words, that every process is composed of two pieces, the deliberate or intelligent, and the random. With this belief in mind, I began searching for one of the most deliberate

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1 Goodman, Jonathan. "Stochastic Calculus Lecture 7." New York University http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc/notes/l9.pdf (accessed November 16, 2011)

processes we know of, but a process that I could still remove any sign of forced movement from if applied properly, Human Movement.

CHAPTER II

HISTORY OF BROWNIAN MOTION

§Applications Today

Prior to two years ago, I had never even heard of Brownian Motion, much less known what it was, and from the looks of my classmates, I was not the only one. As humans the first thing we always want to know with a new piece of knowledge is "When can I use this?" Brownian Motion is very frequently used and has found its way into almost every aspect of our lives.

Fractional Brownian Motion, a concept we will discuss later, is the most widely used method for determining irregularities in cloud formations while simultaneously allowing us to better predict weather patterns. In dealing with weather, multiple aspects come into play, such as temperature and humidity. Meteorologists make several basic assumptions when dealing with forecasting weather patterns:

1) *The heat index, a number determined using temperature and humidity, is separable and linear;*
2) *Temperature forecasts are normally distributed;*

3) Statistics of forecast uncertainty are independent of current location and flow direction.

As the reader will see, subject to these assumptions, weather patterns can be described using Brownian Motion and thus be predicted within a smaller percentage of error.[2]

In medicine, this same concept, **Fractional Brownian Motion**, is used in conjunction with ultrasonic imaging to detect abnormalities in the size or shape of organs, leading to faster and more accurate detection of cancerous tumors.[3] Cancer is classified as an abnormal growth in the organism, in our case the human body. When taking an image through ultrasonic imaging, the picture that results is fuzzy because of the body's movement, both systematic and random, passing fluids' varying density, or even minor changes in the intensity of the waves sent out. The solution to this is to break every image down into its individual pixels and to map their movement instead of viewing the image as a whole. For each pixel, we then determine its most probable location through the use of **Fractional Brownian Motion**. This provides a much more precise placement of each individual pixel, and as a result, a more accure entire image.[4]

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2 Brix, Anders, Stephen Jewson, and Christine Ziehmann. Weather Derivative Valuation: The Meteorological, Statistical, Financial, and *Mathematical Foundations.* Cambridge, United Kingdom: Cambridge University Press, 2005

3 Grupo de Fisica Matematica da Universidade de Lisboa. “Uses of Brownian Motion in Brain Imaging and Neuroscience.” GFM Seminar. http://gfm.cii.fc.ul.pt/events/seminars/20070614-lori (accessed October 10, 2011)

4 Chen, C. C., J. S. Daponte, and M. D. Fox. "Fractal feature analysis and classification in medical imaging." IEEE Trans Med Imaging. http://www.ncbi.nlm.nih.gov/pubmed/18230510 (accessed October 24, 2011

Fractional Brownian Motion is a continuous process with expected value zero for its integral. This motion is a continuous Gaussian process, and is also considered to be the only self-similar Gaussian Process we currently know of. That is, a mapping of Fractional Brownian Motion would appear the same regardless of what section is currently being viewed, or if a portion was enlarged, it would also appear the same as the rest of the mapping.

Chemical solutions follow **Einstein's Diffusivity Model**, which will be further explained in Chapter III and derived in Chapter IV, and allow toxic gasses to be properly mapped and contained. This is because of the tendency for gaseous and liquid solutions to continue moving out in every direction until they are evenly distributed and at equilibrium with their surroundings. However, it is impossible to say that the introduced solution will move outward as a perfect sphere because of multiple variables such as the inherent Brownian Motion already present in the surroundings or any motion the surroundings input can be taking as a whole, such as wind or flowing water.[5]

Even in our own economics, Brownian Motion has been used to describe certain phenomena that exist in stock markets. In 1959, the Naval Research Laboratory in Washington, D.C. discovered that the stock market, as well as other financial markets, moves in the same general form as suspended molecules do. At first glance, this would appear to be false, but there are a few basic rules that the stock market must operate by, the most useful in this case being that any rise or fall must occur as a multiple of 1/8 of

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[5] Imperial College of London. "Applications of Brownian Motion: Article Two." Surprise 95. http://www.doc.ic.ac.uk/~nd/surprise_95/journal/vol2/ykl/article2.html (accessed October 10, 2011)

a dollar at the time that this paper was written. This turns a formerly fluid problem into a discrete one, setting it up for the first comparison to molecules. Secondly, the number of trades in a day must be finite. Furthermore, an average value can be assumed for the number of trades in a day, though the actual number can certainly go above or below. Using these pieces of information, the researchers at NRL began to construct what was initially an inductive proof by taking many samples of the closing sales data. Once the data were analyzed and a Brownian Motion was established, they began work on proving this deductively. The final conclusion was that buyers and sellers operate off of an average 1:1 ratio that can easily go one way or the other on a daily basis.[6]

Brownian Motion is all around us and is becoming increasingly useful with every new detail we learn about it, but now we are left with the obvious question of "What is Brownian Motion?". Unfortunately that is a long answer, and to properly describe it we must start at the beginning with the man who discovered it.

§Founders

Since its discovery, there have been many great leaps in our understanding, but throughout the last two centuries, there are three men whose work has stood out the most: Robert Brown, the discoverer, Albert Einstein, the man who was first able to

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[6] US Naval Research Laboratory. "Brownian Motion In The Stock Market." Operations Research.http://www.e-m-h.org/Osbo59.pdf (accessed October 10, 2011)

describe the motion, and Norbert Wiener, a mathematician that, while not known for his work on this subject, made the most advancements in one-dimensional Brownian Motion. In this section, we will provide a small amount of background on these men.

Brownian Motion, named after its discoverer, was first described by an English botanist named Robert Brown (1773-1858) in the year 1827. A scientist at heart, Robert Brown spent most of his life in the medical field, first as a field surgeon in 1795, and then as the first person to discover the "naked ovule in the gymnospermae".[7] While watching the pollen suspended in air, Brown noticed that the particles were in a constant state of seemingly random motion. When curiosity overtook him, Brown set the pollen in a still glass of water and noticed that even there, the pollen was in a similar state of constant motion. Since his first experiment involved "living plant specimens he was led to ask whether it persisted in plants that were dead." Eleven months later, Brown used grains that had been preserved in alcohol and were undoubtedly devoid of life. The same results ensued, leading the scientist to try the same experiment with a variety of stones that were pulverized into a fine powder. Once more, the same random and constant motion was viewed. Unfortunately, because of his current technology and the lack of development of certain mathematical and physical theories at the time, Robert Brown was unable to provide further insight.

Albert Einstein (1879-1955), one of the main founders of Brownian Motion, was born in Germany, but moved to Switzerland in 1896 where he enrolled at the Swiss Federal

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[7] Brian J. Ford, "Brownian Movement in Clarkia Pollen: A Reprise of the First Observations." http://www.sciences.demon.co.uk/wbbrowna.htm (accessed October 10, 2011)

Polytechnic School in Zurich. He went there in order to become a teacher in physics and mathematics, and in 1901, Einstein earned his diploma, as well as gained Swiss citizenship. Even though he had the degree, Albert Einstein was unable to procure a teaching position and instead found a job in the Swiss Patent Office, all the while still attending school for his graduate education. In 1905, a year commonly referred to as his “Miracle Year”, Einstein received his doctorate. Over the next few decades, Einstein moved from university to university, always taking a position in the department of physics. His jobs ranged from Professor to Director. In 1933, Albert Einstein emigrated to the United States and once again acquired a position as a physics professor at Princeton. At the end of World War II, shortly after he had retired from his teaching position, Albert Einstein was offered Presidency of the State of Isreal, a position that he declined, choosing instead to establish the Hebrew University of Jerusalem.[8]

Norbert Wiener (1894-1964) was considered one of the few child prodigies in mathematics. Born in Columbia, Missouri, Wiener earned his Ph.D. from Harvard at the age of 18. He then went on to study a variety of subjects such as philosophy, logic, and mathematics while attending school in Cambridge, England. In 1919, Norbert Wiener gained a teaching position for Mathematics at MIT (Massachusetts Institute of Technology), gaining his assistant professorship by 1929, and becoming a full professor in 1931 at the young age of 37. Norbert Wiener worked on a variety of projects, ranging from anti-aircraft devices in 1940 to his work on Brownian Motion in 1930. However, he

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[8] "Albert Einstein-Biography." Nobelprize.org http://www.nobelprize.org/nobel_prizes/physics/laureates/1921/einstein-bio.html (accessed November 29, 2011)

is most known for his work in Cybernetics as well as the theories behind data transfer. In regards to Brownian Motion, Norbert Wiener made his name with the **Wiener Process,** the process summarized in Chapter 1.[9]

[9] Robert Vallee. "Norbert Wiener-Biography." International Society for the Systems Sciences. http://www.isss.org/lumwiener.htm (accessed November 28, 2011)

CHAPTER III

EINSTEIN'S NEW APPROACH

§Einstein's Paper

Though many mathematicians tried to conquer Brownian Motion since its discovery, the true breakthrough did not come until 1905 when Albert Einstein arrived at a solution in his dissertation. Until Einstein began his work on Brownian Motion, the majority of the methods tried were aimed at describing the motion of each individual particle through velocities.[10] They were working in this manner because they were working off of the Equipartition Theorem which informally states that "energy is shared equally amongst all energetically accessible degrees of freedom of a system."[11] During the 1870's, a number of scientists and mathematicians attempted to use the **Kinetic Theory of Heat** to explain the movement, much like Einstein did, but were unable to overcome a counterargument presented by Karl von Nageli in 1879 who, after using the **Equipartition Theorem**, showed that because the mass of the particles were so high in comparison to the surrounding liquid particles, the velocities of the particles would have to be "vanishingly

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[10] John Stachel, Einstein's Miraculous Year (Princeton: Princeton University Press, 2005)

[11] "The Equipartition Theorem." Department of Chemistry, Oxford University. http://vallance.chem.ox.ac.uk/pdfs/Equipartition.pdf (accessed October 3, 2011)

small." (Stachel 2005,75) Instead, Einstein chose to focus on the **Osmotic Pressure** and the **Mean-Square Displacement** of particles under observation rather than their individual motions. Finally, he related the motions to the **Molecular Theory of Heat** and the **Macroscopic Theory of Dissipation**. In his "Second Paper" as it is known since it was the second paper during his "Miraculous Year", Einstein explained why these theories could be used, writing that "According to this theory, a dissolved molecule differs from a suspended body in size only, and it is difficult to see why suspended bodies should not produce the same osmotic pressure as an equal number of dissolved molecules." (Stachel 2005, 86)

To provide a better basis for our analysis of Einstein's methods, the **Molecular Theory of Heat**, a model most commonly used to describe gaseous or suspended particles, begins with five assumptions:

1) A macroscopic volume contains a large number of particles.

2) The separation between particles is relatively large

3) There are no forces between particles except when they collide.

4) All collisions are elastic

5) The particles' directions are random

Also, the **Macroscopic Theory of Dissipation** states that :

> *The tendency to maximize multiplicity predicts that when there is inequity in a given quantity (e.g. concentration of particles), there will be a net movement of this quantity from areas of higher concentration to areas of lower concentration until an equilibrium is reached.*[12]

With these two theorems in mind, Einstein began his work. In a paper he wrote to Conrad Habicht, a prominent physicist of the time and one of his close friends, Einstein stated that:

> …on the assumption of the molecular theory of heat, bodies on the order of magnitude 1/1000mm, suspended in liquids, must already perform an observable random motion that is produced by thermal motion; in fact, physiologists have observed <unexplained> motions of suspended small, inanimate, bodies, which motions they designate as "Brownian molecular motion. (Stachel 2005, 78)

His next step was to show that the movements of particles were independent of one another. He did this by assuming that at some point the liquid must reach a point of equilibrium, i.e. the solution will be stagnant and of a single temperature. Since it could happen to one glass, it could happen to another. So long as the number of particles contained in each glass was the same, Einstein showed that their velocities were independent of volume, position, and by extension, time (Einstein 1956, 8). The exact calculations and method will be explained further in Chapter IV when we begin construction of the diffusivity constant.

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[12] Maria Spies. "Macroscopic Theory of Dissipation." University of Illinois http://www.life.illinois.edu/biophysics/401/Files/083111notes.pdf (accessed October 10, 2011)

§*Post Dissertation Discoveries*

Though Einstein did manage to prove the irrefutable existence of Brownian Motion and create an expression to describe it as well, there was still work to be done. Only one year later, Einstein released another paper on "Brownian Movement".[13] This was brought about because of work done by Professor Gouy who had direct observations that showed "…Brownian Motion is caused by the irregular thermal movements of the molecules of the liquid." (Einstein 1956, 19) Einstein, after reading Gouy's paper, began work on considering the rotation of the particles as well as their positional displacement.

The first case he worked on was considering the liquid to be at thermal equilibrium, or rather the heat entering the solution equaled the heat leaving to the surroundings. Following a similar process to before, Einstein showed that adding a rotational component to the particles does not affect the probability that the particle will be found in a certain area. Though this was a simple concept to show, it had enormous implications. Not only did it show that a rotational spin leaves the solution relatively unaffected, but it also played a part when energy was being transferred into or out of the liquid. (Einstein 1956, 22)

Einstein assumed that the heat was working on an indefinitely small portion of the liquid. This preserved the generality of adding heat while simultaneously allowing for unimaginably complex systems of heat transfer. However, by sticking to this method, the

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[13] Albert Einstein. *Investigations on the Theory of the Brownian Movement.* New York, NY: Dover Publications, 1956

equation was affected only by a constant in regards to the liquid at equilibrium for each partition. "This relation, which corresponds exactly with the exponential law frequently used by Boltzmann in his investigations in the theory of gases, is characteristic of the molecular theory of heat." (Einstein 1956, 23)

Einstein continued on with his work, discovering new properties for unique situations such as un-dissociated solvents in liquids, relatively large particles in small amounts of liquid, and with every step he continued to incorporate all previous characteristics. However, the next paper of his that we will cover here is one submitted two years later in January of 1907. After reading Svedberg's "*Zeit. f. Elektrochemie"*, Einstein wrote a paper to "point out some properties of this motion indicated by the molecular theory of heat." (Einstein 1956, 63) He summarized these properties into what he called Theoretical Observations.

The first theory involved the ability to "calculate the mean value of the instantaneous velocity" of a particle at any particular temperature. This implies, much like his earlier solutions showed, that the velocity of a particle is independent of all other extrinsic values, such as its own size, as well as all extrinsic and intrinsic characteristics of the solution in which the particle is suspended in. (Einstein 1956, 63)

The second theory was that the velocity of these particles would be impossible to determine through observation with the ultra microscope. This is because every suspended particle of this nature has continuous and random impulses that force it to continue moving, but in no particular or consistent direction. Because Brownian Motion is independent of the time intervals we use to describe it, then the particles could in theory be changing their velocities at immeasurably small times. (Einstein 1956, 65)

The third theory was that the velocity of a particle has no limiting factor. When looked at over a particular path length, any velocity ensued would appear as an instantaneous move and could not be measured, even if the system could be observed. This is because, much like above in the second theory, the velocity of the particle is independent of the time segment, but as velocity is the change in position over a time, then as the time interval decreases, the velocity inversely increases without limit. It should be noted here that, though the velocity is increasing inversely to time, the distance travelled is decreasing proportionally with it. This places a restriction on time never decreasing below the minimum interval of time required for a molecule to make a single movement, as we will see in the next chapter. During Einstein's time, this extremely small interval could never be measured; however, times have changed. (Einstein 1956, 67)

Physicists from University of Texas and Institut de Physique de la Matiere Complexe in Switzerland have recently been able to witness and track the velocity of a single particle obeying Brownian Motion. In March, 2011, Huang et al. released a paper detailing the experiment that was created for this particular purpose.[14]

Using a 75 MHz bandwidth optical trap with sub-angstrom spatial precision, as well as a dichroic mirror and condenser lenses to increase the precision of the sensor, a single particle was tracked and observed through its various impulse velocities, each of which was mapped instantly with a velocity autocorrelation function. The results were that

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[14] Huang, Rongxin, I. Chavez, K. Tuate, B. Lukic, S. Jeney, M. Raisen, And E. Florin. "Direct observation of the full transition from ballistic to diffusive Brownian Motion in a liquid". Nature Physics. http://chaos.utexas.edu/wp-uploads/2011/02/27.-2011-Nature-Physics.pdf (accessed October 31, 2011)

Einstein's descriptions were extremely accurate, but with the technology now at our disposal, we can now make even more precise equations to describe this random movement.

CHAPTER IV

DEVELOPMENT OF THE DIFFUSIVITY CONSTANT

§ List of Symbols

E	Energy	T	Temperature
F	Free Energy	t	Time
k	Viscosity	τ	Time Interval
n	Number of Suspended Particles	V	Volume
N	Number of Molecules	v	Particle Density in Solution
ρ	Osmotic Pressure	μ	Mass of a Particle
S	Entropy	λ_x	Displacement of Particle
R	Gas Law Constant	z	Molecules
P	Radius of Particle (Assumed Spherical)		

§Einstein's Diffusivity Coefficient

In this section we will begin construction of the diffusivity coefficient discovered by Einstein in his 1905 dissertation. We will follow his work closely, though since much of the theory was discussed in the previous chapter, we will be working more with the applied portions. All work in this section is adapted directly from Albert Einstein's *Investigations on the theory of the Brownian Movement.*

Our first step is to assume that we have a liquid with a semi-permeable wall and that only one side of this solution has suspended particles. We will call the volume of this side of the solution *V**. If the ratio of space to particles, *V*/z*, is sufficiently large, then we have the General Gas Law:

$$\mathrm{V}^{*}\rho = RTz \tag{1}$$

When looking at this same situation in a stagnant solution, i.e. one with no flow or fluid movement, then by the classical theory of thermodynamics, there should be no force acting against our partition. Note here that we can ignore the force of gravity because we have already determined that this force is negligible in this situation since our particles are in a perpetually suspended state.

According to the molecular theory of heat, a "dissolved molecule is differentiated from a suspended body solely by its dimensions,…" (Einstein 1956, 3) Thus, it is only logical that suspended particles would create the same amount of osmotic pressure as the dissolved molecules. Einstein argues that then the particles must not be stationary. In other words, the suspended particles must be in a constant state of irregular motion, even if this motion is a very slow one. Taking this into account, we can now relate (1) in terms of the number of molecules *V** possesses in a unit volume, *n/V*=v*. Because we are dealing with suspended particles, we can now assume that there is some amount of separation between them, because if there were not then they would be the same particle. We now arrive at:

$$\rho = \frac{RTn}{NV^{*}} = \frac{RTv}{N} \tag{2}$$

where N is the number of molecules contained in a gram-molecule.

Next we will define an arbitrary list of quantities, $p_1, p_2, \ldots, p_m$, which completely define the instantaneous conditions, such as location and velocity of all particles, for our system. Since these conditions are instantaneous, then they must have some dependency on time. Writing:

$$\frac{\partial p_v}{\partial t} = \Phi_v(p_1, \ldots, p_m) \quad (v = 1, 2, \ldots m) \tag{3}$$

leads to:

$$\Sigma \frac{\partial \Phi_v}{\partial p_v} = 0 \tag{4}$$

It then follows that the entropy, defined by Webster's Dictionary as a function of thermodynamic variables, such as temperature, pressure, or composition, that is a measure of the energy that is not available for work during a thermodynamic process, is given by:

$$S = \frac{\bar{E}}{T} + 2x\ln\left[\int e^{-\left(\frac{E}{2xT}\right)} dp_1, \ldots, dp_m\right] \tag{5}$$

where T is absolute temperature, $\bar{E}$ is the energy of the system, E is the energy as a function of p_v, and x is related to N by the relation $2xN = R$. (Einstein 1956,5)

According to Gibbs Free Energy, Einstein states:

$$F(P', T) = \bar{E} + P'V - TS$$

where P' is pressure, T is temperature, V is volume, S is our entropy, and $\bar{E}$ is the internal energy of the system. Substituting, we have:

$$F = -\frac{R}{N} T \ln\left[\int e^{-\left(\frac{EN}{RT}\right)} dp_1, \ldots, dp_m\right] = -\frac{RT}{N} \ln B \tag{6}$$

Here B is simply replacing our integral. Note that the internal energy in our entropy and in our free equation cancel. Also, because the V in PV is in regards to the total volume occupied by the suspended particles, and as we stated earlier that volume is negligible, this term also vanishes to 0.

Since B replaced our integral in (6) then B is dependent upon the volume of the affected portion of the system, recall V^* is separated from V by a semi permeable wall, as well as the number of suspended particles, n. We can note here that the volume of the suspended particles is negligible when compared to V^*, so this system will be entirely defined by our quantities $p_1, \ldots, p_m$.

At this point, Einstein begins using rectangular coordinates, and as we are following his work, we will do the same. Moreover, he uses the location of each particle's center of gravity as that particle's location, which is allowable since their volumes are negligible. Thus, each particle is located at (x_i, y_i, z_i), $1 \leq i \leq n$, and each set of coordinates along with the change in each coordinate, dx_i, dy_i, dz_i, must be entirely contained in V^*. We can then write the differential of B as

$$dB = dx_1 dy_1 dz_1 \ldots dx_n dy_n dz_n \cdot J \tag{7}$$

where J is all remaining characteristics of dB, and is independent of each dx_i, dy_i, dz_i, as well as V^*, and therefore of the semi permeable wall as well. Because of this, J is also independent of the magnitude of V^* as well as the positions of each of the particles. For example, if there was a second system of equal number of particles, denoted dB', and:

$$dx_1 dy_1 dz_1 \ldots dx_n\, dy_n\, dz_n = dx'_1 dy'_1 dz'_1 \ldots dx'_n\, dy'_n\, dz'_n \tag{8}$$

then it follows that:

$$\frac{dB}{dB\prime} = \frac{J}{J'} \tag{9}$$

If we assume that the movement of a single particle is independent of any other particle and that the solution is stagnant, i.e. the system is at equilibrium with no forces exerted on the particles contained within it, then it follows from (8) that *B* and *B'* are equal, yielding:

$$\frac{dB}{B} = \frac{dB'}{B} \tag{10}$$

It follows from (10) that:

$$\frac{dB}{dB'} = 1 = \frac{J}{J'}$$

meaning that:

J'=J

Therefore *J* must be independent of both *V** and the location of each particle. By using integration, we then arrive at:

$$B = J\int dx_1 dy_1 dz_1 \ldots dx_n dy_n dz_n = JV^{*}n \tag{11}$$

Equation (6) implies:

$$F = -\frac{RT}{N}\{\ln(J) + nln(V^{*})\} \tag{12}$$

and from equation (2):

$$\rho = \frac{RTn}{NV^*} = \frac{RT\nu}{N} = -\frac{\partial F}{\partial V^*} \qquad (13)$$

Equations (12) and (13) together show that "…solute molecules and suspended particles are, according to this theory, identical in their behaviour at great dilution." (Einstein 1956, 9)

Next we will take the change of our free energy equation (12) with respect to a change in location of the particles, δF, already substituting 0 in for *V*, yielding:

$$\delta F = \delta \bar{E} - T\delta S = 0 \qquad (14)$$

Note that the above expression equals 0. This is because the quantity of free energy does not change with the position of the particles, and therefore the change of free energy must equal 0.

Next, we will assume that our problem is bounded on an interval with length *L*. For simplicity, we will consider a one dimensional problem. We can do this because even though the problem is in three dimensions, and occasionally higher with more advanced mathematical problems, each particle's change in position can be viewed as a one dimensional line from the point of starting to the point of conclusion.

We will also assume that there is a force, *K*, that is acting on each individual particle. This can be thought of as the effect of an internal flow in a liquid or a ceiling fan circulating the air in a sealed room. In both cases, there is no outside force acting on the room as a whole, but the internal force is affecting the suspended particles. From this, we then have:

$$\delta\bar{E} = -\int_0^L Kv\,\delta x dx$$

where *v* is our solution particle density. Also:

$$\delta S = \int_0^L \frac{Rv}{N}\frac{\partial \delta x}{\partial x}dx = -\frac{R}{N}\int_0^L \frac{\partial v}{\partial x}\delta x dx$$

where δx is some virtual arbitrary displacement. Substituting the above two expressions in (14), considering our system to be at equilibrium, then yields:

$$0 = -Kv + \frac{RT}{N}\frac{\partial v}{\partial x} = Kv - \frac{\partial \rho}{\partial x} \tag{15}$$

The above equation "states that equilibrium with the force *K* is brought about by osmotic pressure forces." (Einstein 1956, 10) This then gives us that this system's dynamic equilibrium is composed of two opposite forces, the movements of the suspended particles, each affected individually by *K*, and the process of diffusion which will force the particles away from one another and towards a near even spread.

If we now consider our particles to have radius *P* centered about their center of gravity, or in other words, if we assume our particles are spheres, we then have a velocity:

$$\frac{K}{6\pi kP} \tag{16}$$

imparted upon each particle, where k is the viscosity of the solution that the particles are suspended in. This leads to a flow of:

$$Kv6\pi kP \tag{17}$$

per unit area per unit time.

Because we are working also with the diffusion of the particles as one of the forces acting on each particle, let *D* be the diffusivity coefficient, and μ the mass of a single particle. Then diffusion will force:

$$-D\,\frac{\partial(\mu v)}{\partial x} \tag{18}$$

grams of mass through a unit area over a unit time. But since we are thinking of this diffusion in terms of solid particles with a separation between each, we can instead view (18) in terms of number of particles, essentially taking μ=1, which yields:

$$-D\,\frac{\partial(v)}{\partial x}\ . \tag{19}$$

Now that we have the velocity imparted on each particle through both diffusion and the force *K*, we can substitute these values in equation (15).

$$\frac{Kv}{6\pi kP} - D\,\frac{\partial v}{\partial x} = 0 \tag{20}$$

Solving for *D* with the assistance of (15) and (20) yields:

$$D = \frac{RT}{N}\frac{1}{6\pi kP} \tag{21}$$

This means that the diffusivity coefficient of our suspended particles is dependent only upon the viscosity of the liquid and the size of the particles. However, given that *NP*, the number of particles multiplied by the size of the particles, must always be a constant, and that the viscosity and temperature are constants in a solution at equilibrium, *D* must always be a constant for each system we view.

Even though we have found the coefficient of diffusion for the most basic of Brownian Motions, we have yet to actually describe the spread of the particles over a time. Therefore, we will now work on a function to determine the location of a particle at a given time. Seeing as how the very nature of Brownian Motion is the randomness that each of its particles possesses, we will not try to map the movement of each individual particle, but instead will create a probability distribution that will describe the likelihood of finding some particle in some location at some time.

An example of this can be shown in Figure 4.1 with its three time-varying images. The first image shows an initial distribution, with probability much greater to the left of the solution than to the right. However, by the third image, the probability of locating a solute particle in a given area is relatively close to the same value throughout the solution. This is the process we will be using.

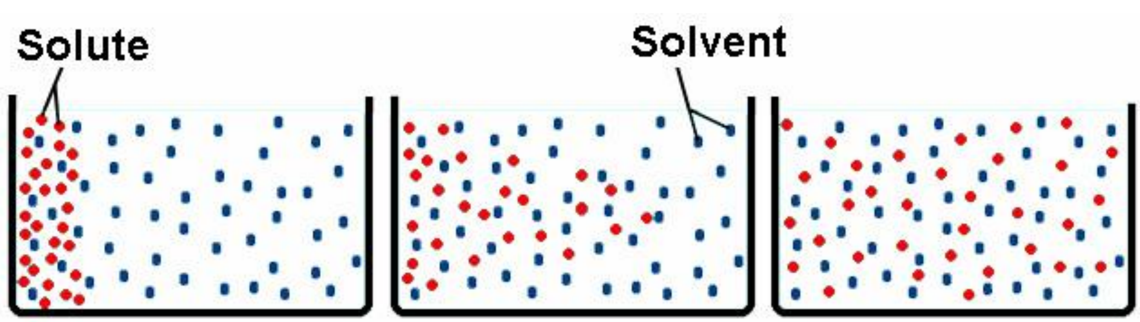

Figure 4.1 Description: Example of Particle Diffusion

For this, we will now introduce a time-interval, τ, and will take τ to be extremely small when compared to an observable amount of time, but still large enough to wholly encompass a single movement by a suspended particle. This restriction makes the movement of a particle in some time period τ_1 independent of the movement of that same particle in some other time period τ_2.

We will also assume once again that the number of particles in our solution is *n*. We will also take advantage that in a bounded time interval, a particle can only move a finite distance, Δ. Note that we are still considering our particle's motion to be one-dimensional, and also that Δ can be different for each particle. Take *dn* to be the number of particles which experience a change in location of order of magnitude lying between Δ and $\Delta + d\Delta$ over the time interval τ, then:

$$dn = n\Phi(\Delta)d\Delta \tag{22}$$

where Φ is an even probability function, based upon Δ, distance, always being positive, and:

$$\int_{-\infty}^{\infty} \Phi(\Delta)d\Delta = 1 \quad . \tag{23}$$

Because of (23) we know that $\Phi(\Delta)$ can only differ from 0 for small values of Δ and that $\lim_{\Delta\to\infty} \Phi(\Delta) = 0$. (Einstein 1956, 13)

Recall that v was the number of particles per unit volume. Now instead of viewing the system as a whole, we will look at a single unit volume. From (22) we know that v is dependent only upon location x and time t. Thus define v=f(x,t), and will use this to determine the particle distribution at time t+τ in regards to the distribution at time t.

Because of (23) we are able to determine the number of particles between the limits x and dx, to be:

$$f(x,t+\tau)dx = dx \cdot \int_{\Delta\to-\infty}^{\Delta\to\infty} f(x+\Delta,t)\Phi(\Delta)d\Delta \qquad (24)$$

Also, since τ is extremely small in regards to *t*:

$$f(x,t+\tau) = f(x,t) + \tau\frac{\partial f}{\partial t} \qquad (25)$$

Using a Taylor Series expansion, we can also rewrite *f(x+Δ,t)* as:

$$f(x+\Delta,t) \approx f(x,t) + \sum_{i=1}^{\infty}\frac{\Delta^i}{i!}\frac{\partial^i f(x,t)}{\partial x^i} \qquad (26)$$

Recall that only small values of Δ contribute to the summation, allowing us to bring the summation under an integral. Thus, from (25) and (26) we have:

$$f + \tau\frac{\partial f}{\partial t} = f\int_{-\infty}^{\infty}\Phi(\Delta)d\Delta + \sum_{i=1}^{\infty}\left[\frac{1}{i!}\frac{\partial^i f}{\partial x^i}\int_{-\infty}^{\infty}\Delta^i\Phi(\Delta)d\Delta\right] \qquad (27)$$

Since *Φ(x)=Φ(-x)*, because *Φ* is an even function, then all even terms on the right side of (27) vanish while the odd terms are diminishing rapidly. Because of this, we will only take into account the first and third terms of the right hand side of (27). Also, recall the condition we placed on (23) and place:

$$\tau\int_{-\infty}^{\infty}\frac{\Delta^2}{2!}\Phi(\Delta)d\Delta = D$$

Doing this yields:

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \qquad (28)$$

It should be noted at this point that the above equation is a well known partial differential equation used to map one-dimensional diffusion, namely the heat equation.

Through this entire chapter we have described each particle's movement in terms of the same coordinate system, much like Einstein has done through the majority of the paper that we're following, *On the Movement of Small Particles Suspended in a Stationary Liquid Demanded by the Molecular-Kinetic Theory of* Heat, contained in *Investigations on the Theory of the Brownian Movement*. However, "this is unnecessary, since the movements of the single particles are mutually independent." (Einstein 1956, 15) Instead we will now proceed with all of our particles beginning at a single point.

Imagine if you will an Alka-Seltzer being dropped into a glass of water. While it's true that all of the bubbles that rise from the tablet rise to the surface of the water, they have one of the best viewable spreads of this concept. If one was to take a snapshot of these bubbles while the tablet is still dissolving, then the height of the glass can be thought of as time and the cross sectional planes the area. Near the base, the bubbles are all clustered together, but as they reach the top, they will all spread out in a more even distribution. At the same time though, there is a level of randomness for where the bubbles will meet the surface.

Now that we are all on the same page in terms of how diffusion works, we will begin playing with the equations once again. In this case, we will start with all of our particles at a generally centralized location, denote that location to be our origin, and call this time $t_0=0$. Since we cannot actually place a great number of particles at a specific point, we will instead use the origin to denote their center of gravity.

Now f(x,t)dx, first brought up for use in (24), will "give the number of particles whose x-coordinate has increased between t=0 and t=t by a value between x and x+dx." (Einstein 1956, 16) We then have:

$$f(x,0) = 0 \text{ and } \int_{-\infty}^{\infty} f(x,0)dx = n \tag{29}$$

for $x \neq 0$ and t=0.

With (28) and (29), we now have a second order partial differential equation with initial conditions. Solving this, we arrive at the solution:

$$f(x,t) = \frac{n}{\sqrt{4\pi D}} \frac{e^{\left(-x^2/_{4Dt}\right)}}{\sqrt{t}} \tag{30}$$

Now that we have the above function, which is the probability distribution for a system of suspended particles, we use it in conjunction with the second part of (29) to determine the **Mean-Square Displacement**:

$$\lambda = \sqrt{\overline{x^2}} \tag{31}$$

Solving for $\overline{x^2}$ is completed as follows: (Einstein 1956, 101)

$$\overline{x^2} = \frac{1}{n}\int_{-\infty}^{\infty} f(x,t)x^2 dx = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} e^{\left(\frac{-x^2}{4Dt}\right)} x^2\, dx$$

$$= \frac{4Dt}{\sqrt{\pi}}\int_0^{\infty} \sqrt{y}e^{-y} dy = \frac{4Dt}{\sqrt{\pi}}\left\{\left[-\sqrt{y}e^{-y}\right]_0^{\infty} + \frac{1}{2}\int_0^{\infty} \frac{e^{-y}}{\sqrt{y}} dy\right\}$$

$$= \frac{4Dt}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2} du = 2Dt$$

where $y = (x^2/4Dt)$. Thus:

$$\lambda = \sqrt{2Dt} \tag{32}$$

Substituting (21) into the above equation yields:

$$\lambda = \sqrt{\frac{RTt}{N3\pi kP}} \tag{33}$$

for our **Mean-Square Displacement.**

To summarize, in this chapter we followed Einstein's 1905 dissertation on Brownian Motion and concluded with three very important findings: the diffusivity coefficient, *D*; the displacement probability function, *f*; and the mean-square displacement, λ. They are listed below respectively.

$$D = \frac{RT}{N}\frac{1}{6\pi kP} \tag{21}$$

$$f(x,t) = \frac{n}{\sqrt{4\pi D}}\frac{e^{-x^2/_{4Dt}}}{\sqrt{t}} \tag{30}$$

$$\lambda = \sqrt{\frac{RTt}{N3\pi kP}} \tag{33}$$

CHAPTER V

ADAPTING EINSTEIN'S DIFFUSIVITY COEFFICIENT

§Need for Adaptation

Now that we have Einstein's diffusivity coefficient, Chapter IV (21), we arrive at a very distinct difference between his system and the one this thesis is based on: Humans are not randomly floating particles. In Einstein's coefficient, there are constants that are only applicable to mindless molecules, such as the gas law constant R. In fact, the only constant that we can confidently keep the same is the number of particles N. Because of this, we will need to adapt his constant to a function of our own.

In the next few sections, we will break down the meaning and effect behind each of the constants given in Einstein's paper. We will then view similar effects that play into how people go about their daily lives. The most difficult part of this will be determining exactly where intelligent movement comes into play and whether it actually affects our data or not.

§Size of the Particle

In Einstein's paper, there was only one restriction in regards to size, and that was that the size of the particle must be very small compared to the volume of the system.

This was so the particles could move with a great amount of freedom without hitting a wall and having that influence their movement. It also can be thought of in conjunction with the number of particles in the system. If the size of the particles is not extremely small and there are a large number of particles in the volume, then the probability that two particles will collide with one another and alter their paths is greatly increased.

Now we must consider this same concept in regards to our own problem. In the problem of pedestrian intersections, our volume is not necessarily the size of the intersection, but rather the size of the paths leaving it. This follows from the same concept, that only so many people can fit down a pathway before they bump into one another, an occurrence most of us prefer to avoid. Though as stated in the previous section, the number of people will not require any adapting, it can be implied that the number of people traveling down a large pathway is going to be greater than those traveling down a slim path. At least it would be if human intelligence would mind its own business.

The main drive behind where we go and what paths we take is time. This idea will crop up a few times throughout this chapter, and so we will explain the concept here. Most humans who are walking in the types of conditions we are applying this thesis to, college campus, city streets, etc., are walking with a purpose. For example, they have to go to work, get to class, or accomplish errands before stores close. In this respect, there are two quotes that stand out greatly:

Benjamin Franklin: *"Time is money."*

***Author Unknown*:** *"A watch is nothing more than a leash meant to bind a man's life."*

This mindset drives the large majority to take the swiftest route, which in the case of most intersections, implies the one of shortest distance. At most, the flow of pedestrians down a path may be slowed, but the quantity taking that path will be the same. Remember that we are, in essence, ignoring the time it takes for a person to choose a path, and instead are only concerned with the path that they chose.

If the above is true, then our size, or rather what Einstein's size constant relates to in our settings, will still remain a constant, though it will definitely depend upon the individuals taking the paths. As the reader will see later, when we put our theories to the test, the size can be approximated for each group within a small degree of error. Therefore, we will adapt our P to a new P', where P' is an average size of the individuals passing through our intersection.

§Viscosity of Pedestrian Movement

Viscosity, denoted above by *k*, is definitely a more concrete thought in regards to liquids than pathways. Viscosity is formally defined as: a resistance to flow in a fluid or semi fluid. This comes about from the nature of the fluid, such as oil versus water. Oil is much denser than water and is much more difficult to move through, especially because its cohesion, the force holding individual molecules together, is greater than that of water.

In terms of pedestrians, we must now ponder the thought of what slows our movement. In truth there are a variety of things, such as the slope of the terrain or whether a pathway is improved or not. Take a moment to consider crossing a muddy path. Most people will be careful with every step both for the sake of their shoes and so they don't accidentally step into a semi-filled hole and sink down to their knees. As a result, their progress is greatly slowed in regards to somebody taking a stroll along a sidewalk on a sunny day. This would be a great difficulty for us, except that our choices are not that free.

The intersections that we are choosing are all of one general type, in most cases paved. This allows us to ignore the condition of one pathway compared to another and instead only be concerned with that which affects all pathways, for example the weather.

While we have dealt with the issue of terrain type though, we have yet to deal with the difficult of travelling a sloped pathway. It is also only logical that, like water, humans would take the path of least resistance. This means that it would make more sense for us to travel around a hill instead of going over it. In some situations, this does occur, which is why we will break this problem into a few cases in order to get a grasp of how this problem will work itself out. Before that though, it is important to introduce an elementary physics equation:

$$W = F(x_f - x_i)$$

where W is the work, $x_f - x_i$ is the change in location, and F is the force exerted. In our case it can be thought of in terms of change of elevation of the final location versus the initial point.

Case I

The first case is that the person must reach a location on the far side of the hill, we will assume the "hill" is a not a mountain, or that if it is then the person walking it does not have to be anywhere anytime soon. Also note that even though we are using the term "hill", it could equally mean depression since the person walking through will still need to walk up and down the slopes. From the above equation for work, it is clear that it will take the same amount of work to travel regardless of which path the walker takes. Keep in mind though that this only applies to a hill whose height is non-negligible, but at the same time is still able to be travelled somewhat casually by foot. For example, few people on their way to work will climb a wall instead of walking around, while if the angle of incline of the hill is extremely small, then few people would waste the time walking around.

Considering these two extremes, we can infer that there is a middle range of slope where the probability of traveling either way is equally likely, anywhere below that slope people will walk over, while at an angle above they will walk around. The beauty of this case is in its complexity though. Because there is a middle range, a range above, and a range below, the probability of each choice is relatively equal if we know nothing of the surrounding terrain. Moreover, since we can consider hills to be depressions in this case, I conjecture that our mapping of probability for pedestrians choosing the hill or not based upon angle will most likely resemble a modified cosine function.

Case II

Our next case is when the location is on some elevation of the hill. If the location is sufficiently close to the far side, then the situation can be thought of as the same as Case I, so we will ignore that situation. Equally, if the location is on the near side of the hill, then the path of least resistance, in regards to which path will take the least work and energy for the person travelling it, is the path going up the hill. Therefore, the only case that we need to consider is if the location is on the far side of the hill, but still sufficiently removed from the base of the far side. Once again, we will ignore the two ranges of slope that skew the Case I probability one way or the other and instead only view the middle ground. In this situation, I claim that it will be quicker to travel over the hill than to walk around it, but that at the same time this path will take a slightly higher amount of energy from the pedestrian. We are now left with the question of "Is the probability equal at this point or is something else affecting it?"

Recall that in the **Size** section of this chapter we described how deliberate human movement is usually dependent, in some way, on time. In this situation we can now use this concept, which predicts that the pedestrians will take the path over the hill because it is a shorter path of travelable slope, and therefore will allow them to arrive in a shorter amount of time.

Using Case I and Case II, we now have a sufficient grasp on what viscosity would translate to in terms of walking. The predicted conclusion from these cases is that whether the pathway goes up a hill or not can be ignored when observing which way the humans will travel. This, taken along with the fact that the pathways we will be observing in each case will all be of one type, in terms of improved or unimproved, allows us to describe our new viscosity, k', as a constant. I predict that this constant will depend upon the weather as well as the terrain that all pathways will share.

§Temperature

Our next term to adapt is temperature. At first glance, it would appear that we could leave this alone, but unfortunately this will be one of the more difficult concepts to adapt.

Not just a mild novelty for molecules, temperature determines their velocity. Part of the **Kinetic Molecular Theory** is that the average kinetic energy of a molecule is:

$$\frac{3kT}{2}$$

where *k* is Boltzmann's constant and *T* is the temperature in degrees Kelvin.[15] Kinetic energy, a basic physics concept, can be calculated by the equation:

$$KE = \frac{1}{2}mv^2$$

where *m* is mass and *v* is velocity. Since mass is an intrinsic property, then we have shown that temperature affects the square of the velocity proportionally.

Now we are left with a question much like we had with viscosity: What affects how rapidly a person moves? While the obvious answer is a vehicle, that falls outside of our jurisdiction, so instead we will consider the exact opposite of our viscosity concept. That is, we will now look at all of the things that will not slow a person down, but will instead motivate them to move faster. Once again, our friend time rears its ugly head. Just as time provided enough motivation for our person to travel over the hill, making that problem easier, it will make this problem much harder as we must now create a function dependent upon time. What does this mean for us? This means that our diffusivity coefficient is no longer a constant for the system, but will instead be a function that varies

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15 Blauch, David N. "Kinetic Molecular Theory." Davidson University. http://www.chm.davidson.edu/vce/kineticmoleculartheory/BasicConcepts.html (accessed November 19, 2011)

over time, which may or may not affect the function, (Chapter IV, Equation (30)), Einstein found over a century ago.

Our first move will be to try and determine a time interval, τ, such that $f(x,t) = f(x,t+\tau)$, that is, the two functions are time independent. We will start with the same method Einstein did and add in a few other traits of humans in order to make the solution more manageable. Einstein restricted his τ in such a manner that a particle would have sufficient time to make an entire jump from one location to another. (Einstein 1956, 13) Therefore, we will need to make our τ large enough so that it covers all actions and reactions that might be taken. To do this, we will start from small time intervals and work our way up.

The lowest possible case, using our own Texas State University as a basis, would be a full class period. This allows students to rush to class as others are exiting, for those students to walk to their next location, a lull in traffic after class starts, and finally it all starts over again. Now we'll make an assumption, a simple counterexample that I have video data to back up the result of. Wednesday mornings at Texas State are a mixture of 80 minute classes and 50 minute classes. If our first case is true, then the mixture of this overlapping time period should match up with that of when there are only 80 minute classes running. However, we can immediately see that this is not the case. This is because now there is less traffic and more time during this passing period for those in the 50 minute classes going to an 80 minute class, allowing them to move more casually, while simultaneously getting to their destination early enough that the pathways are still clear for those just leaving their 80 minute class who must immediately go to a new destination with little time to spare.

Now we take a step up to days, and while this may look tempting for a work week, we must also take weekends into account. Granted, the diminished numbers are not exactly what we're opposing here, but what does it mean to be a weekend on campus? It means that most buildings are closed and the destinations for those walking around are severely depleted. This will lead to an obvious skew in the paths pedestrians will choose. For example, at the center of the quad there are four directions to go, however nothing going up the hill is open, and the only open building in the direction of the stallions is the recreation center on the far side of campus. Therefore, it would be much easier to predict the movement of somebody standing in the center of that intersection: they would either walk down the hill towards the dining facility or the city, or they will walk towards the bus loop which is a pathway towards other off-campus shops and residential areas.

Now we must go one more step up to the week level. In reality, the week and month level can be excluded by the same method that the days were, mainly because we have vacations that come up at regular times each year. Also, we could eliminate the annual cycle because of long running events such as construction projects. The conclusion is that no matter how long we make the cycle, there will always be some reaction to an action that was taken in the period before it. This is because of the deliberate and adaptable nature of humans, and as such, will never be a pure Brownian Motion.

§New Restrictions

At this point, it would be perfectly common to call this thesis a failure, and it is true that the original all-encompassing idea has been proven false. However, we will not stop here. Instead, we will now narrow down our strategy to a particular situation, but before that, it is important to recount what this false-hood actually means.

If we back up to the very meaning behind a path's intersection, the reader will understand why this concept will never be an all-encompassing one. An intersection in terms of travel is a point in the road where multiple pathways can be chosen. Throughout this chapter, I have been vaguely placing them in terms of sidewalks, using my own university campus as an example for when something must be proven false. However, there are more than those few options available to any who think to take them.

A perfect example comes from when I was hiking up a mountain in Colorado last year with a friend. In that situation, we were generally presented with two continuous options, follow the trail or turn back. Yes, occasionally there would be other trails that branched off, but for the majority of the time, my friend and I only had two options. That last statement is clearly false, and I will explain why. While I admit that it is not always the safest option, we chose to break away from the path where it was winding down along the riverside and hiked straight up the mountain where the map told us we would eventually meet another path. And now like magic those reading this have just seen what many might never understand, that there are "pedestrian intersections" everywhere, whether we know to take them or not.

It may look at first like this contradicts the assumption of data collection that we made in the section of Viscosity, but in reality there was no change. Both the path along the river and our own trail up the side of the mountain were unimproved, frequently cluttered with trees and rocks, leaving the only real difference to be the slope. This concept can be applied in a large majority, though not all, situations because sharp changes in terrain are extremely uncommon. This holds true for the jagged mountains slowly transitioning to rolling hills before hitting the flatlands, for deserts becoming tundra as well as the dense clustered network of a city thinning out into more spacious towns before becoming residential and finally rural.[16] Even though we have solidly proven that, in general, pedestrian intersections cannot be classified as Brownian Motions, I conjecture that there is a location where it is.

While the above paragraph may seem out of place, there is a necessary concept in there that will allow us to redirect our focus and continue on with our work. The concept is the fading of types of climates and conditions. It is true that the nature side of the claim was serving only as an illustration, yet it was a necessary one to allow the complete understanding of our next step, to choose a location that acts as a point of, for lack of a better term, "climate diffusion". Short of a person whose entire life is spent on the mountain with no desire to interact with other humans, the path chosen at the top of a mountain will have a high probability of either following a trail, giving only two choices,

[16] Short, Nicholas M. "Vegetation Applications." NASA. http://rst.gsfc.nasa.gov/Sect3/Sect3_1.html (accessed November 20, 2011)

or of being directed back towards society, which also severely limits choices. Thus we shall now focus only on the case of a large metropolis, a city that never sleeps.

As we now look at a concrete jungle, it may appear that we will need to revisit all of the work we have already completed in terms of altering Einstein's coefficient, but in reality the work for all but one has already been completed. While the average size of a person may vary from location to location, we have already accepted this and have acknowledged that it will be dependent upon the system. Thus narrowing down our choices for systems will not affect this portion. The viscosity is now even easier to view as a constant in this location as well. Because in the high density portion of a city all of the side-walks are either paved or are greatly improved paths, not to mention that cities historically select areas that are relatively flat for ease of transporting supplies and goods for commerce, it is clear that our viscosity now falls under the most simplistic form that we described in its above section.

The final term that requires attention is once again, temperature. We will continue along with the concept of using time as the major component that affects pedestrian speed, and we will also continue on with the same method for selecting our time interval, τ. Before we dive too deeply into this new area, I find it necessary to explain why this situation will survive where the general state failed.

The main killer of our example, which eliminated everything below the town level of density, was that there were too many rest periods for small intervals of time. For large intervals of time, annual and above, we then had to take account of construction projects that were eliminating a path choice in one interval, and then not only opening it up for the next, but also acting as another main pathway, extremely skewing results and destroying the concept of time-independent.

It should be noted at this point that there are now standardized distinguishing characteristics or classifications separating what a city or a town is, so we will proceed with using the term city with the concept of permanent establishments of civilization with high-rise buildings and high density populations. For an example, consider places such as Dallas, TX or Denver, CO. Now that we have established a basis for the terms we are using, let us continue on with constructing our τ.

A main defining factor of a city is its night life. Even if a building is closed, it is usually still occupied by security or custodians. An example of this situation would be establishments such as banks or warehouses. Note that businesses such as bars or nightclubs which are only open during the night fall under this same category as well, just with a slightly translated time with respect to the our first examples. On the other hand, if it remains open, then there are all those individuals as well as the normal workers on the night-shift, as well as the occasional customer. A perfect example of this would be supermarkets, convenient stores, hotels, and even gas stations.

Secondly, a city does not take weekends off. While large skyscrapers filled with businessmen deciding a company's future are generally closed on the weekends, the entertainment sections of the city are not. If anything, they are even more active on the weekends to accommodate those who have the day off.

Finally, a city is more or less a clustered mess. No city simple springs from the ground in a nice orderly fashion, it is built up over decades, if not longer. In that long stretch of time, buildings are constructed wherever they can find space, normally on the edge of town, civilization grows up around those buildings, and then the process continues with yet another construction. Meanwhile, the establishments in the middle of the city would occasionally rise and fall as new owners bought out the previous ones. Once large enough, cities will usually create zones for industrial, commercial, entertainment, and residential areas, but all of the buildings already located in a zone that they were not supposed to be in are protected under grandfather clauses which permit them to remain where they are.[17] For further information on this concept, read Ernest Burgess' *The Growth of the City: An Introduction to a Research Project*.

Using these three conditions together, I hypothesize that the movement of a pedestrian at an intersection in the domain of a city is of the form of a Brownian Motion. I claim this because there will always be motion throughout the city, accepting that the number of persons passing over a time will fluctuate, and that there is no destination in the city that makes any single direction more likely to be chosen than the rest. Furthermore, I put forth that our τ need only be large enough for a person within the realm of a single intersection to have sufficient time to exit. Note that even though τ must be large enough to give the person a chance to leave, it does not mean that every person in our intersection will leave by the end of that time interval, only that they have the chance to.

The reader may note at this point that this is almost exactly the same restriction that Einstein placed on his system with the floating particles in the previous chapter (refer to

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[17] Burgess, Ernest W. *The Growth of the City: An Introduction to a Research Project*. Boston, MA: Ardent Media, 1967

page 20). Furthermore, this claim would imply that, regardless of time, if we were to take a snapshot of an intersection, a second photo only a few moments later, and only view the pedestrians contained inside as particles, i.e. ignore the direction that they are facing, then there will be equal probability of choosing any path leaving that intersection. Therefore, our "temperature" factor, T', will also be a constant.

§Rate

We will go ahead and adapt the gas law constant, $R \rightarrow R'$, here as it will provide a small example of our methods. R is most known for its use in the basic chemistry equation for ideal gasses:

$$PV = NRT$$

where P is pressure, V is volume, T is temperature, and N is the molar quantity of a substance. In other words, N is the mass of our substance divided by the mass in one mole of that same substance. However, this equation can be adapted away from R and to a well known constant, Boltzmann's Constant.

In this case, we can rewrite the above equation as:

$$PV = N_A k_b T$$

where N_A is Avogadro's Number, i.e. the number of molecules in a mole, and k_b is Boltzmann's Constant. By making this change, we now eliminate the molar quantity portion of the units, allowing us to view the problem in terms of energy per degree Kelvin.

Now we will apply what we learned in the previous section on adapting temperature. In the previous section we concluded that our temperature relates to time in our new system, and that one unit of our time will be the smallest time interval, τ, such that a person in the intersection has sufficient time to leave that intersection. Therefore, one degree Kelvin will relate to a single increment of τ.

Next we will view the amount of energy exerted through walking. We will view this in terms of average daily walking distance versus the average distance to a mass transit system. This may seem like a strange move as both terms are distances, however the first determines that the average person has enough energy and physical stamina to conquer a distance, A, while the second claims that a pedestrian will only walk a distance B. Therefore A divided by B will allow us to have a view of the amount of energy that a person possesses in terms of walking.

According to research done by the Fairfax County Department of Transportation, the average distance that a pedestrian will tolerate walking is less than .5 miles, with the majority of people refusing to walk much farther than .3 miles. As such, most cities plan for mass transit systems, i.e. subways and bus stops, to be no more than .3 miles from any high traffic areas. For example, downtown Manhattan has subway entrances spaced an average of .17 miles apart.[18]

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[18] Planning Commission Committee. "Walking Distance Research." Fairfax County Department of Transportation. http://www.fairfaxcounty.gov/planning/tod_docs/walking_distance_abstracts.pdf (accessed November 20,2011)

According to a research study completed in 2008 by the American Diabetes Association, a free-living American, i.e. an American with no notable daily physical regimen, walks an average of 7 miles a day.[19] This value is obviously much greater than that of the distance to a mass transit system, presented in the above paragraph, at a factor of being 14 times greater. Therefore, this implies that a person will normally have more than enough energy to make it from their location to a form of transportation.

The reason we had to consider such a large distance in regards to the size of an intersection is because our average pedestrian could be anywhere in from just beginning their walk to being half a mile in. It was then necessary to account for their energy at this point. However, since less than 8% of the average daily distance is being covered in this short span, we can now consider the energy of each person to be a constant. Therefore the energy of our system, which is composed of many people, can also be estimated by a constant.

At this point, we have accounted for all conditions contained inside of Einstein's famous diffusivity coefficient. To summarize, $k \rightarrow k'$, $P \rightarrow P'$, $T \rightarrow T'$, $R \rightarrow R'$, $N \rightarrow N$, viscosity, size, temperature or driving force, rate or energy, and number respectively, are all constants. This forces our new $D \rightarrow D'$:

$$D = \left[\frac{RT}{N}\frac{1}{6\pi kP}\right] \rightarrow \left[\frac{R'T'}{N}\frac{1}{6\pi k'P'}\right] = D'$$

to also be a constant only dependent upon the unchanging characteristics of the system.

[19] American Diabetes Association. "The Role of Free-Living Daily Walking in Human Weight Gain and Obesity: Results" Medscape Today News. http://www.medscape.com/viewarticle/572997_3 (accessed November 20, 2011)

§University

We will continue with the analysis of the data for the city intersections in the next chapter, but before that there is another issue to be discussed, and that is the benefits of restrictions, as well as the doors they open for us. Notice that in order to have Brownian Motion be true in theory, we had to restrict its location. However, if we apply another restriction, we can continue to apply it to many more situations, even if on a grand scale these situations are very few in number.

The next case to be discussed will be in regards to what many of my examples thus far have been , a university campus. In the section on Temperature in Chapter 5 we used the universities to show that Brownian Motion does not hold true for all times. Also, we used an example of mountains to show that Brownian Motion does not apply to rural situations. In order to apply this concept to university campuses, we will need to make both restrictions, location and time. We will restrict our location to only university campuses, considering a university to be a collection of colleges in one general location.

It can be noted that much like small towns still early in their development, universities are somewhat organized, and yet because of unforeseen construction projects, there are still anomalies caused either from available land restricting area, or the technology of the time being impractical for a certain terrain type. Therefore, there is a certain amount of chaos in their overall design.

Secondly, we must return to the same concept that forced our original case into failure, and that is time. Brownian Motion will only have a chance of survival if over its entire time domain, the probability distributions are time independent. We have already shown that because of the vast quantity of closed, entirely unmanned structures in a university at night, not to mention the holidays and breaks that come around regularly, a proper mapping could never be achieved for all times. However, if we restrict our time domain as well, then we eliminate this problem all together.

For this new case, we will be restricting our location to university campuses and our time to being no earlier than half-way through the first class cycle of the day, and no later than normal business closing hours. For example, in terms of our own Texas State University-San Marcos, we will restrict the time domain to between the hours of 0900 and 1700 (9am to 5pm). For the same reasons as explained before, rate, viscosity, size, and number will all remain as constants. This leaves us with showing that once again our temperature will relate to a constant.

As we have claimed previously, we will continue to have temperature dependent upon some form of time, but what about all of the other events that occur on campus? When Final Exam week nears, the library sees more students than it normally does. At the same time though, the number of students setting up group study sessions at other locations across campus increases as well, not to mention the quantity coming in to see their professors. Therefore, if we consider our restriction to exclude the week of finals, but not the week before when this sudden surge of studying arrives, then I conjecture that nothing other than the number of students has changed.

Another distinguishing factor between cities and universities is the number of group activities per area. If we exclude classes, jobs, and meetings, then what we are left with are the extra activities that a person participates in outside of their home. This could include anything from going golfing, to joining a bowling league, or even practicing with a marching band. As a university is a concentration of as many activities as possible in order to broaden and expose a person's mind, then it is clear that there will always be more to do outside of the normal routine of the work day. However, the nature of human cohesiveness eliminates any true difference between these regular activities and a class or job, a claim that we will show at this time.

It is typical that in order for a large number of people to come together more than once, then there is a set location and a set time. The more meetings a group has, the more rigid that schedule becomes. Diversely, the fewer meetings there are, the more random the meetings become. To illustrate this concept, we will provide examples of the two extremes.

Assume that there is an organization which meets more than once. We will make no other assumptions on this case, allowing them to meet anywhere from every hour to once a year. We can now create a collection of subsets, indexed by meeting location. Thus if there is only one meeting location, then the subset will be the entire set of meetings that this club holds. On the other hand, if there are a variety of locations that this group meets at, then there will be more than one subset. At this point, we can make a very important observation, that the number of locations will always be finite. One might ask the question of why is this move legal, and it is because if there is any change in the meeting of the group, then we can consider it an entirely different group. We simple choose one of the variations, in this case location, as our basis for selection.

We can now simplify the problem to a group meeting at a variety of times in a single location. We will now narrow this down more by eliminating all meetings that occur outside of our time domain of 0900 to 1700, and will consider only one cycle, i.e. one day. In most situations this would remove a great deal of meetings as it is easier for students to match their schedules after the school day has concluded, but for the purpose of this paper, we will still assume that there are multiple meetings within that time frame.

The best case scenario is that the meetings are regular to the point of being synonymous to a class, in which case it can be ignored. Life is rarely that kind, so we will look at the worst case, sporadic unpredictable meeting times. Note that every other formation of meetings must be a subset of this cluster, thus if this situation holds then we are finished.

In order to properly analyze this case, we will create a set E comprised of time intervals $\{a_1,a_2,\ldots a_n\}$ where n is the number of meetings held in this location held between 0900 to 1700, and each a_i is the time interval during which the meeting is held. Recall here that E is only spans one day. Note that each a_i is a closed interval ranging from the start of the meeting till the end. We will make our set even smaller now by placing a restriction on our a_i's. If $a_i \cap a_j$ is not empty for any i,j between 1 and n, then we will denote $a_i' = a_i U a_j$. Furthermore, let τ be the time increment defined before. Denote $a_i = [x_i, y_i]$ for each a_i in E. If for any x_i greater than y_j, the length of the interval between them, $|x_i - y_j|$, is less than τ, then they are sufficiently close together to constitute one meeting. Therefore, we will denote $a_j' = [x_j, y_i]$. Our last change will be if a_i has length less than τ, then we will remove it from the set.

We will denote our new set, *E* after applying the above alterations as many times as is necessary, as *E'*. At this point, *E'* is a set of closed, bounded, disjoint intervals of length greater than and separation strictly greater than 0. These alterations greatly simplified a formerly indescribable set into a greatly restricted one without losing generality. Since there is a separation between meetings, then there is time for those involved in the meetings to leave and return. If they choose not to leave the meeting location for whatever reason, then they are not part of our data set and can be ignored.

We will now view the individuals leaving the meeting location. They can be doing a variety of things, from leaving for the day, to grabbing lunch, and even to heading to another class. While it is true that the more we know about the individual the better we could predict where that one person is going, but if we consider him a random particle at some intersection, then we can say nothing in regards to which direction he is heading.

Also, since there is sufficient separation of time between meetings, then we can also view those heading towards the meetings. Much like those heading to classes, they would be walking to some location on campus. If they are walking as individuals, then the data set is not altered as they are no different from any other student walking to class. However, let us assume that they are walking as a massive group.

A large influx of people leaving an intersection through the exact same pathway would upset the data for that intersection. However, short of their continued and repetitive movement through that single intersection, the group would only appear as a singularity in the data set, one that as time continued on, the movement of the other students would sufficiently compensate for as time grows sufficiently large. Note that beyond the time frame that the group moving together interrupts the intersection, there is factor affecting the normal usage of that intersection.

Therefore, since meetings being held at unspecified times for an unspecified duration in the day can only, at most, affect the motion of an intersection for the duration that they move through it as a unified front, and since all motion before and after is independent of that influx, then group meetings will not affect the possibility or form of the Brownian Motion Diffusivity Coefficient. It should be noted at this point that we made no specifications as to the location, the duration, or the number of members attending the meeting. In this manner, we can then classify all other activities that may gain some notoriety, for example a school art fair or a football game, as meetings and thus conclude that they will also not affect the diffusivity coefficient of a Brownian Motion being applied to the intersections, at least not so long as they are considered within the same restrictions that we applied to the meetings. Recall that the restrictions are the location being on campus and the time being contained between 0900 till 1700. Therefore our temperature will once again be a constant that can be related to time.

At this point, we have already shown that the factors of viscosity, number, size, and temperature will all be transferred to constants in our new D' for our university case, but there is still one more issue, rate. Recall that for rate, we showed that we could show a relation of average distance willing to travel versus the average distance to public transportation, and that if this ratio is sufficiently large, then we can consider our new rate, R', as a constant.

All of the universities in America with population above 10,000, as determined by CollegeStat.org, have a public transit system composed mainly of busses. According to Huan Li and Robert Bertini of Portland State University, the optimal bus stop spacing is every 930 feet[20], although the average for the nation is 802. To plan for the longer of the two cases, we will use our optimal bus stop spacing for the calculations. Then if the numbers imply a constant value for this case, then it must hold true for the average which is lower.

According to American College of Sports Medicine, the average distance walked daily by a normal student at a university is 4.17 kilometers a day with an average distance of 1.43 kilometers at a time without rest[21]. These distances correspond to 2.59 miles and 0.89 miles respectively. Note that 930 feet is equivalent to 0.176 miles. This means that of the average distance walked casually at a time, it would only require approximately 20% of the distance to reach a bus stop. We can therefore assume that, like before, there will be no great energy difference that a person contains in regards to the distance of the intersection with respect to the bus stop. Thus, our rate, R', will also be a constant.

To summarize, every one of the constants that constructs Einstein's Diffusivity Coefficient is converted to some other constant for our own diffusivity coefficient in

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[20] Huan Li, Robert Bertini. "Optimal Bus Stop Spacing for Minimizing Transit Operation Cost." Department of Civil and Environmental Engineering, Portland State University. http://www.its.pdx.edu/upload_docs/1248894217RWAeWSDYZ9.pdf (accessed November 25, 2011)

[21] David R. Bassett Jr., Andra L. Cureton, Barbara E. Ainsworth. "Measurement of daily walking distance-questionnaire versus pedometer." Medicine and Science in Sports and Exercise. http://journals.lww.com/acsm-msse/Abstract/2000/05000/Measurement_of_daily_walking.21.aspx (accessed November 25, 2011)

regards to the scenario of a university campus during normal business days on a time span from 0900 till 1700. Therefore, our diffusivity coefficient for this case will also be a constant that is dependent only upon the conditions of the intersection in question.

CHAPTER VI

DATA GATHERING

§ City

In this section, we will review the methods for gathering data to supplement our theory of Brownian Motion in city intersections. Video data is the most reliable source for analyzing this data, and as such we will utilize the internet. A variety of data can always be found on the internet, and so we will utilize multiple search engines to locate video footage of full traffic intersections, or at the very least a full view of one corner of the intersection. It is equally applicable to disregard the scale of the system because, much like in Fractional Brownian Motion, if it is true for the individual corner of the intersection, which itself is a pedestrian intersection, then it must also be true for the traffic intersection as a whole which is simply an enlarged version of our single corner intersection.

After locating a large variety of video footage made public online, links of which will be listed along with each trial in Chapter 7 as well as being logged with our sources, we will then begin narrowing our search to only those locations with multiple video logs, and of those, only the ones that are in a major city. These two restrictions are the most important, the latter for obvious reasons. The first restriction is important because if we only have one log of an intersection, then there is nothing to say if a spike on one side is regular or a random occurrence. The minimum number we will use to test this is three videos, though more are greatly desired. We will then do a simple headcount as people cross a set line. This line can vary from city to city, but should remain as close to constant as possible when viewing the same intersection. As it is unlikely that any actual divider will be present in all of our footage, the default measurement we will use is when the person or persons pass beyond the corner of the surrounding buildings.

Furthermore, we will also be looking at the conditions of the time that the video was taken. While determining exact outdoor temperature and humidity at the time will be next to impossible, it should quickly become clear as the videos are observed whether the time is night or day, as well as if it is raining, clear, or overcast. This will be important to us as we have already determined that the coefficient may depend upon the weather, however it should not play a major role in the directions pedestrians choose. In other words, even though we may have insufficient evidence to concur exactly what the equation to map the movement will be, there will be enough data to determine if the movement is or is not a Brownian Motion.

§ University

For the university portion of this thesis, we will be using my own campus to collect data. To attempt to compensate for this limit in location types on a larger scale, we will be pursuing multiple locations on campus.

Unlike before where we were primarily dependent upon the data collected and posted by others, this scenario will be much easier to observe. As we will be there in person, there is no difference between marking down the people as we see them versus recording the motion and watching it later. However, we will continue to video tape the intersections as it allows for pauses and reviewing at later dates. Also, as we are already on scene, it will be much easier to more accurately describe all weather conditions that day, from the temperature to how hard the wind is blowing.

Once again, we will be using the corner of buildings as our default measuring tool to determine whether a person has exited the intersection or not. Also, as we will clearly have more access to observational resources, we will raise the minimum footage of one area from three to five clips, as well as placing a minimum ten minute restriction on each clip. This will assist us greatly in not only confirming or disproving Brownian Motion is present, but if it is present, it will also provide a clearer example of the mapping of a Brownian Motion. For example, if we analyze the data in a cumulative manner, then not only should we see relatively even percentages at the end of the trials, but the percentages should start skewed and then form to the estimated percentage, which in every case will be the inverse of the number of options a pedestrian can take. Make note that if even one direction has an incorrect percentage, then at least two of the directions must be skewed, the one we first detected, as well as at least one other to compensate.

CHAPTER VII

DATA ANALYSIS

As all data collected and analyzed will require a large amount of space, we will be utilizing both the full data as well as a summarized explanation of each intersection as a whole. We will be analyzing the data not by the number of people who chose an intersection at any given time, but rather what percentage of people leaving the intersection chose that direction. This will allow a greater cohesiveness between times of traffic and emptier periods.

Brownian Motion, though random by its very nature, is one of the most difficult concepts to map. As such, we will analyze the data in a three step process. First, we will create a running evaluation which will show how the numbers behave as time carries on. While this alone can never be solid proof of whether our hypotheses are true or false, it will provide an intuitive basis for how to proceed with that test. Also, should the hypothesis prove to be false, then this portion of the analyzing process will be vital in explaining why.

Step two of analyzing the data is attempting to prove our hypotheses false. In reality, to prove a Brownian Motion exists at the intersection, we will need to create a normalized function with average equal to the inverse number of options available, as well as determining how to vary coefficients to match the setting. Because this is extremely time-consuming, it would be more beneficial for us to first determine if the hypothesis is incorrect. We will do this by assuming that our hypothesis is true, that Brownian Motion is present in intersections, and then use a computer program known as R to determine the validity of this.

Unfortunately, this process could never confirm that our hypotheses are true, but proving something false can be done in a matter of seconds. To do this, we will perform a t-test, which is used in statistics to determine if a sample follows a t-distribution. A t-distribution is much like a normalized distribution, and actually provides a more flexible fit for sample data. As our number of samples increase, our t-distribution will tend towards our normalized function, if one exists. Therefore, if our sample data cannot satisfy the t-test, then it cannot satisfy the more stringent Brownian Motion requirements. In our t-test, we will be assuming that the distribution is symmetric across an average equal to the inverse of the number of options available.

Our third step, if step two fails to reject our hypothesis, will be to break down each video log into the most specific subcategories we possibly can. This will include pairing first intersections, then matching time of day, weather conditions, nearby entertainment areas, transportation options, as well as any other conditions that become apparent during this process. While this is possible for our University case, because we are using observational type data, it will be extremely difficult for our City case because our data there is retrospective. We will then determine the normalized function for each, matching up the coefficients detailed in Chapter 5, and then attempting to solve for the others.

§ City

In regards to pedestrian movements at intersections that exist even in major cities where movement in general is a constant, both step one and two results have shown that Brownian Motion is not present. However, it does not suffice to simply say that, so we will instead look at the two cities that we were able to collect the most data on and explain why those two in particular failed. From those, we will also draw a tentative conclusion as to why Brownian Motion is not present in even major cities.

In New York, particularly Manhattan where the majority of our films came from, the streets are not simple cells with random constructions on every side. Instead, the streets and city blocks are formed into extremely long and narrow blocks only large enough for two buildings to occupy width wise. While this poses no irregularities for the middle of the street, it does limit the number of entrances contained on the narrow side of the block. In particular, the entrances on those sides are generally for either extremely small shops, or for service entrances to the larger complexes.

Furthermore, there is an extremely complicated system of subway stations that, for the most part, run straight along the major streets, crossing over the minor ones occasionally only to reach hubs or other high traffic areas. This implies that if a person has a long distance to travel, that they are more likely to travel up along the major streets to reach a subway entrance. However, there is still the issue of the entrance locations to the subways. Unlike in London, where the entrances are generally at the intersection, New York subway stations are located along the narrow portions of the surrounding blocks, occupying all four directions around the intersection in question. At first glance, it would appear that this would only strengthen the theory of Brownian Motion being present, but in reality, it is a hindrance to more intersections than it assists. While we have no data to establish if there is Brownian Motion at the intersections where the subway entrances are located as our data is retrospectively collected, we can only make conclusions on the majority of the intersections in the city, which have no subway entrance.

For the answer to the obvious question of "Why?", the answer is one that we've already established and used multiple times. With a subway entrance located on the narrow side, it is faster for a person to walk along the major road they exited their previous location from, and then turn only when they've reached the point where they are on the same street as the subway entrances. This leads to a large difference in the number

Figure 7.1 Manhattan Subway Map

of people travelling laterally through a city intersection, giving those numbers instead to the North-South directions.

Also, keeping in line with our idea that modes of transportation are important to how humans plan their paths to and from a location, it is important to note busses. Bus stops in New York occur along major roads as well, and are generally located close to a subway entrance, however the busses themselves, which we were not concerned with while collecting data, run a variety of routes through the city, transporting the pedestrian to any number of locations. We are only concerned with our one intersection in particular though, so where the bus travels is not nearly as important as where it stops. Because the stops are on the major roads and near subway entrances, it is clearly more likely for a person to walk along that major road, though where they cross over remains far more random than in the case with the subways. That is, until we consider their major difference from subways.

Because busses require only roads and bus stops, they are able to be placed much more frequently. In fact, while there are only four subway routes running in Manhattan, there are bus routes and bus stops along nearly every major road, and not only that, but they also cross over laterally every few blocks. Thus, if the person has a relatively long distance to travel, which from the data collected before was well under a quarter mile for the average human, they are likely to seek a bus stop if they are not already within range of a subway entrance.

Next, we will look at Shibuya Station, Tokyo, Japan. In particular we will be observing the corner of the intersection that is directly next to the main entrance to the station. Initially, this didn't appear to be a problem as there were entrances on every corner of the intersection, but this assumption was wrong. The majority of the pedestrians entering our corner came from across the intersections from other corners, and still they chose to enter the main entrance instead of using the side entrances to the station. This happened regardless of whether it was night or day.

Secondly, there was a high skew in the numbers that was time dependant. Shibuya was the only one that I noticed this trend in, though after a small amount of investigation, I believe that it is only because of the lack of data we possessed on the other cities to

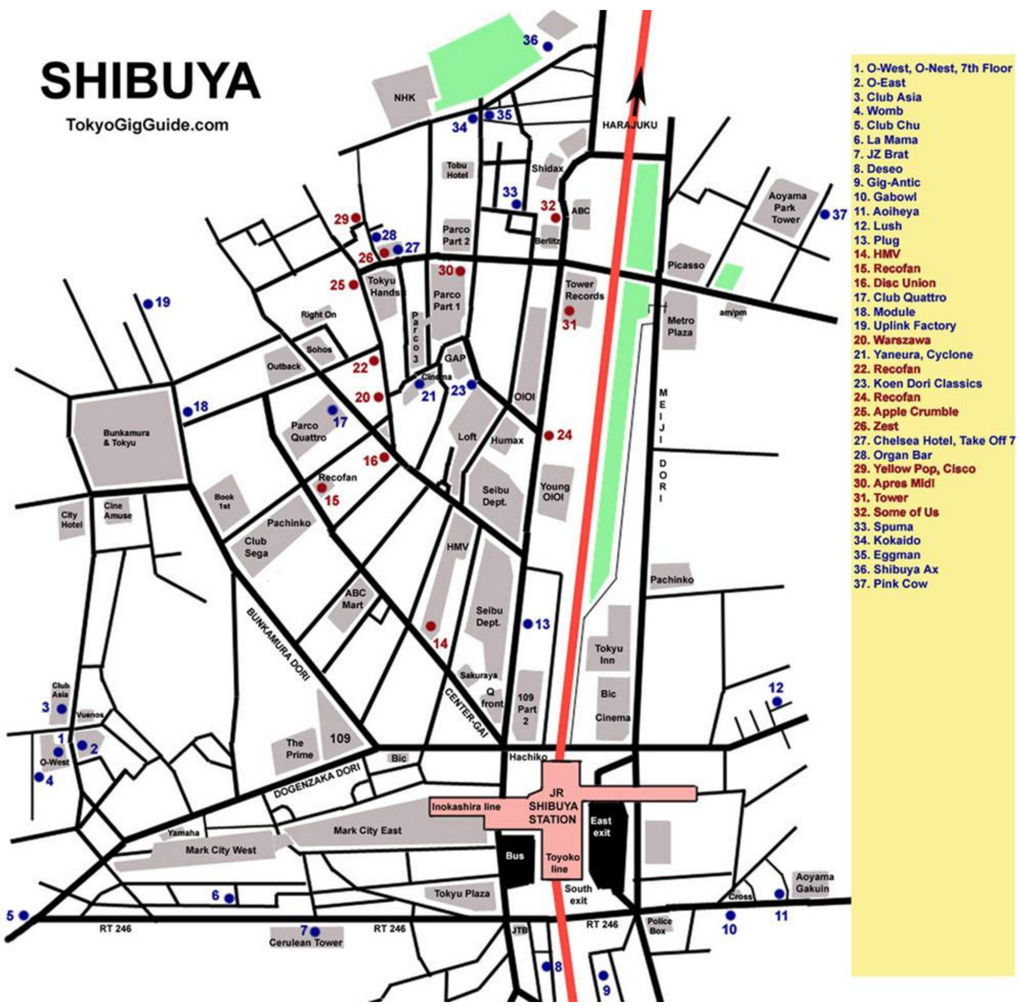

Figure 7.2 Map of Shibuya

match Shibuya's conditions. While the southern choice was consistently low, there were major consistent flows to the north during the day time, and to the west during the night. With looking at the one intersection as a single self contained unit, there is no reason for this flux of people, however the surrounding area answers all of the questions.

To the north of Shibuya Station is Shibuya's famous shopping district. In this region, people of all ages come together in what can only be described as a giant outdoor mall. Much like 6th Avenue in New York, which is a major tourist attraction and shopping center that leads into Time Square, Shibuya's shopping district is equally visited, if not more so for the already high density that Tokyo possesses. Also, just beyond the shopping area is a large park. While this park may not match Central Park in any sense, having a small patch of nature in the middle of a city will always have a magnetic pull on the populous. With all of the family friendly events to the north, it is easy to see why that route is chosen so frequently during the day light hours.

During the night, it all shifts to the west though. As shops and restaurants to the north close, the western area known as Shibuya's Dogenzaka area start their nightly debut. Dogenzaka is easily the night life of this region of Tokyo, and is most well known for the multitude of night clubs, bars, and even "Love Hotels".[22] Dogenzaka has even gone so far as to be called the backbone to Shibuya's nightlife scene. There can be no doubt that this is an area of Tokyo that flourishes at night, and as a result, the flow of people travelling there must increase at night.

Finally, there is a question to both of these instances which is, "What lies beyond Dogenzaka and the shopping district?". The answer is that we don't care. Because of

Tokyo's high population density and its major reliance on public transportation instead of everyone owning their own vehicle, their train stations are placed extremely close together. In fact, just on the other side of Dogenzaka is a train station, as well as just north of Shibuya's market district where the park starts to make its debut. Because of the intersection that we are viewing in Tokyo, it is clear that any person we witness is likely to have come by train, and are equally likely to leave in the same fashion. With this in mind, there is no reason for them to travel beyond those train stations. On the other hand, if their objective was on the other side of those stations, then they would not have left the station at Shibuya and would never be considered in our data set.

Now is the time when we must say what we've learned from the above. First and foremost, let me restate that our hypothesis, that pedestrian movement in cities followed Brownian Motion, is not correct. However, with every failure comes greater understanding, and what we've learned from this is how to predict where a person will go based solely upon where they are. Before now, I have always thought that it would be near impossible to determine where a person would go if you only knew where they were, but that's not true anymore.

From the above, it is clear that a pedestrian is follows a loose set of priorities when travelling, and by that, these object will hold a higher probability of being chosen in almost nearly this order. First, there's the need for faster and easier transportation. This can come in the form of busses, trains, subways, or even walking to a car-lot to regain

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[22] Jack Song. "Dogenzaka in Shibuya: Tokyo's Hippest Entertainment District." Asiagate. http://goldsea.com/Asiagate/Entertainment/tokyo.html (accessed January 30, 2012)

one's own vehicle. From there, the pedestrian has much more freedom to move while exerting the least amount of energy. Second is entertainment. While entertainment is certainly not second on most individuals' lists, those areas do have a higher density than places where only those working would travel. This is because both those seeking entertainment, whether it be for shopping or heading to a pub, as well as the people who work at these locations are present.

Taking the above information in conjunction with knowledge about which side of an intersection a person entered from, thus eliminating any number of a finite quantity of choices, prediction of where a person will go is greatly increased. Using this concept over and over again, we could ideally map exactly where a pedestrian leaving one area would travel to, however that task will have to be left to another. In the end, while Brownian Motion may not be present in the cities, the void it has left is equally useful if not more so.

§ University

For our university scenario, we used our own Texas State University-San Marcos campus as a trial to base our initial results upon. Three locations were chosen at random with the only criteria being that they had to be in a relatively centralized location on campus and that there had to be a minimum of four pathways leaving the intersection. For this purpose, the three intersections chosen were by the Bus Loop, the main intersection of the Quad, and at the base of Alkek Library. The results in all three cases showed false for the first two tests.

At this point, with all results pointing to false on all pathways of all intersections, we must accept the conclusion that Brownian Motion is not present in pedestrian intersections on a university campus. However, much like in the city situation, we must go one step farther and explain why this is the case. To do this we will break down all three intersections, analyze where their paths might lead, and ideally arrive at a reasonable explanation for why the probabilities were so skewed from the hypothesized value.

To allow the reader to gain a better understanding of our intersections with reference to the rest of campus, included below is a map of both the central area that we restricted our observations to (Figure 7.3), a necessary task given the findings outlined in chapter 5, as well as an overall map of the campus (Figure 7.4).

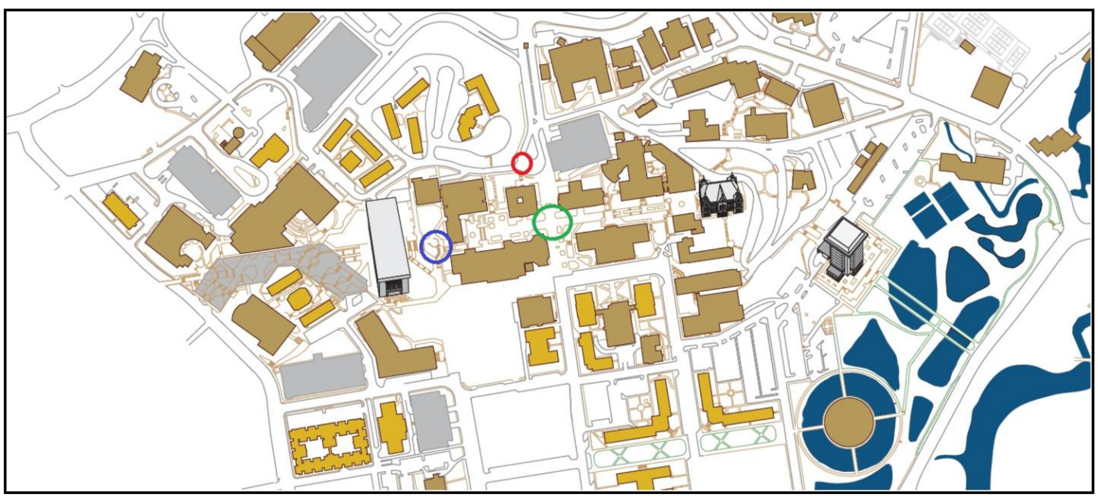

Figure 7.3 Map of Texas State Core
Description: Bus Loop (Red Circle); Alkek Base (Blue Circle); Quad (Green Circle)

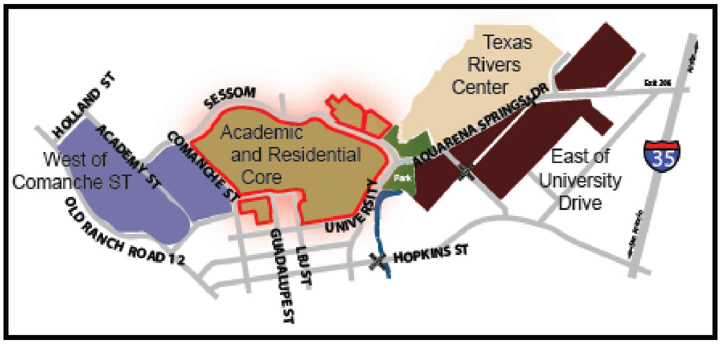

Figure 7.4 Map of Texas State Sections
Description: Figure 7.3 is bordered in red

The Bus Loop was the most extensively studied of the group, but at the same time is the most reasonable to arrive at the conclusion we have drawn. All of our perspectives will be specified in reference to the video's view, but to state them plainly: Right led to the line of busses commuting students to off campus locations; Up consisted of the sidewalk and roadway leading up the hill towards Derrick Hall; Stairs was a pathway leading off to the left; Ramp was a pathway also headed to the left, but whose exit was


Figure7.5 Bus Loop Image

distinctly different from Stairs. We will use these names in place of explaining the pathway direction each time. It should be noted at this time that, though the percentages may have varied from test to test, as did the population, the ranking of each probability in terms of highest to lowest almost always remained the same.

Right was the clear choice for the majority of the population. As the name I have given to the intersection suggests, the main reason that students would visit this pathway is to access the busses for transportation purposes. At first, one might assume that the probability would be the same since roughly the same number of students entering campus by this method will also leave campus by this same method, however because this is a four path intersection, that is not the case. All of the students entering by busses have three unique options of where to go, while those preparing to leave campus only have one pathway from this intersection that will end at the desired location. Beyond the purpose of mass transportation, there is no other conceivable reason why a student would choose this pathway, but as we saw in Chapter 5, transportation is a major influence on where a person will walk.

The second strongest probability pathway went to the Ramp. This pathways was the closest to reaching the desired percentage, and was close enough that we could not reject the possibility of the true mean, or rather the average of all people to select that pathway, to be twenty-five percent. However, because all four are required to meet this restriction to be considered for a Brownian Motion, the test as a whole still failed.

The possible causes for Ramp being second were considerably less obvious based upon the possible outcomes of the closely related Stairs, however upon closer inspection the unexpected reasons become clear. Choosing the Ramp pathway is the shortest distance to almost half of the buildings on campus. It also provides the shortest route to a major commuter parking area by the tennis courts, a wide gently sloped pathway to the Quad, though notably not the shortest, but most importantly, it is the quickest path to the strip of busses serving all on-campus locations. From these busses, the students may travel to the outlying commuter parking, multiple locations all over campus, the small patch of green on campus, Sewell Park, and the Student Recreation Center. At this point, from the earlier argument that most students entering campus by these busses are not very likely to leave by some other method, the possibility of needing to travel to a commuter lot cannot be ruled out simply by personal experience of having to leave my own car at campus overnight for a variety of reasons. The secondary busses are still the most important factor in this situation though, and it is because of the entertainment portion.

As was put forth as a hypothesis in the above City section of the data analysis, the second most travelled pathway leads to a source of entertainment, we must view the few locations the campus busses can lead to, the park and the recreation center. Whether it's a game of soccer, lifting weights, or relaxing by the river, these two places offer just about any outdoor fun that can be desired inside of a large town. The Ramp pathway also leads to the music building, which is frequently holding concerts, the photography and art labs for those who enjoy taking photos, and my personal favorite, is the shortest pathway to the off-campus coffee shops located just north of the traffic intersection.

The clear third choice in the ranking of the Bus Loop is the Up path. This pathway is notably the largest simply to compensate for those students who would prefer to walk in a rarely travelled road, which is unfortunately a common occurrence on our campus, than continue on the distance farther to the stairs. Up provides the shortest distance to the majority of Derrick Hall, a large grouping of dorm buildings, as well as every building west of Alkek Library. If walking was the only method that had to be considered, then this path would have been the clear second, but after analyzing the Ramp path, it is clearer why the probabilities for this pathway were so severely decreased. Even though the Up path does provided the shortest walking distance to half of campus, the Student Rec Center, and a major parking area, only a few of the campus buildings can be reached by walking from here easier than by travelling via a campus bus. Due to traffic on the busses, walking is frequently the only option most students are willing to take, but the traffic itself is proof enough that students are more willing to ride the bus if possible and convenient enough.

The last pathway on the probability ranking for the Bus Loop was the Stairs path. This pathway, while providing a shorter distance to the base of Derrick Hall as well as all buildings in the main portion of campus east of Alkek, still fell far too short to be competitive. I believe this is because:

1) *The proportion of buildings are far too small to compare with Ramp and Up.*
2) *Stairs has very few or no sources of entertainment or places of mass transportation.*

Our next analysis will be of the intersection at the base of Alkek Library, which we will call Alkek Base from here on. On the next page, figure 7.6, is an image of the intersection at a single point in time during our observations.

Figure 7.6 Alkek Base Image

From the perspective of the footage, Up is the pathway leading towards Derrick Hall, Right is the path directed down towards the Stallions statue and the Quad, Left leads to the stairs headed up towards the library, and Down is a sidewalk towards a construction area and will be the first pathway we analyze.

The Down pathway ends in a three-way intersection. To the left is a narrow sidewalk that leads only to the Evans building, a building that can be reach much faster by choosing Right instead. To the right of the t-intersection is yet another narrow pathway, though this one leads up a set of stairs that snakes around to the other side of Alkek Library. Due to construction, the stairs do not lead anywhere else, and so it is clear to see that choosing Left would accomplish the same thing at a much shorter distance. This being said, it is easy to see why Down was consistently the lowest ranking probability choice. The second lowest was Up, and the reason is much easier to explain. Up is a path that ends in MCS and Derrick Hall with only minor side paths that weave around the building to the road behind. This same conclusion can be reached much faster and easier by taking Left and then turning at another intersection closer to the library stairs.

The first and second highest probabilities of this intersection are the only ones in our entire data set that varied enough to alter rankings from set to set. The Left and Right pathways cumulative percentages swapped locations, and so to actually rank them farther we must look at the averages. By this method, Right falls in third place, though this is the first anomaly we have encountered. Using our earlier observations, it would appear that, with this intersection being near center of campus and both pathways leading towards half of the buildings on campus, that the decision would fall to the transportation and entertainment aspects. However, with both the bus loop and the park planted to the right, and only the Student Recreation Center and a bus stop, that is much farther away than the bus loop, to the left, this decision clearly points to Right ranking above Left. This prediction contradicts observed results though, leading to a probable combination of one or more of three likely conclusions.

1) *There is an unknown factor present on university campuses that is able to sway population flow.*

2) *The results of the sample set of percentages of Right fell below the average of the true population average.*

3) *The results of the sample set of percentages of Left fell above the average of the true population average.*

In any case, there is currently insufficient data from this intersection to draw a possible conclusion of exactly what the cause is.

The final of our three intersections is the Quad. Of all three, this was the most closely related to being a pure Brownian Motion, but even so, this intersection failed to meet the most basic requirements. To provide the reader with a view of exactly where our paths lead: Up leads towards Old Main; Right leads to Commons and The Square; Left leads to the Bus Loop; Down leads towards The Stallions and Alkek Library. See figure 7.7 below.

Left, the obvious choice to be our top pathway because of the extremely close Bus Loop, actually had the lowest average, however its rankings were more frequently in

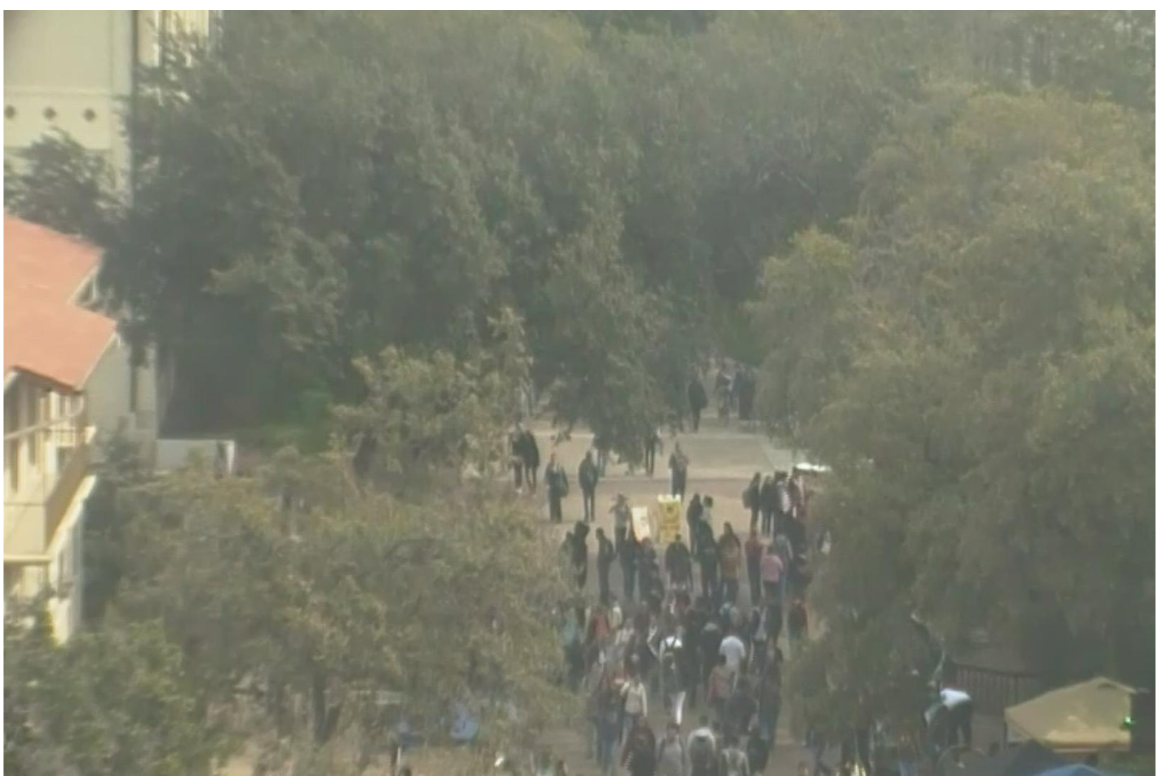

Figure 7.7 The Quad Image

third place. This could imply that, on a university campus, buildings hold a larger pull than transportation does on the students within our restricted times.

Up was ranked third on the list, though as can be implied from above, it was actually ranked fourth in two of the three periods. In this intersection, this pathway provides the shortest distance to roughly half of the academic buildings as well as one dining facility. It is also the shortest pathway down to the Tennis Courts, which have a commuter parking area, to Sewell Park, which was listed before as being a source of entertainment, and to Bobcat Stadium, which houses the largest commuter parking area. With this pathway meeting both of our major influences, it once again places doubt on our conjectured influences, but places an even larger question in our minds. Exactly what is pulling college students down one pathway or another?

The Down pathway was the most unlike a pure Brownian Motion because of its unique consistency. Regardless of time, place, or weather conditions, this pathway was always chosen the most frequently by a large margin. In two of our three observational sessions, each of which lasted in excess of fifteen minutes, the difference in cumulative percentages between this and the second highest rate was eleven percent. On the third test, this difference jumped to twenty one percent. Besides being the shortest pathway to approximately half of the buildings on campus, both academic and residential, none of our other aforementioned influences are accessed via this route. At least, not at first glance.

The only noticeable pull that this pathway might have is the large quantity of booths and stands. The Quad is frequently home to a variety of student organizations trying to gain new members, hosting fundraisers, or gaining support for a variety of events and projects. While these booths may not be entertainment themselves, though occasionally some are, they do provide a link to entertaining and motivational things. As students one of our largest priorities should be education, and almost every organization, whether it be fraternities, sororities, ROTC, or religious groups, stress education first and have minimum GPA requirements. This is mostly due to the students needing to remain in good standing with the university in order to remain in the groups, but still the motivation is there.

Also, there is rarely a better way to meet new people and have fun gatherings than to join an organization with other like-minded people. Therefore, even though these booths themselves may not provide entertainment, I believe that the human mind is advanced enough to realize that these booths provide a gamble for greater future experiences, allowing us to label entertainment as a pull on this intersection.

Our second ranked intersection is the Right pathway, and of all four this intersection had the highest likelihood of being a pure Brownian Motion. Its average percentage fell at 25.38 and its t* value was such that we could not fully reject its true average percentage. However, as was stated before, in order for a Brownian Motion to rule our intersection, all of the pathways must meet this requirement. As such, this pathway still fails.

Right leads to mostly residential buildings and a few dining facilities, but its largest pull is likely to come from The Square. The Square is an area of San Marcos known to have a very large variety of restaurants, stores, and shops. While it's true that we exempted food from entertainment, this is could still be a major pull on students for two main reasons. The first reason is that walking to town for sustenance provides a much larger variety, as well as a higher quality, of food choices. The second reason is that, disregarding those who bring their own lunches from home, the options off campus provide larger portions for a lower cost.

A perfect example of this would be the Blimpie's on campus. Each meal trade costs approximately five dollars, varying only slightly depending upon the size chosen, and one meal trade earns a six inch sub, chips, and a soda. Meanwhile, there is a Subway just south of campus that provides a foot long sub, chips, and a soda for only seven dollars and twenty five cents. Furthermore, the Subway allows free refills while the Blimpie's does not permit refills. Thus, for a one hundred percent increase in the size of the meal it only took a fifty percent increase in cost. For a size to size match, Subway's six inch subs, chips, and a drink only cost five dollars as well, but once again they permit free refills. College students spend their days in class, their nights studying, and any free time they have is split between sleep and work. It is here that we are likely to find the third pull on college students that we could not determine in the City section, which I believe is cost.

We will dwell here for a moment as we revisit a pathway from our first intersection, the Ramp pathway in the Bus Loop intersection. While describing that pathway, we made note that it was the fastest way to go off campus, and just like south of campus, the north side has plenty of reasonably priced restaurants. Also like our Right pathway in the Quad, the Ramp in the Bus Loop fell second in its ranking percentages, and neither could reject the possibility of the true mean being twenty five percent, the desired average for both intersections. The similarities between the two paths cannot be ignored, and if the conjecture of cost being the third influence, it is here that we would determine it. However, there are still too many variables to say exactly what pulls the students off campus with a large amount of certainty. One example of this would be bookstores. Both north and south of campus contain bookstores within walking distance, and both locations sell books at a lower price than the university's. In this case, was it the cost pulling the students, or could it actually have been the books? If we put forth the books as being third in pull and cost, though still influential, farther down on our unknown list, then it provides a much larger pull on the Down path of the Quad as well as explaining why the Left path of the Alkek Base intersection rose above the Right pathway. In both cases, Alkek Library and the University Bookstore can be reached with the least energy by choosing these pathways. Adding in this possible influencer begins to bring a greater cohesion into the results of our observations, providing greater support for adding books into our hypothesized affecters.

§Conclusion

In this thesis, we explored the discovery of Brownian Motion, its founders, and even some of its applications. We then moved into studying how Albert Einstein solved the question of how to describe the motion, and viewed how his new perspective was developed into a mathematical representation. We then moved into adapting his Diffusivity Coefficient into our own theoretical one, showing how each constant related to a constant in our own situation. Finally, we collected and analyzed our data, proving false all three starting hypotheses:

1) In general, pedestrian movement at an intersection is not a Brownian Motion.

2) Pedestrian movement at intersections in high density cities is a Brownian Motion.

3) Pedestrian movement at intersections on a university campus during normal business hours on normal business days is a Brownian Motion.

We then moved on to attempt to explain why the data moved the way it did. From the city data, we were able to create two possible influencers, transportation and entertainment, that seemed to explain the majority of the results. Once we viewed the university scenario, we continued with the two city influencers, but were forced to create two others, cost and books, to explain abnormalities that our earlier affecters could not account for.

Due to restrictions in time and data, we were not able to place numbers for exactly how much these influencers pull pedestrian movement down one pathway or another. In the university case we were unable to even determine an appropriate ranking for which have the greatest pull and which are only mildly important. Future research could alleviate this problem, and therein create a simple pedestrian movement mapping that could label where a random person is most likely to walk to and with what probability. This could even branch out to explaining animal movement, and why some animals will choose one pathway to migrate one year, and then entirely changes for the next. Only one thing can be assured from this research, that the future holds infinite possibilities.

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VITA

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