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ON THE WAVE EQUATIONS WITH MEMORY IN NONCYLINDRICAL DOMAINS

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Abstract. In this paper we prove the exponential and polynomial decays rates in the case $n > 2$, as time approaches infinity of regular solutions of the wave equations with memory

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0 \quad \text{in } \widehat{Q}$$

where $\widehat{Q}$ is a non cylindrical domains of $\mathbb{R}^{n+1}$, $(n \geq 1)$. We show that the dissipation produced by memory effect is strong enough to produce exponential decay of solution provided the relaxation function g also decays exponentially. When the relaxation function decay polynomially, we show that the solution decays polynomially with the same rate. For this we introduced a new multiplier that makes an important role in the obtaining of the exponential and polynomial decays of the energy of the system. Existence, uniqueness and regularity of solutions for any $n \geq 1$ are investigated. The obtained result extends known results from cylindrical to non-cylindrical domains.

1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n$ containing the origin and having C^2 boundary. Let $\gamma : [0, \infty[\to \mathbb{R}$ be a continuously differentiable function. See hypothesis (1.11)–(1.13) on γ. Consider the family of subdomains $\{\Omega_t\}_{0 \leq t < \infty}$ of $\mathbb{R}^n$ given by

$$\Omega_t = T(\Omega), \quad T : y \in \Omega \mapsto x = \gamma(t)y$$

whose boundaries are denoted by Γ_t, and let the noncylindrical domain of $\mathbb{R}^{n+1}$ be

$$\widehat{Q} = \cup_{0 \leq t < \infty} \Omega_t \times \{t\}$$

with lateral boundary

$$\widehat{\Sigma} = \cup_{0 \leq t < \infty} \Gamma_t \times \{t\}.$$

Let us consider the Hilbert space $L^2(\Omega)$ endowed with the inner product

$$(u, v) = \int_\Omega u(x)v(x)dx$$

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and corresponding norm $\|u\|^2_{L^2(\Omega)} = (u,u)$. We also consider the Sobolev space $H^1(\Omega)$ endowed with the scalar product

$$(u,v)_{H^1(\Omega)} = (u,v) + (\nabla u, \nabla v).$$

We define the subspace of $H^1(\Omega)$, denoted by $H^1_0(\Omega)$, as the closure of $C^\infty_0(\Omega)$ in the strong topology of $H^1(\Omega)$. By $H^{-1}(\Omega)$ we denote the dual space of $H^1_0(\Omega)$. This space endowed with the norm induced by the scalar product

$$((u,v))_{H^1_0(\Omega)} = (\nabla u, \nabla v)$$

is a Hilbert space; due to the Poincaré inequality

$$\|u\|^2_{L^2(\Omega)} \le C\|\nabla u\|^2_{L^2(\Omega)}.$$

For $1 \le p < \infty$, we define

$$\|u\|^p_{L^p(\Omega)} = \int_\Omega |u(x)|^p dx,$$

and for $p = \infty$,

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{esssup}_{x\in\Omega} |u(x)|.$$

In this work we study the existence and uniqueness of strong global solutions, as well the exponential and polynomial decays of the energy for the wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0 \quad \text{in } \widehat{Q}, \tag{1.1}$$

$$u = 0 \quad \text{on } \widehat{\sum}, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega_0, \tag{1.3}$$

where u is the transverse displacement. The method used for proving existence and uniqueness is based on transforming our problem into another initial boundary value problem defined over a cylindrical domain, whose sections are not time-dependent. This is done using a suitable change of variable. Then we show the existence and uniqueness for this new problem. Our existence result on non-cylindrical domains will follows using the inverse transformation. That is, using the diffeomorphism $\tau : \widehat{Q} \to Q$ defined by

$$\tau : \widehat{Q} \to Q, \quad (x,t) \in \Omega_t \mapsto (y,t) = (\frac{x}{\gamma(t)}, t) \tag{1.4}$$

and $\tau^{-1} : Q \to \widehat{Q}$ defined by

$$\tau^{-1}(y,t) = (x,t) = (\gamma(t)y, t). \tag{1.5}$$

Denoting by v the function

$$v(y,t) = u \circ \tau^{-1}(y,t) = u(\gamma(t)y, t) \tag{1.6}$$

the initial boundary value problem (1.1)–(1.3) becomes

$$v_{tt} - \gamma^{-2}\Delta v + \int_0^t g(t-s)\gamma^{-2}(s)\Delta v(s)ds - A(t)v + a_1 \cdot \nabla\partial_t v + a_2 \cdot \nabla v = 0 \quad \text{in } Q, \tag{1.7}$$

$$v|_\Gamma = 0, \tag{1.8}$$

$$v|_{t=0} = v_0, \quad v_t|_{t=0} = v_1 \quad \text{in } \Omega, \tag{1.9}$$

where

$$A(t)v = \sum_{i,j=1}^{n} \partial_{y_i}(a_{ij}\partial_{y_j} v)$$

and

$$\begin{gathered} a_{ij}(y,t) = -(\gamma'\gamma^{-1})^2 y_i y_j \quad (i,j = 1,\dots,n), \\ a_1(y,t) = -2\gamma'\gamma^{-1} y, \\ a_2(y,t) = -\gamma^{-2} y(\gamma''\gamma + (\gamma')^2(n-1)). \end{gathered} \tag{1.10}$$

To show the existence of strong solution we will use the following hypotheses:

$$\gamma' \le 0 \quad \text{if } n > 2, \quad \gamma' \ge 0 \quad \text{if } n \le 2, \tag{1.11}$$

$$\gamma(\cdot) \in L^\infty(0,\infty), \quad \inf_{0\le t<\infty} \gamma(t) = \gamma_0 > 0, \tag{1.12}$$

$$\gamma' \in W^{2,\infty}(0,\infty) \cap W^{2,1}(0,\infty). \tag{1.13}$$

Note that the assumption (1.11) means that $\widehat{Q}$ is decreasing if $n > 2$ and increasing if $n \le 2$ in the sense that when $t > t'$ and $n > 2$ then the projection of $\Omega_{t'}$ on the subspace $t = 0$ contain the projection of Ω_t on the same subspace. The opposite holds in the case $n \le 2$.

The above method was introduced by Dal Passo and Ughi [14] to study certain class of parabolic equations in non cylindrical domains.

We only obtained the exponential and polynomial decays of solution for our problem for the case $n > 2$. The main difficulty to prove the exponential and polynomial decays for the case $n \le 2$ are in the Lemma 3.3, 3.4 and 3.5, where appears the boundary terms, since we worked directly in $\widehat{Q}$. To control those terms we used the hypothesis (1.11). Therefore the case $n \le 2$ is an important open problem.

The equation (1.1) can be seen as a model of propagation of seismic waves, where the function g represents the medium of propagation of waves. In the considered case, the medium is elastic.

The wave equations with dissipation was studied by several authors. All of them consider essentially two types of dissipative mechanisms (or a combination of them):

(a) The frictional dissipation, obtained by introducing a frictional damping that can be acting either on the boundary or in a neighborhood of the boundary;

(b)The viscoelastic dissipation given by the memory effects as in [11, 16, 17, 18].

The frictional damping is the simplest dissipative mechanism when one is working either in the whole domain Ω or over a strategic part of the domain (locally). It was proved by [1, 2, 3, 4, 7, 12, 13, 19, 20] that the first-order energy decays exponentially to zero as time goes to infinity.

Finally, the memory effect produces a suitable dissipative mechanism which depends on the ralaxation function (see [16, 17, 18]). They proved that the energy decays uniformly exponentially or algebraically with the same rate of decay as the relaxation function.

In a non cylindrical domain, the problem of existence, uniqueness and exponential decay of the solutions for the wave equations with memory and weak damping was studied by Ferreira and Santos [5]. They proved that the energy decays uniformly exponentially to zero as time goes to infinity.

The main result of this paper is to extend the result obtained by Ferreira and Santos [5]. That is, to remove the term u_t of the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + u_t = 0$$

in [5].

The main technical difficulty it is to control the term $\int_{\Omega_t} |u_t|^2 dx$ of the total energy of the system (1.1)–(1.3). To solve this problem we introduced a new multiplier $(g * u)_t$, that makes key role to obtain the exponential and polynomial decays.

The present paper extends the results from cylindrical to non cylindrical domains.

To see the dissipative properties of the system we have to construct a suitable functional whose derivative is negative and is equivalent to the first order energy. This functional is obtained using the multiplicative technique following Komornik [6], Lions [8] and Rivera [10].

The notation we use in this paper are standard and can be found in Lion's book [8, 9]. In the sequel by C (sometimes $C_1, C_2, \dots$) we denote various positive constants which do not depend on t or on the initial data. This paper is organized as follows. In section 2 we prove the existence, regularity and uniqueness of solutions. We use Galerkin approximation, Aubin-Lions theorem, energy method introduced by Lions [8] and some technical ideas to show existence regularity and uniqueness of regular solution for the problem (1.1)–(1.3). Finally, in the section 3 and 4, we establish the results on the exponential and polynomial decays of the regular solution to the problem (1.1)–(1.3). We use the technique of the multipliers introduced by Kormornik [6], Lions [8] and Rivera [11] coupled with some technical lemmas and some technical ideas.

2. Existence and Regularity

In this section we shall study the existence and regularity of solutions for the system (1.1)–(1.3). For this we assume that the kernel $g : \mathbb{R}_+ \to \mathbb{R}_+$ is in $W^{2,1}(0,\infty)$ and satisfy

$$g, -g' \geq 0, \tag{2.1}$$

$$\gamma_1^{-2} - \int_0^\infty g(s)\gamma^{-2}(s)ds = \beta > 0, \tag{2.2}$$

where $\gamma_1 = \sup_{0<t<\infty} \gamma(t)$. Note that (2.2) implies

$$\beta \leq \gamma(t)^{-2} - \int_0^t g(s)\gamma^{-2}(s)ds \leq \frac{1}{\gamma_0^2}.$$

On the other hand, since all the dissipation of the system is contained only in the memory term, we also have to require that $g \neq 0$, and this explains (2.2).

Typical examples of functions γ and g satisfying (1.11)–(1.13) and (2.1)–(2.2) are

$$\gamma(t) = e^{-\sigma_0 t} + \sigma_1, \quad g(t) = \sigma_2 e^{-\sigma_3 t}$$

where σ_i, $i = 0, 1, 2, 3$, are positive constants. To simplify our analysis, we define the binary operator

$$g\Box\frac{\nabla u(t)}{\gamma(t)} = \int_0^t g(t-s)\gamma^{-2}(s)\int_\Omega |\nabla u(t) - \nabla u(s)|^2 dx\, ds.$$

With this notation we have the following statement.

Lemma 2.1. *For $v \in C^1(0,T:H^1(\Omega))$ and $g \in C^1(0,\infty)$ we have*

$$
\begin{aligned}
\int_\Omega \int_0^t \frac{g(t-s)}{\gamma^2(s)} \nabla v ds \cdot \nabla v_t dx &= -\frac{1}{2}\frac{g(t)}{\gamma^2(0)} \int_\Omega |\nabla v|^2 dx + \frac{1}{2} g' \Box \frac{\nabla v}{\gamma} \\
&\quad - \frac{1}{2}\frac{d}{dt}\big[g \Box \frac{\nabla v}{\gamma} - (\int_0^t \frac{g(t-s)}{\gamma^2(s)} ds) \int_\Omega |\nabla v|^2 dx\big] \\
&\quad + \int_0^t g(t-s) \frac{\gamma'(s)}{\gamma^3(s)} \int_\Omega |\nabla u|^2 dx\, ds.
\end{aligned}
$$

The proof of this lemma follows by differentiating the term

$$
g \Box \frac{\nabla u(t)}{\gamma(t)} - \int_0^t \frac{g(t-s)}{\gamma^2(s)} \int_\Omega |\nabla u|^2 dx ds.
$$

The well-posedness of system (1.7)-(1.9) is given by the following theorem.

Theorem 2.2. *Let us take $v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $v_1 \in H_0^1(\Omega)$ and let us suppose that assumptions* (1.11)–(1.13) *and* (2.1)–(2.2) *hold. Then there exists a unique solution v of the problem (1.7)-(1.9) satisfying*

$$
\begin{gathered}
v \in L^\infty(0,\infty : H_0^1(\Omega) \cap H^2(\Omega)), \\
v_t \in L^\infty(0,\infty : H_0^1(\Omega)), \\
v_{tt} \in L^\infty(0,\infty : L^2(\Omega)).
\end{gathered}
$$

Proof. The main idea is to use the Galerkin method. To do this let us take a basis $\{w_j\}_{j\in\mathbb{N}}$ to $H_0^1(\Omega) \cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$ and we represent by V_m the space generated by $w_1, w_2, \dots, w_m$. Let us denote by

$$
v_0^m = \sum_{j=1}^m (v_0, w_j) w_j, \quad v_1^m = \sum_{j=1}^m (v_1, w_j) w_j.
$$

Note that for any $(v_0, v_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$, we have $v_0^m \to v_0$ strong in $H_0^1(\Omega) \cap H^2(\Omega)$ and $v_1^m \to v_1$ strong in $H_0^1(\Omega)$.

Standard results on ordinary differential equations imply the existence of a local solution v^m of the form

$$
v^m(t) = \sum_{j=1}^m g_{jm}(t) w_j,
$$

to the system

$$
\begin{aligned}
&\int_\Omega v_{tt}^m w_j dy - \gamma^{-2} \int_\Omega \Delta v^m w_j dy + \int_\Omega \int_0^t g(t-s) \gamma^{-2}(s) \nabla v^m(s) \cdot \nabla w_j ds dy \\
&+ \int_\Omega A(t) v^m w_j dy + \int_\Omega a_1 \cdot \nabla v_t^m w_j dy + \int_\Omega a_2 \cdot \nabla v^m w_j dy = 0, \quad (j = 1, \dots, m),
\end{aligned} \tag{2.3}
$$

$$
v^m(x,0) = v_0^m, \quad v_t^m(x,0) = v_1^m. \tag{2.4}
$$

The extension of this solution to the whole interval $[0,\infty)$ is a consequence of the a priori estimate below.

First estimate. Multiplying the equation (2.3) by $g'_{jm}(t)$, summing up the product result in $j = 1, 2, \dots, m$, and using the Lemma 2.1 we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\pounds_1^m(t, v^m) + \int_\Omega A(t)v^m v_t^m dy + \int_\Omega a_1 \cdot \nabla v_t^m v_t^m dy + \int_\Omega a_2 \cdot \nabla v^m v_t^m dy \\
&= -\frac{1}{2}\frac{g(t)}{\gamma^2(0)}\|\nabla v^m\|^2_{L^2(\Omega)} + \frac{1}{2}g'\Box\frac{\nabla v^m}{\gamma} \\
&\quad - \frac{\gamma'}{\gamma^3}\|\nabla v^m\|^2_{L^2(\Omega)} + \int_0^t g(t-s)\gamma'(s)\gamma^{-3}(s)ds\int_\Omega |\nabla v^m|^2 dx,
\end{aligned}$$

where

$$\pounds_1^m(t, v^m) = \|v_t^m\|^2_{L^2(\Omega)} + \big(\frac{1}{\gamma^2(t)} - \int_0^t \frac{g(t-s)}{\gamma^2(s)}ds\big)\|\nabla v^m\|^2_{L^2(\Omega)} + g\Box\frac{\nabla v^m}{\gamma}.$$

Taking into account (1.11), (1.13), (2.1) and (2.2) we obtain

$$\frac{1}{2}\frac{d}{dt}\pounds_1^m(t, v^m) \le C(|\gamma'| + |\gamma''|)\pounds_1^m(t). \tag{2.5}$$

Integrating the inequality (2.5), taking account (1.13) and using Gronwall's Lemma we get

$$\pounds_1^m(t, v^m) \le C, \quad \forall m \in \mathbb{N},\ \forall t \in [0, T]. \tag{2.6}$$

Second estimate. From equation (2.3) we get

$$\|v_{tt}^m(0)\|^2_{L^2(\Omega)} \le C, \quad \forall m \in \mathbb{N}. \tag{2.7}$$

Differentiating the equation (2.3) with respect to the time, we obtain

$$\begin{aligned}
&\int_\Omega v_{ttt}^m w_j dy - \gamma^{-2}\int_\Omega \Delta v_t^m w_j dy + 2\frac{\gamma'}{\gamma^3}\int_\Omega \Delta v^m w_j dy - \frac{g(0)}{\gamma^2(0)}\int_\Omega \Delta v_0^m w_j dy \\
&+ \int_\Omega\int_0^t g'(t-s)\gamma^{-2}(s)\nabla v^m(s)\cdot\nabla w_j ds dy + \int_\Omega \frac{d}{dt}(A(t)v^m)w_j dy \\
&+ \int_\Omega \frac{d}{dt}(a_1\cdot\nabla v_t^m)w_j dy + \int_\Omega \frac{d}{dt}(a_2\cdot\nabla v^m)w_j dy = 0.
\end{aligned} \tag{2.8}$$

Multiplying (2.8) by $g''_{jm}(t)$, summing up the product result in $j = 1, 2, \dots, m$ and using similar arguments as (2.6) we obtain

$$\pounds_1^m(t, v_t^m) + \int_0^t \|v_{ss}^m(s)\|^2_{L^2(\Omega)}ds \le C, \quad \forall t \in [0, T],\ \forall m \in \mathbb{N}. \tag{2.9}$$

The first and second a priori estimates allow us to obtain a subsequence of (v_m) which from now on will be also denoted by (v_m) and a function $v : \Omega \times (0, \infty) \to \mathbb{R}$ satisfying:

$$\begin{gathered}
v^m \to v \quad \text{weak star in } L^\infty(0, \infty; H_0^1(\Omega)) \\
v_t^m \to v \quad \text{weak star in } L^\infty(0, \infty; H_0^1(\Omega)) \\
v_{tt}^m \to v_{tt} \quad \text{weak star in } L^\infty(0, \infty; L^2(\Omega)).
\end{gathered}$$

The above convergence allows us to pass to the limit in the problem (2.3)-(2.4).

Letting $m \to \infty$ in the equation (2.3) we conclude that

$$v_{tt} - \gamma^{-2}\Delta v + \int_0^t g(t-s)\gamma^{-2}(s)\Delta v(s)ds - A(t)v + a_1\cdot\nabla\partial_t v + a_2\cdot\nabla v = 0$$

in $L^\infty(0,\infty : L^2(\Omega))$. Therefore, using the elliptic regularity, we have that

$$v \in L^\infty(0,\infty : H_0^1(\Omega) \cap H^2(\Omega)).$$

Uniqueness. Suppose we have two solutions v and $\widehat{v}$ in the conditions of Theorem 2.2. Then $\phi = v - \widehat{v}$ satisfies the same conditions and $\phi(0) = 0$, $\phi_t(0) = 0$. Let us prove that $\phi = 0$ on $\Omega \times [0,\infty[$.

Multiplying the equations (1.7) by ϕ_t, summing up the product result and using the Lemma 2.1 we get

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_1(t,\phi) \le C(|\gamma'| + |\gamma''|)\mathcal{L}_1(t),$$

where

$$\mathcal{L}_1(t,\phi) = \|\phi_t\|^2_{L^2(\Omega)} + (\frac{1}{\gamma^2(t)} - \int_0^t \frac{g(t-s)}{\gamma^2(s)}ds)\|\nabla\phi\|^2_{L^2(\Omega)} + g\Box\frac{\nabla\phi}{\gamma}.$$

Integrating with respect to the time the above inequality and applying Gronwall's inequality we conclude that $\phi = 0$ on $\Omega \times [0,\infty[$. $\square$

To show the existence in non cylindrical domains, we return to our original problem in the non cylindrical domains by using the change variable given in (1.4) by $(y,t) = \tau(x,t)$, $(x,t) \in \widehat{Q}$. Let v be the solution obtained from Theorem 2.2 and u defined by (1.6), then u belongs to the class

$$u \in L^\infty(0,\infty : H_0^1(\Omega_t)), \tag{2.10}$$

$$u_t \in L^\infty(0,\infty : H_0^1(\Omega_t)), \tag{2.11}$$

$$u_{tt} \in L^\infty(0,\infty : L^2(\Omega_t)). \tag{2.12}$$

Denoting by

$$u(x,t) = v(y,t) = (v \circ \tau)(x,t),$$

from (1.6) it is easy to see that u satisfies

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0 \quad \text{in} \quad L^\infty(0,\infty : L^2(\Omega_t)). \tag{2.13}$$

Using regularity elliptic, we obtain

$$u \in L^\infty(0,\infty : H_0^1(\Omega_t) \cap H^2(\Omega_t)). \tag{2.14}$$

Let u_1, u_2 be two solutions to (1.1), and v_1, v_2 be the functions obtained through the diffeomorphism τ given by (1.4). Then v_1, v_2 are the solutions to (1.7). By the uniqueness result Theorem 2.2, we have $v_1 = v_2$, so $u_1 = u_2$. Therefore, we have the following result.

Theorem 2.3. *Let us take $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$ and let us suppose that assumptions* (1.11)–(1.13) *and* (2.1)–(2.2) *hold. Then there exists a unique solution u of the problem* (1.1)–(1.3) *satisfying (2.10)-(2.14).*

3. Exponential Rate of Decay

In this section we show that the solution of system (1.1)–(1.3) decays exponentially. To this end we will assume that the memory g satisfies:

$$g'(t) \le -C_1 g(t) \tag{3.1}$$

for all $t \ge 0$, with positive constant C_1. Additionally, we assume that the function $\gamma(\cdot)$ satisfies the conditions

$$\gamma' \le 0, \quad t \ge 0, \quad n > 2, \tag{3.2}$$

$$0 < \max_{0 \le t < \infty} |\gamma'(t)| \le \frac{1}{d}, \tag{3.3}$$

where $d = \mathrm{diam}(\Omega)$. The condition (3.3) implies that our domains is "time like" in the sense that

$$|\underline{\nu}| < |\overline{\nu}|$$

where $\underline{\nu}$ and $\overline{\nu}$ denote the t-component and x-component of the outer unit normal of $\widehat{\Sigma}$. To facilitate our calculations we introduce the notation

$$(g \Box \nabla u)(t) = \int_{\Omega_t} \int_0^t g(t-s) |\nabla u(t) - \nabla u(s)|^2 ds\, dx.$$

First, we prove the following two lemmas that will be used in the sequel.

Lemma 3.1. *Let $F(\cdot,\cdot)$ be the smooth function defined in $\Omega_t \times [0,\infty[$, $(t \in [0,\infty[$. Then*

$$\frac{d}{dt} \int_{\Omega_t} F(x,t) dx = \int_{\Omega_t} \frac{d}{dt} F(x,t) dx + \frac{\gamma'}{\gamma} \int_{\Gamma_t} F(x,t) (x \cdot \overline{\nu}) d\Gamma_t, \tag{3.4}$$

where $\overline{\nu}$ is the x-component of the unit normal exterior ν.

Proof. By a change variable $x = \gamma(t) y$, $y \in \Omega$, we have

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega_t} F(x,t) dx &= \frac{d}{dt} \int_{\Omega} F(\gamma(t) y, t) \gamma^n(t) dy \\
&= \int_{\Omega} (\frac{\partial F}{\partial t}) \gamma^n(t) dy + \sum_{i=1}^n \int_{\Omega} \frac{\gamma'}{\gamma} x_i (\frac{\partial F}{\partial t}) \gamma^n(t) dy \\
&\quad + n \int_{\Omega} \gamma'(t) \gamma^{n-1}(t) F(\gamma(t) y, t) dy.
\end{aligned}$$

If we return at the variable x, we get

$$\frac{d}{dt} \int_{\Omega_t} F(x,t) dx = \int_{\Omega_t} \frac{\partial F}{\partial t} dx + \frac{\gamma'}{\gamma} \int_{\Omega_t} x \cdot \nabla F(x,t) dx + n \frac{\gamma'}{\gamma} \int_{\Omega_t} F(x,t) dx.$$

Integrating by part in the last equality we obtain the formula (3.4). □

Lemma 3.2. *For any functions $g \in C^1(\mathbb{R}_+)$ and $u \in C^1((0,T) : H^2(\Omega_t))$, we have*

$$\begin{aligned}
&\int_{\Omega_t} \int_0^t g(t-s) \nabla u(s) \cdot \nabla u_t ds\, dx \\
&= -\frac{1}{2} g(t) \int_{\Omega_t} |\nabla u(t)|^2 dx + \frac{1}{2} g' \Box \nabla u - \frac{1}{2} \frac{d}{dt} \Big[g \Box \nabla u - (\int_0^t g(s) ds) \int_{\Omega_t} |\nabla u|^2 \Big] \\
&\quad + \frac{\gamma'}{2\gamma} \int_{\Gamma_t} \int_0^t g(t-s) |\nabla u(t) - \nabla u(s)|^2 (\overline{\nu} \cdot x) d\Gamma_t
\end{aligned}$$

$$-\frac{\gamma'}{2\gamma}\int_{\Gamma_t}\int_0^t g(t-s)|\nabla u(t)|^2(\overline{\nu}\cdot x)d\Gamma_t.$$

Proof. Differentiating the term $g\Box\nabla u$ and applying the lemma 3.1 we obtain

$$\begin{aligned}\frac{d}{dt}g\Box\nabla u &= \int_{\Omega_t}\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2 ds\,dx\\ &\quad -2\int_{\Omega_t}\int_0^t g(t-s)\nabla u(s)\cdot\nabla u_t ds\,dx\\ &\quad +\Big(\int_0^t g(t-s)ds\Big)\int_{\Omega_t}\frac{d}{dt}|\nabla u(t)|^2dx\\ &\quad +\frac{\gamma'}{\gamma}\int_{\Gamma_t}\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2(x\cdot\overline{\nu})dsd\Gamma_t.\end{aligned}$$

From where it follows that

$$\begin{aligned}&2\int_{\Omega_t}\int_0^t g(t-s)\nabla u(s)\cdot\nabla u_t ds\,dx\\ &= -\frac{d}{dt}\{g\Box\nabla u-\int_0^t g(t-s)ds\int_{\Omega_t}|\nabla u(t)|^2dx\}\\ &\quad +\int_{\Omega_t}\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2ds\,dx - g(t)\int_{\Omega_t}|\nabla u(t)|^2dx\\ &\quad +\frac{\gamma'}{\gamma}\int_{\Gamma_t}\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2(x\cdot\overline{\nu})dsd\Gamma_t\\ &\quad -\frac{\gamma'}{2\gamma}\int_{\Gamma_t}\int_0^t g(t-s)|\nabla u(t)|^2(\overline{\nu}\cdot x)d\Gamma_t.\end{aligned}$$

The proof is complete. □

Let us introduce the functional

$$E(t)=\|u_t\|^2_{L^2(\Omega_t)}+\Big(1-\int_0^t g(s)ds\Big)\|\nabla u\|^2_{L^2(\Omega_t)}+g\Box\nabla u.$$

Lemma 3.3. *Let us take $u_0\in H_0^1(\Omega_0)\cap H^2(\Omega_0)$, $u_1\in H_0^1(\Omega_0)$ and let us suppose that assumptions* (1.11)–(1.13) *and* (2.1)–(2.2) *hold. Then any regular solution of system* (1.1)–(1.3) *satisfies*

$$\begin{aligned}&\frac{d}{dt}E(t)-\int_{\Gamma_t}\frac{\gamma'}{\gamma}(\overline{\nu}\cdot x)(|u_t|^2+|\nabla u|^2)d\Gamma_t\\ &-\int_{\Gamma_t}\frac{\gamma'}{\gamma}(\overline{\nu}\cdot x)\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2dsd\Gamma_t\\ &=-\frac{1}{2}\int_{\Omega_t}g(t)|\nabla u|^2dx+\frac{1}{2}g'\Box\nabla u.\end{aligned}$$

Proof. Multiplying the equation (1.1) by u_t and integrating over Ω_t we get

$$\frac{1}{2}\int_{\Omega_t}\frac{d}{dt}|u_t|^2dx+\frac{1}{2}\int_{\Omega_t}\frac{d}{dt}|\nabla u|^2dx-\int_{\Omega_t}\int_0^t g(t-s)\nabla u(s)\cdot\nabla u_t ds\,dx=0.$$

Using Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}
&\frac{d}{dt}E(t) - \frac{\gamma'}{2\gamma}\int_{\Gamma_t}(\overline{\nu}\cdot x)|u_t|^2 d\Gamma_t - \frac{\gamma'}{2\gamma}\Big(1-\int_0^t g(s)ds\Big)\int_{\Gamma_t}(\overline{\nu}\cdot x)|\nabla u|^2 d\Gamma_t \\
&- \frac{\gamma'}{2\gamma}\int_{\Gamma_t}\int_0^t(\overline{\nu}\cdot x)g(t-s)|\nabla u(\cdot,t)-\nabla u(\cdot,s)|^2 ds d\Gamma_t \\
&= -\frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx + \frac{1}{2}g'\Box\nabla u.
\end{aligned}$$

The proof is complete. □

For the estimate of the term $\int_{\Omega_t}|u_t|^2dx$ we introduced the functional

$$\varphi(t) = -\int_{\Omega_t} u_t(g*u)_t dx + \frac{1}{2}\int_\Omega |g*\nabla u|^2 dx$$

where $(g*u)_t = g(0)u + g'*u$.

Lemma 3.4. *Let us take $u_0 \in H_0^1(\Omega_0)\cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$ and let us suppose that assumptions* (1.11)–(1.13) *and* (2.1)–(2.2) *hold. Then any regular solution of system* (1.1)–(1.3) *satisfies*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\varphi(t) - \frac{\gamma'}{2\gamma}\int_{\Gamma_t}(\overline{\nu}\cdot x)|g*\nabla u|^2 d\Gamma_t \\
&\le -\frac{g(0)}{2}\int_{\Omega_t}|u_t|^2dx + \frac{3g(0)}{2}\int_{\Omega_t}|\nabla u|^2dx + g(t)\int_{\Omega_t}|\nabla u|^2 dx \\
&\quad + \frac{(\int_0^t|g'(s)|ds)}{2g(0)}|g'|\Box\nabla u + \frac{|g'(t)|^2}{g(0)}\int_{\Omega_t}|u_0|^2dx \\
&\quad - g(t)\int_{\Omega_t}|u_t|^2dx + \frac{(\int_0^t|g'(s)|ds)}{g(0)}|g'|\Box u_t.
\end{aligned}$$

Proof. From the equation (1.1) and using the fact that $u=0$ on the boundary we get

$$\begin{aligned}
&-\frac{d}{dt}\int_{\Omega_t}u_t(g*u)_t dx \\
&= \int_{\Omega_t}(-\Delta u + g*\Delta u)(g*u)_t dx - g(0)\int_{\Omega_t}|u_t|^2dx - \int_{\Omega_t}u_t(g'*u)dx \\
&= g(0)\int_{\Omega_t}|\nabla u|^2dx + \int_{\Omega_t}\nabla u\cdot(g'*\nabla u)dx - \frac{1}{2}\int_{\Omega_t}\frac{d}{dt}|g*\nabla u|^2dx \\
&\quad - g(0)\int_{\Omega_t}|u_t|^2dx - \int_{\Omega_t}u_t(g'*u)_t dx.
\end{aligned} \tag{3.5}$$

Using Lemma 3.1 we have

$$\frac{1}{2}\int_{\Omega_t}\frac{d}{dt}|g*\nabla u|^2dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}|g*\nabla u|^2dx - \frac{\gamma'}{2\gamma}\int_{\Gamma_t}(\overline{\nu}\cdot x)|g*\nabla u|^2d\Gamma_t. \tag{3.6}$$

Define

$$I_1 := \int_{\Omega_t}\nabla u\cdot\int_0^t g'(t-s)\nabla u(s)ds\,dx;$$

then we have

$$
\begin{aligned}
I_1 &= \int_{\Omega_t} g(t)|\nabla u|^2 dx + \int_{\Omega_t} \nabla u \cdot \int_0^t g'(t-s)(\nabla u(\cdot,t) - \nabla u(\cdot,s)) ds\, dx \\
&\le (\int_{\Omega_t} |\nabla u|^2 dx)^{1/2} (\int_0^t |g'(s)| ds)^{1/2} (|g'|\Box \nabla u)^{1/2} + g(t) \int_{\Omega_t} |\nabla u|^2 dx.
\end{aligned}
\tag{3.7}
$$

Define

$$
\begin{aligned}
I_2 &:= -\int_{\Omega} u_t (g * u)_t dx \\
&= -\int_{\Omega_t} u_t (g'(t) u_0 + g' * u_t) dx \\
&= -\int_{\Omega_t} g'(t) u_t u_0 dx - \int_{\Omega_t} g(t) |u_t|^2 dx \\
&\quad - \int_{\Omega_t} u_t \int_0^t g'(t-s)(u_t(\cdot,s) - u_t(\cdot,t)) ds\, dx;
\end{aligned}
$$

then thanks to the Young inequality, we obtain

$$
|I_2| \le \frac{g(0)}{2} \int_{\Omega_t} |u_t|^2 dx + \frac{|g'|^2}{g(0)} \int_{\Omega_t} |u_0|^2 dx - \int_{\Omega_t} |u_t|^2 dx + \frac{|g(t) - g(0)|}{g(0)} |g'|\Box u_t. \tag{3.8}
$$

Substituting the inequalities (3.6), (3.7) and (3.8) into (3.5) we obtain the conclusion of lemma. $\square$

For the estimate of the term $\int_{\Omega_t} |g * \nabla u|^2 dx$ we introduced the following functional

$$
\eta(t) := \frac{1}{2} \int_{\Omega_t} g \Box u_t dx - \int_{\Omega_t} \Big(\int_0^t g(s) ds \Big) |u_t|^2 dx - \frac{1}{2} \int_{\Omega_t} |g * \nabla u|^2 dx.
$$

Lemma 3.5. *Let us take $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$ and let us suppose that assumptions (1.11)–(1.13) and (2.1)–(2.2) hold. Then any regular solution of system (1.1)–(1.3) satisfies*

$$
\begin{aligned}
&\frac{d}{dt}\eta(t) + \frac{\gamma'}{2\gamma} \int_{\Gamma_t} (\overline{\nu} \cdot x) |g * \nabla u|^2 d\Gamma_t \\
&- \frac{\gamma'}{2\gamma} \int_{\Gamma_t} \int_0^t (\overline{\nu} \cdot x) g(t-s) |u_t(\cdot,t) - u_t(\cdot,s)|^2 ds d\Gamma_t \\
&+ \frac{\gamma'}{2\gamma} \int_{\Gamma_t} (\overline{\nu} \cdot x) (\int_0^t g(s) ds) |u_t|^2 d\Gamma_t \\
&\le -g(t) \int_{\Omega_t} |u_t|^2 dx + g'\Box u_t - \frac{g(0)}{2} \int_{\Omega_t} |\nabla u|^2 dx \\
&\quad + g(t) \int_{\Omega_t} |\nabla u|^2 dx + \frac{(\int_0^t |g'(s)| ds)}{2g(0)} |g'|\Box \nabla u.
\end{aligned}
$$

Proof. Multiplying the equation (1.1) by $g * u_t$ and using similar argument as in the lemma 3.4 we obtain

$$\begin{aligned}
&\frac{d}{dt}\eta(t) + \frac{\gamma'}{2\gamma}\int_{\Gamma_t}(\overline{\nu}\cdot x)|g*\nabla u|^2 d\Gamma_t \\
&- \frac{\gamma'}{2\gamma}\int_{\Gamma_t}\int_0^t(\overline{\nu}\cdot x)g(t-s)|u_t(\cdot,t)-u_t(\cdot,s)|^2 ds d\Gamma_t \\
&+ \frac{\gamma'}{2\gamma}\int_{\Gamma_t}(\overline{\nu}\cdot x)(\int_0^t g(s)ds)|u_t|^2 d\Gamma_t \\
&= -\frac{1}{2}g(t)\int_{\Omega_t}|u_t|^2dx + \frac{1}{2}g'\Box u_t - g(0)\int_{\Omega_t}|\nabla u|^2dx \\
&\quad + g(t)\int_{\Omega_t}|\nabla u|^2dx + \int_{\Omega_t}\nabla u\int_0^t g'(t-s)(\nabla u(\cdot,s)-\nabla u(\cdot,t))ds\,dx
\end{aligned} \tag{3.9}$$

Noting that

$$\begin{aligned}
&\int_{\Omega_t}\nabla u\int_0^t g'(t-s)(\nabla u(\cdot,s)-\nabla u(\cdot,t))ds\,dx \\
&\le \Big(\int_{\Omega_t}|\nabla u|^2dx\Big)^{1/2}\Big(\int_0^t|g'(s)|ds\Big)^{1/2}(|g'|\Box\nabla u)^{1/2},
\end{aligned}$$

considering the inequality $ab \le \frac{a^2}{2}+\frac{b^2}{2}$ and using the Cauchy-Schwarz inequality we deduce that

$$\begin{aligned}
&\frac{d}{dt}\eta(t) + \frac{\gamma'}{2\gamma}\int_{\Gamma_t}(\overline{\nu}\cdot x)|g*\nabla u|^2 d\Gamma_t \\
&- \frac{\gamma'}{2\gamma}\int_{\Gamma_t}\int_0^t(\overline{\nu}\cdot x)g(t-s)|u_t(\cdot,t)-u_t(\cdot,s)|^2 ds d\Gamma_t \\
&+ \frac{\gamma'}{2\gamma}\int_{\Gamma_t}(\overline{\nu}\cdot x)(\int_0^t g(s)ds)|u_t|^2 d\Gamma_t \\
&\le -\frac{1}{2}g(t)\int_{\Omega_t}|u_t|^2dx + \frac{1}{2}g'\Box u_t - \frac{g(0)}{2}\int_{\Omega_t}|\nabla u|^2dx \\
&\quad + g(t)\int_{\Omega_t}|\nabla u|^2dx + \frac{(\int_0^t|g'(s)|ds)}{2g(0)}|g'|\Box\nabla u.
\end{aligned}$$

The proof is complete. $\square$

To estimate the term $(1-\int_0^t g(s)ds)\int_{\Omega_t}|\nabla u|^2dx$, we introduced the functional

$$\psi(t) = \int_{\Omega_t} u_t u dx.$$

Lemma 3.6. *Let us take $u_0 \in H_0^1(\Omega_0)\cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$ and let us suppose that assumptions (1.11)–(1.13) and (2.1)–(2.2) hold. Then any regular solution of system (1.1)–(1.3) satisfies*

$$\begin{aligned}
\frac{d}{dt}\psi(t) &\le -(1-\int_0^t g(s)ds)\int_{\Omega_t}|\nabla u|^2dx + g(0)\int_{\Omega_t}|\nabla u|^2dx \\
&\quad + \frac{(\int_0^t g(s)ds)}{4g(0)}g\Box\nabla u + \int_{\Omega_t}|u_t|^2dx.
\end{aligned}$$

Proof. From (1.1) we get

$$\frac{d}{dt}\psi(t) = -\int_{\Omega_t} |\nabla u|^2 dx + \int_{\Omega_t} \nabla u \int_0^t g(t-s)\nabla u(\cdot, s) ds\, dx + \int_{\Omega_t} |u_t|^2 dx.$$

Considering the inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$, making use of the Cauchy-Schwarz inequality and using similar arguments as in the lemmas 3.4 and 3.5, follows the conclusion of lemma. □

The following lemma is the key to obtain exponential decay.

Lemma 3.7. *Let f be a real positive function of class C^1. If there exists positive constants γ_0, γ_1 and C_0 such that*

$$f'(t) \le -\gamma_0 f(t) + C_0 e^{-\gamma_1 t},$$

then there exist positive constants γ and C such that

$$f(t) \le (f(0) + C)e^{-\gamma t}.$$

Proof. First, suppose that $\gamma_0 < \gamma_1$. Define

$$F(t) := f(t) + \frac{C_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t}.$$

Then

$$F'(t) = f'(t) - \frac{\gamma_1 C_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t} \le -\gamma_0 F(t).$$

Integrating from 0 to t we arrive at

$$F(t) \le F(0)e^{-\gamma_0 t} \quad \Rightarrow \quad f(t) \le \Big(f(0) + \frac{C_0}{\gamma_1 - \gamma_0}\Big)e^{-\gamma_0 t}.$$

Now, we shall assume that $\gamma_0 \ge \gamma_1$. In this conditions we get

$$f'(t) \le -\gamma_1 f(t) + C_0 e^{-\gamma_1 t} \quad \Rightarrow \quad \{e^{\gamma_1 t} f(t)\}' \le C_0.$$

Integrating from 0 to t we obtain

$$f(t) \le \big(f(0) + C_0 t\big)e^{-\gamma_1 t}.$$

Since $t \le (\gamma_1 - \epsilon)e^{(\gamma_1 - \epsilon)t}$ for any $0 < \epsilon < \gamma_1$ we conclude that

$$f(t) \le \{f(0) + C_0(\gamma_1 - \epsilon)\}e^{-\epsilon t}.$$

This completes the proof. □

Let us introduce the functional

$$\mathcal{L}(t) = N_1 E(t) + N_2 \eta(t) + \epsilon\psi(t) + \varphi(t), \tag{3.10}$$

with $N_1 > N_2 > 0$ and $\epsilon > 0$ small enough. It is not difficult to see that $\mathcal{L}(t)$ verifies

$$k_0 E(t) \le \mathcal{L}(t) \le k_1 E(t), \tag{3.11}$$

for k_0 and k_1 positive constants. Now we are in a position to show the main result of this paper.

Theorem 3.8. *Let us take $u_0 \in H_0^1(\Omega_0)$, $u_1 \in L^2(\Omega_0)$ and let us suppose that assumptions* (1.12), (1.13), (2.1), (2.2), (3.2) *and* (3.3) *hold. Then any regular solution of system* (1.1)–(1.3) *satisfies*

$$E(t) \le Ce^{-\xi t}E(0), \quad \forall t \ge 0$$

where C and ξ are positive constants.

Proof. We shall prove this result for strong solutions, that is, for solutions with initial data $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$. Our conclusion will follows by standard density arguments. Taking N_1, N_2 large enough, with $N_1 > N_2$, $\epsilon > 0$ small enough and using the lemmas (3.3), (3.4), (3.5) and (3.6), we conclude that there exist positive constants α_0 and C_0 such that

$$\frac{d}{dt}\mathcal{L}(t) \le -\alpha_0 \mathcal{L}(t) + C_0 g^2(t) E(0).$$

Using the lemma 3.7 we obtain

$$\mathcal{L}(t) \le \{\mathcal{L}(0) + C\} e^{-\alpha_1 t}$$

where C and α_1 are positive constants. From equivalence relation (3.11) our conclusion follows. $\square$

4. Polynomial Rate of Decay

In this section we assume that the memory g satisfies:

$$g'(t) \le -C_1 g^{1+\frac{1}{p}}(t) \tag{4.1}$$

$$\alpha := \int_0^\infty g^{1-\frac{1}{p}}(s) ds < \infty \tag{4.2}$$

for some $p > 1$ and $t \ge 0$, with positive constant C_1. The following lemmas will play an important role in the sequel.

Lemma 4.1. *Suppose that g and h are continuous functions, $g \in L^{1+\frac{1}{q}}(0,\infty) \cap L^1(0,\infty)$ and $g^r \in L^1(0,\infty)$ for some $0 \le r < 0$. Then*

$$\begin{aligned} &\int_0^t |g(t-s)h(s)| ds \\ &\le \Big\{ \int_0^t |g(t-s)|^{1+\frac{1-r}{q}} |h(s)| ds \Big\}^{\frac{q}{q+1}} \Big\{ \int_0^t |g(t-s)|^r |h(s)| ds \Big\}^{\frac{1}{q+1}}. \end{aligned}$$

Proof. Without loss of generality we can suppose that $g, h \ge 0$. Note that for any fixed t we have

$$\int_0^t g(t-s)h(s) ds = \lim_{\|\Delta s_i\| \to 0} \sum_{i=1}^m g(t-s_i) h(s_i) \Delta s_i.$$

Letting

$$I_m^r := \sum_{j=1}^m g^r(t-s_j) h(s_j) \Delta s_j,$$

we may write

$$\sum_{i=1}^m g(t-s_i) h(s_i) \Delta s_i = \sum_{i=1}^m \varphi_i \theta_i,$$

where

$$\varphi_i = (g^{1-r}(t-s_i) I_m^r), \quad \theta_i = \big(\frac{g^r(t-s_i)h(s_i)\Delta s_i}{I_m^r}\big).$$

Since the function $F(z) := |z|^{1+\frac{1}{q}}$ is convex, it follows that

$$F(\sum_{i=1}^m g(t-s_i)h(s_i)\Delta s_i) = F(\sum_{i=1}^m \varphi_i\theta_i) \le \sum_{i=1}^m \theta_i F(\varphi_i),$$

so, we have

$$\Big\{\sum_{i=1}^{m} g(t-s_i)h(s_i)\Delta s_i\Big\}^{1+\frac{1}{q}} \le |I_m^r|^{\frac{1}{q}} \sum_{i=1}^{m} g^{1+\frac{1-r}{q}}(t-s_i)h(s_i)\Delta s_i. \tag{4.3}$$

In view of

$$\lim_{\|\Delta s_i\|\to 0} I_m^r = \int_0^t g^r(t-s)h(s)ds,$$

letting $\|\Delta s_i\| \to 0$ in (4.3), we get

$$\Big\{\int_0^t g(t-s)h(s)ds\Big\}^{1+\frac{1}{q}} \le \Big\{\int_0^t g^r(t-s)h(s)ds\Big\}^{1+q}\Big\{\int_0^t g^{1+\frac{1-r}{q}}\Big\},$$

from which our result follows. $\square$

Lemma 4.2. *Let $w \in C(0,T;H_0^1(\Omega_t))$ and g be a continuous function satisfying hypothesis (4.1)–(4.2). Then for $0<r<1$ we have*

$$g\Box\nabla w \le 2\Big\{\int_0^t g^r ds\|w\|_{C(0,T;H_0^1(\Omega_t))}\Big\}^{\frac{1}{1+(1-r)p}}\Big\{g^{1+\frac{1}{p}}\Box\nabla w\Big\}^{\frac{(1-r)p}{1+(1-r)P}},$$

while for $r=0$ we have

$$g\Box\nabla w \le 2\Big\{\int_0^t \|w(s)\|^2_{H_0^1(\Omega_t)}ds + t\|w(t)\|^2_{H_0^1(\Omega_t)}\Big\}\Big\{g^{1+\frac{1}{p}}\Big\}^{\frac{p}{1+p}}.$$

Proof. ¿From hypotheses on w and Lemma 4.1 we get

$$\begin{aligned}
g\Box\nabla w &= \int_0^t g(t-s)h(s)ds \\
&\le \Big\{\int_0^t g^r(t-s)h(s)ds\Big\}^{\frac{1}{1+p(1-r)}}\Big\{\int_0^t g^{1+\frac{1}{p}}(t-s)h(s)ds\Big\}^{\frac{(1-r)p}{1+p(1-r)}} \\
&\le \{g^r\Box\nabla w\}^{\frac{1}{1+p(1-r)}}\{g^{1+\frac{1}{p}}\Box\nabla w\}^{\frac{(1-r)p}{1+p(1-r)}}
\end{aligned} \tag{4.4}$$

where

$$h(s) = \int_{\Omega_t} |\nabla w(t) - \nabla w(s)|^2 ds.$$

For $0<r<1$, we have

$$g^r\Box\nabla w = \int_{\Omega_t}\int_0^t g^r(t-s)|\nabla w(t)-\nabla w(s)|^2 ds\,dx \le 4\int_0^t g^r(s)\|w\|^2_{C(0,T:H^1_(\Omega_t))},$$

from which the first inequality of Lemma 4.2 follows. To prove the last part, let us take $r=0$ in Lemma 4.1 to get

$$\begin{aligned}
1\Box\nabla w &= \int_{\Omega_t}\int_0^t |\nabla w(t)-\nabla w(s)|^2 ds\,dx \\
&\le 2t\|w(t)\|^2_{H_0^1(\Omega_t)} + 2\int_0^t \|w(s)\|^2_{H_0^1(\Omega_t)}.
\end{aligned}$$

Substitution of the above inequality into (4.4) yields the second inequality. The proof is complete. $\square$

Lemma 4.3. *Let f be a non-negative C^1 function satisfying*

$$f'(t) \leq -k_0[f(t)]^{1+\frac{1}{p}} + \frac{k_1}{(1+t)^{p+1}},$$

for some positive constants k_0, k_1 and $p > 1$. There exists a positive constant C_1 such that

$$f(t) \leq C_1 \frac{pf(0) + 2k_1}{(1+t)^p}$$

Proof. Let $h(t) := \frac{2k_1}{p(1+t)^{p+1}}$ and $F(t) := f(t) + h(t)$. Then

$$\begin{aligned}
F'(t) &= f'(t) - \frac{2k_1}{(1+t)^{p+1}} \\
&\leq -k_0[f(t)]^{1+\frac{1}{p}} - \frac{k_1}{(1+t)^{p+1}} \\
&\leq -k_0\Big\{[f(t)]^{1+\frac{1}{p}} + \frac{p^{1+\frac{1}{p}}}{2k_0k_1^{\frac{1}{p}}}[h(t)]^{1+\frac{1}{p}}\Big\}.
\end{aligned}$$

From which it follows that there exists a positive constant C_1 such that

$$F'(t) \leq -C_1\{[f(t)]^{1+\frac{1}{p}} + [h(t)]^{1+\frac{1}{p}}\},$$

which gives the required inequality. $\square$

Theorem 4.4. *Let us take $u_0 \in H_0^1(\Omega_0)$, $u_1 \in L^2(\Omega_0)$ and let us suppose that assumptions* (1.12), (1.13), (2.1) (2.2), (4.1) *and* (4.2) *hold. Then any regular solution of system* (1.1)–(1.3) *satisfies*

$$E(t) \leq CE(0)(1+t)^{-p},$$

where C is a positive constant and $p > 1$.

Proof. We shall prove this result for strong solutions, that is, for solutions with initial data $u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$. Our conclusion will follows by standard density arguments. From the Lemmas 3.3, 3.4, 3.5 and 3.6 we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\leq -C_0\Big\{\int_{\Omega_t}|u_t|^2dx + \big(1 - \int_0^t g(s)ds\big)\int_{\Omega_t}|\nabla u|^2dx - g'\Box\nabla u\Big\} \\
&\quad + C_1g^2(t)\int_{\Omega_0}|u_0|^2dx.
\end{aligned}$$

Using hypothesis (4.1) we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\leq -C_0\Big\{\int_{\Omega_t}|u_t|^2dx + \big(1 - \int_0^t g(s)ds\big)\int_{\Omega_t}|\nabla u|^2dx\Big\} \\
&\quad - C_1g^{1+\frac{1}{p}}(t)\Box\nabla u + C_2g^2(t)\int_{\Omega_0}|u_0|^2dx.
\end{aligned}$$

for some positive constants C_0, C_1 and C_2. Let us define the functional

$$\mathcal{N}(t) = \int_{\Omega_t}|u_t|^2dx + \int_{\Omega_t}|\nabla u|^2dx.$$

Since the total energy is bounded, Lemma 4.2 implies

$$\mathcal{N}(t) \geq C_2\mathcal{N}(t)^{\frac{(1+(1-r)p)}{(1-r)p}},$$

$$g^{1+\frac{1}{p}}(t)\Box\nabla u \geq C_2 \{\ g\Box\nabla u\}^{\frac{(1+(1-r)p)}{(1-r)p}}.$$

It is not difficult to see that for N_1, N_2 large enough, with $N_1 > N_2$, and ϵ small enough the inequality

$$CE(t) \leq \mathcal{L}(t) \leq C_3\{\mathcal{N}(t) + g\Box\nabla u\} \leq C_4 E(t)$$

holds. From this follows that

$$\frac{d}{dt}\mathcal{L}(t) \leq -C_5\mathcal{L}(t)^{\frac{(1+(1-r)p)}{(1-r)p}} + C_2 g^2(t)\int_{\Omega_0} |u_0|^2 dx.$$

Using Lemma 4.3, we obtain

$$\mathcal{L}(t) \leq C\{\mathcal{L}(0) + C_6\}\frac{1}{(1+t)^{p(1-r)}}$$

where C and C_6 are positive constants independent on the initial data. From which it follows that the energy decay to zero uniformly.

Using Lemma 4.2 for $r = 0$ we get

$$\mathcal{N}(t) \geq C_2\mathcal{N}(t)^{\frac{(1+p)}{p}},$$
$$g^{1+\frac{1}{p}}(t)\Box\nabla u \geq C_2\big\{g\Box\nabla u\big\}^{\frac{1}{p}}.$$

Repeating the same reasoning as above, we obtain

$$\mathcal{L}(t) \leq C\{\mathcal{L}(0) + C_6\}\frac{1}{(1+t)^{p}}$$

From which our result follows. The proof is now complete. □

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References

[1] M. Assila, M. M. Cavalcanti and J. A. Soriano, *Asymptotic stability and energy decay rates for solutions of the wave equation with memory in a star-shaped domain.* Siam J. Control Optim., Vol. 38, N 5, 1581-1602 (2000).

[2] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. S. Prates Filho & J. A. Soriano, *Existence and uniform decay rates for viscoelastic problems with non-linear boundary damping.* Differential and Integral Equations, 14(1), 85-116 (2001).

[3] M. M. Cavalcanti, V. N. Domingos Cavalcanti & J. Ferreira, *Existence and uniform decay for a non-linear viscoelastic equatioin with strong damping.* Math. Meth. Appl. Sci. 24, 1043-1053 (2001).

[4] F. Conrad and B. Rao, *Decay of solutios of the wave equation in a star-shaped domain with nonlinear boundary feedback.* Asymptotic Analysis, 7, 159-177 (1993).

[5] J. Ferreira and M. L. Santos, *Asymptotic behaviour for wave equations with memory in a noncylindrical domains.* Comunications on Pure and Applied Analysis, 2(4), 511-520 (2003).

[6] V. Komornik, *Exact controllability and stabilization. The Multiplier Method.* John Wiley and Sons-Masson, Paris (1994).

[7] I. Lasiecka and D. Tataru, *Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping.* Lectures Notes in Pure and Appl. Math. 142, Dekker, New York, (1993).

[8] J. L. Lions, *Quelques Métodes de Résolution des Problémes aux Limites Non Linéaires.* Dunod, Paris (1969).

[9] J. L. Lions & E. Magenes, *Problémes aux Limites non Homog'enes, Aplicatons.* Dunod, Paris, Vol. 1 (1968).

[10] J. E. Muñoz Rivera, *Energy decay rates in linear thermoelasticity.* Funkc. Ekvacioj, Vol. 35(1), 19-30 (1992).

[11] J. E. Munõz Rivera, *Global solutions on a quasilinear wave equation with memry.* Bolletino Unione Matematica Italiana, 7(8-B), 289-303 (1994).

[12] M. Nakao, *Asymptotic stability of the bounded or almost periodic solution of the wave equation with a nonlinear dissipative term.* J. Math. Anal. Appl., 58, 336-343 (1977).

[13] M. Nakao, *Energy decay for the wave equation with a nonlinear weak dissipation.* Differential Integral Equations, 8, 681-688 (1995).

[14] R. D. Passo & M. Ughi, *Problèmes de Dirichlet pour une classe d'équations paraboliques non linéaires dans des ouverts non cylindriques.* C. R. Acad. Sc. Paris, 308, 555-558 (1989).

[15] L. R. Tcheugoué Tébou, *Stabilization of the wave equation with localized nonlinear damping.* J. Differential Equations, 145, 502-524 (1998).

[16] M. L. Santos, *Asymptotic behavior of solutions to wave equtions with a memory condition at the bounadry.* Electronic J. Differential Equations, Vol. 2001, no. 73, 1-11, (2001).

[17] M. L. Santos, *Decay rates for solutions of a system of wave equations with memory.* Elect. Journal of Differential Equations, Vol. 2002, no. 38, 1-17 (2002).

[18] M. L. Santos and J. Ferreira, *Stability for a system of wave equations of Kirchhoff with coupled nonlinear and boundary conditions of memory type.* Advances in Differential Equations, 8(7), 873-896 (2003).

[19] J. Vancostenoble and P. Martinez, *Optimality of energy estimates for the wave equation with nonlinear boundary velocity feedbacks.* Siam J. Control Optim. Vol. 39, no. 3, 776-797 (2000).

[20] E. Zuazua, *Uniform stabilization of the wave equation by nonlinear boundary feedback.* Siam J. Control Optim., 28, 466-477 (1990).

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