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STOCHASTIC ATTRACTOR BIFURCATION FOR THE TWO-DIMENSIONAL SWIFT-HOHENBERG EQUATION WITH MULTIPLICATIVE NOISE

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Abstract. This article concerns the dynamical transitions of the stochastic Swift-Hohenberg equation with multiplicative noise on a two-dimensional domain $(-L, L) \times (-L, L)$. With α and L regarded as parameters, we show that the approximate reduced system corresponding to the invariant manifold undergoes a stochastic pitchfork bifurcation near the critical points, and the impact of noise on stochastic bifurcation of the Swift-Hohenberg equation. We find the approximation representation of the manifold and the corresponding reduced systems for stochastic Swift-Hohenberg equation when L_2 and $\sqrt{2}L_1$ are close together.

1. Introduction

The Swift-Hohenberg equation was initially proposed by Swift and Hohenberg ([27]) in 1977 as a simple model for the Rayleigh-Bénard instability of roll waves, which takes the form

$$\frac{\partial u}{\partial t} = \alpha u - (1 + \frac{\partial^2}{\partial x^2})^2 u - u^3. \tag{1.1}$$

This equation plays an important role in the study of various phenomena in pattern formation, see [4, 9]. It has been studied a great deal, both analytically and numerically. These fields include the Rayleigh-Bénard problem of convection in a horizontal fluid layer in the gravitational field [28], Taylor-Couette flow [15], some chemical reactions [25] and large-scale flows and spiral core instabilities [1]. These are effects which relate to systems far from equilibrium. In optics, this equation has been considered in relation to spatial structures in large aspect lasers, and synchronously pumped optical parametric oscillators. In [34], attractor bifurcation and asymptotic behavior of the real Swift-Hohenberg equation and the generalized Swift-Hohenberg equation with Dirichlet boundary condition and periodic boundary condition are investigated, in which the techniques are based on the results given in [20, 21].

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In [22, 23, 24], the authors studied the asymptotic behavior of the solutions of the Cauchy-Dirichlet problem for the Swift-Hohenberg equation on the domain $(0, L)$,

$$\begin{gathered}
\frac{\partial u}{\partial t} = \alpha u - (1 + \frac{\partial^2}{\partial x^2})^2 u - u^3, \quad \text{for } 0 < x < L, t > 0, \\
u = 0, \quad \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{at } x = 0, L, \\
u(x, 0) = u_0(x), \quad \text{for } 0 < x < L,
\end{gathered} \tag{1.2}$$

where the initial function u_0 is a smooth function, α and σ are positive numbers. With α and the length of the domain L regarded as bifurcation parameters, different types of structures in the bifurcation diagrams are presented when the bifurcation points are closer. We have studied the asymptotic behavior of the solutions of the Cauchy-Dirichlet problem for the Swift-Hohenberg equation with quintic nonlinearity [29, 30]. In [32], we have considered bifurcation of a modified Swift-Hohenberg equation in two spatial dimension with periodic boundary condition. There has been some research in the optimal distributed control for the modified Swift-Hohenberg equation, see [26].

The dynamical behavior of solutions to stochastic differential equations and stochastic partial differential equations, such as long time behavior, ergodicity, and periodicity, has been studied in [8, 10, 16, 33]. In the recent two decades, there has been some research in the impacts of noise on the stochastic dynamics, see [5, 15]. The study of the asymptotic behavior of the following stochastic equation driven by multiplicative noise in Stratonovich sense

$$du = (L_\alpha u + G(u))dt + \sigma u \circ dW_t$$

has an extensive literature, see [5, 6, 7, 11], and the references therein. Here L_α is a linear operator parameterized by a parameter $\alpha \in R$, $G(u)$ represents the nonlinear terms, W_t is a two-sided one-dimensional Winner process, and $\sigma \in R$ gives a measure of the amplitude of the noise.

The dynamics of the stochastic Swift-Hohenberg equation has attracted much attention in recent years. In [14], the authors studied the dynamic transitions of the two-dimensional Swift-Hohenberg equation with multiplicative noise, and showed that the approximate reduced system corresponding to the invariant manifold undergoes a stochastic pitchfork bifurcation. Li ([17]) studied the dynamic transitions of the two-dimensional Swift-Hohenberg equation with multiplicative noise, the study is based on the stochastic parameterizing manifolds developed by Chekroun, Liu and Wang ([6, 7]). They both considered α as a parameter to study the dynamic transitions. Approximation representation of parameterizing manifold and non-Markovian reduced systems for a stochastic Swift-Hohenberg equation with additive noise has been investigated in [12]. There are some other results for approximation of manifolds for stochastic Swift-Hohenberg equation with multiplicative noise in Stratonovich sense [3, 18, 27].

In [31], we studied the dynamical transitions of the stochastic Swift-Hohenberg equation with multiplicative noise on a one-dimensional domain $(0, L)$. With α and the length of the domain L regarded as parameters, we showed that the approximate reduced system corresponding to the invariant manifold undergoes a stochastic pitchfork bifurcation near the critical points, and the impact of noise on stochastic bifurcation of the Swift-Hohenberg equation.

In this paper, we consider the stochastic attractor bifurcation of two dimensional Swift-Hohenberg equation (SHE) on the domain $Q = (-L, L) \times (-L, L)$ with multiplicative noise in Stratonovich sense

$$du = (\alpha u - (1 + \Delta)^2 u - u^3)dt + \sigma u \circ dW_t, \quad (x, y) \in Q,\ t > 0, \tag{1.3}$$

with boundary conditions

$$u(x, y, t) = u(x, y + 2L, t) = u(x + 2L, y, t), \quad (x, y) \in Q,\ t \geq 0, \tag{1.4}$$

$$u(x, y, t) = -u(-x, -y, t), \quad (x, y) \in Q,\ t \geq 0, \tag{1.5}$$

and the SHE with multiplicative noise in Ito sense

$$\begin{gathered} du = (\alpha u - (1 + \Delta)^2 u - u^3)dt + \sigma u dW_t, \quad (x, y) \in Q,\ t > 0, \\ u(x, y, t) = u(x, y + 2L, t) = u(x + 2L, y, t), \quad (x, y) \in Q,\ t \geq 0, \\ u(x, y, t) = -u(-x, -y, t), \quad (x, y) \in Q,\ t \geq 0, \end{gathered} \tag{1.6}$$

where the initial function u_0 is a smooth function, α and σ are positive numbers, in particular, W_t is a two-sided one-dimensional Winner process. With α and the length of the domain L regarded as parameters, we study the dynamic transitions of the the stochastic Swift-Hohenberg equation. One main objective of this paper is to extend the work in [22, 23, 24, 32] to the two-dimensional stochastic Swift-Hohenberg equation, we will consider the stochastic attractor bifurcation of the Swift-Hohenberg equation near the critical points, and the case when the bifurcation points nearly coincide. With α and the length of the domain L regarded as parameters, we will study the dynamical transitions of the stochastic Swift-Hohenberg equation and the impact of noise on the stochastic dynamics of the Swift-Hohenberg equation.

One standard technique in the analysis of deterministic dynamical systems is the center manifold reduction. However, the study of the reduction problem of SPDE to its corresponding stochastic invariant manifolds is much less, one reason is the incompatibility with large excursions of SPDE solutions caused by white noise. The above-mentioned difficulty can be overcome by using stochastic parameterizing manifolds developed ([6, 7]). This approach is based on approximate parameterizations of the small scales by the large ones via the concept of stochastic parameterizing manifolds, where the latter are random manifolds aiming to improve the partial knowledge of the full SPDE¡¯s solution in mean square error when compared with its projection onto the resolved modes. Approximate parameterizing manifolds can be obtained by representing the modes with high wave numbers as a pullback limit depending on the time-history of the nodes with low wave numbers for the corresponding backward-forward systems.

This article is organized as follows. In section 2, we recall some results of deterministic the Swift-Hohenberg equation, and introduce some mathematical settings. In sections 3, 4, and 5, we analyze stochastic attractor bifurcation of the Swift-Hohenberg equation near the points $\sqrt{m^2 + n^2} L_1$ and $\sqrt{m^2 + n^2} L_2$ under three cases. The approximation representation of manifold and the corresponding reduced systems for stochastic Swift-Hohenberg equation when L_2 and $\sqrt{2} L_1$ are close together are obtained in section 6.

2. Mathematical setting

We will recall in this section some results of deterministic the Swift-Hohenberg equation, and introduce some mathematical settings.

Let us introduce the following spaces.

$$H = \{u \in L^2(Q) : u \text{ satisfies (1.4)-(1.5) and } \int_Q u\,dx\,dy = 0\},$$

$$H_1 = H^4(Q) \cap H,$$

and with the inner product

$$\langle u, v\rangle = \frac{1}{4L^2}\int_0^L\int_0^L uv\,dx\,dy.$$

Notice that H and H_1 are Hilbert spaces, and $H_1 \hookrightarrow H$ is a dense and compact inclusion. We consider the nonlinear evolution equations

$$\frac{du}{dt} = L_\alpha u + G(u), \tag{2.1}$$

$$u(0) = u_0, \tag{2.2}$$

where $u : [0,\infty) \to H$ is the unknown function, $\alpha \in R$ is the system parameter, and $L_\alpha : H_1 \hookrightarrow H$ are parameterized linear completely continuous fields continuously depending on $\alpha \in R$, which satisfy

$$L_\alpha = A + B_\alpha \quad \text{is a sectorial operator,} \tag{2.3}$$

where $A : H_1 \hookrightarrow H$ is a linear homeomorphism, $B_\alpha : H_1 \hookrightarrow H$ are parameterized linear compact operators. Furthermore, the nonlinear term $G(u) = -u^3$ is a bounded operator such that

$$G(u,\alpha) = o(\|u\|_{H_1}), \quad \forall \alpha \in R.$$

Hence the stochastic Swift-Hohenberg equation can be written as

$$du = (L_\alpha u + G(u))dt + \sigma u \circ dW_t. \tag{2.4}$$

Let

$$\mathfrak{Z} = \{(m,n) : m \in \mathbb{N}, n \in \mathbb{Z}\} \cup \{(0,n) : n \in \mathbb{N}\}. \tag{2.5}$$

Then the eigenvalues of the following eigenvalue problem on H_1,

$$L_\alpha \varphi = \lambda \varphi,$$

are

$$\lambda_{m,n} = P(\frac{\sqrt{m^2+n^2}\pi}{L}), \quad \text{where } P(\xi) = \alpha - (\xi^2-1)^2,$$

with the corresponding eigenfunctions

$$e_{k,l} = \sqrt{2}\sin\frac{\pi(kx+ly)}{L}. \tag{2.6}$$

for all (k,l) which satisfies $k^2+l^2 = m^2+n^2$, and $(k,l) \neq (0,0)$, $(k,l) \in \mathfrak{Z}$. We can see that $\{e_{m,n}, (m,n) \in \mathfrak{Z}\}$ forms a basis of H, and $\langle e_{m,n}, e_{m,n}\rangle = 1$.

Throughout this article, we consider α positive and small enough, and we let $\alpha \in (0,1)$, then $P(\xi)$ has two positive zeros:

$$\xi_- = (1-\sqrt{\alpha})^{1/2}, \quad \xi_+ = (1+\sqrt{\alpha})^{1/2}.$$

This implies that $\lambda_{m,n} > 0$, when $L \in (\sqrt{m^2+n^2}L_1, \sqrt{m^2+n^2}L_2)$, where $L_1 = \pi/\xi_+$, $L_2 = \pi/\xi_-$. If $L \in (0, L_1)$, then

$$\frac{\sqrt{m^2+n^2}\pi}{L} > \frac{\sqrt{m^2+n^2}\pi}{L_1} \geq \frac{\pi}{L_1}$$

for all $(m,n) \in \mathfrak{Z}$. This implies that

$$\lambda_{m,n} = P(\frac{\sqrt{m^2+n^2}\pi}{L}) > P(\xi_+) = 0, \quad \text{for all } (m,n) \in \mathfrak{Z}.$$

Thus, if $L < L_1$, the trivial solution is asymptotically stable.

When L is larger than L_1, we denote

$$I_{m,n} = \big\{L > 0 : P\big(\frac{\sqrt{m^2+n^2}\pi}{L}\big) \geq 0\big\},$$

then $I_{m,n} = [\sqrt{m^2+n^2}L_1, \sqrt{m^2+n^2}L_2]$ for every $(m,n) \in \mathfrak{Z}$.

For α small,

$$L_1(\alpha) \sim \pi - \frac{\pi}{2}\sqrt{\alpha}, \quad L_2(\alpha) \sim \pi + \frac{\pi}{2}\sqrt{\alpha}, \quad \text{as } \alpha \to 0^+. \tag{2.7}$$

If α is small enough, the intervals $I_{m,n}$ will not overlap, but be separated by intervals in which $\lambda_{m,n} > 0$.

Suppose there is a gap between the intervals $I_{m,n}$. Then we define

$$\Pi_{m,n} = (0, \sqrt{m^2+n^2}L_2) - \cup_{1\leq i\leq m, 1\leq j\leq n} I_{i,j}.$$

From the Fourier series of the solution, as discussed in our previous work [29, 30], we have the following theorem.

Theorem 2.1. *Suppose there is a gap between the intervals $I_{m,n}$, and let $u(t)$ be the solution of Problem* (1.2)*. Then for all $L \in \Pi_{m,n}$, we have $u(t) \to 0$ as $t \to \infty$.*

Remark 2.2. For the deterministic case of Problem (1.3), we can conclude that the equation undergoes a supercritical bifurcation at $L = \sqrt{m^2+n^2}L_1$ and a subcritical bifurcation at $L = \sqrt{m^2+n^2}L_2$.

Note that for fixed K, the number of solutions (m,n) to $K = m^2+n^2$ may be lager than two, such as $m^2+n^2 = 25$ has 6 different solutions $(m,n) = (5,0)$, $(0,5)$, $(3,4)$, $(3,-4)$, $(4,3)$, and $(4,-3)$. Here we consider only $m \geq 0$.

In the following two sections, we consider stochastic attractor bifurcation of the Swift-Hohenberg equation near the points $\sqrt{m^2+n^2}L_1$ and $\sqrt{m^2+n^2}L_2$. This is done in two cases: first we assume that $K = m^2+n^2$ has only two solutions $(m,n) = (k,k)$ and $(k,-k)$, this case will be discussed in section 3. Then, in section 4, we consider the case when $K = m^2+n^2$ has only two solutions $(m,n) = (k,0)$ and $(0,k)$.

3. Analysis of the case $(m,n) = (k,k)$ and $(k,-k)$

In this section, we consider the attractor bifurcation near the points $\sqrt{m^2+n^2}L_1$ and $\sqrt{m^2+n^2}L_2$, in the case the intervals $I_{m,n}$ do not overlap, and $K = m^2+n^2$ has only two solutions $(m,n) = (k,k)$ and $(k,-k)$. That is, we consider the attractor bifurcation near the points $\sqrt{2}kL_1$ and $\sqrt{2}kL_2$.

In this case, the space H_1 and H can be decomposed into

$$H_1 = H_1^c \oplus H_1^s, \quad H = H_1^c \oplus \widetilde{H}^s,$$

where

$$H_1^c = \mathrm{span}\{e_{k,k}, e_{k,-k}\},$$

and $\widetilde{H}^s$ is the closure of H_1^s in H.

We will present a stochastic reduction procedure based on parameterizing manifolds (PM) associated with (1.3). A stochastic parameterizing manifolds [6, 7], as the graph of a random continuous function $h_\alpha(\xi,\omega)$ from H_1^c to $\widetilde{H}^s$, and for each realization ω, the function is defined for $\xi \in H_1^c$.

The PM-based reduced equation for the resolved modes is

$$du_c = (L_\alpha^c u_c + P_c G(u_c + u_s))dt + \sigma u_c \circ dW_t, \tag{3.1}$$

where $\xi \in H_1^c$. However, it's more involved to give an explicit expression of $h_\alpha(\xi,\omega)$, the key idea ([6, 7])is to provide an approximation of $h_\alpha(\xi,\omega)$ via the pullback characterization

$$h_\alpha(\xi,\omega) := \lim_{T\to+\infty} u_s^{(2)}(\xi)(T, \theta_{-T}\omega; 0)).$$

Indeed it is too cumbersome to use the above pullback characterization to approximate the vector field $P_c G(\xi + h_\alpha(\xi, \theta_t\omega))$ as ξ varies in H_1^c. We adopt instead a "Lagragian approach" which consists of approximating "on the fly" this vector along a trajectory $\xi(t,\omega)$ of interest, as the time t flows.

So this is much more manageable and leads naturally to consider, instead of (3.1), we consider the reduced equation

$$\begin{gathered} d\xi_t = (L_\alpha^c \xi_t + P_c G(\xi_t + u_s^{(2)}[\xi(t,\omega)](t+T, \theta_{-T}\omega; 0)))dt + \sigma \xi_t \circ dW_t, \\ \xi(0,\omega) = \phi, t > 0, \end{gathered} \tag{3.2}$$

where the notation ξ_t emphasized the t-dependence of the variable ξ_t, $\phi = P_c u_0$, and $u_s^{(2)}$ can be used to approximate the stochastic inertial manifold and is obtained from the following backward-forward systems (3.3)-(3.5).

For a given $t > 0$ and T sufficiently large, let us consider the following 2-layer auxiliary backward-forward system.

$$du_c^{(1)} = L_\alpha^c u_c^{(1)} d\tau + \sigma u_c^{(1)} \circ dW_\tau, \tau \in [t-T, t], \tag{3.3}$$

$$du_c^{(2)} = (L_\alpha^c u_c^{(2)} + P_c G(u_c^{(1)})d\tau + \sigma u_c^{(2)} \circ dW_\tau, \tau \in [t-T, t], \tag{3.4}$$

$$du_s^{(2)} = (L_\alpha^s u_s^{(2)} + P_s G(u_c^{(2)}(\tau - T, \omega)))d\tau + \sigma u_s^{(2)} \circ dW_{\tau-T}, \quad \tau \in [t, t+T], \tag{3.5}$$

with

$$\begin{gathered} u_c^{(1)}(\tau,\omega)\big|_{\tau=t} = \xi(t,\omega), \\ u_c^{(2)}(\tau,\omega)\big|_{\tau=t} = \xi(t,\omega), \\ u_s^{(2)}(\tau,\theta_{-T}\omega)\big|_{\tau=t} = 0, \end{gathered}$$

where $u_s^{(2)}$ can be used to approximate the stochastic inertial manifold, $L_\alpha^c = P_c L_\alpha$, $L_\alpha^s = P_s L_\alpha$. In the systems above, the initial value of $u_c^{(1)}$ and $u_c^{(2)}$ are prescribed in fiber $\theta_t\omega$, and the the initial value of $u_s^{(2)}$ is prescribed in fiber $\theta_{t-T}\omega$. The solution of system (3.3)-(3.5) is obtained by using a backward-forward integration procedure made possible because of the partial coupling between the equations constituting this system. Here $u_c^{(1)}$ and $u_c^{(2)}$ emanate backward from ξ in H_1^c force the evolution equation of $u_s^{(2)}$ to depend naturally on ξ but not reciprocally. Here,

θ_t is an element of a metric dynamical system, and ω is a given realization, $\theta_{t-T}\omega$ is called a fiber, see Arnold [2].

Since $u_c^{(1)} \in H_1^c$, we write

$$u_c^{(1)}(\tau,\omega) = x_1^{(1)}(\tau,\omega)e_{k,k} + x_2^{(1)}(\tau,\omega)e_{k,-k}, \tag{3.6}$$

$$\xi(\tau,\omega) = \xi_1(\tau,\omega)e_{k,k} + \xi_2(\tau,\omega)e_{k,-k}. \tag{3.7}$$

Then by projecting equation (3.2) onto H_1^c, we obtain

$$dx_1^{(1)} = P(\frac{\sqrt{2}k\pi}{L})x_1^{(1)}d\tau + \sigma x_1^{(1)} \circ dW_\tau, \tau \in [t-T,t], \tag{3.8}$$

$$dx_2^{(1)} = P(\frac{\sqrt{2}k\pi}{L})x_2^{(1)}d\tau + \sigma x_2^{(1)} \circ dW_\tau, \tau \in [t-T,t], \tag{3.9}$$

with

$$x_1^{(1)}(\tau,\omega)\big|_{\tau=t} = \xi_1(t,\omega), \tag{3.10}$$

$$x_2^{(1)}(\tau,\omega)\big|_{\tau=t} = \xi_2(t,\omega). \tag{3.11}$$

After simple calculation, we can obtain

$$\langle P_cG(u_c^{(1)}), e_{k,k}\rangle = -\frac{3}{2}(x_1^{(1)})^3 - 3x_1^{(1)}(x_2^{(1)})^2, \tag{3.12}$$

$$\langle P_cG(u_c^{(1)}), e_{k,-k}\rangle = -\frac{3}{2}(x_2^{(1)})^3 - 3(x_1^{(1)})^2x_2^{(1)}. \tag{3.13}$$

We write

$$u_c^{(2)}(\tau,\omega) = x_1^{(2)}(\tau,\omega)e_{k,k} + x_2^{(2)}(\tau,\omega)e_{k,-k}.$$

Then we have

$$\begin{aligned} dx_1^{(2)} &= (P(\frac{\sqrt{2}k\pi}{L})x_1^{(2)} - \frac{3}{2}(x_1^{(1)})^3 - 3x_1^{(1)}(x_2^{(1)})^2)d\tau \\ &\quad + \sigma x_1^{(2)} \circ dW_\tau, \tau \in [t-T,t], \end{aligned} \tag{3.14}$$

$$\begin{aligned} dx_2^{(2)} &= (P(\frac{\sqrt{2}k\pi}{L})x_2^{(2)} - \frac{3}{2}(x_2^{(1)})^3 - 3(x_1^{(1)})^2x_2^{(1)})d\tau \\ &\quad + \sigma x_2^{(2)} \circ dW_\tau, \tau \in [t-T,t], \end{aligned} \tag{3.15}$$

with

$$x_1^{(2)}(s,\omega)\big|_{s=t} = \xi_1(t,\omega), \tag{3.16}$$

$$x_2^{(2)}(s,\omega)\big|_{s=t} = \xi_2(t,\omega). \tag{3.17}$$

Notice that

$$\begin{aligned} G(u_c^{(2)}) &= -2\sqrt{2}[(x_1^{(2)})^3(\frac{3}{4}\sin\frac{k\pi(x+y)}{L} - \frac{1}{4}\sin\frac{3k\pi(x+y)}{L}) \\ &\quad + \frac{3}{2}(x_1^{(2)})^2x_2^{(2)}(\sin\frac{k\pi(x-y)}{L} - \frac{1}{2}\sin\frac{k\pi(3x+y)}{L} \\ &\quad + \frac{1}{2}\sin\frac{k\pi(x+3y)}{L}) + \frac{3}{2}x_1^{(2)}(x_2^{(2)})^2(\sin\frac{k\pi(x+y)}{L} \\ &\quad + \frac{1}{2}\sin\frac{k\pi(x-3y)}{L} - \frac{1}{2}\sin\frac{k\pi(3x-y)}{L}) \\ &\quad + (x_2^{(2)})^3(\frac{3}{4}sin\frac{k\pi(x-y)}{L} - \frac{1}{4}sin\frac{3k\pi(x-y)}{L})]. \end{aligned} \tag{3.18}$$

Then we have

$$\langle P_sG(u_c^{(2)}), e_{3k,3k}\rangle = \frac{1}{2}(x_1^{(2)})^3, \tag{3.19}$$

$$\langle P_sG(u_c^{(2)}), e_{3k,k}\rangle = \frac{3}{2}(x_1^{(2)})^2 x_2^{(2)}, \tag{3.20}$$

$$\langle P_sG(u_c^{(2)}), e_{k,3k}\rangle = -\frac{3}{2}(x_1^{(2)})^2 x_2^{(2)}, \tag{3.21}$$

$$\langle P_sG(u_c^{(2)}), e_{3k,-k}\rangle = \frac{3}{2}x_1^{(2)}(x_2^{(2)})^2, \tag{3.22}$$

$$\langle P_sG(u_c^{(2)}), e_{k,-3k}\rangle = -\frac{3}{2}x_1^{(2)}(x_2^{(2)})^2, \tag{3.23}$$

$$\langle P_sG(u_c^{(2)}), e_{3k,-3k}\rangle = \frac{1}{2}(x_2^{(2)})^3, \tag{3.24}$$

$$\langle P_sG(u_c^{(2)}), e_{m,n}\rangle = 0, (m,n) \in \widetilde{\mathfrak{Z}}, \tag{3.25}$$

where $\widetilde{\mathfrak{Z}} = \mathfrak{Z} - \{(k,k),(k,-k),(3k,3k),(3k,k),(k,3k),(3k,-k),(k,-3k),(3k,-3k)\}$.

Since $u_s^{(2)} \in \widetilde{H}^s$, we set

$$\begin{aligned} u_s^{(2)} &= y_1 e_{3k,3k} + y_2 e_{3k,k} + y_3 e_{k,3k} + y_4 e_{3k,-k} + y_5 e_{k,-3k} + y_6 e_{3k,-3k} \\ &\quad + \sum_{(m,n)\in\widetilde{\mathfrak{Z}}} y_{m,n} e_{m,n}. \end{aligned} \tag{3.26}$$

By projecting (3.5) onto $e_{m,n}$ ($(m,n) \in \{(3k,3k),(3k,k),(k,3k),(3k,-k),$ $(k,-3k),(3k,-3k)\}$), and with (3.19)-(3.25), for $\tau \in [t, t+T]$, we obtain

$$dy_1 = (P(\frac{3\sqrt{2}k\pi}{L})y_1 + \frac{1}{2}(x_1^{(2)})^3)d\tau + \sigma y_1 \circ dW_{\tau-T}, \tag{3.27}$$

$$dy_2 = (P(\frac{\sqrt{10}k\pi}{L})y_2 + \frac{1}{2}(x_1^{(2)})^3)d\tau + \sigma y_2 \circ dW_{\tau-T}, \tag{3.28}$$

$$dy_3 = (P(\frac{\sqrt{10}k\pi}{L})y_3 + \frac{1}{2}(x_1^{(2)})^3)d\tau + \sigma y_3 \circ dW_{\tau-T}, \tag{3.29}$$

$$dy_4 = (P(\frac{\sqrt{10}k\pi}{L})y_4 + \frac{1}{2}(x_1^{(2)})^3)d\tau + \sigma y_4 \circ dW_{\tau-T}, \tag{3.30}$$

$$dy_5 = (P(\frac{\sqrt{10}k\pi}{L})y_5 + \frac{1}{2}(x_1^{(2)})^3)d\tau + \sigma y_5 \circ dW_{\tau-T}, \tag{3.31}$$

$$dy_6 = (P(\frac{3\sqrt{2}k\pi}{L})y_6 + \frac{1}{2}(x_1^{(2)})^3)d\tau + \sigma y_6 \circ dW_{\tau-T}, \tag{3.32}$$

$$dy_{m,n} = P(\frac{\sqrt{m^2+n^2}\pi}{L})y_{m,n}d\tau + \sigma y_{m,n} \circ dW_{\tau-T}, \tag{3.33}$$

where $(m,n) \in \{(3k,3k),(3k,k),(k,3k),(3k,-k),(k,-3k),(3k,-3k)\}$, and

$$u_{m,n}^{(2)}(s, \theta_{-T}\omega)|_{s=t} = 0, \quad (m,n) \in \widetilde{\mathfrak{Z}}. \tag{3.34}$$

It is easy to show that

$$y_{m,n} = 0, \quad (m,n) \in \widetilde{\mathfrak{Z}}. \tag{3.35}$$

Hence, from (3.26) and (3.35), we have

$$
\begin{aligned}
&u_s^{(2)}[\xi_t](s,\theta_{-T}\omega;0)\\
&= y_1[\xi_t](s,\theta_{-T}\omega;0)e_{3k,3k} + y_2[\xi_t](s,\theta_{-T}\omega;0)e_{3k,k} + y_3[\xi_t](s,\theta_{-T}\omega;0)e_{k,3k}\\
&\quad + y_4[\xi_t](s,\theta_{-T}\omega;0)e_{3k,-k} + y_5[\xi_t](s,\theta_{-T}\omega;0)e_{k,-3k}\\
&\quad + y_6[\xi_t](s,\theta_{-T}\omega;0)e_{3k,-3k}.
\end{aligned}
\tag{3.36}
$$

From the form of $u_s^{(2)}$, we now can consider equation (3.2), by writing

$$
\xi(t,\omega) = \xi_1(t,\omega)e_{k,k} + \xi_2(t,\omega)e_{k,-k}.
$$

Notice that

$$
\begin{aligned}
G(\xi + u_s^{(2)}) = -(&\xi_1 e_{k,k} + \xi_2 e_{k,-k} + y_1 e_{3k,3k} + y_2 e_{3k,k} + y_3 e_{k,3k}\\
&+ y_4 e_{3k,-k} + y_5 e_{k,-3k} + y_6 e_{3k,-3k})^3.
\end{aligned}
\tag{3.37}
$$

Hence by projecting (3.2) onto $e_{k,k}$ and $e_{k,-k}$ respectively, and with (3.37), we have

$$
\begin{aligned}
&d\xi_1\\
&= (P(\frac{\sqrt{2}k\pi}{L})\xi_1 - \frac{3}{2}\xi_1^3 - 3\xi_1\xi_2^2 - 3\xi_1 y_1^2 - 3\xi_1 y_2^2 - 3\xi_1 y_3^2 - 3\xi_1 y_4^2 - 3\xi_1 y_5^2\\
&\quad - 3\xi_1 y_6^2 + \frac{3}{2}\xi_1^2 y_1 + \frac{3}{2}\xi_2^2 y_4 - \frac{3}{2}\xi_2^2 y_5 + 3\xi_1\xi_2 y_2 - 3\xi_1\xi_2 y_3 + 3\xi_1 y_4 y_5\\
&\quad - 3\xi_2 y_1 y_2 + 3\xi_2 y_1 y_3 - 3\xi_2 y_2 y_4 - 3\xi_2 y_3 y_5 - 3\xi_2 y_4 y_6 + 3\xi_2 y_5 y_6\\
&\quad - 3y_1 y_2 y_3 - 3y_1 y_4 y_5 - 3y_2 y_3 y_4 + 3y_2 y_3 y_5 - 3y_2 y_5 y_6 - 3y_3 y_4 y_6)dt\\
&\quad + \sigma\xi_1 \circ dW_t,
\end{aligned}
\tag{3.38}
$$

$$
\begin{aligned}
d\xi_2 = (&P(\frac{\sqrt{2}k\pi}{L})\xi_2 - \frac{3}{2}\xi_2^3 - 3\xi_1^2\xi_2 - 3\xi_2 y_1^2 - 3\xi_2 y_2^2 - 3\xi_2 y_3^2 - 3\xi_2 y_4^2\\
&- 3\xi_2 y_5^2 - 3\xi_2 y_6^2 + \frac{3}{2}\xi_1^2 y_2 - \frac{3}{2}\xi_1^2 y_3 + \frac{3}{2}\xi_2^2 y_6 + 3\xi_1\xi_2 y_4 - 3\xi_1\xi_2 y_5\\
&- 3\xi_1 y_1 y_2 + 3\xi_1 y_1 y_3 - 3\xi_1 y_2 y_4 - 3\xi_1 y_3 y_5\\
&- 3\xi_1 y_4 y_6 + 3\xi_1 y_5 y_6 + 3\xi_2 y_2 y_3 - 3y_1 y_2 y_5 - 3y_1 y_3 y_4\\
&- 3y_2 y_3 y_6 - 3y_2 y_4 y_5 + 3y_3 y_4 y_5 - 3y_4 y_5 y_6)dt + \sigma\xi_2 \circ dW_t.
\end{aligned}
\tag{3.39}
$$

To extract information from the PM reduction, we show that the asymptotic behavior can be completely captured by a sufficiently small neighborhood of the origin. Using the Ito formula, similar as the proposition 1 in [17], we have the following result.

Proposition 3.1. *For L near $\sqrt{2}kL_1$ and $L > \sqrt{2}kL_1$, there exists a random closed ball $B_L(\omega)$ such that for every bounded set $B \subset H$ and almost every ω, there exists a time $T_B(\omega) > 0$ so that*

$$
\phi(t,\theta_{-t}\omega)B \subset B_L(\omega), \quad \forall t > T_B(\omega).
$$

It is known from Chekroun et al [6] that the center manifold reduction for the system (1.3) holds in deterministic neighborhood of the origin. By considering only those ω such that $B_L(\omega)$ is small enough so that the center manifold reduction holds, we expect that the reduced system (3.38)-(3.39) has a good description of the dynamics of (1.3), then we turn to the study of the reduced systems (3.38)-(3.39). In the study of dynamic transition near $\sqrt{2}kL_1$, higher-order terms can be dropped, and we can consider the reduced model, up to the leading order.

Now we give the solutions of some above-mentioned equations. From (3.8)-(3.11), we have

$$\begin{aligned} x_1^{(1)}(\tau,\omega) &= \xi_1(t,\omega)e^{P(\frac{\sqrt{2}k\pi}{L})(\tau-t)+\sigma(W_\tau(\omega)-W_t(\omega))}, \quad \tau\in[t-T,t],\\ x_2^{(1)}(\tau,\omega) &= \xi_2(t,\omega)e^{P(\frac{\sqrt{2}k\pi}{L})(\tau-t)+\sigma(W_\tau(\omega)-W_t(\omega))}, \quad \tau\in[t-T,t]. \end{aligned}$$

From (3.14)-(3.17), we have

$$\begin{aligned} &x_1^{(2)}(\tau,\omega)\\ &= x_1^{(1)}(\tau,\omega) - \int_\tau^t e^{P(\frac{\sqrt{2}k\pi}{L})(\tau-\rho)+\sigma(W_\tau(\omega)-W_\rho(\omega))}[\frac{3}{2}(x_1^{(1)})^3+3x_1^{(1)}(x_2^{(1)})^2]d\rho,\\ &x_2^{(2)}(\tau,\omega)\\ &= x_2^{(1)}(\tau,\omega) - \int_\tau^t e^{P(\frac{\sqrt{2}k\pi}{L})(\tau-\rho)+\sigma(W_\tau(\omega)-W_\rho(\omega))}[3(x_1^{(1)})^2x_2^{(1)}+\frac{3}{2}(x_2^{(1)})^3]d\rho. \end{aligned}$$

From (3.27)-(3.34), we have

$$\begin{aligned} &y_1(\tau,\theta_{-T}\omega) = \frac12\int_t^\tau e^{P(\frac{3\sqrt{2}k\pi}{L})(\tau-\rho)+\sigma(W_{\tau-T}(\omega)-W_{\rho-T}(\omega))}(x_1^{(2)}(\rho-T,\omega))^3d\rho,\\ &y_2(\tau,\theta_{-T}\omega)\\ &= \frac32\int_t^\tau e^{P(\frac{\sqrt{10}k\pi}{L})(\tau-\rho)+\sigma(W_{\tau-T}(\omega)-W_{\rho-T}(\omega))}(x_1^{(2)}(\rho-T,\omega))^2x_2^{(2)}(\rho-T,\omega)d\rho,\\ &y_3(\tau,\theta_{-T}\omega)\\ &= -\frac32\int_t^\tau e^{P(\frac{\sqrt{10}k\pi}{L})(\tau-\rho)+\sigma(W_{\tau-T}(\omega)-W_{\rho-T}(\omega))}(x_1^{(2)}(\rho-T,\omega))^2x_2^{(2)}(\rho-T,\omega)d\rho,\\ &y_4(\tau,\theta_{-T}\omega)\\ &= \frac32\int_t^\tau e^{P(\frac{\sqrt{10}k\pi}{L})(\tau-\rho)+\sigma(W_{\tau-T}(\omega)-W_{\rho-T}(\omega))}x_1^{(2)}(\rho-T,\omega)(x_2^{(2)}(\rho-T,\omega))^2d\rho,\\ &y_5(\tau,\theta_{-T}\omega)\\ &= -\frac32\int_t^\tau e^{P(\frac{\sqrt{10}k\pi}{L})(\tau-\rho)+\sigma(W_{\tau-T}(\omega)-W_{\rho-T}(\omega))}x_1^{(2)}(\rho-T,\omega)(x_2^{(2)}(\rho-T,\omega))^2d\rho,\\ &y_6(\tau,\theta_{-T}\omega) = \frac12\int_t^\tau e^{P(\frac{3\sqrt{2}k\pi}{L})(\tau-\rho)+\sigma(W_{\tau-T}(\omega)-W_{\rho-T}(\omega))}(x_2^{(2)}(\rho-T,\omega))^3d\rho. \end{aligned}$$

From computations, we observe that

$$u_s^{(2)}[\xi_t](s,\theta_{-T}\omega) = O(\|\,(\xi_1,\xi_2)\,\|^3), \tag{3.40}$$

so for the reduced equation (3.38)-(3.39), we obtain the leading order equations

$$d\xi_1 = (P(\frac{\sqrt{2}k\pi}{L})\xi_1 - \frac32\xi_1^3 - 3\xi_1\xi_2^2)dt + \sigma\xi_1\circ dW_t, \tag{3.41}$$

$$d\xi_2 = (P(\frac{\sqrt{2}k\pi}{L})\xi_2 - \frac32\xi_2^3 - 3\xi_1^2\xi_2)dt + \sigma\xi_2\circ dW_t. \tag{3.42}$$

For L near $\sqrt{2}kL_1$ and $L > \sqrt{2}kL_1$, this system has the following 8 nontrivial solutions:

$$
\begin{gathered}
S_L^{1,2}(\theta_t\omega) = (\pm\sqrt{\frac{1}{3}}a_L(\theta_t\omega), 0),\\
S_L^{3,4}(\theta_t\omega) = (0, \pm\sqrt{\frac{1}{3}}a_L(\theta_t\omega)),\\
S_L^{5}(\theta_t\omega) = (\frac{1}{3}a_L(\theta_t\omega), \frac{1}{3}a_L(\theta_t\omega)),\\
S_L^{6}(\theta_t\omega) = (-\frac{1}{3}a_L(\theta_t\omega), -\frac{1}{3}a_L(\theta_t\omega)),\\
S_L^{7}(\theta_t\omega) = (\frac{1}{3}a_L(\theta_t\omega), -\frac{1}{3}a_L(\theta_t\omega)),\\
S_L^{8}(\theta_t\omega) = (-\frac{1}{3}a_L(\theta_t\omega), \frac{1}{3}a_L(\theta_t\omega)),
\end{gathered}
$$

where

$$
a_L(\omega) = (\int_{-\infty}^{0} e^{2P(\frac{\sqrt{2}k\pi}{L})\tau + 2\sigma W_\tau(\omega)} d\tau)^{-1/2}.
$$

For the system, we have the following theorems, whose proof is essentially the same as in [6].

Theorem 3.2. *The reduced system* (3.41)-(3.42) *undergoes a stochastic supercritical bifurcation at $L = \sqrt{2}kL_1$ in the pullback sense. More precisely, when L near $\sqrt{2}kL_1$,*

(a) *For $L < \sqrt{2}kL_1$, the origin is globally asymptotically stable.*

(b) *For $L > \sqrt{2}kL_1$, the random compact set*

$$
A_L(\omega) = \{S_L^i(\omega),\ i = 1, 2, \dots, 8\}
$$

is a random pullback attractor.

As in [17], let ψ denote the flow associated with the system, and $(\xi_1(t), \xi_2(t)) = \psi(t, \theta_{-t}\omega)(\xi_1^0, \xi_2^0)$ be a solution of the SDE with initial condition (ξ_1^0, ξ_2^0). Using standard arguments in ODE given by [17], we have

$$
\lim_{t\to+\infty} \psi(t, \theta_{-t}\omega)(\xi_1^0, \xi_2^0) = \begin{cases} (\pm\sqrt{\frac{1}{3}}a_L(\omega), 0), & \text{if } \xi_1^0 > \xi_2^0, \\ (\frac{1}{3}a_L(\omega), \frac{1}{3}a_L(\omega), & \text{if } \xi_1^0 = \xi_2^0. \end{cases}
$$

The other cases can be obtained in a similar fashion. In a similar manner, we have the following results.

Theorem 3.3. *The reduced system* (3.41)-(3.42) *undergoes a stochastic subcritical bifurcation at $L = \sqrt{2}kL_2$ in the pullback sense. More precisely, when L near $\sqrt{2}kL_2$,*

(a) *For $L < \sqrt{2}kL_2$, the random compact set $A_L(\omega)$ is a random pullback attractor.*

(b) *For $L > \sqrt{2}kL_2$, the origin is globally asymptotically stable.*

Remark 3.4. We can conclude that if α is small enough, the equation undergoes stochastic supercritical bifurcation at $L = \sqrt{2}kL_1$, and stochastic subcritical bifurcation at $L = \sqrt{2}kL_2$.

Now we consider the stochastic Swift-Hohenberg equation with multiplicative noise in Ito sense. As for the stochastic Swift-Hohenberg equation with multiplicative noise in Stratonovich sense, we can obtain the reduced equation, up to the leading order,

$$d\xi_1 = (P(\frac{\sqrt{2}k\pi}{L})\xi_1 - \frac{3}{2}\xi_1^3 - 3\xi_1\xi_2^2)dt + \sigma\xi_1 dW_t, \tag{3.43}$$

$$d\xi_2 = (P(\frac{\sqrt{2}k\pi}{L})\xi_2 - \frac{3}{2}\xi_2^3 - 3\xi_1^2\xi_2)dt + \sigma\xi_2 dW_t. \tag{3.44}$$

For L near $\sqrt{2}kL_1$ and $L > \sqrt{2}kL_1$, this system has the following 8 nontrivial solutions:

$$\begin{gathered}
\tilde{S}_L^{1,2}(\theta_t\omega) = (\pm\sqrt{\frac{1}{3}}\tilde{a}_L(\theta_t\omega), 0), \\
\tilde{S}_L^{3,4}(\theta_t\omega) = (0, \pm\sqrt{\frac{1}{3}}\tilde{a}_L(\theta_t\omega)), \\
\tilde{S}_L^{5}(\theta_t\omega) = (\frac{1}{3}\tilde{a}_L(\theta_t\omega), \frac{1}{3}\tilde{a}_L(\theta_t\omega)), \\
\tilde{S}_L^{6}(\theta_t\omega) = (-\frac{1}{3}\tilde{a}_L(\theta_t\omega), -\frac{1}{3}\tilde{a}_L(\theta_t\omega)), \\
\tilde{S}_L^{7}(\theta_t\omega) = (\frac{1}{3}\tilde{a}_L(\theta_t\omega), -\frac{1}{3}\tilde{a}_L(\theta_t\omega)), \\
\tilde{S}_L^{8}(\theta_t\omega) = (-\frac{1}{3}\tilde{a}_L(\theta_t\omega), \frac{1}{3}\tilde{a}_L(\theta_t\omega)),
\end{gathered}$$

where

$$\tilde{a}_L(\omega) = \Big(\int_{-\infty}^0 e^{2(P(\frac{\sqrt{2}k\pi}{L}) - \frac{\sigma^2}{2})\tau + 2\sigma W_\tau(\omega)} d\tau\Big)^{-1/2}. \tag{3.45}$$

Theorem 3.5. *When L is near $\sqrt{2}kL_1$, for the reduced system* (3.43)-(3.44)*, we have:*

(a) *If $L < \sqrt{2}kL_1$, the origin is globally asymptotically stable.*
(b) *If $L > \sqrt{2}kL_1$, and $P(\frac{\sqrt{2}k\pi}{L}) < \frac{\sigma^2}{2}$, the origin is globally asymptotically stable.*
(c) *If $L > \sqrt{2}kL_1$, and $P(\frac{\sqrt{2}k\pi}{L}) > \frac{\sigma^2}{2}$, then the random compact set*
$$\tilde{A}_L(\omega) = \{\tilde{S}_L^i(\omega), i = 1, 2, \dots, 8\}$$
is a random pullback attractor.

In a similar manner, we have the following results.

Theorem 3.6. *For the reduced system* (3.43)-(3.44)*, we have:*

(a) *If $L < \sqrt{2}kL_2$, $P(\frac{\sqrt{2}k\pi}{L}) > \frac{\sigma^2}{2}$, then the random compact set $\tilde{A}_L(\omega)$ is a random pullback attractor.*
(b) *If $L < \sqrt{2}kL_2$, and $P(\frac{\sqrt{2}k\pi}{L}) < \frac{\sigma^2}{2}$, then the origin is globally asymptotically stable.*
(c) *If $L > \sqrt{2}kL_2$, then the origin is globally asymptotically stable.*

Remark 3.7. For the deterministic Swift-Hohenberg equation, we can conclude that the equation undergoes a supercritical bifurcation at $L = \sqrt{2}kL_1$ and subcritical bifurcation at $L = \sqrt{2}kL_2$. However, for the stochastic Swift-Hohenberg equation with multiplicative noise in Ito sense, we have the above theorem, that is to say, the noise may destroy the bifurcation, the size of parameter interval may be shortened.

From Theorems 3.2, 3.3, 3.5, and 3.6, we see the impact of noise in Stratonovich sense and Ito sense on the stochastic dynamics respectively, the multiplicative noise may destroy or induce bifurcations for different stochastic systems.

4. Analysis of the case $(m,n)=(k,0)$ and $(0,k)$

In this section, we consider the attractor bifurcation near the points $\sqrt{m^2+n^2}L_1$ and $\sqrt{m^2+n^2}L_2$ in the case the intervals $I_{m,n}$ do not overlap, and $K=m^2+n^2$ has only two solutions $(m,n)=(k,0)$ and $(0,k)$; that is when the attractor bifurcates near the points kL_1 and kL_2.

Notice that the space H_1 and H can be decomposed into

$$H_1=H_2^c\oplus H_2^s,\quad H=H_2^c\oplus \widetilde{H}^s,$$

where $H_2^c=\operatorname{span}\{e_{k,0},e_{0,k}\}$ and $\widetilde{H}^s$ is the closure of H_2^s in H.

We will present a stochastic reduction procedure based on parameterizing manifolds (PM) associated with (1.3). A stochastic parameterizing manifolds ([6, 7]), as the graph of a random continuous function $\widetilde{h}_\alpha(\xi,\omega)$ from H_1^c to $\widetilde{H}^s$, and for each realization ω, the function is defined for $\xi\in H_1^c$.

Projecting equation (2.4) onto the subspace H_1^c, we obtain

$$d\widetilde{u}_c=(\widetilde{\mathcal{L}}_\alpha^c\widetilde{u}_c+\widetilde{\mathcal{P}}_cG(\widetilde{u}_c+\widetilde{u}_s))dt+\sigma\widetilde{u}_c\circ dW_t,$$

where $\widetilde{u}_s=\widetilde{\mathcal{P}}_su$ is the unresolved variable, and $\widetilde{\mathcal{P}}_s$, $\widetilde{\mathcal{P}}_c$ are respectively canonical projections from H to H_1^c and $\widetilde{H}^s$. To obtain a closed form of the above equation, the unresolved variables $\widetilde{u}_s$ is parameterized in terms of the resolved variables $\widetilde{u}_c$ through a random continuous function $\widetilde{h}_\alpha(\xi,\omega):H_1^c\times\Omega\to\widetilde{H}^s$.

The PM-based reduced equation for the resolved modes is

$$d\xi=(\widetilde{\mathcal{L}}_\alpha^c\xi+\widetilde{\mathcal{P}}_cG(\xi+\widetilde{h}_\alpha(\xi,\theta_t\omega)))dt+\sigma\xi\circ dW_t, \tag{4.1}$$

where $\xi\in H_1^c$.

Instead of (4.1), we consider the reduced equation

$$\begin{gathered}d\xi_t=(\widetilde{\mathcal{L}}_\alpha^c\xi_t+\widetilde{\mathcal{P}}_cG(\xi_t+\widetilde{u}_s^{(2)}[\xi(t,\omega)](t+T,\theta_{-T}\omega;0)))dt+\sigma\xi_t\circ dW_t,\\ \xi(0,\omega)=\phi,t>0,\end{gathered} \tag{4.2}$$

where the notation ξ_t emphasized the t-dependence of the variable ξ_t, $\phi=\widetilde{\mathcal{P}}_cu_0$, and $\widetilde{u}_s^{(2)}$ can be used to approximate the stochastic inertial manifold and is obtained from the following backward-forward systems (4.3)-(4.5).

We now consider approximation representation for stochastic parameterizing manifold as pullback limits of backward-forward systems. For a given $t>0$ and T sufficiently large,

$$d\widetilde{u}_c^{(1)}=\widetilde{\mathcal{L}}_\alpha^c\widetilde{u}_c^{(1)}ds+\sigma\widetilde{u}_c^{(1)}\circ dW_\tau,\quad \tau\in[t-T,t], \tag{4.3}$$

$$d\widetilde{u}_c^{(2)}=(\widetilde{\mathcal{L}}_\alpha^c\widetilde{u}_c^{(2)}+\widetilde{\mathcal{P}}_cG(\widetilde{u}_c^{(1)}(\tau-T,\omega)))d\tau+\sigma\widetilde{u}_c^{(2)}\circ dW_{\tau-T},\quad \tau\in[t,t+T], \tag{4.4}$$

$$d\widetilde{u}_s^{(2)}=(\widetilde{\mathcal{L}}_\alpha^s\widetilde{u}_s^{(2)}+\widetilde{\mathcal{P}}_sG(\widetilde{u}_c^{(2)}(\tau-T,\omega)))d\tau+\sigma\widetilde{u}_s^{(2)}\circ dW_{\tau-T},\quad \tau\in[t,t+T]. \tag{4.5}$$

with

$$\widetilde{u}_c^{(1)}(\tau,\omega)\big|_{\tau=t}=\xi(t,\omega),$$
$$\widetilde{u}_c^{(2)}(\tau,\omega)\big|_{\tau=t}=\xi(t,\omega),$$
$$\widetilde{u}_s^{(2)}(\tau,\theta_{-T}\omega)\big|_{\tau=t}=0,$$

where $\widetilde{\mathcal{L}}_\alpha^c=\widetilde{\mathcal{P}}_c\mathcal{L}_\alpha$ and $\widetilde{\mathcal{L}}_\alpha^s=\widetilde{\mathcal{P}}_s\mathcal{L}_\alpha$.

Since $\widetilde{u}_c^1,\widetilde{u}_c^2\in H_2^c$, we write

$$\widetilde{u}_c^{(1)}(\tau,\omega)=\widetilde{x}_1^{(1)}(\tau,\omega)e_{k,0}+\widetilde{x}_2^{(1)}(\tau,\omega)e_{0,k}, \tag{4.6}$$
$$\widetilde{u}_c^{(2)}(\tau,\omega)=\widetilde{x}_1^{(2)}(\tau,\omega)e_{k,0}+\widetilde{x}_2^{(2)}(\tau,\omega)e_{0,k}, \tag{4.7}$$
$$\xi(\tau,\omega)=\widetilde{\xi}_1(\tau,\omega)e_{k,0}+\widetilde{\xi}_1(\tau,\omega)e_{0,k}. \tag{4.8}$$

As in section 3, we obtain the reduced model, up to the leading order, which has a good description of the dynamics of (1.3). By computations, the leading order reduced equation is given below so for the reduced equation (4.3)-(4.5), we obtain the leading order equations

$$d\widetilde{\xi}_1=(P(\frac{k\pi}{L})\widetilde{\xi}_1-\frac{3}{2}\widetilde{\xi}_1^3-3\widetilde{\xi}_1\widetilde{\xi}_2^2)dt+\sigma\widetilde{\xi}_1\circ dW_t, \tag{4.9}$$
$$d\widetilde{\xi}_2=(P(\frac{k\pi}{L})\widetilde{\xi}_2-\frac{3}{2}\widetilde{\xi}_2^3-3\widetilde{\xi}_1^2\widetilde{\xi}_2)dt+\sigma\widetilde{\xi}_2\circ dW_t. \tag{4.10}$$

When L is near kL_1 and $L>kL_1$, this system has the following 8 nontrivial solutions:

$$\widetilde{S}_L^{1,2}(\theta_t\omega)=(\pm\sqrt{\frac{1}{3}}\widetilde{a}_L(\theta_t\omega),0),$$
$$\widetilde{S}_L^{3,4}(\theta_t\omega)=(0,\pm\sqrt{\frac{1}{3}}\widetilde{a}_L(\theta_t\omega)),$$
$$\widetilde{S}_L^{5}(\theta_t\omega)=(\frac{1}{3}\widetilde{a}_L(\theta_t\omega),\frac{1}{3}\widetilde{a}_L(\theta_t\omega)),$$
$$\widetilde{S}_L^{6}(\theta_t\omega)=(-\frac{1}{3}\widetilde{a}_L(\theta_t\omega),-\frac{1}{3}\widetilde{a}_L(\theta_t\omega)),$$
$$\widetilde{S}_L^{7}(\theta_t\omega)=(\frac{1}{3}\widetilde{a}_L(\theta_t\omega),-\frac{1}{3}\widetilde{a}_L(\theta_t\omega)),$$
$$\widetilde{S}_L^{8}(\theta_t\omega)=(-\frac{1}{3}\widetilde{a}_L(\theta_t\omega),\frac{1}{3}\widetilde{a}_L(\theta_t\omega)),$$

where

$$\widetilde{a}_L(\omega)=\Big(\int_{-\infty}^0 e^{2P(\frac{k\pi}{L})\tau+2\sigma W_\tau(\omega)}d\tau\Big)^{-1/2}.$$

For the system, we have the following theorems.

Theorem 4.1. *The reduced system* (4.9)-(4.10) *undergoes a stochastic supercritical bifurcation at $L=kL_1$ in the pullback sense. More precisely, for L near kL_1,*

(a) *If $L<kL_1$, the origin is globally asymptotically stable.*
(b) *If $L>kL_1$, the random compact set*

$$\widetilde{A}_L(\omega)=\{\widetilde{S}_L^i(\omega),\ i=1,2,\dots,8\}$$

is a random pullback attractor.

Theorem 4.2. *The reduced system* (4.9)-(4.10) *undergoes a stochastic subcritical bifurcation at $L = kL_2$ in the pullback sense. More precisely, for L near kL_2,*

(a) *If $L < kL_2$, the random compact set $\widetilde{A}_L(\omega)$ is a random pullback attractor.*
(b) *If $L > kL_2$, the origin is globally asymptotically stable.*

Now we consider the stochastic Swift-Hohenberg equation with multiplicative noise in Ito sense. As the case for the stochastic Swift-Hohenberg equation with multiplicative noise in Stratonovich sense, we can get the reduced equation, up to the leading order,

$$d\widetilde{\xi}_1 = (P(\frac{k\pi}{L})\widetilde{\xi}_1 - \frac{3}{2}\widetilde{\xi}_1^3 - 3\widetilde{\xi}_1\widetilde{\xi}_2^2)dt + \sigma\widetilde{\xi}_1 dW_t, \tag{4.11}$$

$$d\widetilde{\xi}_2 = (P(\frac{k\pi}{L})\widetilde{\xi}_2 - \frac{3}{2}\widetilde{\xi}_2^3 - 3\widetilde{\xi}_1^2\widetilde{\xi}_2)dt + \sigma\widetilde{\xi}_2 dW_t. \tag{4.12}$$

When L is near kL_1 and $L > kL_1$, this system has the following 8 nontrivial solutions:

$$\begin{gathered}
\mathfrak{S}_L^{1,2}(\theta_t\omega) = (\pm\sqrt{\frac{1}{3}}\mathfrak{a}_L(\theta_t\omega), 0), \\
\mathfrak{S}_L^{3,4}(\theta_t\omega) = (0, \pm\sqrt{\frac{1}{3}}\mathfrak{a}_L(\theta_t\omega)), \\
\mathfrak{S}_L^{5}(\theta_t\omega) = (\frac{1}{3}\mathfrak{a}_L(\theta_t\omega), \frac{1}{3}\mathfrak{a}_L(\theta_t\omega)), \\
\mathfrak{S}_L^{6}(\theta_t\omega) = (-\frac{1}{3}\mathfrak{a}_L(\theta_t\omega), -\frac{1}{3}\mathfrak{a}_L(\theta_t\omega)), \\
\mathfrak{S}_L^{7}(\theta_t\omega) = (\frac{1}{3}\mathfrak{a}_L(\theta_t\omega), -\frac{1}{3}\mathfrak{a}_L(\theta_t\omega)), \\
\mathfrak{S}_L^{8}(\theta_t\omega) = (-\frac{1}{3}\mathfrak{a}_L(\theta_t\omega), \frac{1}{3}\mathfrak{a}_L(\theta_t\omega)).
\end{gathered}$$

where

$$\mathfrak{a}_L(\omega) = \Big(\int_{-\infty}^0 e^{2(P(\frac{k\pi}{L}) - \frac{\sigma^2}{2})\tau + 2\sigma W_\tau(\omega)} d\tau\Big)^{-1/2}.$$

We have the following results.

Theorem 4.3. *For L near kL_1, then for the reduced system* (4.11)-(4.12)*, we have:*

(a) *If $L < kL_1$, the origin is globally asymptotically stable.*
(b) *If $L > kL_1$, and $P(\frac{k\pi}{L}) < \frac{\sigma^2}{2}$, the origin is globally asymptotically stable.*
(c) *If $L > kL_1$, and $P(\frac{k\pi}{L}) > \frac{\sigma^2}{2}$, then the random compact set*

$$\mathfrak{A}_L(\omega) = \{\mathfrak{S}_L^i(\omega), i = 1, 2, \dots, 8.\}.$$

is a random pullback attractor.

Theorem 4.4. *For L near kL_2, then for the reduced system* (4.11)-(4.12)*, we have:*

(a) *If $L < kL_2$, $P(\frac{k\pi}{L}) > \frac{\sigma^2}{2}$, then the random compact set $\mathfrak{A}_L(\omega)$ is a random pullback attractor.*
(b) *If $L < kL_2$, and $P(\frac{k\pi}{L}) < \frac{\sigma^2}{2}$, then the origin is globally asymptotically stable.*
(c) *If $L > kL_2$, then the origin is globally asymptotically stable.*

Remark 4.5. For the deterministic Swift-Hohenberg equation, we can conclude that the equation undergoes a supercritical bifurcation at $L = kL_1$ and subcritical bifurcation at $L = kL_2$. However, for the stochastic Swift-Hohenberg equation with multiplicative noise in Ito sense, we have the above theorem; that is to say, the noise may destroy the bifurcation, the size of parameter interval may be shortened. We may expect that the noise may induce bifurcation for some systems.

5. Analysis of the case $(m, \pm n)$ and $(n, \pm m)$

In this section, we consider the attractor bifurcation near the points $\sqrt{m^2+n^2}L_1$ and $\sqrt{m^2+n^2}L_2$ in the case the intervals $I_{m,n}$ do not overlap, and for fixed K. $K = m^2+n^2$ has only four solutions $(m, \pm n)$ and $(n, \pm m)$ in $\mathfrak{Z}$, e.g. $5 = m^2+n^2$ has four solutions $(1, \pm 2)$ and $(2, \pm 1)$ in $\mathfrak{Z}$. Here $m \neq n$, $mn \neq 0$, as these two cases have been discussed in the previous two sections.

In this case, the space H_1 and H can be decomposed into

$$H_1 = H_3^c \oplus H_3^s, \quad H = H_3^c \oplus H^s,$$

where

$$H_3^c = \operatorname{span}\{e_{m,n}, e_{m,-n}, e_{n,m}, e_{n,-m}\}.$$

Projecting equation (2.4) onto the subspace H_3^c, we obtain

$$dv_c = (\mathcal{L}_\alpha^c v_c + \mathcal{P}_c G(v_c + u_s))dt + \sigma u_c \circ dW_t,$$

where $v_c = \mathcal{P}_c u$, and $v_s = \mathcal{P}_s u$ is the unresolved variable, and $\mathcal{P}_c$, $\mathcal{P}_s$ are respectively canonical projections from H to H_3^c and H^s. To obtain a closed form of the above equation, the unresolved variables u_s is parameterized in terms of the resolved variables v_c through a random continuous function $h_\alpha(\xi,\omega) : H_3^c \times \Omega \to H^s$.

The PM-based reduced equation for the resolved modes is

$$d\xi = (\mathcal{L}_\alpha^c \xi + \mathcal{P}_c G(\xi + h_\alpha(\xi, \theta_t\omega)))dt + \sigma\xi \circ dW_t,$$

where $\xi \in H_3^c$. Using an approximation of $h_\alpha(\xi,\omega)$ via the pullback characterization as in section 3, we instead consider the reduced equation

$$\begin{gathered} d\xi_t = (\mathcal{L}_\alpha^c \xi_t + \mathcal{P}_c G(\xi_t + v_s^{(4)}[\xi(t,\omega)](t+T, \theta_{-T}\omega; 0)))dt + \sigma\xi_t \circ dW_t, \\ \xi(0,\omega) = \varphi, t > 0, \end{gathered} \tag{5.1}$$

where T is sufficiently large, $\varphi = \mathcal{P}_c u_0$, $v_s^{(4)}$ is used to approximate the stochastic inertial manifold and is obtained from the following backward-forward systems (5.2)-(5.6).

For a given $t > 0$ and T sufficiently large, let us consider the 4-layer auxiliary backward-forward system

$$dv_c^{(1)} = \mathcal{L}_\alpha^c v_c^{(1)} d\tau + \sigma v_c^{(1)} \circ dW_\tau, \quad \tau \in [t-T, t], \tag{5.2}$$

$$dv_c^{(2)} = (\mathcal{L}_\alpha^c v_c^{(2)} + \mathcal{P}_c G(v_c^{(1)})d\tau + \sigma v_c^{(2)} \circ dW_\tau, \quad \tau \in [t-T, t], \tag{5.3}$$

$$dv_c^{(3)} = (\mathcal{L}_\alpha^c v_c^{(4)} + \mathcal{P}_c G(v_c^{(2)})d\tau + \sigma v_c^{(3)} \circ dW_\tau, \quad \tau \in [t-T, t], \tag{5.4}$$

$$dv_c^{(4)} = (\mathcal{L}_\alpha^c v_c^{(4)} + \mathcal{P}_c G(v_c^{(3)})d\tau + \sigma v_c^{(4)} \circ dW_\tau, \quad \tau \in [t-T, t], \tag{5.5}$$

$$dv_s^{(4)} = (\mathcal{L}_\alpha^s v_s^{(4)} + \mathcal{P}_c G(v_c^{(4)}(\tau - T, \omega)))d\tau + \sigma v_s^{(4)} \circ dW_{\tau - T}, \quad \tau \in [t, t+T]. \tag{5.6}$$

with

$$v_c^{(1)}(\tau,\omega)\big|_{\tau=t} = \xi(t,\omega),$$

$$
\begin{aligned}
v_c^{(2)}(\tau,\omega)\big|_{\tau=t} &= \xi(t,\omega),\\
v_c^{(3)}(\tau,\omega)\big|_{\tau=t} &= \xi(t,\omega),\\
v_c^{(4)}(\tau,\omega)\big|_{\tau=t} &= \xi(t,\omega),\\
v_s^{(4)}(\tau,\theta_{-T}\omega)\big|_{\tau=t} &= 0.
\end{aligned}
$$

Since $\xi \in H_3^c$, we write

$$
\begin{aligned}
\xi(\tau,\omega) = \xi_1(\tau,\omega)e_{m,n} + \xi_2(\tau,\omega)e_{m,-n} + \xi_3(\tau,\omega)e_{n,m}\\
+ \xi_4(\tau,\omega)e_{n,-m}.
\end{aligned} \tag{5.7}
$$

As in section 3, by computations, we obtain the reduced model on the PM, up to the leading order,

$$
d\xi_1 = (P(\frac{\sqrt{m^2+n^2}\pi}{L})\xi_1 - \frac{3}{2}\xi_1^3 - 3\xi_1\xi_2^2 - 3\xi_1\xi_3^2 - 3\xi_1\xi_4^2)dt + \sigma\xi_1 \circ dW_t, \tag{5.8}
$$

$$
d\xi_2 = (P(\frac{\sqrt{m^2+n^2}\pi}{L})\xi_2 - \frac{3}{2}\xi_2^3 - 3\xi_1^2\xi_2 - 3\xi_2\xi_3^2 - 3\xi_2\xi_4^2)dt + \sigma\xi_2 \circ dW_t, \tag{5.9}
$$

$$
d\xi_3 = (P(\frac{\sqrt{m^2+n^2}\pi}{L})\xi_3 - \frac{3}{2}\xi_3^3 - 3\xi_1^2\xi_3 - 3\xi_2^2\xi_3 - 3\xi_3\xi_4^2)dt + \sigma\xi_3 \circ dW_t, \tag{5.10}
$$

$$
d\xi_4 = (P(\frac{\sqrt{m^2+n^2}\pi}{L})\xi_4 - \frac{3}{2}\xi_4^3 - 3\xi_1^2\xi_4 - 3\xi_2^2\xi_4 - 3\xi_3^2\xi_4)dt + \sigma\xi_4 \circ dW_t. \tag{5.11}
$$

For L near $\sqrt{m^2+n^2}L_1$ and $L > \sqrt{m^2+n^2}L_1$, this system has the following 80 nontrivial solutions:

$$
\begin{aligned}
\mathcal{S}_L^{1,2}(\theta_t\omega) &= (\pm\sqrt{\frac{1}{3}}\mathrm{a}_L(\theta_t\omega), 0, 0, 0),\\
\mathcal{S}_L^{3,4}(\theta_t\omega) &= (0, \pm\sqrt{\frac{1}{3}}\mathrm{a}_L(\theta_t\omega), 0, 0),\\
\mathcal{S}_L^{5,6}(\theta_t\omega) &= (0, 0, \pm\sqrt{\frac{1}{3}}\mathrm{a}_L(\theta_t\omega), 0),\\
\mathcal{S}_L^{7,8}(\theta_t\omega) &= (0, 0, 0, \pm\sqrt{\frac{1}{3}}\mathrm{a}_L(\theta_t\omega)),\\
\mathcal{S}_L^{9,10}(\theta_t\omega) &= (\frac{1}{3}\mathrm{a}_L(\theta_t\omega), \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0, 0),\\
\mathcal{S}_L^{11,12}(\theta_t\omega) &= (-\frac{1}{3}\mathrm{a}_L(\theta_t\omega), \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0, 0),\\
\mathcal{S}_L^{13,14}(\theta_t\omega) &= (\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0, \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0),\\
\mathcal{S}_L^{15,16}(\theta_t\omega) &= (-\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0, \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0),\\
\mathcal{S}_L^{17,18}(\theta_t\omega) &= (\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0, 0, \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega)),\\
\mathcal{S}_L^{19,20}(\theta_t\omega) &= (-\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0, 0, \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega)),\\
\mathcal{S}_L^{21,22}(\theta_t\omega) &= (0, \frac{1}{3}\mathrm{a}_L(\theta_t\omega), \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0),\\
\mathcal{S}_L^{23,24}(\theta_t\omega) &= (0, -\frac{1}{3}\mathrm{a}_L(\theta_t\omega), \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0),
\end{aligned}
$$

$$
\begin{gathered}
\mathcal{S}_L^{25,26}(\theta_t\omega) = (0, \frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0, \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{27,28}(\theta_t\omega) = (0, -\frac{1}{3}\mathrm{a}_L(\theta_t\omega), 0, \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{29,30}(\theta_t\omega) = (0, 0, \frac{1}{3}\mathrm{a}_L(\theta_t\omega), \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{31,32}(\theta_t\omega) = (0, 0, -\frac{1}{3}\mathrm{a}_L(\theta_t\omega), \pm\frac{1}{3}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{33,34}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0), \\
\mathcal{S}_L^{35,36}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0), \\
\mathcal{S}_L^{37,38}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0), \\
\mathcal{S}_L^{39,40}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0), \\
\mathcal{S}_L^{41,42}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0, \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{43,44}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0, \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{45,46}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0, -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{47,48}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0, -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{49,50}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0, \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{51,52}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0, -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{53,54}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0, \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{55,56}(\theta_t\omega) = (\pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), 0, -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{57,58}(\theta_t\omega) = (0, \pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{59,60}(\theta_t\omega) = (0, \pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{61,62}(\theta_t\omega) = (0, \pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{63,64}(\theta_t\omega) = (0, \pm\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{15}}\mathrm{a}_L(\theta_t\omega)), \\
\mathcal{S}_L^{65,66}(\theta_t\omega) = (\pm\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega)),
\end{gathered}
$$

$$\mathcal{S}_L^{67,68}(\theta_t\omega) = (\pm\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega)),$$
$$\mathcal{S}_L^{69,70}(\theta_t\omega) = (\pm\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega)),$$
$$\mathcal{S}_L^{71,72}(\theta_t\omega) = (\pm\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega)),$$
$$\mathcal{S}_L^{73,74}(\theta_t\omega) = (\pm\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega)),$$
$$\mathcal{S}_L^{75,76}(\theta_t\omega) = (\pm\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega)),$$
$$\mathcal{S}_L^{77,78}(\theta_t\omega) = (\pm\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), \sqrt{-\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega)),$$
$$\mathcal{S}_L^{79,80}(\theta_t\omega) = (\pm\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega), -\sqrt{\frac{1}{21}}\mathrm{a}_L(\theta_t\omega)),$$

where

$$\mathrm{a}_L(\omega) = (\int_{-\infty}^0 e^{2P(\frac{\sqrt{m^2+n^2}\pi}{L})\tau + 2\sigma W_\tau(\omega)} d\tau)^{-1/2}. \tag{5.12}$$

Let Φ denote the flow associated with the system, and

$$(\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)) = \Phi(t, \theta_{-t}\omega)(\xi_1^0, \xi_2^0, \xi_3^0, \xi_4^0)$$

be a solution of the SDE with initial condition satisfying $\xi_1^0 \geq \xi_2^0 > \xi_3^0 > \xi_4^0 > 0$. Using standard argument in ordinary differential equations, we have

$$\lim_{t\to+\infty} \Phi(t, \theta_{-t}\omega)(\xi_1^0, \xi_2^0, \xi_3^0, \xi_4^0) = \begin{cases} (\pm\sqrt{\frac{1}{3}}a_L(\omega), 0, 0, 0), & \text{if } \xi_1^0 > \xi_2^0, \\ (\frac{1}{3}a_L(\omega), \frac{1}{3}a_L(\omega), 0, 0), & \text{if } \xi_1^0 = \xi_2^0. \end{cases}$$

The other cases can be obtained in a similar fashion. In a similar manner, we have the following results.

Theorem 5.1. *The reduced system* (5.8)-(5.11) *undergoes a stochastic supercritical bifurcation at $L = \sqrt{m^2+n^2}L_1$ in the pullback sense. More precisely, when L near $\sqrt{m^2+n^2}L_1$,*

- (a) *For $L < \sqrt{m^2+n^2}L_1$, the origin is globally asymptotically stable.*
- (b) *For $L > \sqrt{m^2+n^2}L_1$, the random compact set*

$$\mathcal{A}_L(\omega) = \{\mathcal{S}_L^i(\omega),\ i = 1, 2, \dots, 80.\}$$

is a random pullback attractor.

Theorem 5.2. *The reduced system* (5.8)-(5.11) *undergoes a stochastic subcritical bifurcation at $L = \sqrt{m^2+n^2}L_2$ in the pullback sense. More precisely, when L near $\sqrt{m^2+n^2}L_2$,*

- (a) *For $L < \sqrt{m^2+n^2}L_2$, the random compact set $\mathcal{A}_L(\omega)$ is a random pullback attractor.*
- (b) *For $L > \sqrt{m^2+n^2}L_2$, the origin is globally asymptotically stable.*

For the stochastic Swift-Hohenberg equation with multiplicative noise in Ito sense, we have similar results to the previous cases; here we do not state them.

6. Attractor bifurcation analysis when $\sqrt{2}L_1$ and L_2 coincide

In this section, we analyze the stochastic attractor bifurcation when the bifurcation points L_2 and $\sqrt{2}L_1$ are close together, i.e., when $\alpha = \frac{1}{9} + \varepsilon$ where $\alpha = \frac{1}{9}$ satisfies $L_2(\alpha) = \sqrt{2}L_1(\alpha)$ and ε is positive and small. In this case, the space H_1 and H can be decomposed into

$$H_1 = H_4^c \oplus H_4^s, \quad H = H_4^c \oplus H^s,$$

where

$$H_4^c = \mathrm{span}\{e_{1,0}, e_{0,1}, e_{1,1}, e_{1,-1}\}.$$

Projecting the above equation onto the subspace H_4^c, we obtain

$$d\widetilde{v}_c = (\mathcal{L}_\alpha^c \widetilde{v}_c + \mathcal{P}_c G(\widetilde{v}_c + \widetilde{v}_s))dt + \sigma \widetilde{v}_c \circ dW_t,$$

where $\widetilde{v}_c = \mathcal{P}_c u$, and $\widetilde{v}_s = \mathcal{P}_s u$ is the unresolved variable, and $\mathcal{P}_c$, $\mathcal{P}_s$ are respectively canonical projections from H to H_4^c and H^s. To obtain a closed form of the above equation, the unresolved variables $\widetilde{v}_s$ is parameterized in terms of the resolved variables $\widetilde{v}_c$ through a random continuous function $h_\alpha(\xi,\omega) : H_4^c \times \Omega \to H^s$.

The PM-based reduced equation for the resolved modes is

$$d\xi = (\mathcal{L}_\alpha^c \xi + \mathcal{P}_c G(\xi + h_\alpha(\xi, \theta_t\omega)))dt + \sigma\xi \circ dW_t,$$

where $\xi \in H_4^c$. Using an approximation of $h_\alpha(\xi,\omega)$ via the pullback characterization as in section 3, we instead consider the reduced equation

$$\begin{gathered} d\xi_t = (\mathcal{L}_\alpha^c \xi_t + \mathcal{P}_c G(\xi_t + v_s^{(4)}[\xi(t,\omega)](t+T, \theta_{-T}\omega; 0)))dt + \sigma\xi_t \circ dW_t, \\ \xi(0,\omega) = \varphi, t > 0, \end{gathered} \tag{6.1}$$

where T is sufficiently large, $\varphi = \mathcal{P}_c u_0$, $\widetilde{v}_s^{(4)}$ is used to approximate the stochastic inertial manifold and is obtained from the following backward-forward systems (6.2)-(6.6).

For a given $t > 0$ and T sufficiently large, let us consider the following 4-layer auxiliary backward-forward system

$$d\widetilde{v}_c^{(1)} = \mathcal{L}_\alpha^c \widetilde{v}_c^{(1)} d\tau + \sigma \widetilde{v}_c^{(1)} \circ dW_\tau, \quad \tau \in [t-T, t], \tag{6.2}$$

$$d\widetilde{v}_c^{(2)} = (\mathcal{L}_\alpha^c \widetilde{v}_c^{(2)} + \mathcal{P}_c G(\widetilde{v}_c^{(1)})d\tau + \sigma \widetilde{v}_c^{(2)} \circ dW_\tau, \quad \tau \in [t-T, t], \tag{6.3}$$

$$d\widetilde{v}_c^{(3)} = (\mathcal{L}_\alpha^c \widetilde{v}_c^{(4)} + \mathcal{P}_c G(\widetilde{v}_c^{(2)})d\tau + \sigma \widetilde{v}_c^{(3)} \circ dW_\tau, \quad \tau \in [t-T, t], \tag{6.4}$$

$$d\widetilde{v}_c^{(4)} = (\mathcal{L}_\alpha^c \widetilde{v}_c^{(4)} + \mathcal{P}_c G(\widetilde{v}_c^{(3)})d\tau + \sigma \widetilde{v}_c^{(4)} \circ dW_\tau, \quad \tau \in [t-T, t], \tag{6.5}$$

$$d\widetilde{v}_s^{(4)} = (\mathcal{L}_\alpha^s \widetilde{v}_s^{(4)} + \mathcal{P}_c G(\widetilde{v}_c^{(4)}(\tau - T, \omega)))d\tau + \sigma \widetilde{v}_s^{(4)} \circ dW_{\tau - T}, \quad \tau \in [t, t+T] \tag{6.6}$$

with

$$\begin{gathered} \widetilde{v}_c^{(1)}(\tau,\omega)\big|_{\tau=t} = \xi(t,\omega), \\ \widetilde{v}_c^{(2)}(\tau,\omega)\big|_{\tau=t} = \xi(t,\omega), \\ \widetilde{v}_c^{(3)}(\tau,\omega)\big|_{\tau=t} = \xi(t,\omega), \\ \widetilde{v}_c^{(4)}(\tau,\omega)\big|_{\tau=t} = \xi(t,\omega), \\ \widetilde{v}_s^{(4)}(\tau,\theta_{-T}\omega)\big|_{\tau=t} = 0. \end{gathered}$$

Since $\xi \in H_4^c$, we write

$$\xi(\tau,\omega) = \xi_1(\tau,\omega)e_{1,0} + \xi_2(\tau,\omega)e_{0,1} + \xi_3(\tau,\omega)e_{1,1} + \xi_4(\tau,\omega)e_{1,-1}. \tag{6.7}$$

By computations we obtain the reduced model on the PM, up to the leading order,

$$d\xi_1 = (P(\frac{\pi}{L})\xi_1 - \frac{3}{2}\xi_1^3 - 3\xi_1\xi_2^2 - 3\xi_1\xi_3^2 - 3\xi_1\xi_4^2 - 3\xi_1\xi_3\xi_4)dt + \sigma\xi_1 \circ dW_t,$$
$$d\xi_2 = (P(\frac{\pi}{L})\xi_2 - \frac{3}{2}\xi_2^3 - 3\xi_1^2\xi_2 - 3\xi_2\xi_3^2 - 3\xi_2\xi_4^2 + 3\xi_2\xi_3\xi_4)dt + \sigma\xi_2 \circ dW_t.$$
$$d\xi_3 = (P(\frac{\sqrt{2}\pi}{L})\xi_3 - \frac{3}{2}\xi_3^3 - 3\xi_1^2\xi_3 - 3\xi_2^2\xi_3 - 3\xi_3\xi_4^2 - \frac{3}{2}\xi_1^2\xi_4 + \frac{3}{2}\xi_2^2\xi_4)dt + \sigma\xi_3 \circ dW_t,$$
$$d\xi_4 = (P(\frac{\sqrt{2}\pi}{L})\xi_4 - \frac{3}{2}\xi_4^3 - 3\xi_1^2\xi_4 - 3\xi_2^2\xi_4 - 3\xi_3^2\xi_4 - \frac{3}{2}\xi_1^2\xi_3 + \frac{3}{2}\xi_2^2\xi_3)dt + \sigma\xi_4 \circ dW_t.$$

From the above, we obtain the approximation representation of manifold and the corresponding reduced systems for stochastic Swift-Hohenberg equation when L_2 and $\sqrt{2}L_1$ are close together. The performances achieved by the above reduced system can approximate dynamics on the H_4^c modes in modeling of the pathwise SPDE (1.3). The dynamical behavior of the above reduced system is not easily to analyze because of its complex structure. In fact, it is possible to achieve good modeling performances of solution from these results.

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