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BARRIERS ON CONES FOR DEGENERATE QUASILINEAR ELLIPTIC OPERATORS

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Abstract. Barrier functions $w = |x|^{\lambda}\Phi(\omega)$ are constructed for the first boundary value problem as well as for the mixed boundary value problem for quasilinear elliptic second order equation of divergent form with triple degeneracy on the n-dimensional convex circular cone:

$$\frac{d}{dx_i}(|x|^{\tau}|u|^{q}|\nabla u|^{m-2}u_{x_i}) = \mu|x|^{\tau}|u|^{q-1}\mathrm{sgn}\ u|\nabla u|^{m}\,,$$

where $-1 < \mu \le 0$, $q \ge 0$, $m > 1$, $\tau > m - n$.

Introduction

Lately many mathematicians have been considering nonlinear problems for elliptic degenerate equations; see e.g. [1] and its extensive bibliography. In the present paper, we take a first step on the investigation of the behaviour of solutions of boundary value problems for quasilinear elliptic second-order equations with triple degeneracy. We study the problem

$$\begin{gathered} Lu \equiv \frac{d}{dx_i}(|x|^{\tau}|u|^{q}|\nabla u|^{m-2}u_{x_i}) = \mu|x|^{\tau}|u|^{q-1}\mathrm{sgn}\ u|\nabla u|^{m}, \quad x \in G_0\,, \\ -1 < \mu \le 0, \quad q \ge 0, \quad m > 1, \quad \tau > m - n\,, \end{gathered} \tag{1}$$

where G_0 is an n-dimensional *convex* circular cone with its vertex at the origin O, having Γ_0 as lateral area; and Ω is a domain, on the unit sphere, with a smooth boundary $\partial\Omega$. We shall construct functions playing a fundamental role in the study of the behaviour of solutions to elliptic boundary value problems in the neighbourhood of the irregular boundary point; see e.g. [2–6]. The special structure of the solution near a conical point is of particular interest in physical applications, [7–9]. It is also used for improving numerical algorithms, [10–12].

The proof of the estimates for the solution is based on the observation that the function $r^{\lambda}\Phi(\omega)$ is usable as barrier in the above problem. By the weak comparison principle in [2, chapt. 10], it is possible to verify that the assumptions of this principle are fulfilled. Since (1) is equivalent to

$$\begin{gathered} \frac{d}{dx_i}\left(|\nabla u|^{m-2}u_{x_i}\right) + \tau|x|^{-2}|\nabla u|^{m-2}(x\nabla u) + (q-\mu)|u|^{-1}\mathrm{sgn}\ u|\nabla u|^{m} = 0, \\ x \in G_0, \quad -1 < \mu \le 0, \quad q \ge 0, \quad m > 1, \quad \tau > m - n \end{gathered}$$

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on the set where $u \neq 0$, one can obtain the bound for solution near conical boundary point. In this setting, finding of exact value of the exponent λ is very important and very difficult. For the case of a planar bounded domain with corner boundary points, the exact value of the exponent λ will be calculated explicitly.

Let us transfer the above problem to spherical coordinates with the pole at the point O.

$$\begin{aligned} x_1 &= r\cos\omega_1, \\ x_2 &= r\cos\omega_2\sin\omega_1, \\ &\vdots \\ x_{n-1} &= r\cos\omega_{n-1}\sin\omega_{n-2}\dots\sin\omega_1, \\ x_n &= r\sin\omega_{n-1}\dots\sin\omega_1, \end{aligned}$$

where $r = |x| > 0$, $0 < \omega_i < \pi$ for $i = 1, \dots, n-2$, and $0 < \omega_{n-1} < 2\pi$.

The differential operator L takes the form

$$Lu = \frac{1}{J}\sum_{i=1}^{n}\frac{d}{d\xi_i}\left(r^\tau|u|^q|\nabla u|^{m-2}\frac{J}{H_i^2}\frac{\partial u}{\partial\xi_i}\right),$$

where $J = r^{n-1}\sin^{n-2}\omega_1, \dots, \sin\omega_{n-2}$; $H_1 = 1$; $\xi_1 = r$; $\xi_{i+1} = \omega_i$ and $H_{i+1} = r\sqrt{q_i}$, for $i = \{\overline{1, n-1}\}$; $q_1 = 1$; $q_i = (\sin\omega_1\dots\sin\omega_{i-1})^2$ for $i = \{\overline{2, n-1}\}$.

We shall seek the solution of (1) as a function of the form $u = r^\lambda\Phi(\omega)$ with $\Phi(\omega) \geq 0$. Then $\Phi(\omega)$ satisfies the equation

$$\begin{aligned} &\frac{1}{j(\omega)}\sum_{k=1}^{n-1}\frac{d}{d\omega_k}\left(\frac{j(\omega)}{q_k}(\lambda^2\Phi^2 + |\nabla_\omega\Phi|^2)^{(m-2)/2}|\Phi|^q\frac{\partial\Phi}{\partial\omega_k}\right) \\ &\qquad + \lambda[\lambda(q+m-1)+\tau+n-m](\lambda^2\Phi^2+|\nabla_\omega\Phi|^2)^{(m-2)/2}\Phi|\Phi|^q \\ &\qquad\qquad = \mu\Phi|\Phi|^{q-2}(\lambda^2\Phi^2+|\nabla_\omega\Phi|^2)^{m/2}, \omega \in \Omega, \end{aligned} \tag{2}$$

where $|\nabla_\omega\Phi|^2 = \sum_{j=1}^{n-1}\frac{1}{q_j}(\frac{\partial\Phi}{\partial\omega_j})^2$, and $j(\omega) = \sin^{n-2}\omega_1\dots\sin\omega_{n-2}$.

The Dirichlet problem

First we consider the Dirichlet problem for (1) when $u|_{\Gamma_0} = 0$. On multiplying (2) by $\Phi(\omega)$ and integrating by parts over Ω we obtain

$$\begin{aligned} &\int_\Omega \left(\lambda^2\Phi^2+|\nabla_\omega\Phi|^2\right)^{(m-2)/2}|\Phi|^q|\nabla_\omega\Phi|^2 d\Omega \\ &= \lambda[\lambda(q+m-1)+\tau+n-m]\int_\Omega\left(\lambda^2\Phi^2+|\nabla_\omega\Phi|^2\right)^{(m-2)/2}|\Phi|^{q+2}d\Omega \\ &\quad - \mu\int_\Omega|\Phi|^q\left(\lambda^2\Phi^2+|\nabla_\omega\Phi|^2\right)^{m/2}d\Omega \\ &\equiv \int_\Omega\left(\lambda^2\Phi^2+|\nabla_\omega\Phi|^2\right)^{(m-2)/2}|\Phi|^q\times \\ &\quad \left\{\lambda[\lambda(q+m-1)+\tau+n-m]\Phi^2 - \mu(\lambda^2\Phi^2+|\nabla_\omega\Phi|^2)\right\}d\Omega\,. \end{aligned}$$

Hence, it follows that

$$
\begin{aligned}
&(1+\mu)\int_\Omega \left(\lambda^2\Phi^2+|\nabla_\omega\Phi|^2\right)^{(m-2)/2}|\Phi|^q|\nabla_\omega\Phi|^2 d\Omega \\
&\quad = \lambda[\lambda(q+m-1-\mu)+\tau+n-m]\int_\Omega \left(\lambda^2\Phi^2+|\nabla_\omega\Phi|^2\right)^{(m-2)/2}|\Phi|^{q+2}d\Omega\,.
\end{aligned}
$$

Since $\Phi(\omega)\not\equiv 0$ and $\mu>-1$, we have

$$
\lambda[\lambda(q+m-1-\mu)+\tau+n-m]>0\,. \tag{*}
$$

We shall consider the case of $\Phi(\omega)$ not depending on $\omega_2,\dots,\omega_{n-1}$; so that Φ is a function of a single angular coordinate $\omega_1=\omega\in(-\omega_0/2,\omega_0/2)$, $0<\omega_0<\pi$. Such function $\Phi(\omega)$ satisfies the boundary value problem for ordinary differential equation

$$
\begin{aligned}
&[(m-1)\Phi'^2+\lambda^2\Phi^2]\Phi\Phi''+(\lambda^2\Phi^2+\Phi'^2)\times\\
&\left\{(q-\mu)\Phi'^2+\lambda[\lambda(q+m-1-\mu)+\tau+n-m]\Phi^2+(n-2)\Phi\Phi'\cot\omega\right\}\\
&+(m-2)\lambda^2\Phi^2\Phi'^2=0,\qquad \omega\in(-\omega_0/2,\omega_0/2)\\
&\Phi(-\omega_0/2)=\Phi(\omega_0/2)=0\,.
\end{aligned}
\tag{ODE}
$$

By making the substitution $y=\Phi'/\Phi$ and $y'+y^2=\Phi''/\Phi$, we arrive to

$$
\begin{aligned}
&[(m-1)y^2+\lambda^2]y'+(m-1+q-\mu)(y^2+\lambda^2)^2\\
&\qquad +[\lambda(\tau+n-m)+(n-2)y\cot\leq](y^2+\lambda^2)=0;\quad \omega\in(-\frac{\omega_0}{2},\frac{\omega_0}{2}).
\end{aligned}
\tag{3}
$$

Let us now verify that

$$
\begin{aligned}
\Phi(-\omega)=\Phi(\omega),\quad y(-\omega)=-y(\omega),\quad y'(-\omega)=y'(\omega),\\
\Phi'(-\omega)=-\Phi'(\omega),\quad \forall\omega\in(-\frac{\omega_0}{2},\frac{\omega_0}{2})\,.
\end{aligned}
$$

Putting $\omega=0$ we obtain $y(0)=0$. Therefore, it is sufficient to consider the equation only on the interval $(0,\omega_0/2)$. Since $\cot\omega>0$ on $(0,\omega_0/2)$, from (3) and (*) it follows that

$$
[(m-1)y^2+\lambda^2]y'+(n-2)y(y^2+\lambda^2)\cot\omega<0\,,\quad \omega\in(0,\frac{\omega_0}{2})\,. \tag{4}
$$

Let us solve the Cauchy problem

$$
\begin{aligned}
[(m-1)\overline{y}^2+\lambda^2]\overline{y}'+(n-2)\overline{y}(\overline{y}^2+\lambda^2)\cot\omega=0,\quad \omega\in(0,\frac{\omega_0}{2});\\
\overline{y}(0)=0\,.
\end{aligned}
$$

We obtain

$$
\int\frac{(m-1)\overline{y}^2+\lambda^2}{\overline{y}(\overline{y}^2+\lambda^2)}d\overline{y}=-(n-2)\int\cot\omega d\omega+\text{const}
$$

which implies

$$\overline{y}(\overline{y}^2+\lambda^2)^{(m-2)/2} = C\sin^{2-n}\omega$$
$$\overline{y}(0)=0\,.$$

This, in turn, implies $C=0$ and $\overline{y}\equiv 0$.

Comparing the solution of (4) with that of the Cauchy problem, we deduce that $y(\omega)\le 0$. Since $\cot\omega>0$ and $y\le 0$ on our interval, by (3) we have

$$\begin{aligned}[(m-1)y^2+\lambda^2]y' + [(m-1+q-\mu)(\lambda^2+y^2)+\lambda(\tau+n-m)](\lambda^2+y^2)\\ = -(n-2)y(y^2+\lambda^2)\cot\omega \ge 0, \quad \omega\in(0,\frac{\omega_0}{2})\,.\end{aligned}$$

Thus, for $(0,\omega_0/2)$ we have,

$$[(m-1)y^2+\lambda^2]y' \ge -[(m-1+q-\mu)(\lambda^2+y^2)+\lambda(\tau+n-m)](\lambda^2+y^2)$$
$$y(0)=0\,.$$

Similarly by the comparison theorem, we obtain $y(\omega)\ge z(\omega)$, where $z(\omega)$ with $\omega\in(0,\omega_0/2)$ is the solution to Cauchy problem

$$[(m-1)z^2+\lambda^2]z' = -[(m-1+q-\mu)(\lambda^2+z^2)+\lambda(\tau+n-m)](\lambda^2+z^2),$$
$$z(0)=0\,.$$

On solving the latter, we obtain the expression for z in the implicit form

$$\begin{aligned}\frac{\frac{m-1}{m-1+q-\mu}+\lambda\frac{m-2}{\tau+n-m}}{\sqrt{\lambda^2+\lambda\frac{\tau+n-m}{m-1+q-\mu}}}\arctan\frac{z}{\sqrt{\lambda^2+\lambda\frac{\tau+n-m}{m-1+q-\mu}}}\\ +\omega+\frac{m-2}{m-n-\tau}\arctan(\frac{z}{\lambda})=0\,.\end{aligned} \tag{5}$$

By combining the obtained results, we conclude that

$$0\ge y(\omega)\ge z(\omega)\,. \tag{6}$$

Let us now return to the equation for $y(\omega)$. On making the substitution $\varphi=\ln\Phi$, $w(\varphi)=y^2(\varphi)$,

$$w'(\varphi)=2yy'(\varphi)=2y\frac{d\omega}{d\varphi}y'(\omega)=2y'(\omega)\,,$$

we obtain

$$\begin{aligned}\frac{1}{2}[(m-1)w+\lambda^2]w' + [(m-1+q-\mu)(\lambda^2+w)+\lambda(\tau+n-m)](\lambda^2+w)\\ -(n-2)\sqrt{w}(w+\lambda^2)\cot\omega=0\,,\end{aligned}$$

where we have used $y=\pm\sqrt{w}$ and $y<0$. As we did above, we obtain a differential inequality for w,

$$\frac{1}{2}[(m-1)w+\lambda^2]w' + [(m-1+q-\mu)(\lambda^2+w)+\lambda(\tau+n-m)](\lambda^2+w) > 0\,.$$

Integrating the respective differential equation

$$\frac{1}{2}[(m-1)\overline{w}+\lambda^2]\overline{w}'+[(m-1+q-\mu)(\lambda^2+\overline{w})+\lambda(\tau+n-m)](\lambda^2+\overline{w})=0$$

we obtain

$$\begin{aligned}&\lambda\frac{m-2}{m-n-\tau}\ln(\lambda^2+\overline{w})\\&+\left(\frac{m-1}{m-1+q-\mu}+\lambda\frac{m-2}{\tau+n-m}\right)\ln((m-1+q-\mu)(\lambda^2+\overline{w})+\lambda(\tau+n-m))\\&\qquad+2\ln\Phi=\ln C\,.\end{aligned}$$

Solving the latter expression with the respect to Φ we obtain

$$\begin{aligned}\Phi^2(\omega)=C^2&\left(\frac{(m-1+q-\mu)(\lambda^2+\overline{w})+\lambda(\tau+n-m)}{\lambda^2+\overline{w}}\right)^{\lambda(m-2)/(m-n-\tau)}\times\\&[(m-1+q-\mu)(\lambda^2+\overline{w})+\lambda(\tau+n-m)]^{-(m-1)/(m-1+q-\mu)}.\end{aligned}$$

Now it is evident that $\overline{w}=z^2(\varphi)$ and $w=y^2(\varphi)$. From (6) and $w\le\overline{w}$, it follows that

$$\Phi^2(\omega)=C^2(z^2+\lambda^2)^{\frac{1-m}{m-1+q-\mu}}\left(m-1+q-\mu+\frac{\lambda(\tau+n-m)}{(z^2+\lambda^2)}\right)^{\frac{\lambda(m-2)}{m-n-\tau}-\frac{m-1}{m-1+q-\mu}}.$$

Whence it follows that

$$\Phi(\omega)\sim|z|^{-\frac{(m-1)}{m-1+q-\mu}}\quad\text{as}\quad|z|\to+\infty\,.$$

Since $y^2\le z^2$, it follows that $1/z^2\le 1/y^2$, and it is now clear that

$$\lim_{\omega\to(\omega_0/2)-0}z(\omega)=-\infty$$

(since $\Phi(\omega_0/2)=0$). Furthermore, since $y=\frac{\Phi'}{\Phi}<0$ and $\Phi>0$ on $(0,\omega_0/2)$, $\Phi'<0$. i.e., $\Phi(\omega)$ decreases on $(0,\omega_0/2)$ from the positive value $\Phi(0)$ to $\Phi(\omega_0/2)=0$. Φ does not vanish anywhere else in $(0,\omega_0/2)$, otherwise it should increase somewhere. From this equation we have

$$\begin{aligned}y'=&-[(m-1+q-\mu)(y^2+\lambda^2)+\lambda(\tau+n-m)]\frac{(y^2+\lambda^2)}{(m-1)y^2+\lambda^2}\\&-(n-2)y\frac{y^2+\lambda^2}{(m-1)y^2+\lambda^2}\cot\omega\to-\infty\quad\text{as}\quad y\to-\infty\,.\end{aligned}$$

That is to say $y(\omega)$ decreases in the vicinity of the point $-\omega$, when $y\to-\infty$. It is possible only at $-\omega=\omega_0/2$, (when passing $\omega\to\frac{\omega_0}{2}-0$). On performing the passage to the limit $\omega\to\frac{\omega_0}{2}-0$ in (5), and taking into account that $z\to-\infty$, we obtain

$$\begin{aligned}&\frac{\omega_0}{\pi}+\frac{m-2}{\tau+n-m}\\&=\left(\frac{m-1}{m-1+q-\mu}+\lambda\frac{m-2}{\tau+n-m}\right)\left(\frac{\lambda[\lambda(m-1+q-\mu)+\tau+n-m]}{m-1+q-\mu}\right)^{-1/2}.\end{aligned}$$

Hence we obtain an explicit expression for λ,

$$\lambda = \frac{\pi}{2\omega_0(m-1+q-\mu)} \times$$
$$\left\{ \frac{m(m-2)-2(m-2)t-t^2}{t+2(m-2)} + \frac{\sqrt{[t^2+2(m-2)t+m^2][t^2+2(m-2)t+(m-2)^2]}}{t+2(m-2)} \right\}, \tag{7}$$

where $t = (\tau + n - m)\omega_0/\pi$.

In the case of $n = 2$, $\tau = 0$ *we obtain*

$$\lambda = \frac{(m-1)}{(m-1-\mu+q)} + \frac{(\pi-\omega_0)[m(\pi-\omega_0)+\sqrt{(m-2)^2(\pi-\omega_0)^2+4(m-1)\pi^2}]}{2\omega_0(2\pi-\omega_0)(m-1-\mu+q)}. \tag{8}$$

In the case of $\tau = \mu = q = 0$, $n = 2$ *we get the result of [13]. If* $m = n = 2$, *from (7) we get*

$$\lambda = \frac{\sqrt{\left(\frac{2\pi}{\omega_0}\right)^2 + \tau^2} - \tau}{2(1+q-\mu)}. \tag{9}$$

For the case $n = 3$, we assume that $\tau = 0$. We shall seek a solution of the form $u = r^\lambda \Phi(\omega)\sin^\lambda \Phi$, with $\Phi \in (0,\pi)$ and $\omega \in (-\omega_0/2, \omega_0/2)$. Then we obtain for $\Phi(\omega)$ a problem which coincides with (ODE) with $n = 2$, and so for λ we have the Expression (8).

The mixed boundary value problem

Now we consider the mixed boundary value problem in the planar domain $G_0 = \{(r,\omega) \mid r > 0,\ 0 < \omega < \omega_0 < \pi\}$, with a corner boundary point,

$$\frac{d}{dx_i}\left(|u|^q|\nabla u|^{m-2}u_{x_i}\right) = \mu|u|^{q-1}\operatorname{sgn} u|\nabla u|^m, \quad x \in G_0\,,$$
$$u\Big|_{\omega=\omega_0} = 0\,, \qquad \frac{\partial u}{\partial x_2}\Big|_{\omega=0} = 0\,,$$

where ω_0 is the angle with the vertex at the point O. By a process analogous to the one above, we come to the expression

$$\lambda = \frac{(m-1)}{(m-1-\mu+q)} + \frac{(\pi-2\omega_0)[m(\pi-2\omega_0)+\sqrt{(m-2)^2(\pi-2\omega_0)^2+4(m-1)\pi^2}]}{8\omega_0(\pi-\omega_0)(m-1-\mu+q)}. \tag{10}$$

Obviously, this expression coincides with (8) for the Dirichlet problem, if in the latter we put $2\omega_0$ instead of ω_0.

Therefore, barrier functions $w = r^\lambda\Phi(\omega)$ *have been constructed for the first boundary value problem for the equation (1), and for the mixed boundary value problem for (1) with* $\tau = 0$.

References

1. Drábek P., Kufner A., Nicolosi F., *Nonlinear elliptic equations*, Universität of West Bohemia at Pilsen, 1996.
2. Gilbarg D., Trudinger N., *Elliptic partial differential equations of second order*, Springer, 1983.
3. Miller K., *Barriers on cones for uniformly elliptic operators*, Ann. mat. pura appl. **76, No. 4** (1967), 93–105.
4. Tolksdorf P., *On the Dirichlet problem for quasilinear elliptic equations in domains with conical boundary points*, Comm. Part. Diff. Eqns. **8, No. 7** (1983), 773–817.
5. Dobrowolski M., *On quasilinear elliptic equations in domains with conical boundary points*, J. Reine und Angew. Math. **394** (1989), 186–195.
6. Borsuk M.V., *Estimates of solutions of Dirichlet problem for elliptic nondivergence second-order equations in a neighbourhood of a conical boundary point*, Differ. Uravnenija **30, No.1** (1994), 104–108.
7. Sommerfeld A., *Asymptotische Integration der Differentialgleichung der Thomas-Fermischen atoms*, Z. für Physik **78** (1932), 283–308.
8. Jackson J.D., *Classical Electrodinamics*, John Wiley & Sons, NY, 1975.
9. Benilan Ph., Brézis H., *Some variational problems of the Thomas-Fermi type*, in "Variational inequalities" (Cottle, Gianessi, Lions, Eds.), (1980), Wiley, NY, 53–73.
10. Babuska J., Kellogg R.B., *Numerical solution of the neutron diffusion equation in the presence of corners and interfaces*, in: "Numerical reactor calculations" (1972), Intern. Atomic Energy Agency, Vienna, 473–486.
11. Strang G. & Fix J., *An analysis of the finite element method*, Prentice-Hall, Englewood Cliffs, 1973.
12. Dobrowolski M., *Numerical approximation of elliptic interface and corner problems*, Habilitationsschrift, Universität Bonn (1981); Z.A.A.M. **64** (1984), 270–271.
13. Borsuk M.V., Plesha M.I., *Estimates of generalized solutions of Dirichlet problem for quasilinear elliptic second-order equations in a domain with conical boundary point*, Ukrainer Math. journal (to appear).

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