Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 129, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MEMORY BOUNDARY FEEDBACK STABILIZATION FOR SCHRÖDINGER EQUATIONS WITH VARIABLE COEFFICIENTS

ABDESSELAM NAWEL, KHALED MELKEMI

Communicated by Jesus Ildefonso Diaz

Abstract. First we consider the boundary stabilization of Schrödinger equations with constant coefficient memory feedback. This is done by using Riemannian geometry methods and the multipliers technique. Then we explore the stabilization limits of Schrödinger equations whose elliptical part has a variable coefficient. We established the exponential decay of solutions using the multipliers techniques. The introduction of dissipative boundary conditions of memory type allowed us to obtain an accurate estimate on the uniform rate of decay of the energy for Schrödinger equations.

1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^n$ with boundary $\Gamma := \partial\Omega$. It is assumed that Γ consists of two parts Γ_0 and Γ_1 such that $\Gamma_0, \Gamma_1 \neq \emptyset$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. We consider the mixed problem for Schrödinger equation

$$y_t - \mathbf{i}Ay = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.1}$$

$$y(0,x) = y_0(x) \quad \text{in } \Omega, \tag{1.2}$$

$$y = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}_+, \tag{1.3}$$

$$\frac{\partial y}{\partial \nu_A} = u \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \tag{1.4}$$

where

$$Ay = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big),$$

the functions $a_{ij} = a_{ji}$ are C^∞ functions in $\mathbb{R}^n$,

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial y}{\partial x_i}\nu_i,$$

2010 *Mathematics Subject Classification.* 93D15, 35J10, 35B40, 53C17.
Key words and phrases. Schrödinger equation; exponential stabilization; boundary condition of memory type; Riemannian geometry.

Submitted February 20, 2017. Published May 11, 2017.

$\nu = (\nu_1, \nu_2, \dots, \nu_n)$ is the unit normal of Γ pointing toward the exterior of Ω, $\nu_A = A\nu$, and $A = (a_{ij})$ is a matrix function. We assume that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi_j} > \alpha_1 \sum_{i=1}^{n} \xi_i^2 \quad \forall x \in \Omega,\ \zeta \in \mathbb{C}^n,\ \zeta \neq 0, \tag{1.5}$$

for some positive constant α_1, and

$$u = -\int_0^t k(t-s)y_s(s)ds - by_t\,.$$

Then we transform the boundary condition in Γ_1. Suppose that $y_0 = 0$ on Γ_1, by integration by parts

$$\begin{aligned}\int_0^t k(t-s)y_t(s)ds &= [k(t-s)y(s)]_0^t + \int_0^t k'(t-s)y(s)ds\\ &= \int_0^t k'(t-s)y(s)ds + k(0)y(t) - k(t)y_0\\ &= \int_0^t k'(t-s)y(s)ds + k(0)y(t).\end{aligned}$$

Throughout the paper we assume that

$$u = -\int_0^t k'(t-s)y(s)ds - k(o)y(t) - by_t. \tag{1.6}$$

where $k : \Gamma_1 \times \mathbb{R}_+ \to \mathbb{R}_+ \in C^2(\mathbb{R}^+, L^\infty(\Omega))$ and $b : \Gamma_1 \to \mathbb{R}_+ \in L^\infty(\Omega)$. We define the corresponding energy functional by

$$2E(t) := \int_\Omega |\nabla_g y|_g^2\, d\Omega + \int_{\Gamma_1} k|y|^2\, d\Gamma_1 - \int_0^t \int_{\Gamma_1} k'(t-s)|y(t) - y(s)|\, d\Gamma_1 ds. \tag{1.7}$$

Our goal is to stabilize the system (1.1)–(1.4) and (1.6); to find a suitable feedback $u = F(x, y_t)$ such that the energy (1.7) decays to zero exponentially as $t \to +\infty$ for every solution y of which $E(0) < +\infty$. The approach adopted uses Riemannian geometry. This method was first introduced to boundary-control problems by Yao [10] for the exactly controllability of wave equations.

The stabilization of partial differential equations has been considered by many authors. The asymptotic behaviour of the wave equation with memory and linear feedbacks with constant coefficients has been studied by Guesmia [4], and by Aassila et al [1] in the nonlinear case. This study has been generalised by Chai & Guo [2] for variable coefficients by using a very different method, namely, the Riemannian geometry method.

On the other hand, the stabilization of the Schrödinger equations has been studied by Machtyngier & Zuazua [9] in the Neumann boundary conditions, and by Cipolatti et al [3] with nonlinear feedbacks. This study has been considered by Lasiecka & Triggani [8] with constant coefficients acting in the Dirichlet boundary conditions.

The objective of this work, we consider the boundary stabilization for system (1.1)–(1.4) and (1.6) with variable coefficients and memory feedbacks by using multipliers techniques.

Our paper is organized as follows. In subsection 1.1, we introduce some notation and results on Riemannian geometry. Our main results are studied in section 2. Section 3 is devoted to the proof of the main results.

1.1. **Notation.** The definitions here are standard and classical in the literature, see Hebey [5]. Let $A(x)$ and $G(x)$, respectively, be the coefficient matrix and its inverse

$$A(x) = (a_{ij}(x)), \quad G(x) = [A(x)]^{-1} = (g_{ij}(x)) \tag{1.8}$$

for $i, j = 1, \dots, n$; $x \in \mathbb{R}^n$.

Euclidean metric on $\mathbb{R}^n$. Let $(x_1, \dots, x_n)$ be the natural coordinate system in $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, denote

$$X \cdot Y = \sum_{i=1}^n \alpha_i \beta_i, \quad |X|_0^2 = X \cdot X, \quad \text{for } X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in T_x\mathbb{R}^n .$$

For $f \in C^1(\overline{\Omega})$ and $X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}$, we denote by

$$\nabla_0 f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \quad \text{and} \quad \operatorname{div}_0(X) = \sum_{i=1}^n \frac{\partial \alpha_i(x)}{\partial x_i} \tag{1.9}$$

the gradient of f and the divergence of X in the Euclidean metric.

Riemannian metric. For each $x \in \mathbb{R}^n$, define the inner product and the corresponding norm on the tangent space $T_x\mathbb{R}^n$ by

$$g(X,Y) = \langle X, Y \rangle_g = X \cdot G(x)Y = \sum_{i,j=1}^n g_{ij}(x)\alpha_i\beta_j$$

$$|X|_g^2 = \langle X, Y \rangle_g \quad \text{for } X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in T_x\mathbb{R}^n$$

Then $(\mathbb{R}^n, g)$ is a Riemannian manifold with a Riemannian metric g. Denote the Levi-Cevita connection in metric g by D. Let H be a vector field on $(\mathbb{R}^n, g)$. The covariant differential DH of H determines a bilinear form on $T_x\mathbb{R}^n \times T_x\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, by

$$DH(X,Y) = \langle D_X H, Y \rangle_g, \forall X, Y \in T_x\mathbb{R}^n$$

where $D_X H$ is the covariante derivative of H with respect to X. The following lemma provides some useful equalities.

Lemma 1.1 ([10, lemma 2.1])**.** *Let f, $h \in C^1(\overline{\Omega})$ and let H, X be a vector field on $\mathbb{R}^n$. Then using the above notation, we have*

(i)
$$\langle H(x), A(x)X(x) \rangle_g = H(x)X(x), \quad \forall x \in \mathbb{R}^n \tag{1.10}$$

(ii) *The gradient $\nabla_g f$ of f in the Riemannian metric g is given by*
$$\nabla_g f(x) = \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij}(x) \frac{\partial f}{\partial x_j} \Big) \frac{\partial}{\partial x_i} = A(x)\nabla_0 f. \tag{1.11}$$

(iii)
$$\frac{\partial y}{\partial \nu_A} = (A(x)\nabla_0 y).\nu = \nabla_g y.\nu. \tag{1.12}$$

(iv)
$$\langle \nabla_g f, \nabla_g H\rangle_g = \nabla_g f(h) = \nabla_0 f \cdot A(x)\nabla_0 h. \tag{1.13}$$
(v)
$$\begin{aligned} &\langle \nabla_g f, \nabla_g H(f)\rangle \\ &= DH(\nabla_g f, \nabla_g f) + \frac{1}{2}\operatorname{div}_0(|\nabla_g f|_g^2 H)(x) - \frac{1}{2}|\nabla_g f|_g^2 \operatorname{div}_0(H) \quad x \in \mathbb{R}^n. \end{aligned} \tag{1.14}$$
(vi)
$$\begin{aligned} Ay &= -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial y}{\partial x_j}) \\ &= -\operatorname{div}_0(A(x)\nabla_0 y) = -\operatorname{div}_0(\nabla_g y), \quad y \in C^2(\Omega). \end{aligned} \tag{1.15}$$

2. Statement of main result

To obtain the boundary stabilization of problem (1.1)–(1.4) and (1.6), we assumed that there exists a vector field H on the Riemannian manifold $(\mathbb{R}^n, g)$ such that:
$$\forall X \in T_x\mathbb{R}^n,\ \exists a > 0, \quad \langle D_X H, X\rangle_g \geq a|X|_g^2, \tag{2.1}$$
$$H(x)\cdot \nu < 0 \quad \text{on } \Gamma_0, \tag{2.2}$$
$$H(x)\cdot \nu \geq 0 \quad \text{on } \Gamma_1. \tag{2.3}$$
$k \geq 0$ and $k' \leq 0$, on $\Gamma_0 \times \mathbb{R}_+$. Moreover,
$$\varphi = \inf_{(x,t)\in\Gamma_0\times\Re^+}(-k') \neq 0, \tag{2.4}$$
$$k'' \geq 0\,. \tag{2.5}$$

We have the following result of existence and uniqueness of weak solution to (1.1)–(1.4) and (1.6).

Theorem 2.1. *For all initial data $y_0 \in V = H^1_{\Gamma_0}(\Omega) = \{y \in H^1(\Omega);\ y = 0 \text{ on } \Gamma_0\}$, problem (1.1)–(1.4) and (1.6) admits a unique global weak solution $y \in C(\mathbb{R}^+, V)$. Furthermore, if $y_0 \in H^3(\Omega)\cap H^1_{\Gamma_0}(\Omega)$ and $\frac{\partial y_0}{\partial \nu_A} = -\frac{1}{2}ky(0)$ on Γ_1, then the solution has the regularity*
$$y \in C^1(\mathbb{R}^+, V).$$

Proof. Existence of a solution is proved using Galerkin method [6].

Suppose y is the unique global weak solution of problem (1.1)–(1.4) and (1.6). The variational formulation of the problem is
$$\begin{aligned} \int_\Omega y_t\overline{v}\,d\Omega &= \mathbf{i}\int_\Omega Ay\overline{v}\,d\Omega \\ &= \mathbf{i}\sum_{i,j=1}^n\int_\Omega \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big)\overline{v}\,d\Omega \\ &= \mathbf{i}\sum_{i,j=1}^n\int_{\Gamma_1} a_{ij}(x)\frac{\partial y}{\partial x_i}\nu_i\overline{v}\,d\Gamma_1 - \mathbf{i}\sum_{i,j=1}^n\int_\Omega a_{ij}(x)\frac{\partial y}{\partial x_i}\frac{\partial \overline{v}}{\partial xj}\,d\Omega \\ &= \mathbf{i}\sum_{i,j=1}^n\int_{\Gamma_1}\frac{\partial y}{\partial \nu_A}\overline{v}\,d\Gamma_1 - \mathbf{i}\sum_{i,j=1}^n\int_\Omega a_{ij}(x)\frac{\partial y}{\partial x_i}\frac{\partial \overline{v}}{\partial xj}\,d\Omega \end{aligned}$$

for $v \in V$, and

$$\begin{aligned}\int_\Omega y_t \overline{v}\, d\Omega = \mathbf{i} \int_{\Gamma_1} (-\int_0^t k'(t-s)y(s)ds - k(0)y(t) - by_t)\overline{v}\ d\Gamma_1 \\ - \mathbf{i} \sum_{i,j=1}^n \int_\Omega a_{ij}(x) \frac{\partial y}{\partial x_i} \frac{\partial \overline{v}}{\partial xj}\, d\Omega.\end{aligned}$$

We introduce the following notation: $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\Omega)$,

$$a(y,v) = \sum_{i,j=1}^n \int_\Omega a_{ij}(x) \frac{\partial y}{\partial x_i} \frac{\partial \overline{v}}{\partial xj}\, d\Omega,$$

$$\beta(t,y,v) = \int_{\Gamma_1} (-\int_0^t k'(t-s)y(s)ds - k(0)y(t))\overline{v}\, d\Gamma_1 \quad \beta(0,y,v) = 0.$$

We have

$$\langle y_t, v\rangle + \mathbf{i}a(y,v) + \mathbf{i}\beta(t,y,v) = -\mathbf{i}\langle by_t, v\rangle$$

Uniqueness. Let y and z be two solutions. then $w = y - z$ satisfies

$$w_t - \mathbf{i}Aw = 0 \quad \text{in }]0,+\infty) \times \Omega \tag{2.6}$$

$$w(0,x) = 0 \quad \text{in } \Omega \tag{2.7}$$

$$w = 0 \quad \text{on }]0,+\infty) \times \Gamma_0 \tag{2.8}$$

$$\frac{\partial w}{\partial \nu_A} = -\int_0^t k'(t-s)w(s)ds - k(0)w(t) - bw_t, \quad \text{on }]0,+\infty[\times\Gamma_1\,. \tag{2.9}$$

Multiplying (2.6) by $\overline{w}_t$, integrating over Ω. Applying the Green formula, we obtain

$$\int_\Omega |w_t|^2\, d\Omega - \mathbf{i} \sum_{i,j=1}^n \int_{\Gamma_1} a_{ij}(x) \frac{\partial w}{\partial x_i} \nu_i \overline{w_t}\, d\Gamma + \mathbf{i} \sum_{i,j=1}^n \int_\Omega a_{ij}(x) \frac{\partial w}{\partial x_i} \frac{\partial \overline{w_t}}{\partial xj}\, d\Omega = 0\,.$$

Taking the imaginary part of above equation,

$$\begin{aligned}\mathrm{Re} \int_{\Gamma_1} \frac{\partial y}{\partial \nu_A} \overline{w_t}\, d\Gamma_1 &= \mathrm{Re} \int_\Omega \nabla_g w \nabla_g \overline{w}_t\, d\Omega \\ &= \frac{1}{2} \int_\Omega \big[\frac{d}{dt} |\nabla_g w|_g^2\big]\, d\Omega \\ &= \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla_g w(t)|_g^2\, d\Omega\,.\end{aligned}$$

For $t = 0$, we have

$$\int_\Omega |\nabla_g w|_g^2\, d\Omega + \int_{\Gamma_1} k|w|^2\, d\Gamma_1 - \int_0^t \int_{\Gamma_1} k'(t-s)|w(t) - w(s)|^2\, d\Gamma_1\, ds = 0$$

so

$$\int_\Omega |\nabla_g w|_g^2\, d\Omega = -\int_{\Gamma_1} k|w|^2\, d\Gamma_1 + \int_0^t \int_{\Gamma_1} k'(t-s)|w(t) - w(s)|^2\, d\Gamma_1 ds.$$

We deduce

$$\int_\Omega |\nabla_g w|_g^2\, d\Omega \le 0$$

and so $\nabla_g w = 0$, this with the condition at limit, $w = 0$ □

2.1. **Existence of solutions.** Let $(e_n)_{n\in\aleph}$ be a set of functions in V that form an orthonormal basis for $L^2(\Omega)$. Let V_m be the space generated by $(e_1,\dots,e_m)$, and

$$y^m(t)=\sum_{i=1}^{m}\alpha_{im}(t)e_{i.}$$

be a solution of the Cauchy problem

$$\langle y_t^m,v\rangle+\mathbf{i}a(y^m,v)+\mathbf{i}\beta(t,y^m,v)=-\mathbf{i}\langle by_t^m,v\rangle\,.$$

With $v=y_t^m$, we have

$$\langle y_t^m,y_t^m\rangle+\mathbf{i}a(y^m,y_t^m)+\mathbf{i}\beta(t,y^m,y_t^m)=-\mathbf{i}\langle by_t^m,y_t^m\rangle\,.$$

Taking the imaginaire part, it results

$$\begin{aligned}
&\operatorname{Re}\Big(\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x)\frac{\partial y^m}{\partial x_i}\frac{\partial\overline{y_t^m}}{\partial xj}\,d\Omega\Big)\\
&=-\int_{\Gamma_1}b|y_t^m|^2\,d\Gamma_1-\operatorname{Re}\Big(\int_{\Gamma_1}\int_0^t k'(t-s)y^m(s)\overline{y_t^m}(t)ds\,d\Gamma_1\Big)\\
&\quad-\operatorname{Re}\Big(\int_{\Gamma_1}\int_0^t k(0)y^m(t)\overline{y_t^m}(t)ds\,d\Gamma_1\Big)\,.
\end{aligned}$$

However,

$$\begin{aligned}
&-\int b|y_t^m|^2\,d\Gamma_1-\operatorname{Re}\Big(\int_{\Gamma_1}\int_0^t k'(t-s)y^m(s)\overline{y_t^m}(t)ds\,d\Gamma_1\Big)\\
&-\operatorname{Re}\Big(\int_{\Gamma_1}\int_0^t k(0)y^m(t)\overline{y_t^m}(t)\,d\Gamma_1\Big)\\
&=-\int_{\Gamma_1}b|y_t^m|^2\,d\Gamma_1+\frac12\int_{\Gamma_1}k'|y^m|^2\,d\Gamma_1\\
&\quad-\frac12\int_{\Gamma_1}\int_0^t k''(t-s)|y^m(t)-y^m(s)|^2\,ds\,d\Gamma_1\\
&\quad+\frac12\frac{d}{dt}\big[\int_{\Gamma_1}k|y^m|^2\,d\Gamma_1\big]+\frac12\frac{d}{dt}\Big[\int_{\Gamma_1}\int_0^t k'(t-s)|y^m(t)-y^m(s)|^2\,ds\,d\Gamma_1\Big],
\end{aligned}$$

so

$$\begin{aligned}
&\operatorname{Re}\Big(\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x)\frac{\partial y^m}{\partial x_i}\frac{\partial\overline{y_t^m}}{\partial xj}\,d\Omega\Big)\\
&=-\int b|y_t^m|^2\,d\Gamma_1-\frac12\int_{\Gamma_1}k'|y^m|^2\,d\Gamma_1-\frac12\int_{\Gamma_1}\int_0^t k''(t-s)|y^m(t)-y^m(s)|^2\,ds\,d\Gamma_1\\
&\quad-\frac12\frac{d}{dt}\Big[\int_{\Gamma_1}k|y^m|^2\,d\Gamma_1\Big]+\frac12\frac{d}{dt}\Big[\int_{\Gamma_1}\int_0^t k'(t-s)|y^m(t)-y^m(s)|^2\,ds\,d\Gamma_1\Big]\,.
\end{aligned}$$

We have

$$-\int_{\Gamma_1}b|y_t^m|^2\,d\Gamma_1+\frac12\int_{\Gamma_1}k'|y^m|^2\,d\Gamma_1-\frac12\int_{\Gamma_1}\int_0^t k''(t-s)|y^m(t)-y^m(s)|^2\,ds\,d\Gamma_1<0,$$

thus

$$\begin{aligned}
&\operatorname{Re}\Big(\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}(x)\frac{\partial y^m}{\partial x_i}\frac{\partial \overline{y_t^m}}{\partial xj}\,d\Omega\Big)\\
&\leq -\frac{1}{2}\frac{d}{dt}\Big[\int_{\Gamma_1} k|y^m|^2\,d\Gamma_1\Big]+\frac{1}{2}\frac{d}{dt}\Big[\int_{\Gamma_1}\int_0^t k'(t-s)|y^m(t)-y^m(s)|^2\,ds\,d\Gamma_1\Big].
\end{aligned}$$

We deduce that

$$\operatorname{Re}\Big(\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}(x)\frac{\partial y^m}{\partial x_i}\frac{\partial \overline{y_t^m}}{\partial xj}\,d\Omega\Big)\leq 0\,.$$

On the other hand,

$$\begin{aligned}
\operatorname{Re}\Big(\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}(x)\frac{\partial y^m}{\partial x_i}\frac{\partial \overline{y_t^m}}{\partial xj}\,d\Omega\Big)&=\frac{1}{2}\frac{d}{dt}\Big(\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}(x)\frac{\partial y^m}{\partial x_i}\frac{\partial \overline{y^m}}{\partial xj}\,d\Omega\Big)\\
&=\frac{1}{2}\frac{d}{dt}\Big(\int_{\Omega}|\nabla_g y^m|_g^2\,d\Omega\Big).
\end{aligned}$$

Therefore,

$$\frac{1}{2}\int_{\Omega}|\nabla_g y^m(T)|_g^2\,d\Omega\leq\frac{1}{2}\int_{\Omega}|\nabla_g y^m(0)|_g^2\,d\Omega,$$

from where

$$|y^m|_V^2\leq|y^{0m}|_V^2$$

and $v^m\to v$ in V, or

$$\langle y_t^m,v^m\rangle+\mathbf{i}a(y^m,v^m)+\mathbf{i}\beta(t,y^m,v^m)=-\mathbf{i}\int_{\Gamma_1} b y_t^m\overline{v^m}\,d\Gamma_1. \tag{2.10}$$

For $\zeta\in D(]0,T[)$, we put

$$\psi^m=\zeta v^m\quad\text{and}\quad\psi=\zeta v\,;$$

so we have

$$\psi^m\to\psi\quad\text{in }L^2(0,T,V)\,.$$

Multiplying (2.10) by ζ and integrating over $]0,T[$, we find

$$\begin{aligned}
&\int_0^T\langle y_t^m,\psi^m\rangle dt+\mathbf{i}\int_0^T a(y^m,\psi^m)dt+\mathbf{i}\int_0^T\beta(t,y^m,\psi^m)dt\\
&=-\mathbf{i}\int_0^T\int_{\Gamma_1} b y_t^m\overline{\psi^m}\,d\Gamma_1 dt\,.
\end{aligned}$$

Passing to the limit,

$$\int_0^T\langle y,\psi_t\rangle dt+\mathbf{i}\int_0^T a(y,\psi)dt+\mathbf{i}\int_0^T\beta(t,y,\psi)dt=-\mathbf{i}\int_0^T\int_{\Gamma_1} b y\overline{\psi_t}\,d\Gamma_1 dt$$

for all $v\in V$ and all $\zeta\in D(]0,T[)$. $y\in C]0,T[,V)$.

Regularity. We have

$$
\begin{aligned}
&\langle y_{tt}^m, v\rangle + \mathbf{i}\int_{\Gamma_1}\Big(-\int_0^t k''(t-s)y^m(s)ds - k(0)y_t^m(t)\Big)\overline{v}\ d\Gamma_1\\
&+\mathbf{i}\sum_{i,j=1}^n\int_\Omega a_{ij}(x)\frac{\partial y_t^m}{\partial x_i}\frac{\partial \overline{v}}{\partial xj}\,d\Omega - \mathbf{i}\int_{\Gamma_1} b y_{tt}^m v\,d\Gamma_1 = 0\,.
\end{aligned}
$$

Putting $v = y_{tt}^m$,

$$
\begin{aligned}
&\langle y_{tt}^m, y_{tt}^m\rangle + \mathbf{i}\int_{\Gamma_1}(-\int_0^t k''(t-s)y^m(s)ds - k(0)y_t^m(t))\overline{y_{tt}^m}\ d\Gamma_1\\
&+\mathbf{i}\sum_{i,j=1}^n\int_\Omega a_{ij}(x)\frac{\partial y_t^m}{\partial x_i}\frac{\partial \overline{y_{tt}^m}}{\partial xj}\,d\Omega\ - \mathbf{i}\int_{\Gamma_1} b y_{tt}^m \overline{y_{tt}^m}\,d\Gamma_1 = 0\,.
\end{aligned}
$$

Taking the imaginary part,

$$
\begin{aligned}
&\mathrm{Re}\Big(\sum_{i,j=1}^n\int_\Omega a_{ij}(x)\frac{\partial y_t^m}{\partial x_i}\frac{\partial \overline{y_{tt}^m}}{\partial xj}\,d\Omega\Big)\\
&= -\int_{\Gamma_1} b|y_{tt}^m|^2\,d\Gamma_1 - \mathrm{Re}\Big(\int_{\Gamma_1}\int_0^t k'(t-s)y_t^m(s)\overline{y_{tt}^m}(t)ds\,d\Gamma_1\Big)\\
&\quad - \mathrm{Re}\Big(\int_{\Gamma_1}\int_0^t k(0)y_t^m(t)\overline{y_{tt}^m}(t)ds\,d\Gamma_1\Big),
\end{aligned}
$$

we have

$$
\begin{aligned}
&-\int_{\Gamma_1} b|y_{tt}^m|^2\,d\Gamma_1 - \mathrm{Re}(\int_{\Gamma_1}\int_0^t k''(t-s)y^m(s)\overline{y_{tt}^m}(t)ds\,d\Gamma_1)\\
&-\mathrm{Re}(\int_{\Gamma_1}\int_0^t k(0)y_t^m(t)\overline{y_{tt}^m}(t)\,d\Gamma_1)\\
&= -\int b|y_{tt}^m|^2\,d\Gamma_1 - \frac{1}{2}\int_{\Gamma_1} k''|y_t^m|^2\,d\Gamma_1\\
&\quad+\frac{1}{2}\int_{\Gamma_1}\int_0^t k'''(t-s)|y_t^m(t)+y^m(s)|^2\,ds\,d\Gamma_1\\
&\quad-\frac{1}{2}\frac{d}{dt}\Big[-\int_{\Gamma_1} k'|y_t^m|^2\,d\Gamma_1 - \frac{1}{2}\int_{\Gamma_1}\int_0^t k''(t-s)|y_t^m(t)+y^m(s)|^2\,ds\,d\Gamma_1\Big]\,.
\end{aligned}
$$

We deduce that

$$
-\int_{\Gamma_1} b|y_{tt}^m|^2\,d\Gamma_1 - \frac{1}{2}\int_{\Gamma_1} k''|y_t^m|^2\,d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1}\int_0^t k'''(t-s)|y_t^m(t)+y^m(s)|^2\,ds\,d\Gamma_1 \le 0;
$$

so

$$
\begin{aligned}
&\mathrm{Re}(\sum_{i,j=1}^n\int_\Omega a_{ij}(x)\frac{\partial y_t^m}{\partial x_i}\frac{\partial \overline{y_{tt}^m}}{\partial xj}\,d\Omega)\\
&\le -\frac{1}{2}\frac{d}{dt}(-\int_{\Gamma_1} k'|y_t^m|^2\,d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1}\int_0^t k''(t-s)|y_t^m(t)+y^m(s)|^2\,ds\,d\Gamma_1)\,.
\end{aligned}
$$

On the other hand,

$$\mathrm{Re}\Big(\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x)\frac{\partial y_t^m}{\partial x_i}\frac{\partial \overline{y_{tt}^m}}{\partial xj}\,d\Omega\Big) = \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |\nabla_g y_t^m|_g^2\,d\Omega\Big)$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\int_\Omega |\nabla_g y_t^m|_g^2\,d\Omega)\\ &\leq -\frac{1}{2}\frac{d}{dt}\Big(-\int_{\Gamma_1} k'|y_t^m|^2\,d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1}\int_0^t k''(t-s)|y_t^m(t)+y^m(s)|^2 ds\,d\Gamma_1\Big).\end{aligned}$$

Therefore,

$$-\frac{1}{2}\frac{d}{dt}\Big[-\int_{\Gamma_1} k'|y_t^m|^2\,d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1}\int_0^t k''(t-s)|y_t^m(t)+y^m(s)|^2 ds\,d\Gamma_1\Big] \leq 0,$$
$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |\nabla_g y_t^m|_g^2\,d\Omega\Big) \leq 0\,.$$

We have

$$\frac{1}{2}\int_\Omega |\nabla_g y_t^m(T)|_g^2\,d\Omega \leq \frac{1}{2}\int_\Omega |\nabla_g y_t^m(0)|_g^2\,d\Omega,$$
$$|y_0^m|_V^2 \leq |y_0^{0m}|_V^2$$

Our main result is the following theorem.

Theorem 2.2. *Assume* (1.5)–(1.7) *and* (2.1)–(2.5), *and that the problem*

$$\begin{gathered} y_t - \mathbf{i}Ay = 0 \quad \text{on } \Omega\\ y = 0 \quad \text{in } \Sigma_0\\ \frac{\partial y}{\partial \nu_A} = 0 \quad \text{in } \Sigma\end{gathered}$$

has $y = 0$ *as the unique solution. Then for all given initial data* $y_0 \in H^1_{\Gamma_0}(\Omega)$, *there exist two positive constants* M *and* ω *such that*

$$E(t) \leq Me^{-\omega t}E(0), \quad \text{for } t > 0.$$

3. Proof of main result

For simplicity, we assume that y is a weak solution. By a classical density argument, Theorem 2.2 still holds for a weak solution.

Lemma 3.1. *The energy defined by* (1.7) *is decreasing and satisfies*

$$\begin{aligned}E(T) - E(0) = &-\int_0^T\int_{\Gamma_1} |y_t|^2\,d\Gamma_1 dt + \frac{1}{2}\int_0^T\int_{\Gamma_1} k'|y|^2\,d\Gamma_1\,dt\\ &- \frac{1}{2}\int_0^T\int_0^t\int_{\Gamma_1} k''(t-s)|y(t)-y(s)|^2\,d\Gamma_1\,ds\,dt,\end{aligned}$$

whenever $0 < T < \infty$.

Proof. Differentiating the energy $E(\cdot)$ defined by (1.7) and using Green's second theorem, we have

$$2E'(t) = \int_\Omega \nabla_g\overline{y_t}\nabla_g y\,d\Omega + \int_\Omega \nabla_g\overline{y}\nabla_g y_t\,d\Omega + \int_{\Gamma_1} k'|y|^2\,d\Gamma_1$$

$$
\begin{aligned}
&+ 2\operatorname{Re}\int_{\Gamma_1} k y_t y \, d\Gamma_1 - \int_0^t \int_{\Gamma_1} k''(t-s)|y(t)-y(s)|^2 \, d\Gamma_1 \, ds \\
&- \int_0^t \int_{\Gamma_1} k'(t-s)\frac{d}{dt}(y(t)-y(s))(\overline{y(t)-y(s)}) ds \, d\Gamma_1 .
\end{aligned}
$$

Then

$$
\begin{aligned}
E'(t) = \operatorname{Re}\int_\Omega \langle \nabla_g y_t, \nabla_g \overline{y}\rangle_g \, d\Omega + \frac{1}{2}\int_{\Gamma_1} k'|y|^2 \, d\Gamma_1 + \\
&+ \operatorname{Re}\int_{\Gamma_1} k\overline{y_t} y \, d\Gamma_1 - \frac{1}{2}\int_0^t \int_{\Gamma_1} k''(t-s)|y(t)-y(s)|^2 \, d\Gamma_1 ds \\
&- \operatorname{Re}\int_0^t \int_{\Gamma_1} k'(t-s)\overline{y_t}(t)(y(t)-y(s)) \, d\Gamma_1 ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
E'(t) &= \operatorname{Re}\int_{\Gamma_1} \frac{\partial y}{\partial \nu_A}\overline{y_t} \, d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1} k'|y|^2 \, d\Gamma_1 \\
&+ \operatorname{Re}\int_{\Gamma_1} k\overline{y_t} y \, d\Gamma_1 - \frac{1}{2}\int_0^t \int_{\Gamma_1} k''(t-s)|y(t)-y(s)|^2 \, d\Gamma_1 ds \\
&- \operatorname{Re}\int_0^t \int_{\Gamma_1} k'(t-s)\overline{y_t}(t)(y(t)-y(s)) \, d\Gamma_1 ds
\end{aligned}
$$

By the boundary condition, we have $\operatorname{Re}\int_{\Gamma_1} y\overline{y_t}(k - k(0) - \int_0^t k'(t-s)ds) \, d\Gamma_1 = 0$. Then we find that

$$
\begin{aligned}
E'(t) = -\int_{\Gamma_1} b|y_t|^2 \, d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1} k'|y|^2 \, d\Gamma_1 \\
- \frac{1}{2}\int_0^t \int_{\Gamma_1} k''(t-s)|y(t)-y(s)|^2 \, d\Gamma_1 ds.
\end{aligned}
$$

This completes the proof. □

Lemma 3.2. *For all $0 \le S < T < \infty$ we have*

$$
\begin{aligned}
&\int_S^T \int_{\Gamma_1} \big(\frac{\partial y}{\partial \nu_A}\big)^2 \frac{H\cdot\nu}{|\nu_A(x)|_g^2} \, d\Gamma_1 ds - \int_S^T \int_{\Gamma_1} |\nabla_g y|_g^2 H\nu \, d\Gamma_1 ds \\
&+ 2\operatorname{Re}\int_S^T \int_{\Gamma_1} \big(\frac{\partial y}{\partial \nu_A}\big) H(\overline{y}) \, d\Gamma_1 ds + \operatorname{Im}\int_S^T \int_{\Gamma_1} y\overline{y_t} H\upsilon \, d\Gamma_1 ds \\
&= 2\operatorname{Re}\int_S^T \int_\Omega DH(\nabla_g y, \nabla_g y) \, d\Omega \, ds - \int_S^T \int_\Omega |\nabla_g y|_g^2 \operatorname{div} H \, d\Omega \, ds \\
&\quad + \operatorname{Im}\int_\Omega y_t H(y) \, d\Omega\big|_S^T + \operatorname{Im}\int_S^T \int_\Omega \overline{y_t} y \operatorname{div} H \, d\Omega \, ds
\end{aligned}
$$

Proof. We multiply (1.1) by $H\cdot\nabla\overline{y}$ and integrate over $]S,T[\times\Omega$, to obtain

$$
\int_S^T \int_\Omega y_t H\cdot\nabla\overline{y} \, d\Omega \, ds - \mathbf{i}\int_S^T \int_\Omega AyH\cdot\nabla\overline{y} \, d\Omega \, ds = 0 . \tag{3.1}
$$

We have

$$\begin{aligned}
&\int_S^T \int_\Omega y_t H \cdot \nabla \overline{y}\, d\Omega\, ds \\
&= \int_\Omega yH \cdot \nabla \overline{y}\, d\Omega\Big|_S^T - \int_S^T \int_\Omega yH \cdot \nabla \overline{y_t}\, d\Omega\, ds \\
&= \int_\Omega yH \cdot \nabla \overline{y}\, d\Omega\Big|_S^T - \int_S^T \int_\Gamma y\overline{y_t} H \cdot \nu\, d\Gamma ds + \int_S^T \int_\Omega \overline{y_t} \operatorname{div}(y.H)\, d\Omega\, ds.
\end{aligned} \tag{3.2}$$

Substituting (3.2) in (3.1), we obtain

$$\begin{aligned}
&\int_\Omega yH \cdot \nabla \overline{y}\, d\Omega\Big|_S^T - \int_S^T \int_\Gamma y\overline{y_t} H \cdot \nu\, d\Gamma ds \\
&- \mathbf{i}\Big[\int_S^T \int_\Omega (A\overline{y} H \cdot \nabla y + AyH \cdot \nabla \overline{y})\, d\Omega\, ds\Big] + \int_S^T \int_\Omega \overline{y_t} y \operatorname{div} H\, d\Omega\, ds = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&2 \operatorname{Re} \int_S^T \int_\Omega AyH \cdot \nabla \overline{y}\, d\Omega\, ds \\
&= \operatorname{Im} \int_\Omega yH \cdot \nabla \overline{y}\, d\Omega\Big|_S^T - \operatorname{Im} \int_S^T \int_\Gamma y\overline{y_t} H \cdot \nu\, d\Gamma\, ds \\
&\quad + \operatorname{Im} \int_S^T \int_\Omega \overline{y_t} y \operatorname{div} H\, d\Omega\, ds
\end{aligned} \tag{3.3}$$

Using (1.14), we rewrite the integral on the left-hand side of (3.3) as

$$\begin{aligned}
&\int_S^T \int_\Omega AyH \cdot \nabla \overline{y}\, d\Omega\, ds \\
&= \int_S^T \int_\Gamma \frac{\partial y}{\partial \nu_A} H(y)\, d\Gamma ds - \int_S^T \int_\Omega DH(\nabla_g y, \nabla_g \overline{y})\, d\Omega\, ds \\
&\quad - \frac{1}{2} \int_S^T \int_\Omega \operatorname{div}(|\nabla_g y|^2.H)\, d\Omega\, ds \\
&\quad + \frac{1}{2} \int_S^T \int_\Omega |\nabla_g y|_g^2 \operatorname{div} H\, d\Omega\, ds
\end{aligned} \tag{3.4}$$

Recalling the boundary condition (1.2), on Γ we have

$$y = y_t = 0, \quad |\nabla_g y|_g^2 = \frac{1}{|\nu_A(x)|_g^2} (\frac{\partial y}{\partial \nu_A})^2, \quad H(y) = \frac{H \cdot \nu}{|\nu_A(x)|_g^2} (\frac{\partial y}{\partial \nu_A}). \tag{3.5}$$

Thus using this equation and (1.3), we find that this simplifies the sought-after identity. □

Completion of the proof of theorem 2.1. Set $C_0 = \big(\alpha_1 \sup_{x\in\overline{\Omega}} \frac{|H|^2}{|H\cdot\nu|}\big)^2$, $C_1 = \frac{\alpha_1}{2} \sup_{x\in\overline{\Omega}} |\operatorname{div} H|$,

$$C_2 = \frac{\alpha_1}{2\varepsilon} \sup_{x\in\overline{\Omega}} |\nabla_g(\operatorname{div} H)| + \frac{1}{4} \sup_{x\in\overline{\Omega}} |\operatorname{div} H| + \frac{\varepsilon\alpha_1}{2} \sup_{x\in\overline{\Omega}} |H(x)|,$$

$$C_3 = 2\Big[\frac{|k(0)|_{L^\infty(\Gamma_1)}}{e\delta f} + |h|_{L^\infty(\Gamma_1)}\Big]$$

where $h(x) = \frac{k(0)}{\delta(1+ek'(0))}$, $x \in \Gamma_1$. From assumptions (2.1), (2.2) (2.3), we deduce that

$$\begin{aligned}
&2a\int_S^T\int_\Omega |\nabla_g y|_g^2\,d\Omega\,ds \\
&\le -\int_S^T\int_{\Gamma_1} |\nabla_g y|_g^2 H\cdot\nu\,d\Gamma_1 ds + 2\operatorname{Re}\int_S^T\int_{\Gamma_1}\big(\frac{\partial y}{\partial\nu_A}\big)H(\overline{y})\,d\Gamma_1 ds \\
&\quad + \operatorname{Im}\int_S^T\int_{\Gamma_1} y\overline{y_t}H\cdot\nu\,d\Gamma_1 ds + \int_S^T\int_\Omega |\nabla_g y|_g^2 \operatorname{div} H\,d\Omega\,ds \\
&\quad - \operatorname{Im}\int_\Omega yH(y)\,d\Omega\big|_S^T - \operatorname{Im}\int_S^T\int_\Omega \overline{y_t}y\operatorname{div} H dQ
\end{aligned} \tag{3.6}$$

Also for an arbitrary positive constant ε we have the estimates

$$\begin{aligned}
&-\int_S^T\int_{\Gamma_1} |\nabla_g y|_g^2 H\cdot\nu\,d\Gamma_1 ds + 2\operatorname{Re}\int_S^T\int_{\Gamma_1}\big(\frac{\partial y}{\partial\nu_A}\big)H(\overline{y})\,d\Gamma_1 ds \\
&\le C_0\int_S^T\int_{\Gamma_1}\big(\frac{\partial y}{\partial\nu_A}\big)^2\,d\Gamma_1 ds,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\big|\operatorname{Im}\int_S^T\int_{\Gamma_1} y\overline{y_t}H\cdot\nu\,d\Gamma_1 ds - \operatorname{Im}\int_\Omega yH(y)\,d\Omega\big|_S^T\big| \\
&\le \frac{\sup_{x\in\overline{\Omega}}|H|}{2\alpha_1}\int_S^T\int_{\Gamma_1}|y|^2\,d\Gamma_1 ds \\
&\quad + 2\alpha_1\sup_{x\in\overline{\Omega}}|H|\int_S^T\int_{\Gamma_1}|y_t|^2\,d\Gamma_1 ds + \varepsilon\sup_{x\in\overline{\Omega}}|H(x)|E(S) \\
&\quad + \Big(\frac{\varepsilon\alpha_1}{2}\sup_{x\in\overline{\Omega}}|H(x)|\Big)|y|^2_{C([S,T],L^2(\Omega))},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\big|\int_S^T\int_\Omega |\nabla_g y|_g^2 \operatorname{div} H\,d\Omega\,ds - \operatorname{Im}\int_S^T\int_\Omega \overline{y_t}y\operatorname{div} H\,d\Omega\,ds\big| \\
&\le C_1\int_S^T\int_{\Gamma_1}\big(\frac{\partial y}{\partial\nu_A}\big)^2\,d\Gamma_1 ds + \frac{\varepsilon}{2}\int_S^T\int_\Omega |\nabla_g\overline{y}|_g^2\,d\Omega\,ds \\
&\quad + \Big(\frac{\alpha_1}{2\varepsilon}\sup_{x\in\overline{\Omega}}|\nabla_g(\operatorname{div} H)| + \frac{1}{4}\sup_{x\in\overline{\Omega}}|\operatorname{div} H|\Big)\int_S^T\int_{\Gamma_1}|y|^2\,d\Gamma_1 ds,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\big|\int_{\Sigma_1}\big(\frac{\partial y}{\partial\nu_A}\big)^2 d\Sigma_1\big| &\le 3\Big[\int_S^T\int_{\Gamma_1}\Big(-\int_0^t k'(t-s)y(s)ds\Big)^2\,d\Gamma_1 ds \\
&\quad + \int_S^T\int_{\Gamma_1} k(0)|y|^2\,d\Gamma_1 ds + \int_S^T\int_{\Gamma_1}|y_t|^2\,d\Gamma_1 ds\Big].
\end{aligned} \tag{3.10}$$

Now using a compactness-uniqueness argument in the style of [7], we deduce

$$|y|^2_{C([S,T],L^2(\Omega))} \le \int_S^T\int_{\Gamma_1}|y_t|^2\,d\Gamma_1 ds \tag{3.11}$$

Inserting (3.7), (3.8), (3.9) and (3.11) in (3.6), leads to

$$
\begin{aligned}
&2a\int_S^T\int_\Omega |\nabla_g y|_g^2\,d\Omega\,ds\\
&\le 3(C_0+C_1)\Big[\int_S^T\int_{\Gamma_1}(-\int_0^t k'(t-s)y(s)ds)^2\,d\Gamma_1 ds\Big]\\
&\quad + 3(C_0+C_1)\Big[\int_S^T\int_{\Gamma_1}k(0)|y|^2\,d\Gamma_1 ds+\int_S^T\int_{\Gamma_1}|by_t|^2\,d\Gamma_1 ds\Big]\\
&\quad + C_2\int_S^T\int_{\Gamma_1}|y|^2\,d\Gamma_1 ds+\frac{\varepsilon}{2}\int_S^T\int_\Omega|\nabla_g\overline{y}|_g^2\,d\Omega\,ds\\
&\quad + \varepsilon\sup_{x\in\overline{\Omega}}|H(x)|E(S)+2\alpha_1\sup_{x\in\overline{\Omega}}|H|\int_S^T\int_{\Gamma_1}|y_t|^2\,d\Gamma_1 ds\\
&\quad + (\frac{\varepsilon\alpha_1}{2}\sup_{x\in\overline{\Omega}}|H(x)|)|y|^2_{C([S,T],L^2(\Omega))}
\end{aligned}
\tag{3.12}
$$

Choosing ε so that $4a-\varepsilon>0$, we obtain

$$
\begin{aligned}
&(4a-\varepsilon)\int_S^T E(t)dt\\
&\le [3C_3(C_0+C_1)+\varepsilon\sup_{x\in\overline{\Omega}}|H(x)|]E(S)+2\alpha_1\sup_{x\in\overline{\Omega}}|H|\frac{1}{\sup_{x\in\overline{\Gamma_1}}|b|}E(S)\\
&\quad + [3(C_0+C_1)|k(0)|^2_{L^\infty}+C_2]\frac{1}{\varphi}E(S)\\
&\quad + [3(C_0+C_1)+\frac{\varepsilon\alpha_1}{2}\sup_{x\in\overline{\Omega}}|H(x)|]\sup_{x\in\overline{\Gamma_1}}|b|E(S),
\end{aligned}
$$

where

$$
\int_S^T E(t)dt\le CE(S),
$$

$$
\begin{aligned}
C=\frac{1}{(4a-\varepsilon)}&[3C_3(C_0+C_1)+\varepsilon\sup_{x\in\overline{\Omega}}|H(x)|+2\alpha_1\sup_{x\in\overline{\Omega}}|H|\frac{1}{\sup_{x\in\overline{\Gamma_1}}|b|}\\
&+[3(C_0+C_1)+\frac{\varepsilon\alpha_1}{2}\sup_{x\in\overline{\Omega}}|H(x)|]\sup_{x\in\overline{\Gamma_1}}|b|\\
&+[3(C_0+C_1)|k(0)|^2_{L^\infty}+C_2]\frac{1}{\varphi}].
\end{aligned}
$$

Letting $T\to+\infty$, we obtain for every $S\in\mathbb{R}_+$,

$$
\int_S^{+\infty}E(t)dt\le CE(S).
$$

The desired conclusion follows now from Komornik [6, Theorem 8.1].

Acknowledgments. I would like to thank to Prof. S. E. Rebiai for support received about the regularity questions for the Schrodinger equation.

References

[1] M. Aassila, M. M. Cavalcanti, J. A. Soriano; *Asymptoic stability and energy deay rate for solution of wave equation with memory in a star shaped domain*, SIAM J. Control Optim., 38 (2000), 1581-1602.
[2] S. Chai, Y. Guo; *Boundary stabilization of wave equations with variable coefficients and memory*, Differential and Integral Equations, 17 (2004), 669-980.
[3] R. Cipolotti, E. Machtyngier, San Perdo Siqueira; *Nonlinear boundary feedback stabilization for Schrödinger equations*, Differrential and Integral Equations, 9 (1996), 137-148.
[4] A. Guesmia; *Thèse de doctorat (Chapitre 8)*, IRMA, Strasbourg, (2000).
[5] E. Hebey; *Introduction à l'analyse non liné aire sur les variétés*, Editeur Diderot, Paris, 1997.
[6] V. Komornik; *Exact controlability and stabilization. The multiplier method*, Masson, Paris, 1994.
[7] I. Lasiecka, R. Triggiani; *Optimal reglarity, exact controllability and uniform stabilization of Schrödinger equations with Dirichlet control*, Differrential and Integral Equations, 5 (1992), 521-535.
[8] J. L. Lions; *Quelques méthodes de résolution des problèmes aux limites non linèaires*, Dunod, Paris, 1969.
[9] E. Machtyngier, E. Zuazua; *Stabilization of Schrödinger equation*, Portugaliae Mathematica, 51 (1994), 243-255.
[10] P. F. Yao; *Observabiliy inequality for exact controllability of wave equations with variable coefficients*, SIAM J. Control Optim, 37 (1999), 1568-1599.

Abdesselam Nawel
Department of Mathematics, University of Laghouat, Algeria
E-mail address: nawelabedess@gmail.com

Khaled Melkemi
Department of Mathematics, University of Batna II, Algeria
E-mail address: k.melkemi@univ-batna2.dz