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ASYMPTOTIC STABILITY AND BLOW-UP OF SOLUTIONS FOR AN EDGE-DEGENERATE WAVE EQUATION WITH SINGULAR POTENTIALS AND SEVERAL NONLINEAR SOURCE TERMS OF DIFFERENT SIGN

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Abstract. We study the initial boundary value problem of an edge-degenerate wave equation. The operator $\Delta_{\mathbb{E}}$ with edge degeneracy on the boundary ∂E was investigated in the literature. We give the invariant sets and the vacuum isolating behavior of solutions by introducing a family of potential wells. We prove that the solution is global in time and exponentially decays when the initial energy satisfies $E(0) \le d$ and $I(u_0) > 0$. Moreover, we obtain the result of blow-up with initial energy $E(0) \le d$ and $I(u_0) < 0$, and give a lower bound for the blow-up time T^*.

1. Introduction

We consider the following initial-boundary value problem of an edge-degenerate wave equation with singular potentials and several nonlinear source terms of different sign:

$$
\begin{gathered}
\partial_{tt}u - \Delta_{\mathbb{E}}u + \partial_t u + Vu \\
= \sum_{k=1}^{l} a_k |u|^{p_k-1}u - \sum_{j=1}^{s} b_j |u|^{q_j-1}u \quad \text{in } \operatorname{int}(\mathbb{E}) \times (0,T), \\
u(0) = u_0, \quad \partial_t u(0) = u_1 \quad \text{in } \operatorname{int}(\mathbb{E}), \\
u = 0 \quad \text{on } \partial\mathbb{E} \times (0,T).
\end{gathered} \tag{1.1}
$$

where $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $u_1 \in \mathcal{L}_2(\mathbb{E}) \cap \mathcal{L}_2(Q_T)$, $Q_T = \mathbb{E} \times [0,T]$, $T \in (0,\infty)$ and $N = 1+n+h \ge 3$, $a_k > 0$ for $1 \le k \le l$, $b_j > 0$ for $1 \le j \le s$, p_k and q_j satisfy

$$
1 < q_s < q_{s-1} < \dots < q_1 = q < p < p_l < p_{l-1} < \dots < p_1 < \frac{N+2}{N-2} \quad \text{if } N \ge 3,
$$

and V is a positive potential function which can be unbounded on the edge manifold $\mathbb{E}$. Here, X is a closed compact C^∞-smooth submanifold of dimension n embedded in the unit sphere of $\mathbb{R}^{n+1}$ and Y is a bounded subset in $\mathbb{R}^h$ containing the origin. Write $\mathbb{E} = [0,1) \times X \times Y$, which can be regarded as the local model near the

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boundary of stretched edge-manifolds. Denote $\mathbb{E}_0$ for the interior of $\mathbb{E}$, and the boundary of $\mathbb{E}$ by $\partial\mathbb{E} = 0 \times X \times Y$. The edge-Laplacian operator is defined as

$$\Delta_{\mathbb{E}} = \nabla_{\mathbb{E}}^2 = (\omega\partial_\omega)^2 + \partial_{x_1}^2 + \ldots + \partial_{x_n}^2 + (\omega\partial_{y_1})^2 + \ldots + (\omega\partial_{y_h})^2,$$

where $\nabla_{\mathbb{E}} = (\omega\partial_\omega, \partial_{x_1}, \ldots, \partial_{x_n}, \omega\partial_{y_1}, \ldots, \omega\partial_{y_h})$ denotes the gradient operator with edge degeneracy on the boundary $\partial\mathbb{E}$. Moreover, the degenerate elliptic operator $\Delta_{\mathbb{E}}$ is regarded as a spacial case of type degenerate differential operators on a stretched edge manifold.

In the case of edge degenerate, Chen and Liu [3] considered the following initial-boundary value problem for semi-linear parabolic equations with singular potential term:

$$\begin{gathered}
\partial_t u - \Delta_{\mathbb{E}} u - Vu = |u|^{p-1}u \quad \text{in } \operatorname{int}(\mathbb{E}) \times (0,T), \\
u(0) = u_0 \quad \text{in } \operatorname{int}(\mathbb{E}), \\
u = 0 \quad \text{on } \partial\mathbb{E} \times (0,T),
\end{gathered}$$

and derived a threshold of the existence of global solutions with exponential decay and the blow-up in finite time, with low initial energy case and critical initial energy case. Chen et al. [6] first established the corresponding Sobolev inequality and Poincaré inequality on the cone Sobolev spaces, and then proved the existence of non-trivial weak solution for the following Dirichlet boundary value problem for a class of non-linear elliptic equation on manifolds with conical singularities:

$$\begin{gathered}
-\Delta_{\mathbb{B}} u = u|u|^{p-1}, \quad \text{for } 1 < p < \frac{n+2}{n-2} \text{ in } \operatorname{int}(\mathbb{B}), \\
u = 0 \quad \text{on } \partial\mathbb{B} \times (0,T).
\end{gathered}$$

Here $\mathbb{B} = [0,1) \times X$ and X is an $(n-1)$-dimensional closed compact manifold. The local model $\mathbb{B}$ is regarded as the local model near the conical points, and $\partial\mathbb{B} = \{0\} \times X$. Moreover, the operator $\Delta_{\mathbb{B}}$ in the above equation is defined by $(x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$, which is an elliptic operator with conical degeneration on the boundary $x_1 = 0$, and corresponding gradient operator is denoted by $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$. Alimohammady et al. I[1] applied the family of potential wells to the following initial boundary value problem of semilinear hyperbolic equations on the cone Sobolev spaces:

$$\begin{gathered}
u_{tt} - \Delta_{\mathbb{B}} u + V(x)u + \gamma u_t = f(x,u) \quad \text{in } \operatorname{int}(\mathbb{B}) \times (0,T), \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \quad \text{in } \operatorname{int}(\mathbb{B}), \\
u(t,x) = 0 \quad \text{on } \partial\mathbb{B} \times (0,T),
\end{gathered}$$

They not only gave some results of existence and nonexistence of global solutions, but also obtained the vacuum isolating of solutions and showed blow-up in finite time of solutions on a manifold with conical singularities. For different problems which take into account degenerate spaces, we refer to [2, 4, 5, 14].

For a more general problem, Jiang [10] considered equation

$$\phi_{tt} - \Delta\phi + V(x)\phi + \phi_t = \phi|\phi|^{p-1} \quad \text{in } \mathbb{R}^N \times (0,T) \tag{1.2}$$

with initial data

$$\phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = \phi_1(x) \quad \text{in } \mathbb{R}^N,$$

where $1 < p \le \frac{N}{N-2}$ for $N > 2$ ($p > 1,\ N = 1,2$). He established new stable and unstable sets for the initial data, and proved that the solution blows up in finite time when the evolution enters into the unstable set and the solution is global existence when the evolution enters into the stable set. For the case of linear damping and potential terms, Levine [13] showed that the solution of (1.2) with negative initial energy blows up for the abstract version. Zhou [20] proved that the solution of (1.2) with the intial data (ϕ_0, ϕ_1) satisfies $\int_{\mathbb{R}^N} \phi_0\phi_1 \geq 0$ and $V(x) = 0$ blows up in finite time even for vanishing initial energy. Jiang and Zhang [11] considered the Cauchy problem of a nonlinear wave equation with damping and source terms. When the term works as the damping in the case $m = 1$, the equation is equivalent to the problem (1.2) with the case $V(x) = 0$. Galakhov in [7] discussed the semi-linear wave equation with a potential, which is different from (1.2) because of the damping term. For more papers related to the semi-linear hyperbolic problem, you can refer to [8, 12, 18] and the references therein. Liu and Xu [16] studied the following initial boundary value problem of wave equations and reaction-diffusion equations with several nonlinear source terms of different sign:

$$u_{tt} - \Delta u = f(u) \equiv \sum_{k=1}^{l} a_k |u|^{p_k-1}u - \sum_{j=1}^{s} b_j |u|^{q_j-1}u \quad \text{in } \Omega \times [0,T],$$

with initial boundary conditions

$$\begin{gathered} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0. \end{gathered}$$

The authors introduced a family of potential wells and corresponding outside sets and who also proved the invariance of some sets under the flow of the equation and vacuum isolating of solutions. Then, they got the threshold result of existence and nonexistence of global solution.

Motivated by the above results, in this paper, we intend to study the initial boundary value problem for an edge-degenerate wave equation. We consider the existence of global solution by Faedo-Galerkin approximation method [19] and discuss the blow-up of solutions by means of a a convexity argument [15]. The main difficulty in carrying out this paper is considering the problem with singular potentials in an edge type Sobolev space. We introduce stable and unstable sets and construct some functionals. By giving a family of potential wells, we obtain vacuum isolating of solutions. Then we derive a threshold of the existence of global solutions with exponential decay if the initial data are in the stable set, otherwise, we obtain the blow-up result.

The remaining part of our paper is organized as follows. In Section 2, we give some notation and basic facts which are needed for our work. In Section 3, we introduce a family of potential wells, and give the invariance of some sets under the flow of problem (1.1) and the vacuum isolating behavior of the problem. The existence of global solutions and exponential decay result are given in Section 4. In Section 5, we discuss the blow-up in finite time of solution and seek a lower bound for the blow-up time T^*.

2. Edge type Sobolev spaces

In this section, we introduce some definitions and propositions which will be used in this paper. Now, we recall the Edge type weighted p-Sobolev space [3].

Definition 2.1. For $(\omega,x,y)\in\mathbb{R}_+^N$ with $N=n+h+1$, assume $(\omega,x,y)\in\mathcal{D}'(\mathbb{R}_+^N)$, we say that $u(\omega,x,y)\in\mathcal{L}_p(\mathbb{R}_+^N,\,d\sigma)$ if

$$\|u\|_{\mathcal{L}_p}=\Big(\int_{\mathbb{R}_+^N}\omega^N|u(\omega,x,y)|^p\,d\sigma\Big)^{1/p}<+\infty,$$

where $d\sigma=\frac{d\omega}{\omega}dx_1\dots dx_n\frac{dy_1}{\omega}\dots\frac{dy_h}{\omega}$. Moreover, the weighted $\mathcal{L}_p$ spaces with weight $\gamma\in R$ is denoted by $\mathcal{L}_p^\gamma(\mathbb{R}_+^N,\,d\sigma)$, which consists of functions $u(\omega,x,y)$ such that

$$\|u\|_{\mathcal{L}_p^\gamma}=\Big(\int_{\mathbb{R}_+^N}\omega^N|\omega^{-\gamma}u(\omega,x,y)|^p\,d\sigma\Big)^{1/p}<+\infty.$$

Definition 2.2. For $m\in N$, $\gamma\in\mathbb{R}$ and $N=n+h+1$, the spaces $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)$ is defined by

$$\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N):=\Big\{u\in\mathcal{D}'(\mathbb{R}_+^N)|\omega^{\frac{N}{p}-\gamma}(\omega\partial_\omega)^k\partial_x^\alpha(\omega\partial_y)^\beta:u\in\mathcal{L}_p(\mathbb{R}_+^N,\,d\sigma)\Big\},$$

for $k\in N$, multi-index $\alpha\in N^n$, $\beta\in N^q$ with $k+|\alpha|+|\beta|\le m$. Therefore, $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)$ is a Banach space with the following norm:

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)}=\sum_{k+|\alpha|+|\beta|\le m}\Big(\int_{\mathbb{R}_+^N}\omega^N|\omega^{-\gamma}(\omega\partial_\omega)^k\partial_x^\alpha(\omega\partial_y)^\beta u(\omega,x,y)|^p\,d\sigma\Big)^{1/p}.$$

Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^N)$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)$ denotes the closure of $C_0^\infty(\mathbb{R}_+^N)$ in $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)$.

If $u\in\mathcal{L}_p^{\frac{n+1}{p}}(\mathbb{E})$ and $v\in\mathcal{L}_{p'}^{\frac{n+1}{p'}}(\mathbb{E})$, where $p,p'\in(1,\infty)$ and $\frac1p+\frac1{p'}=1$, then we have the following edge type Hölder inequality:

$$\int_{\mathbb{E}}\omega^h|uv|\,d\sigma\le\Big(\int_{\mathbb{E}}\omega^h|u|^p\,d\sigma\Big)^{1/p}\Big(\int_{\mathbb{E}}\omega^h|v|^{p'}\,d\sigma\Big)^{1/p'}.$$

When $p=2$, we get the corresponding edge type Schwarz inequality

$$\int_{\mathbb{E}}\omega^h|uv|\,d\sigma\le\Big(\int_{\mathbb{E}}\omega^h|u|^2\,d\sigma\Big)^{1/2}\Big(\int_{\mathbb{E}}\omega^h|v|^2\,d\sigma\Big)^{1/2}.$$

For convenience we denote

$$(u,v)_2=\int_{\mathbb{E}}\omega^h uv\,d\sigma,\quad \|u\|_{\mathcal{L}_p^{\frac{n+1}{p}}(\mathbb{E})}=Big(\int_{\mathbb{E}}\omega^h|u|^p\,d\sigma\Big)^{1/p}.$$

For $t\in(0,T)$, we introduce the energy functional

$$\begin{aligned}E(t)=&\frac12\int_{\mathbb{E}}\omega^h|\partial_t u|^2\,d\sigma+\frac12\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\frac12\int_{\mathbb{E}}\omega^hV|u|^2\,d\sigma\\&-\sum_{k=1}^l\frac{a_k}{p_k+1}\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma+\sum_{j=1}^s\frac{b_j}{q_j+1}\int_{\mathbb{E}}\omega^h|u|^{q_j+1}\,d\sigma.\end{aligned}$$

Then

$$\begin{aligned}E(0)=&\frac12\int_{\mathbb{E}}\omega^h|u_1|^2\,d\sigma+\frac12\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u_0|^2\,d\sigma+\frac12\int_{\mathbb{E}}\omega^hV|u_0|^2\,d\sigma\\&-\sum_{k=1}^l\frac{a_k}{p_k+1}\int_{\mathbb{E}}\omega^h|u_0|^{p_k+1}\,d\sigma+\sum_{j=1}^s\frac{b_j}{q_j+1}\int_{\mathbb{E}}\omega^h|u_0|^{q_j+1}\,d\sigma.\end{aligned}$$

An elementary calculation shows

$$E(t) = E(0) - \int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau.$$

Next we define the following functionals on the edge Sobolev space $\mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$:

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \frac{1}{2}\int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma - \sum_{k=1}^{l} \frac{a_k}{p_k+1}\int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma \\
&\quad + \sum_{j=1}^{s} \frac{b_j}{q_j+1}\int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma, \\
I(u) &= \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma - \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma \\
&\quad + \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma, \\
I_\delta(u) &= \delta\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \delta\int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma - \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma \\
&\quad + \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma.
\end{aligned}$$

Here, $J(u)$, $I(u)$ and $I_\delta(u)$ are well-defined and belong to $C^1(\mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E}), \mathbb{R})$. Then, we introduce the potential well

$$W = \big\{u \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E}) : I(u) > 0, J(u) < d\big\} \cup \{0\},$$

and the outside set of the corresponding potential well is defined as

$$V = \big\{u \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E}) : I(u) < 0, J(u) < d\big\}.$$

Now, we define

$$\begin{gathered}
\mathcal{N} = \Big\{u \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})) \setminus \{0\} : I(u) = 0,\ \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma \neq 0\Big\}, \\
d = \inf\{J(u), u \in \mathcal{N}\}.
\end{gathered}$$

And for $\delta \geq 0$, we define

$$\begin{gathered}
\mathcal{N}_\delta = \Big\{u \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})) \setminus \{0\} : I_\delta(u) = 0,\ \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma \neq 0\Big\}, \\
d_\delta = \inf\{J(u), u \in \mathcal{N}_\delta\}.
\end{gathered}$$

To facilitate our calculations, we state the following propositions.

Proposition 2.3 ([3]). *For $1 < l < 2^* = \frac{2N}{N-2}$, the embedding $\mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E}) \hookrightarrow \mathcal{H}^{1,\frac{n+1}{l}}_{l,0}(\mathbb{E}) = \mathcal{L}_l^{\frac{n+1}{l}}(\mathbb{E})$ is compact.*

Proposition 2.4 (Edge type Poincaré inequality [3]). *Let $\mathbb{E} \subset [0,1) \times X \times Y$ be a bounded subset in $\mathbb{R}_+^N$ with $N = 1 + n + h$. If $u(\omega, x, y) \in \mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^N)$ for $1 < p < +\infty$, $\gamma \in \mathbb{R}_+$, then*

$$\|u(\omega, x, y)\|_{\mathcal{L}_p^\gamma(\mathbb{E})} \le d_{\mathbb{E}} \|\nabla_{\mathbb{E}} u(\omega, x, y)\|_{\mathcal{L}_p^\gamma(\mathbb{E})},$$

where $d_{\mathbb{E}}$ is the diameter of $\mathbb{E}$.

Proposition 2.5 (Edge type Hardy's inequality [3]). *For all $u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$,*

$$\int_{\mathbb{E}} \omega^h V |u|^2 \, d\sigma \le C \|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

For $u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, we define

$$C^* = \inf \Big\{ \frac{\sqrt{V} \|u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}} : u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \Big\}.$$

Next, we give some preparatory work, so that there are some lemmas which will be used in this paper.

Lemma 2.6 (Sobolev-Poincaré [3]). *Assume that $u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ and that $\|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. If $1 < p < 2^* - 1 = \frac{N+2}{N-2}$ and $N = n + h + 1 \ge 3$, then there exist a constant C_* such that*

$$\|u\|_{\mathcal{L}_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \le C_* \|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

By Proposition 2.3 we can obtain the following constants:

$$C_k = \sup \Big\{ \frac{\|u\|_{\mathcal{L}_{p_k+1}^{\frac{n+1}{p_k+1}}(\mathbb{E})}}{\|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}} ; u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \Big\},$$

$$C_j = \sup \Big\{ \frac{\|u\|_{\mathcal{L}_{q_j+1}^{\frac{n+1}{q_j+1}}(\mathbb{E})}}{\|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}} ; u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \Big\}.$$

3. Invariant sets and vacuum isolating

In this section, we shall introduce a family of Nehari functionals in edge type Sobolev spaces and the family of potential wells sets. Now, we give the corresponding lemmas, which will help us to demonstrate the invariant sets and the vacuum isolating behavior of solutions for the problem.

3.1. **Properties of potential wells.**
In this subsection, we introduce a family of potential wells and give a series of their properties which are used to prove of our main results.

Lemma 3.1. *Assume that $u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ and that $\|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, we have*

1. $\lim_{\lambda \to 0} J(\lambda u) = 0$, *and* $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$*;*
2. *In the interval $0 < \lambda < \infty$, there is a unique $\lambda^* = \lambda^*(u) > 0$, such that $\frac{d}{d\lambda} J(\lambda^* u) = 0$;*

(3) *$J(\lambda u)$ is increasing on $0 \le \lambda < \lambda^*$, decreasing on $\lambda > \lambda^*$ and takes the maximum at $\lambda = \lambda^*$;*

(4) *$I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$ and $I(\lambda^* u) = 0$.*

Proof. (1) From the definition of $J(u)$, we know that

$$\begin{aligned} J(\lambda u) =& \frac{\lambda^2}{2}\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\,d\sigma + \frac{\lambda^2}{2}\int_{\mathbb{E}} \omega^h V|u|^2\,d\sigma - \sum_{k=1}^{l} \frac{a_k\lambda^{p_k+1}}{p_k+1}\int_{\mathbb{E}} \omega^h |u|^{p_k+1}\,d\sigma \\ &+ \sum_{j=1}^{s} \frac{b_j\lambda^{q_j+1}}{q_j+1}\int_{\mathbb{E}} \omega^h |u|^{q_j+1}\,d\sigma, \end{aligned}$$

which gives $\lim_{\lambda\to 0} J(\lambda u) = 0$, and $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$.

(2) An easy calculation shows that

$$\begin{aligned} \frac{d}{d\lambda}J(\lambda u) =& \lambda\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\,d\sigma + \lambda\int_{\mathbb{E}} \omega^h V|u|^2\,d\sigma \\ &- \lambda^{p_k}\sum_{k=1}^{l} a_k\int_{\mathbb{E}} \omega^h |u|^{p_k+1}\,d\sigma + \lambda^{q_j}\sum_{j=1}^{s} b_j\int_{\mathbb{E}} \omega^h |u|^{q_j+1}\,d\sigma = 0, \end{aligned} \tag{3.1}$$

which is equivalent to

$$\begin{aligned} &\sum_{k=1}^{l} a_k\lambda^{p_k-1}\int_{\mathbb{E}} \omega^h |u|^{p_k+1}\,d\sigma - \sum_{j=1}^{s} b_j\lambda^{q_j-1}\int_{\mathbb{E}} \omega^h |u|^{q_j+1}\,d\sigma \\ &= \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\,d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\,d\sigma. \end{aligned}$$

Then we let

$$\begin{aligned} g(\lambda) =& \sum_{k=1}^{l} a_k\lambda^{p_k-1}\int_{\mathbb{E}} \omega^h |u|^{p_k+1}\,d\sigma - \sum_{j=1}^{s} b_j\lambda^{q_j-1}\int_{\mathbb{E}} \omega^h |u|^{q_j+1}\,d\sigma \\ =& \lambda^{p-1}\Big(\sum_{k=1}^{j} a_k\lambda^{p_k-p}\int_{\mathbb{E}} \omega^h |u|^{p_k+1}\,d\sigma - \sum_{j=1}^{s} b_j\lambda^{q_j-p}\int_{\mathbb{E}} \omega^h |u|^{q_j+1}\,d\sigma\Big) \\ =&:\ \lambda^{p-1}g^*(\lambda). \end{aligned}$$

Note that $g^*(\lambda)$ is increasing on $0 < \lambda < \infty$, and that $\lim_{\lambda\to 0^+} g^*(\lambda) = -\infty$, $\lim_{\lambda\to+\infty} g^*(\lambda) = +\infty$. Hence there exists a unique $\lambda_0 > 0$ such that $g^*(\lambda_0) = 0$, therefore $g(\lambda_0) = 0$, $g(\lambda) < 0$ for $0 < \lambda < \lambda_0$, $g(\lambda) > 0$ for $\lambda_0 < \lambda < \infty$ and $g(\lambda)$ is increasing on $\lambda_0 < \lambda < \infty$. Hence for any $\|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \ge 0$, there exists a unique $\lambda^* > \lambda_0$ such that

$$\begin{aligned} &\sum_{k=1}^{l} a_k\lambda^{p_k-1}\int_{\mathbb{E}} \omega^h |u|^{p_k+1}\,d\sigma - \sum_{j=1}^{s} b_j\lambda^{q_j-1}\int_{\mathbb{E}} \omega^h |u|^{q_j+1}\,d\sigma \\ &= \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\,d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\,d\sigma\,. \end{aligned}$$

(3) Note that

$$\frac{d}{d\lambda}J(\lambda u) = \lambda\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\,d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\,d\sigma - g(\lambda)\Big).$$

From the proof of (2), it follows that if $0 < \lambda \le \lambda_0$ then $g(\lambda) \le 0$, if $\lambda_0 < \lambda < \lambda^*$ then

$$0 < g(\lambda) < \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma,$$

if $\lambda^* < \lambda < \infty$ then $g(\lambda) > \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma$. Hence, we have $\frac{d}{d\lambda} J(u) > 0$ for $0 < \lambda < \lambda^*$, $\frac{d}{d\lambda} J(u) < 0$ for $\lambda^* < \lambda < \infty$. From this, the conclusion of (3) follows.

(4) The conclusion follows from the proof of (3) and

$$\begin{aligned} I(\lambda u) =& \lambda^2 \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \lambda^2 \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma - \lambda^{p_k+1} \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1} \, d\sigma \\ &+ \lambda^{q_j+1} \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1} \, d\sigma \\ =& \lambda \frac{d}{d\lambda} J(\lambda u). \end{aligned}$$

□

Lemma 3.2. *Let $\delta > 0$, if $0 < \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \le \gamma^2(\delta)$, then $I_\delta(u) > 0$. In particular, if $0 < \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \le \gamma^2(1)$, then $I(u) > 0$, where $\gamma(\delta)$ is the unique real root of equation $\varphi(\gamma) = \delta$, and*

$$\varphi(\gamma) = \sum_{k=1}^{l} a_k C_*^{p_k+1} \gamma^{p_k-1}.$$

Proof. From $0 < \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \le \gamma^2(\delta)$, we have $\int_{\mathbb{E}} \omega^h |u|^{q_j+1} \, d\sigma > 0$, $1 \le j \le s$ and by

$$\begin{aligned} &\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1} \, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1} \, d\sigma \\ &< \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1} \, d\sigma \\ &< \sum_{k=1}^{l} a_k C_*^{p_k+1} \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^{p_k+1} \, d\sigma \\ &= \varphi\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u| \, d\sigma\Big) \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma \le \delta \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma, \end{aligned}$$

we get $I_\delta(u) > 0$. □

Lemma 3.3. *Let $\delta > 0$, if $I_\delta(u) < 0$, then $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma > \gamma^2(\delta)$. In particular, if $I_\delta(u) > 0$, then $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma > \gamma^2(1)$.*

Proof. From $I_\delta(u) < 0$, we have

$$\begin{aligned} &\delta \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \delta \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \\ &< \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1} \, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1} \, d\sigma \end{aligned}$$

$$< \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma \le \varphi\big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|\, d\sigma\big) \|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})},$$

which implies $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma > \gamma^2(\delta)$. □

Lemma 3.4. *let $\delta > 0$, if $I_\delta(u) = 0$, then $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma > \gamma^2(\delta)$ or*

$$\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma = 0.$$

In particular, if $I(u) = 0$, then

$$\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma > \gamma^2(1)$$

or

$$\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma = 0.$$

Proof. If $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma = 0$, then $I_\delta(u) = 0$. If $I_\delta(u) = 0$ and $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma = 0$, then $I_\delta(u) \neq 0$, then by

$$\begin{aligned}
&\delta\big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma\big)\\
&= \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma\\
&< \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma \le \varphi\big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|\, d\sigma\big) \|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})},
\end{aligned}$$

we get $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma > \gamma^2(\delta)$. □

Theorem 3.5. *The function of δ, $d(\delta)$ possesses the following properties:*

1. *$d(\delta) > a(\delta)\gamma^2(\delta)$ for $0 < \delta < \frac{(p+1)(1+C^{*2})}{2}$ where $a(\delta) = \frac{1}{2}(1 + C^{*2}) - \frac{\delta}{p+1}$.*
2. *$\lim_{\delta\to 0} d(\delta) > 0$ and there exists a unique $\delta_0 > \frac{(p+1)(1+C^{*2})}{2}$ such that $d(\delta_0) = 0$ and $d(\delta) > 0$ for $0 \le \delta < \delta_0$.*
3. *$d(\delta)$ is increasing on $0 \le \delta \le 1$, decreasing on $1 \le \delta \le \delta_0$ and takes the maximum $d = d(1)$ at $\delta = 1$.*

Proof. (1) If $I_\delta(u) = 0$ and $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma \neq 0$, then by Lemma 3.4, we have $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma > \gamma^2(\delta)$ and

$$\begin{aligned}
J(u) =& \frac{1}{2} \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \frac{1}{2} \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma\\
&- \sum_{k=1}^{l} \frac{a_k}{p_k+1} \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma + \sum_{j=1}^{s} \frac{b_j}{q_j+1} \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma\\
\ge& \frac{1}{2}(1 + C^{*2}) \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma\\
&- \frac{1}{p+1}\Big(\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma \Big)
\end{aligned}$$

$$
\begin{aligned}
&=\Big(\frac{1}{2}(1+C^{*2})-\frac{\delta}{p+1}\Big)\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\frac{1}{p+1}I_\delta(u)\\
&=a(\delta)\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma>a(\delta)\gamma^2(\delta).
\end{aligned}
$$

If $I_\delta(u)=0$ and $\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\int_{\mathbb{E}}\omega^hV|u|^2\,d\sigma=0$, by the definition of $d(\delta)$, which is a contradiction.

(2) First $I_0(u)=0$ implies

$$
\sum_{k=1}^{l}\frac{a_k}{p_k+1}\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma=\sum_{j=1}^{s}\frac{b_j}{q_j+1}\int_{\mathbb{E}}\omega^h|u|^{q_j+1}\,d\sigma
$$

and

$$
\begin{aligned}
J(u)=&\frac{1}{2}\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\frac{1}{2}\int_{\mathbb{E}}\omega^hV|u|^2\,d\sigma\\
&-\sum_{k=1}^{l}\frac{a_k}{p_k+1}\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma+\sum_{j=1}^{s}\frac{b_j}{q_j+1}\int_{\mathbb{E}}\omega^h|u|^{q_j+1}\,d\sigma\\
\geq&\frac{1}{2}(1+C^{*2})\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma-\frac{1}{p+1}\sum_{k=1}^{l}a_k\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma\\
&+\frac{1}{q+1}\sum_{j=1}^{s}b_j\int_{\mathbb{E}}\omega^h|u|^{q_j+1}\,d\sigma)\\
=&\frac{1}{2}(1+C^{*2})\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\frac{p-q}{(p+1)(q+1)}\sum_{k=1}^{l}a_k\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma\\
=&\frac{1}{2}(1+C^{*2})\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\frac{p-q}{(p+1)(q+1)}\sum_{k=1}^{l}b_j\int_{\mathbb{E}}\omega^h|u|^{q_j+1}\,d\sigma.
\end{aligned}
$$

Next, we prove that $d(0)>0$. Let

$$
J_1(u)=\frac{1}{2}\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\frac{1}{2}\int_{\mathbb{E}}\omega^hV|u|^2\,d\sigma+\frac{p-q}{(p+1)(q+1)}\sum_{k=1}^{l}a_k\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma,
$$

and $d_1(0)=\inf_{u\in\mathcal{N}_0}J_1(u)$. The fact that $J(u)\geq J_1(u)$ for $u\in\mathcal{N}_0$ implies $d(0)\geq d_1(0)$. In what follows, we prove $d_1(0)>0$. Let

$$
L_1(\nabla_{\mathbb{E}}u,u)=J(u)=\frac{1}{2}|\nabla_{\mathbb{E}}u|^2+\frac{1}{2}V|u|^2+\frac{p-q}{(p+1)(q+1)}\sum_{k=1}^{l}a_k|u|^{p_k+1}.
$$

Then $J_1(u)=\int_{\mathbb{E}}\omega^hL_1(\nabla_{\mathbb{E}}u,u)\,d\sigma$. Since $L_1(\nabla_{\mathbb{E}}u,u)$ is convex in $|\nabla_{\mathbb{E}}u|$ and satisfies

$$
L_1(\nabla_{\mathbb{E}}u,u)>\alpha|\nabla_{\mathbb{E}}u|-\beta
$$

for $\alpha=\frac{1}{2}(1+C^{*2})$ and any $\beta>0$, from the theory of functional minimization, it follows that there exists a $\mu\in\mathcal{N}_0$ such that $d_1(0)=\inf_{u\in\mathrm{N}_0}J_1(u)=J_1(\mu)>0$, hence we have $d(0)>0$. Next, for any $u\in\mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$, $\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\int_{\mathbb{E}}\omega^hV|u|^2\,d\sigma\neq 0$

and $\delta > 0$, we define $\lambda = \lambda(\delta)$ such that

$$\begin{aligned}&\delta\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} \lambda u|^2 \,d\sigma + \int_{\mathbb{E}} \omega^h V |\lambda u|^2 \,d\sigma\Big)\\ &= \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |\lambda u|^{p_k+1} \,d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |\lambda u|^{q_j+1} \,d\sigma.\end{aligned} \tag{3.2}$$

Then $I_\delta(\lambda u) = 0$ and

$$\begin{aligned}&\delta\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \,d\sigma + \int_{\mathbb{E}} \omega^h V |u|^2 \,d\sigma\Big)\\ &= \sum_{k=1}^{l} a_k \lambda^{p_k-1} \int_{\mathbb{E}} \omega^h |u|^{p_k+1} \,d\sigma - \sum_{j=1}^{s} b_j \lambda^{q_j-1} \int_{\mathbb{E}} \omega^h |u|^{q_j+1} \,d\sigma.\end{aligned} \tag{3.3}$$

From the proof of (2) in Lemma 3.1, it follows that for any $\delta > 0$, there exists a unique $\lambda = \lambda(\delta)$ such that (3.2) and (3.3) hold. Again by the proof of Lemma 3.1, we have $\lim_{\delta\to\infty} \lambda(\delta) = +\infty$. Hence, by (1) of Lemma 3.1, we get $\lim_{\delta\to\infty} J(\delta u) = \lim_{\lambda\to\infty} J(\lambda u) = -\infty$. From this and (1) of this theorem, it follows that there exists a unique $\delta_0 > \frac{(p+1)(1+C^{*2})}{2}$ such that $d(\delta_0) = 0$ and $d(\delta) > 0$ for $0 \le \delta < \delta_0$.

(3) We prove that $d(\delta') < d(\delta'')$ for any $0 \le \delta' < \delta'' < 1$ or $1 < \delta'' < \delta' < b$ and any $u \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$, $I_{\delta''}(u) = 0$ and $\|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$ there exists a $v \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$, $I_{\delta'}(v) = 0$ and $\|\nabla_{\mathbb{E}} v\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, a constant $\varepsilon(\delta', \delta'') > 0$ such that $J(v) < J(u) - \varepsilon(\delta', \delta'')$. In fact, for above u, we also define $\lambda(\delta)$ by (3.2), then $I_\delta(\lambda(\delta)u) = 0$, $\lambda(\delta'') = 1$ and (3.3) holds. Assume that $\varphi(\lambda) = J(\lambda u)$, we obtain

$$\begin{aligned}\frac{d}{d\lambda}\varphi(\lambda) =&\frac{1}{\lambda}\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \,d\sigma + \int_{\mathbb{E}} \omega^h V |u|^2 \,d\sigma - \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1} \,d\sigma\\ &+ \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1} \,d\sigma\Big)\\ =&\frac{1}{\lambda}\Big((1-\delta)\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 d\sigma + \int_{\mathbb{E}} \omega^h V |u|^2 \,d\sigma\Big) + I_\delta(\lambda u)\Big)\\ =&(1-\delta)\lambda\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \,d\sigma + \int_{\mathbb{E}} \omega^h V |u|^2 \,d\sigma\Big).\end{aligned}$$

Taking $v = \lambda(\delta')u$, we have $I_{\delta'}(v) = 0$ and $\|\nabla_{\mathbb{E}} u\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. If $0 < \delta' < \delta'' < 1$, then

$$J(u) - J(v) = \varphi(1) - p(\lambda(\delta')) > (1-\delta'')\gamma^2(\delta'')\lambda(\delta')(1-\lambda(\delta')) \equiv \varepsilon(\delta', \delta'').$$

If $1 < \delta'' < \delta' < \delta_0$, then

$$J(u) - J(v) = \varphi(1) - p(\lambda(\delta')) > (\delta''-1)\gamma^2(\delta'')\lambda(\delta')(\lambda(\delta')-1) \equiv \varepsilon(\delta', \delta'').$$

□

Lemma 3.6. *Let $0 < \delta < \frac{(p+1)(1+C^{*2})}{2}$, assume that $J(u) \le d(\delta)$ and $I_\delta(u) > 0$. Then*

$$0 < \|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} < \frac{d(\delta)}{a(\delta)}.$$

In particular, if $J(u) \le d$ and $I_\delta(u) > 0$, one has

$$0 < \|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} < \frac{2(p+1)}{(p-1)(1+C^{*2})} d.$$

The above lemma follows from

$$a(\delta)\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{p+1} I_\delta(u) < J(u) \le d(\delta).$$

Now we define a family of potential wells as follows

$$W_\delta = \big\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) : I_\delta(u) > 0, J(u) < d(\delta)\big\} \cup \{0\}, \quad 0 < \delta < \delta_0;$$
$$\bar{W}_\delta = \big\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) : I_\delta(u) > 0, J(u) < d(\delta)\big\}, \quad 0 < \delta < \delta_0,$$

and

$$V_\delta = \big\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) : I_\delta(u) < 0, J(u) < d(\delta)\big\}, \quad 0 < \delta < \delta_0;$$
$$B_\delta = \big\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) : \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} \lambda u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V |\lambda u|^2 \, d\sigma < \gamma^2(\delta)\big\}, \quad 0 < \delta < \delta_0;$$
$$\bar{B}_\delta = B_\delta \cup \partial B_\delta = \big\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) : \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} \lambda u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V |\lambda u|^2 \, d\sigma \le \gamma^2(\delta)\big\}, \quad 0 < \delta < \delta_0;$$
$$B_\delta^c = \big\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) : \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} \lambda u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V |\lambda u|^2 \, d\sigma > \gamma^2(\delta)\big\}, \quad 0 < \delta < \delta_0.$$

3.2. **Characteristics of solutions.** In this subsection, we state the invariance of some sets under the flow of (1.1) and the vacuum isolating behavior of problem (1.1).

Definition 3.7 (Maximal existence time)**.** Let $u(t)$ be a weak solution of problem (1.1). We define the maximal existence time $T_{\max}$ of $u(t)$ as follows:

1. If $u(t)$ exists for $0 \le t < \infty$, then $T_{\max} = +\infty$.
2. If there exists a $t_0 \in (0, \infty)$ such that $u(t)$ exists for $0 \le t < t_0$, but doesn't exist at $t = t_0$, then $T_{\max} = t_0$.

Now, we discuss the invariance of some sets corresponding to problem (1.1).

Theorem 3.8. *Let $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $u_1 \in \mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})$, and $0 < e < d$, $\delta_1 < \delta_2$ are two roots of equation $d(\delta) = e$. Then:*

1. *All weak solutions of problem* (1.1) *with $0 < J(u_0) \le e$ belong to W_δ for $\delta_1 < \delta < \delta_2$, $0 \le t < T_{\max}$, provided $I(u_0) > 0$ or $\|\nabla_{\mathbb{E}} u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$.*
2. *All weak solutions of problem* (1.1) *with $0 < J(u_0) \le e$ belong to V_δ for $\delta_1 < \delta < \delta_2$, $0 \le t < T_{\max}$, provided $I(u_0) < 0$,*

where $T_{\max}$ is the maximal existence time of $u(t)$.

Proof. (1) Let $u(t)$ be any weak solution of problem (1.1) with $J(u_0) \le e$, $I(u_0) > 0$ or $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$. $T_{\max}$ is the existence time of $u(t)$. If $\|\nabla_{\mathbb{E}} u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$, then $u_0(x) \in W_\delta$. If $I(u_0) > 0$, by the definition of $d(\delta)$, it follows $I_\delta(u_0) > 0$ and $J(u_0) < d(\delta)$. Then $u_0(x) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. Next, we should prove $u(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T_{\max}$. Arguing by contradiction, by the continuity of

$I(u)$ we suppose that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T_{\max})$ such that $u(t_0) \in \partial W_{\delta_0}$, and $I_{\delta_0}(u(t_0)) = 0$, $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$ or $J(u(t_0)) = d(\delta_0)$. From

$$\int_0^t \|u_\tau\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau + \int_0^t \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau + J(u(t)) \le J(u_0) < d(\delta), \tag{3.4}$$

$\delta_1 < \delta < \delta_2$, $0 \le t < T_{\max}$. we can see that $J(u(t_0)) \neq d(\delta_0)$. If $I_{\delta_0}(u(t_0)) = 0$, $\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, then by the definition of $d(\delta)$, we have $J(u(t_0)) \ge d(\delta_0)$, which contradicts (3.4).

(2) Let $u(t)$ be a weak solution of problem (1.1) with $0 < J(u_0) \le e < d$, $I(u_0) < 0$. From $J(u_0) \le e$, $I(u_0) < 0$ and

$$\frac{1}{2}\|u_1\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + J(u_0) = E(0) = d(\delta_1) = d(\delta_2) < d(\delta), \quad \delta_1 < \delta < \delta_2,$$

it follows $I_\delta(u_0) < 0$ and $J(u_0) < d(\delta)$. Then $u_0(x) \in V_\delta$ for $\delta_1 < \delta < \delta_2$. We prove $u(t) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T_{\max}$. Arguing by contradiction, by time continuity of $I(u)$ we suppose that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T_{\max})$ such that $u(t_0) \in \partial V_{\delta_0}$, and $I_{\delta_0}(u(t_0)) = 0$ or $J(u(t_0)) = d(\delta_0)$. By (3.4) we can see that $J(u(t_0)) \neq d(\delta_0)$. Assume $I_{\delta_0}(u(t_0)) = 0$ and t_0 is the first time such that $I_{\delta_0}(u(t)) = 0$, then $I_{\delta_0}(u(t)) < 0$ for $0 \le t < t_0$. By Lemma 3.3 we have $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V |u|^2\, d\sigma > \gamma^2(\delta_0)$ for $0 \le t < t_0$. Hence $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V |u|^2\, d\sigma \ge \gamma^2(\delta_0)$, then $\|u(t_0)\|_{\mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})} \neq 0$. From $u(t_0) \in \mathcal{N}_{\delta_0}$ and $J(u(t_0)) \neq d(\delta_0)$, we have $J(u(t_0)) > d(\delta_0)$, which contradicts to (3.4). □

To discuss the invariance of the solutions with negative level energy, we introduce the following results.

Proposition 3.9. *All nontrivial solutions of problem* (1.1) *with* $J(u_0) = 0$ *belong to*

$$B^c_{\gamma_0} = \{u \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E}) | \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V |u|^2\, d\sigma \ge \gamma_0^2\},$$

where γ_0 *is the unique real root of equation*

$$\sum_{k=1}^{l} a_k C_k^{p_k+1} \gamma^{p_k-1} = \frac{1}{2}. \tag{3.5}$$

Proof. Let $u(t)$ be any solution of problem (1.1) with $J(u_0) = 0$, $T_{\max}$ is the maximal existence time of $u(t)$. From the energy equality

$$\frac{1}{2}\|\partial_t u\|^2 + J(u) \equiv E(0) = 0,$$

we get $J(u) \le 0$ for $0 \le t < T_{\max}$. Hence from

$$\begin{aligned} &\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma \\ &\le \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma \le \sum_{k=1}^{l} a_k C_k^{p_k+1} \|\nabla_{\mathbb{E}} u\|^{p+1}_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}, \quad 0 \le t < T_{\max}, \end{aligned}$$

it follows that either $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma = 0$ or $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \geq \gamma_0^2$. If $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma = 0$, $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \equiv 0$ for $0 \leq t < T_{\max}$. Otherwise, there exists a $t_0 \in (0, T_{\max})$ such that $0 < \int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma < \gamma_0^2$. By a similar argument we can prove that if $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \geq \gamma_0^2$, then $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \geq \gamma_0^2$ for $0 < t < T_{\max}$. □

Theorem 3.10. *Let $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$. Assume that $J(u_0) < 0$ or $J(u_0) = 0$ and $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \neq 0$. Then all solutions of problem (1.1) belong to V_δ for $0 < \delta < \frac{(p+1)(1+C^{*2})}{2}$.*

Proof. Let $u(t)$ be any solution of problem (1.1) with $J(u_0) < 0$ or $J(u_0) = 0$ and $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \neq 0$, $T_{\max}$ is the maximal existence time of $u(t)$. The energy inequality gives

$$\begin{gathered} a(\delta)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{p+1} I_\delta(u) \leq J(u) \leq J(u_0), \\ 0 < \delta < \frac{(p+1)(1+C^{*2})}{2}. \end{gathered} \tag{3.6}$$

From (3.6) it follows that if $J(u_0) < 0$, then $I_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for $0 < \delta < \frac{(p+1)(1+C^{*2})}{2}$; if $J(u_0) = 0$ and $\|\nabla_{\mathbb{E}} u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, then by Proposition 3.9 we have $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq \gamma_0$ for $0 \leq t < T_{\max}$. Again by (3.6) we get $I_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for $0 < \delta < \frac{(p+1)(1+C^{*2})}{2}$. Hence for above two cases we always have $u(t) \in V_\delta$ for $0 < \delta < \frac{(p+1)(1+C^{*2})}{2}$, $0 \leq t < T_{\max}$. □

Corollary 3.11. *Let $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$. Assume that $J(u_0) < 0$ or $J(u_0) = 0$ and $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \neq 0$. Then all weak solutions of problem (1.1) belong to $\bar{B}^c_{\frac{(p+1)(1+C^{*2})}{2}}$.*

Proof. Let $u(t)$ be any weak solution of problem (1.1) with $J(u_0) < 0$ or $J(u_0) = 0$ and $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \neq 0$, $T_{\max}$ is the maximal existence time of $u(t)$. Then Theorem 3.10 gives

$$u(t) \in V_\delta \quad \text{for } 0 < \delta < \frac{(p+1)(1+C^{*2})}{2}, \ 0 \leq t < T_{\max}.$$

From this and Lemma 3.3 we get $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \leq \gamma^2(\delta)$ for $0 < \delta < \frac{(p+1)(1+C^{*2})}{2}$, $0 \leq t < T_{\max}$. Then, letting $\delta \to \frac{(p+1)(1+C^{*2})}{2}$, we obtain $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma \geq \gamma^2(\frac{(p+1)(1+C^{*2})}{2})$ for $0 \leq t < T_{\max}$. □

Next, we discuss the vacuum isolating to problem (1.1) with $J(u_0) < d$.

Theorem 3.12. *Let $e \in (0, d)$. Suppose δ_1, δ_2 are the two solutions of $d(\delta) = e$. Then for all weak solutions of problem (1.1) with $J(u_0) \leq e$, there is a vacuum region*

$$U_e = \big\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) | I_\delta(u) = 0, \|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0, \delta_1 < \delta < \delta_2\big\},$$

such that there is no any weak solution of problem (1.1) in U_e.

Proof. Assume that $u(t)$ is any weak solution of problem (1.1) with $J(u_0) \leq e$, $T_{\max}$ is the maximal existence time of $u(t)$. We only need to prove that if $\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2 \, d\sigma + \int_{\mathbb{E}} \omega^h V |u|^2 \, d\sigma \neq 0$ and $J(u_0) \leq e$, then for all $\delta \in (\delta_1, \delta_2)$, $u(t) \notin N_\delta$, i.e. $I_\delta(u(t)) \neq 0$, for all $t \in [0, T_{\max})$.

At first, it is clear that $I_\delta(u_0) \neq 0$. Since if $I_\delta(u_0) = 0$, then $J(u_0) \geq d(\delta) > d(\delta_1) = d(\delta_2)$, which contradicts with $J(u_0) \leq e$.

Suppose there exists $t_1 > 0$ such that $u(t_1) \in U_e$. Namely, there must exist a $\delta_0 \in (\delta_1, \delta_2)$ such that $u(t_1) \in N_{\delta_0}$. From the definition of $d(\delta)$, we get $J(u_0) \geq J(u(t_1)) \geq d(\delta) > J(u_0)$, which leads to a contradiction. $\square$

4. Global solution and exponential decay

In this section, we prove the existence of global solutions by using the Galerkin approximation technique and the potential well theory. Meanwhile, we give an exponential decay result of the solution.

Theorem 4.1. *Let $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$. Assume that $E(0) \leq d$ and $I(u) > 0$. Then problem (1.1) admits a global weak solution $u \in L^\infty(0,T; \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ with $\partial_t u \in L^\infty(0,T; \mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})) \cap \mathcal{L}_2^{\frac{n+1}{2}}(Q_T)$, where $Q_T = \mathbb{E} \times [0,T]$.*

Proof. (1) Low initial energy case ($E(0) < d$ and $I(u) > 0$): Let $\{\varphi_j\}$ be a sequence of orthogonal basis of $\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, we introduce the following approximate solutions of problem (1.1):

$$u_m(t,\omega,x,y) = \sum_{j=1}^{m} f_{jm}(t)\varphi_j(\omega,x,y) \quad m = 1,2,\dots,$$

which satisfies

$$\begin{aligned}
&\int_{\mathbb{E}} \omega^h \partial_{tt} u_m \cdot \varphi_k \, d\sigma + \int_{\mathbb{E}} \omega^h \nabla_{\mathbb{E}} u_m \cdot \nabla_{\mathbb{E}} \varphi_k \, d\sigma + \int_{\mathbb{E}} \omega^h \partial_t u_m \cdot \varphi_k \, d\sigma \\
&+ \int_{\mathbb{E}} \omega^h V u_m \cdot \varphi_k \, d\sigma \\
&= \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u_m|^{p_k} u_m \cdot \varphi_k \, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u_m|^{q_j} u_m \cdot \varphi_k \, d\sigma
\end{aligned} \tag{4.1}$$

for $k = 1,2,\dots,m$. As $m \to +\infty$,

$$u_m(0,\omega,x,y) = \sum_{j=1}^{m} c_{jn}(0)\varphi_j(\omega,x,y) \to u_0(\omega,x,y) \quad \text{in } \mathcal{H}_{2,0}^{1\frac{N}{2}}, \tag{4.2}$$

$$u_{mt}(0,\omega,x,y) = \sum_{j=1}^{m} d_{jn}(0)\varphi_j(\omega,x,y) \to u_1(\omega,x,y) \quad \text{in } \mathcal{L}_2^{\frac{N}{2}}. \tag{4.3}$$

Multiplying (4.1) by $f'_{im}(t)$ and summing for i, we have

$$\begin{aligned}
&\sum_{i=1}^{m} \int_{\mathbb{E}} \omega^h \partial_{tt} u_m \cdot f'_{im}(t)\varphi_k \, d\sigma + \sum_{i=1}^{m} \int_{\mathbb{E}} \omega^h \nabla_{\mathbb{E}} u_m \cdot f'_{im}(t) \nabla_{\mathbb{E}} \varphi_k \, d\sigma \\
&+ \sum_{i=1}^{m} \int_{\mathbb{E}} \omega^h \partial_t u_m \cdot f'_{im}(t)\varphi_k \, d\sigma + \sum_{i=1}^{m} \int_{\mathbb{E}} \omega^h V u_m \cdot f'_{im}(t)\varphi_k \, d\sigma
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u_m|^{p_k} u_m \cdot f'_{im}(t)\varphi_k \, d\sigma \\
&\quad - \sum_{i=1}^{m}\sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u_m|^{q_j} u_m \cdot f'_{im}(t)\varphi_k \, d\sigma.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{E}} \omega^h \partial_t u_m \cdot \partial_t u_m \, d\sigma + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{E}} \omega^h \nabla_{\mathbb{E}} u_m \cdot \nabla_{\mathbb{E}} u_m \, d\sigma \\
&+ \frac{1}{2}\frac{d}{dt}\int_{\mathbb{E}} \omega^h V |u_m|^2 \, d\sigma - \sum_{k=1}^{l} \frac{a_k}{p_k+1}\frac{d}{dt}\int_{\mathbb{E}} \omega^h |u_m|^{p_k+1} \, d\sigma \\
&+ \sum_{j=1}^{s} \frac{b_j}{q_j+1}\frac{d}{dt}\int_{\mathbb{E}} \omega^h |u_m|^{q_j+1} \, d\sigma + \int_{\mathbb{E}} \omega^h \partial_t u_m \cdot \partial_t u_m \, d\sigma = 0.
\end{aligned}
\tag{4.4}
$$

Integrating (4.4) with respect t from 0 to t, we get

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{E}} \omega^h |\partial_t u_m|^2 \, d\sigma + \frac{1}{2}\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u_m|^2 \, d\sigma + \frac{1}{2}\int_{\mathbb{E}} \omega^h V |u_m|^2 \, d\sigma \\
&- \sum_{k=1}^{l} \frac{a_k}{p_k+1}\int_{\mathbb{E}} \omega^h |u_m|^{p_k+1} \, d\sigma + \sum_{j=1}^{s} \frac{b_j}{q_j+1}\int_{\mathbb{E}} \omega^h |u_m|^{q_j+1} \, d\sigma \\
&+ \int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \, d\tau \\
&= \frac{1}{2}\int_{\mathbb{E}} \omega^h |u_1|^2 \, d\sigma + \frac{1}{2}\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u_0|^2 \, d\sigma + \frac{1}{2}\int_{\mathbb{E}} \omega^h V |u_0|^2 \, d\sigma \\
&\quad - \sum_{k=1}^{l} \frac{a_k}{p_k+1}\int_{\mathbb{E}} \omega^h |u_0|^{p_k+1} \, d\sigma + \sum_{j=1}^{s} \frac{b_j}{q_j+1}\int_{\mathbb{E}} \omega^h |u_0|^{q_j+1} \, d\sigma = E(0).
\end{aligned}
$$

The next result is given to show the invariant sets of the solutions for problem (1.1). By (4.2) and (4.3), we have $E(u_m(0)) \to E(u_0)$. Then for sufficiently large m, we have

$$
\int_0^t \|\partial_t u_m(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \, d\tau + E(u_m) = E(u_0) < d.
$$

From $E(0) < d$, (4.2) and (4.3) we find $E_m(0) < d$ for sufficiently large m. Hence we have

$$
\int_0^t \|\partial_t u_m(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \, d\tau + J(u_m) + \frac{1}{2}\int_{\mathbb{E}} \omega^h |\partial_t u_m|^2 \, d\sigma < d \quad 0 \le t < \infty. \tag{4.5}
$$

For $0 \le t < \infty$, we have $u_m(t) \in W$ with m sufficiently large. In fact, if it is false, then there exists $t_0 > 0$ such that $u_m(t_0) \in \partial W$ which implies that $I(u_m(t_0)) = 0$ or $E(u_m(t_0)) = d$. Since $E(u_m(t)) < E(u_0) < d$, we know that $I(u_m(t_0)) = 0$. From the definition of d, we have that $J(u_m(t_0)) > d$, which is contradiction. Thus for sufficiently large m, $u_m(t) \in W$ and $I(u_m) > 0$. Then we obtain

$$
J(u_m) = \frac{1}{2}\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u_m|^2 \, d\sigma + \frac{1}{2}\int_{\mathbb{E}} \omega^h V |u_m|^2 \, d\sigma - \sum_{k=1}^{l} \frac{a_k}{p_k+1}\int_{\mathbb{E}} \omega^h |u_m|^{p_k+1} \, d\sigma
$$

$$
\begin{aligned}
&+\sum_{j=1}^{s}\frac{b_j}{q_j+1}\int_{\mathbb{E}}\omega^h|u_m|^{q_j+1}\,d\sigma\\
&\geq \frac{1}{2}\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u_m|^2\,d\sigma+\frac{1}{2}\int_{\mathbb{E}}\omega^h V|u_m|^2\,d\sigma\\
&\quad-\frac{1}{p+1}\Big(\sum_{k=1}^{l}a_k\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma-\sum_{j=1}^{s}b_j\int_{\mathbb{E}}\omega^h|u|^{q_j+1}\,d\sigma\Big)\\
&=\big(\frac{1}{2}-\frac{1}{p+1}\big)\Big(\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u_m|^2\,d\sigma+\int_{\mathbb{E}}\omega^h V|u_m|^2\,d\sigma\Big)-\frac{1}{p+1}I(u_m)\\
&\geq \frac{p-1}{2(p+1)}(1+C^{*2})\|\nabla_{\mathbb{E}}u_m\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}.
\end{aligned}
$$

From (4.5) we have that for $0\leq t<\infty$ and sufficiently large m,

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{E}}\omega^h|u_{mt}|^2\,d\sigma+\frac{p-1}{2(p+1)}(1+C^{*2})\|\nabla_{\mathbb{E}}u_m\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}+\int_0^t\|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\,d\tau\\
&\leq \frac{1}{2}\int_{\mathbb{E}}\omega^h|u_{mt}|^2\,d\sigma+J(u_m)+\int_0^t\|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\,d\tau<d.
\end{aligned}
$$

That means

$$
\|\nabla_{\mathbb{E}}u_m\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}<\frac{2(p+1)}{p-1}\big(1+C^{*2}\big)^{-1}d.
$$

From Lemma 2.6 we have

$$
\begin{gathered}
\|u_m\|_{\mathcal{L}_2^{\frac{p_k+1}{2}}}\leq C_k^2\|\nabla_{\mathbb{E}}u_m\|^2<C_k^2\frac{2(p+1)}{p-1}(1+C^{*2})^{-1}d,\\
\|u_m\|_{\mathcal{L}_2^{\frac{q_j+1}{2}}}\leq C_j^2\|\nabla_{\mathbb{E}}u_m\|^2<C_j^2\frac{2(p+1)}{p-1}(1+C^{*2})^{-1}d,\\
\int_{\mathbb{E}}\omega^h V|u_m|^2\,d\sigma\leq C^2\|\nabla_{\mathbb{E}}u_m\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}<C^2\cdot\Big[\frac{2(p+1)}{p-1}(1+C^{*2})^{-1}d\Big]^2,\\
\int_0^t\|\partial_t u_m(\tau)\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\,d\tau<d,\\
\int_{\mathbb{E}}\omega^h|\partial_t u_m|^2\,d\sigma<2d.
\end{gathered}
$$

Hence, there exist u and a subsequence $\{u_m\}$ such that as $m\to\infty$, $u_m\to u$ in $L^\infty(0,\infty;\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ wead star and a.e. in $[0,\infty)\times\mathrm{int}(\mathbb{E})$, $|u_m|^{p_k-1}u_m\to|u|^{p_k-1}u$ in $L^\infty(0,\infty;\mathcal{L}_{\frac{p_1+1}{q}}^{\frac{(n+1)q}{p_1+1}}(\mathbb{E}))$ weak star and a.e. in $[0,\infty)\times\mathrm{int}(\mathbb{E})$, $|u_m|^{q_j-1}u_m\to|u|^{q_j-1}u$ in $L^\infty(0,\infty;\mathcal{L}_{\frac{p_1+1}{q}}^{\frac{(n+1)q}{p_1+1}}(\mathbb{E}))$ wead star and a.e. in $[0,\infty)\times\mathrm{int}(\mathbb{E})$, $V|u_m|^2\to V|u|^2$ in $L^\infty(0,\infty;\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ wead star and a.e. in $[0,\infty)\times\mathrm{int}(\mathbb{E})$, $\partial_t u_m\to\partial_t u$

in $L^2(0,\infty;\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E}))$ wead star. In (4.1) we fixed k, then letting $m\to\infty$, we have

$$\begin{aligned}
&\int_{\mathbb{E}}\omega^h\partial_{tt}u\cdot\varphi_k\,d\sigma+\int_{\mathbb{E}}\omega^h\nabla_{\mathbb{E}}u\cdot\nabla_{\mathbb{E}}\varphi_k d\sigma+\int_{\mathbb{E}}\omega^h\partial_t u\cdot\varphi_k\,d\sigma+\int_{\mathbb{E}}\omega^h Vu\cdot\varphi_k\,d\sigma\\
&=\sum_{k=1}^{l}a_k\int_{\mathbb{E}}\omega^h|u_m|^{p_k}u\cdot\varphi_k\,d\sigma-\sum_{j=1}^{s}b_j\int_{\mathbb{E}}\omega^h|u_m|^{q_j}u\cdot\varphi_k\,d\sigma.
\end{aligned}\tag{4.6}$$

Integrating (4.6) with respect t from 0 to t, we get

$$\begin{aligned}
&(\partial_t u,\varphi_k)_2+\int_0^t(\nabla_{\mathbb{E}}u,\nabla_{\mathbb{E}}\varphi_k)_2\,d\tau+\int_0^t(Vu,\varphi_k)_2\,d\tau+\int_0^t(\partial_t u,\varphi_k)_2\,d\tau\\
&=\sum_{k=1}^{l}a_k\int_0^t(|u|^{p_k}u,\varphi_k)_2\,d\tau-\sum_{j=1}^{s}b_j\int_0^t(|u|^{q_j}u,\varphi_k)_2\,d\tau+(u_1,\varphi_k)_2,
\end{aligned}$$

Moreover, (4.2) and (4.3) give $u(0)=u_0$ in $\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$. By density, one has that u is a global weak solution of problem (1.1) and $u(t)\in W$ where $u\in L^\infty(0,\infty;\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ with $\partial_t u\in L^\infty(0,T;\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E}))\cap\mathcal{L}_2^{\frac{n+1}{2}}(Q_T)$, where $Q_T=\mathbb{E}\times[0,T]$.

(2) Critical initial energy case($E(0)=d$ and $I(u)>0$): First $E(0)=d$ implies that $\|\nabla_{\mathbb{E}}u_0\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\neq 0$. Pick a sequence λ_m such that $0<\lambda_m<1$, $m=1,2,\dots$ and $\lambda_m\to 1$ as $m\to\infty$. Let $u_{0m}(x)=\lambda_m u_0(x)$, we consider the initial conditions

$$u(x,0)=u_{0m}(x)\quad\text{in }\operatorname{int}(\mathbb{E})\tag{4.7}$$

and the corresponding problem (1.1) with (4.7). From $I(u_0)\ge 0$ and Lemma 3.1, we have $\lambda^*=\lambda^*(u_0)\ge 1$. Thus, we get $I(u_{0m})=I(\lambda_m u_0)>0$ and $J(u_{0m}=J(\lambda_m u_0)<J(u_0)=d$. From the low initial case, it follows that for each m, problem (1.1) with (4.7) admits a global weak solution $u_m\in L^\infty(0,T;\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ with $\partial_t u_m\in L^\infty(0,T;\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E}))\cap\mathcal{L}_2^{\frac{n+1}{2}}(Q_T)$, where $Q_T=\mathbb{E}\times[0,T]$ and $u_m(t)\in W$ for $0\le t<\infty$ satisfying

$$\begin{aligned}
&\int_{\mathbb{E}}\omega^h\partial_{tt}u_m\cdot v\,d\sigma+\int_{\mathbb{E}}\omega^h\nabla_{\mathbb{E}}u_m\cdot\nabla_{\mathbb{E}}v\,d\sigma+\int_{\mathbb{E}}\omega^h\partial_t u_m\cdot v\,d\sigma+\int_{\mathbb{E}}\omega^h Vu_m\cdot v\,d\sigma\\
&=\sum_{k=1}^{l}a_k\int_{\mathbb{E}}\omega^h|u_m|^{p_k}u\cdot v\,d\sigma-\sum_{j=1}^{s}b_j\int_{\mathbb{E}}\omega^h|u_m|^{q_j}u\cdot v\,d\sigma,\quad\forall v\in\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}),
\end{aligned}$$

for $t>0$, and

$$\int_0^t\|\partial_t u_m(\tau)\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\,d\tau+J(u_m)+\frac{1}{2}\int_{\mathbb{E}}\omega^h|\partial_t u_m|^2\,d\sigma\le E(u_{0m})<d,\tag{4.8}$$

for $0\le t<\infty$. From (4.8) and

$$\begin{aligned}
J(u_m)&\ge\big(\frac{1}{2}-\frac{1}{p+1}\big)\Big(\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u_m|^2\,d\sigma+\int_{\mathbb{E}}\omega^h V|u_m|^2\,d\sigma\Big)-\frac{1}{p+1}I(u_m)\\
&\ge\frac{p-1}{2(p+1)}(1+C^{*2})\|\nabla_{\mathbb{E}}u_m\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})},
\end{aligned}$$

we get

$$\frac{p-1}{2(p+1)}(1+C^{*2})\|\nabla_{\mathbb{E}}u_m\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}+\int_0^t\|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\,d\tau<d,\quad 0\le t<\infty.$$

The remainder of the proof is similar to the low initial case, then the proof is complete. $\square$

Theorem 4.2. *Let $u_0\in\mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$, $u_0\in W$ and $I(u)>0$, then there exist two positive constants $\widehat{C}$ and ζ independent of t such that $0<E(t)\le\widehat{C}e^{-\zeta t}$ for all $t\ge 0$.*

To prove Theorem 4.2, we need the following lemma:

Lemma 4.3. *Assume that $1<p<\frac{N+2}{N-2}$ and $N=n+h+1\ge 3$, u is the solution of problem (1.1) with*

$$u_0\in\mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E}),\quad u_1\in\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})\cap\mathcal{L}_2^{\frac{n+1}{2}}(Q_T),\quad Q_T=\mathbb{E}\times[0,T].$$

If

$$E(0)\le\frac{2(p+1)}{p-1}\Big(\sum_{k=1}^{l}a_kC_k^{p_1+1}\Big)^{-\frac{2}{p_1-1}}(1+C^{*2}),\tag{4.9}$$

then $u\in W$ on $[0,T]$.

Proof. Assume that there exists some time $T^*>0$ such that $u(t)\in W$, where $0\le t<T^*$ and $u(T^*)\in\partial W$, we can obtain $I(u(T^*))=0$ and $u(T^*)\ne 0$. At the time $I(u(T^*))=0$ and $u(T^*)\ne 0$, notice that

$$\begin{aligned}J(u)&\ge(\frac12-\frac{1}{p+1})\Big(\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\int_{\mathbb{E}}\omega^hV|u|^2\,d\sigma\Big)-\frac{1}{p+1}I(u)\\&\ge\frac{p-1}{2(p+1)}(1+C^{*2})\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}.\end{aligned}$$

Then, by the energy identity $\int_0^t\|\partial_t u(\tau)\|_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\,d\tau+J(u)+\frac12\int_{\mathbb{E}}\omega^h|\partial_t u|^2\,d\sigma\le E(0)$, we have

$$\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\le\frac{2(p+1)}{p-1}(1+C^{*2})^{-1}J(u)\le\frac{2(p+1)}{p-1}(1+C^{*2})^{-1}E(0).\tag{4.10}$$

It follows from the Sobolev-poincaré's inequality, (4.9) and (4.10), that

$$\begin{aligned}&\sum_{k=1}^{l}a_k\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma-\sum_{j=1}^{s}b_j\int_{\mathbb{E}}\omega^h|u|^{q_j+1}\,d\sigma\\&\le\sum_{k=1}^{l}a_k\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma\\&\le\sum_{k=1}^{l}a_kC_k^{p_k+1}\|\nabla_{\mathbb{E}}u\|^{p_k+1}_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\\&\le\sum_{k=1}^{l}a_kC_k^{p_1+1}(\frac{2(p+1)}{p-1}(1+C^{*2})^{-1}E(0))^{\frac{p_1-1}{2}}\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})},\end{aligned}\tag{4.11}$$

which implies that $I(u(T^*))>0$ for $t=T^*$. This completes the proof. $\square$

Proof of Theorem 4.2. From

$$\begin{aligned}
E(t) =&\frac{1}{2}\int_{\mathbb{E}}\omega^h|\partial_t u|^2\,d\sigma+\frac{1}{2}\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\frac{1}{2}\int_{\mathbb{E}}\omega^h V|u|^2\,d\sigma\\
&-\sum_{k=1}^{l}\frac{a_k}{p_k+1}\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma+\sum_{j=1}^{s}\frac{b_j}{q_j+1}\int_{\mathbb{E}}\omega^h|u|^{q_j+1}\,d\sigma\\
\geq&\frac{1}{2}\int_{\mathbb{E}}\omega^h|\partial_t u|^2\,d\sigma+(\frac{1}{2}-\frac{1}{p+1})\Big(\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma+\int_{\mathbb{E}}\omega^h V|u|^2\,d\sigma\Big)\\
&+\frac{1}{p+1}I(u),
\end{aligned}$$

we have $0<E(t)$ for all $t\geq 0$.

The proof of other inequality relies on the construction of a Lyapunov functional. And the Lyapunov functional is performed by a suitable modification of energy. For this purpose, let $\theta>0$, which will be chosen later, we define

$$L(t)=E(t)+\theta\int_{\mathbb{E}}\omega^h u\partial_t u\,d\sigma. \tag{4.12}$$

It is straightforward to see that $L(t)$ and $E(t)$ are equivalent in the sense. So there exist two positive constants β_1 and β_2 depending on θ such that for $t\geq 0$, we have

$$\beta_1 E(t)\leq L(t)\leq \beta_2 E(t). \tag{4.13}$$

By taking the time derivative of the function $L(t)$ defined in (4.12), using equation (1.1) and performing several integration by parts, we get

$$\begin{aligned}
\frac{dL(t)}{dt}&=\frac{d}{dt}E(t)+\theta\int_{\mathbb{E}}\omega^h|\partial_t u|^2\,d\sigma+\theta\int_{\mathbb{E}}\omega^h|u\partial_{tt}u|\,d\sigma\\
&=-\|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}+\theta\|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\\
&\quad+\theta(-\int_{\mathbb{E}}\omega^h|\nabla_{\mathbb{E}}u|^2\,d\sigma-\int_{\mathbb{E}}\omega^h|u\partial_t u|\,d\sigma-\int_{\mathbb{E}}\omega^h V|u|^2\,d\sigma)\\
&\quad+\theta(\sum_{k=1}^{l}a_k\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma-\sum_{j=1}^{s}b_j\int_{\mathbb{E}}\omega^h|u|^{q_j+1}\,d\sigma).
\end{aligned}\tag{4.14}$$

Using Young inequality and Sobolev equality, for any $r>0$, we obtain

$$\begin{aligned}
\int_{\mathbb{E}}\omega^h|u\partial_t u|\,d\sigma&\leq\frac{1}{4r}\int_{\mathbb{E}}\omega^h|\partial_t u|^2\,d\sigma+r\int_{\mathbb{E}}\omega^h|u|^2\,d\sigma\\
&\leq\frac{1}{4r}\int_{\mathbb{E}}\omega^h|\partial_t u|^2\,d\sigma+C^2r\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})},
\end{aligned}\tag{4.15}$$

where C is the Sobolev constant. By Lemma 4.3, we have

$$\begin{aligned}
&\sum_{k=1}^{l}a_k\int_{\mathbb{E}}\omega^h|u|^{p_k+1}\,d\sigma-\sum_{j=1}^{s}b_j\int_{\mathbb{E}}\omega^h|u|^{q_j+1}\,d\sigma\\
&\leq\sum_{k=1}^{l}a_kC_k^{p_k+1}\|\nabla_{\mathbb{E}}u\|^{p+1}_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\\
&\leq\sum_{k=1}^{l}a_kC_k^{p_1+1}\Big(\frac{2(p+1)}{p-1}(1+C^{*2})^{-1}E(0)\Big)^{\frac{p_1-1}{2}}\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}.
\end{aligned}$$

Inserting (4.15) into (4.14), we arrive at

$$
\begin{aligned}
&\frac{dL(t)}{dt}\\
&\le (\theta+\frac{\theta}{4r}-1)\|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}+\Big(\theta\sum_{k=1}^{l}a_kC_k^{p_1+1}\Big(\frac{2(p+1)}{p-1}(1+C^{*2})^{-1}E(0)\Big)^{\frac{p_1-1}{2}}\\
&\quad -\theta+\theta C^2r-\theta C^2\Big)\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}.
\end{aligned}
$$

By Lemma 4.3 we know $\sum_{k=1}^{l}a_kC_k^{p_1+1}\Big(\frac{2(p+1)}{p-1}(1+C^{*2})^{-1}E(0)\Big)^{\frac{p_1-1}{2}}\le 1$. Then we choose $r<1$ such that

$$
\theta\sum_{k=1}^{l}a_kC_k^{p_1+1}\Big(\frac{2(p+1)}{p-1}(1+C^{*2})^{-1}E(0)\Big)^{\frac{p_1-1}{2}}-\theta+\theta C^2r-\theta C^2<0.
$$

From this inequality we may find $\phi>0$, which depends only on r such that

$$
\frac{dL(t)}{dt}\le(\theta+\frac{\theta}{4r}-1)\|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}-\theta\phi\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}.
$$

Consequently using the definition of the energy $E(t)$, for any suitable positive constant M, we obtain

$$
\frac{dL(t)}{dt}\le -M\theta E(t)+\Big(\theta+\frac{\theta}{4r}-1+\frac{\theta M}{2}\Big)\|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}+\theta(\frac{M}{2}-\phi)\|\nabla_{\mathbb{E}}u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}. \tag{4.16}
$$

Then, choosing $M<2\phi$, and θ small enough such that

$$
\theta+\frac{\theta}{4r}-1+\frac{\theta M}{2}<0,
$$

inequality (4.16) becomes

$$
\frac{dL(t)}{dt}\le -M\theta E(t)\quad\text{for all } t\ge 0. \tag{4.17}
$$

On the other hand, by (4.13), setting $\zeta=\frac{M\theta}{\beta_2}$, inequality (4.17) becomes

$$
\frac{dL(t)}{dt}\le -\zeta L(t)\quad\text{for all } t\ge 0.
$$

Integrating this differential inequality between 0 and t gives the following estimate for the function $L(t)$

$$
L(t)\le \dot{C}e^{-\zeta t}\quad\text{for all} t\ge 0.
$$

Consequently, by using (4.13) once again, we conclude that

$$
E(t)\le \widehat{C}e^{-\zeta t}\quad\text{for all } t\ge 0.
$$

The proof is complete. $\square$

5. Finite time blow-up of solution

In this section, we discuss the blow-up in finite time and seek a lower bound for the blow-up time T^* for the solution of (1.1).

Theorem 5.1. *Assume that u is a local solution of problem* (1.1)*, and $u_0 \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$. If $I(u) < 0$ and $E(0) \le d$, and $\int_{\mathbb{E}} \omega^h u_0 u_1 d\sigma > 0$ when $0 \le E(0) < d$, where $d < \frac{p-1}{2(p+1)}(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma)$ then the solution u blows up in a finite time.*

The following three lemmas will be used to prove Theorem 5.1.

Lemma 5.2. *Let u be the unique local solution of problem* (1.1) *and assume that u satisfy $E(0) \le d$, one has $I(u) < 0$ and*

$$\begin{aligned} d <& \frac{p-1}{2(p+1)}\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma\Big) \\ <& \frac{p-1}{2(p+1)}\Big(\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma\Big), \end{aligned}$$

for all $t \in [0, T_{Max})$.

Proof. Since $I(u) < 0$ for all $0 \le t < T^*$, it follows that

$$\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma < \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma. \tag{5.1}$$

For all $0 \le t < T^*$, using the definition of d, we get

$$d < \frac{p-1}{2(p+1)}\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma\Big). \tag{5.2}$$

Then (5.1) and (5.2) imply

$$\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma > \frac{2(p+1)}{p-1} d > 0.$$

By the continuity of $t \mapsto \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma$ we get $u(T^*) \ne 0$. From the definition of $J(u)$, we obtain

$$d \le \frac{p-1}{2(p+1)}\Big(\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u(T^*)|^{p_k+1}\, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u(T^*)|^{q_j+1}\, d\sigma\Big) = J(u(T^*)),$$

which contradicts to $J(u(T^*) \le E(T^*) < d$. Then, we obtain

$$\begin{aligned} d <& \frac{p-1}{2(p+1)}(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma + \int_{\mathbb{E}} \omega^h V|u|^2\, d\sigma) \\ <& \frac{p-1}{2(p+1)}(\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma). \end{aligned}$$

∎

Lemma 5.3. *Assume $M(t) > 0$ is a twice differentiable function, then the inequality*
$$M(t)M''(t) - (1+\alpha)M'(t)^2 \geq 0$$
holds for $t > 0$ and $\alpha > 0$. If $M(0) > 0$ and $M'(0) > 0$, then there exists a time $T^* \leq \frac{M(0)}{\alpha M'(0)}$ such that $\lim_{t\to T^*-} M(t) = \infty$.

Proof of Theorem 5.1. We consider $M : [0, T] \to R_+$ defined by
$$M(t) = \|u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_0^t \|u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau + (T-t)\|u_0\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \kappa(t+T_0)^2,$$
where T and T_0 are positive constants. Furthermore,
$$\begin{aligned} M'(t) &= 2\int_{\mathbb{E}} u\partial_t u dx + \|u(t)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - \|u_0\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + 2\kappa(t+T_0) \\ &= 2\int_{\mathbb{E}} u\partial_t u dx + 2\int_0^1 (u(\tau), \partial_t u(\tau))_2 \, d\tau + 2\kappa(t+T_0), \end{aligned} \tag{5.3}$$
and consequently,
$$\begin{aligned} M''(t) &= 2(\partial_{tt}u, u) + 2\|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + 2(\partial_t u, u) + 2\kappa \\ &= 2\Big[-\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - \int_{\mathbb{E}} \omega^h V|u|^2 \, d\sigma + \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1} \, d\sigma\Big] \\ &\quad - 2\Big[\sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1} \, d\sigma - \|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - \kappa\Big]. \end{aligned}$$
For almost every $t \in [0, T]$, we get
$$\begin{aligned} &M(t)M''(t) - \frac{p+3}{4} M'(t)^2 \\ &= M(t)M''(t) + (p+3)\Big[\eta(t) - \Big(M(t) - (T-t)\|u_0\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\Big) \\ &\quad \times \Big(\|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau + \kappa\Big)\Big], \end{aligned}$$
where $\eta : [0, T] \to R_+$ is the function defined by
$$\begin{aligned} \eta(t) &= \Big(\|u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_0^t \|u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau + \kappa(t+T_0)^2\Big) \\ &\quad \times \Big(\|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau + \kappa\Big) \\ &\quad - \Big(\int_{\mathbb{E}} \omega^h u\partial_t u \, d\sigma + \int_0^t (u, \partial_t u)_2 \, d\tau + \kappa(t+T_0)\Big)^2 \geq 0. \end{aligned}$$
As a consequence, we have differential equality
$$M(t)M''(t) - \frac{p+3}{4} M'(t)^2 \geq M(t)\xi(t). \tag{5.4}$$

For almost every $t \in [0,T]$, $\xi : [0,T] \to \mathbb{R}_+$ is the map defined by

$$
\begin{aligned}
\xi(t) =& M''(t) + \Big(\|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau + \kappa\Big)(p+3) \\
=& 2\Big[- \|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - \int_{\mathbb{E}} \omega^h V |u|^2\, d\sigma + \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k+1}\, d\sigma \Big] \\
& - 2\Big[\sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j+1}\, d\sigma - \|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} - \kappa \Big] \\
& - (p+3)\Big(\|\partial_t u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau + \kappa\Big) \\
\geq & -2(p+1)E(t) + (p-1)\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)\int_{\mathbb{E}} \omega^h V |u|^2\, d\sigma \\
& - (p+3)\int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau - (p+1)\kappa \\
=& -2(p+1)E(0) + (p-1)\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)\int_{\mathbb{E}} \omega^h V |u|^2\, d\sigma \\
& + (p-1)\int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau - (p+1)\kappa.
\end{aligned}
$$

Case 1. If $E(0) < 0$, then

$$
\begin{aligned}
\xi(t) =& -2(p+1)E(0) + (p-1)\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)\int_{\mathbb{E}} \omega^h V |u|^2\, d\sigma \\
& + (p-1)\int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau - (p+1)\kappa \\
\geq & -2(p+1)E(0) + (p-1)(1+C^{*2})\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \\
& + (p-1)\int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau - (p+1)\kappa.
\end{aligned}
$$

Choosing κ satisfying $\kappa \leq -2E(0)$, we have

$$
\xi(t) \geq (p-1)(1+C^{*2})\|\nabla_E u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)\int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau \geq 0. \tag{5.5}
$$

Case 2. If $0 \leq E(0) < d$, then by Lemma 5.2, we have

$$
\begin{aligned}
\xi(t) =& -2(p+1)E(0) + (p-1)\|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)\int_{\mathbb{E}} \omega^h V |u|^2\, d\sigma \\
& + (p-1)\int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau - (p+1)\kappa \\
=& 2(p+1)(d - E(0)) + (p-1)\int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau - (p+1)\kappa \\
\geq & (p-1)\int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\, d\tau - (p+1)\kappa.
\end{aligned}
$$

Choosing $\kappa = 0$, we have $(p-1)\int_0^t \|\partial_t u(\tau)\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})}\,d\tau \geq 0$. Then we obtain

$$\xi(t) \geq 0. \tag{5.6}$$

Therefore, from (5.4) (5.5) and (5.6), we obtain

$$M(t)M''(t) - \frac{p+3}{4}M'(t)^2 \geq 0.$$

By Lemma 5.3 and (5.3), for every $t \in [0,T]$, if $E(0) < 0$, we then choose T_0 sufficiently large such that $M'(0) = 2\int_{\mathbb{E}} \omega^h u_0 u_1\,d\sigma + 2bT_0 > 0$; if $0 \leq E(0) < d$, the condition $\int_{\mathbb{E}} \omega^h u_0 u_1\,d\sigma > 0$ also ensure that $M'(0) > 0$. As $\frac{p+3}{4} > 1$, letting $\alpha = \frac{p-1}{4}$, we get $\lim_{t\to T^*-} M(t) = \infty$ by the convexity argument, which implies that $\lim_{t\to T^*-} \|\nabla_{\mathbb{E}} u\|^2_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} = \infty$. □

Theorem 5.4. *Assume that*

$$1 < p \leq \frac{N}{N-2}, \quad N \geq 3, \quad u_0 \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E}), \quad u_1 \in \mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E}), \quad E(0) < 0,$$

Then the solution blows up at time T^*.

Proof. We define the auxiliary function

$$\Phi(t) := \int_{\mathbb{E}} \omega^h (u_t^2 + |\nabla_{\mathbb{E}} u|^2 + Vu^2)\,d\sigma.$$

Differentiating $\Phi(t)$ and making use of the divergence theorem, we obtain

$$\begin{aligned}
\Phi'(t) &= 2\int_{\mathbb{E}} \omega^h (u_t u_{tt} + |\nabla_{\mathbb{E}} u||\nabla_{\mathbb{E}} u_t| + Vuu_t)\,d\sigma \\
&= 2\int_{\mathbb{E}} \omega^h u_t (u_{tt} + |\Delta_{\mathbb{E}} u| + Vu)\,d\sigma \\
&= 2\Big(\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k} u_t\,d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j} u_t\,d\sigma - \int_{\mathbb{E}} \omega^h |u_t|^2\,d\sigma\Big).
\end{aligned} \tag{5.7}$$

The two terms on the right-hand side of (5.7) can be estimated as follows

$$\begin{aligned}
\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k}\,d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j}\,d\sigma &\leq \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k}\,d\sigma \\
&\leq \sum_{k=1}^{l} a_k C_k^{p_k} \|\nabla_{\mathbb{E}} u\|^{p_k}_{\mathcal{L}_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&\leq C\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\,d\sigma\Big)^{p_1}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&2\Big(\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k} u_t\,d\sigma - \sum_{j=1}^{s} b_j \int_{\mathbb{E}} \omega^h |u|^{q_j} u_t\,d\sigma\Big) \\
&\leq 2\sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{p_k} u_t\,d\sigma
\end{aligned}$$

$$
\begin{aligned}
&\le 2\Big(\int_{\mathbb{E}} \omega^h u_t^2\, d\sigma \sum_{k=1}^{l} a_k \int_{\mathbb{E}} \omega^h |u|^{2p_k}\, d\sigma\Big)^{1/2} \\
&\le 2\sqrt{C}\Big(\int_{\mathbb{E}} \omega^h u_t^2\, d\sigma\Big)^{1/2}\Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma\Big)^{p_1/2} \\
&\le \sqrt{C}\Big\{\int_{\mathbb{E}} \omega^h u_t^2\, d\sigma + \Big(\int_{\mathbb{E}} \omega^h |\nabla_{\mathbb{E}} u|^2\, d\sigma\Big)^{p_1}\Big\} \le \sqrt{C}(\Phi + \Phi^{p_1}),
\end{aligned}
$$

which leads to the differential inequality

$$
\Phi'(t) \le \alpha\Phi + \beta\Phi^{p_1},
$$

where $\alpha = \beta = \sqrt{C}$. This differential inequality may be reduced to a linear differential inequality for $v(t) := \Phi^{1-p_1}$. Integrating this inequality, we obtain

$$
(\Phi(t))^{1-p_1} \ge \{(\Phi(0))^{1-p_1} + 1\}e^{-(p_1-1)\sqrt{C}t} - 1. \tag{5.8}
$$

It follows from (5.8), that $\Phi(t)$ remains bounded for $t \in [0, T)$ with

$$
T^* = \frac{1}{(p_1-1)\sqrt{C}} \log\{1 + (\Phi(0))^{1-p_1}\}.
$$

It gives that T^* is a lower bound for the blow-up time. □

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