Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 23, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

PULLBACK ATTRACTORS FOR A CLASS OF NON-NEWTONIAN MICROPOLAR FLUIDS

GERALDO M. DE ARAUJO, MARCOS A. F. ARAUJO,
FLANK D. M. BEZERRA, MIRELSON M. FREITAS

Abstract. In this article we study the long time behavior of the two-dimensional flow for non-Newtonian micropolar fluids in bounded smooth domains, in the sense of pullback attractors. We prove the existence and upper semicontinuity of the pullback attractors with respect to the viscosity coefficient of the model.

1. Introduction

This article concerns the long time behavior of the two-dimensional flow of a non-Newtonian micropolar fluid in the sense of pullback attractors. We are interested in a class of models of non-Newtonian micropolar fluids, where the relation between the viscous stress tensor and the symmetric component of the gradient (derivative with respect to position) of the flow velocity is nonlinear and it is defined by a class of non-negative and continuously differentiable functions, we consider the following mathematical model of a non-Newtonian micropolar fluid

$$\begin{gathered} \partial_t u - \nabla \cdot \tau(e(u)) + (u\cdot\nabla)u + \nabla p = 2\nu_r \operatorname{rot} w + f(x,t), \quad x\in\Omega,\ t>\tau, \\ \nabla\cdot u = 0, \quad x\in\Omega,\ t>\tau, \\ \partial_t w - \nu_1\Delta w + (u\cdot\nabla)w + 4\nu_r w = 2\nu_r \operatorname{rot} u + g(x,t), \quad x\in\Omega,\ t>\tau, \end{gathered} \tag{1.1}$$

with corresponding initial-boundary condition

$$\begin{gathered} u(x,\tau) = u_\tau(x), \quad w(x,\tau) = w_\tau(x), \quad x\in\Omega, \\ u(x,t) = 0, \quad w(x,t) = 0, \quad x\in\partial\Omega,\ t>\tau, \end{gathered} \tag{1.2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^2$, the positive constants ν_1, ν_r represent viscosity coefficients, $u = (u_1, u_2)$ is the velocity field, p is the pressure, and w is the scalar microrotation field, commonly interpreted as the angular velocity field of rotation of particles, the fields $f = (f_1, f_2)$ and g are external forces and moments, respectively.

The map $\tau : \mathbb{R}^2_{\rm sym} \to \mathbb{R}^2_{\rm sym}$ denotes the extra stress tensor given by

$$\tau(e(u)) = 2\left(\nu + \nu_r + M(|e(u)|^2)\right) e(u), \tag{1.3}$$

where $\mathbb{R}^{2^2}_{\rm sym}$ represents the set of all symmetric 2×2 matrices, $\nu > 0$ represents the usual Newtonian viscosity, $M : (0,+\infty) \to (0,+\infty)$ is a continuously differentiable

---

2010 *Mathematics Subject Classification.* 35B40, 35B41, 35Q30, 76D03, 76D05.
Key words and phrases. Non-Newtonian micropolar fluid; non-autonomous dynamical system; pullback attractor; upper semincontinuity.

Submitted July 21, 2017. Published January 17, 2018.

function which denotes the generalized viscosity function, and $e : \mathbb{R}^2 \to \mathbb{R}^{2^2}_{\text{sym}}$ denotes the symmetric part of the velocity gradient, as well as in [1, 9, 15]; that is,

$$e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right),$$

whose components are defined by

$$e_{ij}(u) = \frac{1}{2}\left(\partial_{x_j} u_i + \partial_{x_i} u_j\right), \quad i, j = 1, 2.$$

Our motivation for considering the equations of micropolar fluid in (1.1) is the works [1, 11, 13, 18, 23]. Many works have been studied the model of micropolar fluids in many theoretical issues; namely, about existence, uniqueness, regularity and stability of solutions, see e.g. [1, 4, 7, 8, 9, 10, 17] and references therein; and on asymptotic behavior of solutions, in the sense of attractors, see e.g. [2, 3, 7, 9, 10, 12, 14, 19, 20, 21, 22] and references therein. Since the operator stress tensor in this paper is given by (1.3) we assume that there exist positive constants c_1, c_2 and c_3 such that for any $t > 0$,

$$c_1(1 + \sqrt{t})^2 \leqslant M(t) \leqslant c_2(1 + \sqrt{t})^2, \tag{1.4}$$

$$0 \leqslant M'(t)\sqrt{t} \leqslant c_3(1 + \sqrt{t}) \tag{1.5}$$

in order to recover embedding theorems for Sobolev spaces, as well as in [9], and consequently to prove the existence and upper semicontinuity of the pullback attractors.

To better present our results we introduce some terminologies. The space V_p is the closure of

$$\mathscr{V} = \{(\varphi_1, \varphi_2) \in (C_0^\infty(\Omega))^2 : \nabla \cdot (\varphi_1, \varphi_2) = 0\}$$

in the space $(W^{1,p}(\Omega))^2$ with norm $\|\nabla u\|_p = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$, $1 \leqslant p < \infty$. For $p = 2$ we denote $V = V_2$ and the inner product and norm in V is denoted, respectively, by $((u, v)) = \sum_{i,j=1}^2 \int_\Omega \partial_{x_j} u_i \partial_{x_j} v_i dx$ and $\|u\| = ((u, u))^{1/2}$.

The space H is the closure of $\mathscr{V}$ in the space $(L^2(\Omega))^2$ with inner product and norm defined, respectively by $(u, v) = \sum_{i=1}^2 \int_\Omega u_i v_i dx$ and $|u| = (u, u)^{1/2}$. Note that V and H are Hilbert spaces, and we have the following embedding $V \hookrightarrow H \hookrightarrow V'$ where the first embedding is compact.

We introduce the bilinear form $a : V \times V \to \mathbb{R}$ defined by

$$a(u, v) = \sum_{i,j=1}^{2} \int_\Omega \partial_{x_j} u_i \partial_{x_j} v_i dx.$$

We also introduce the maps $B : V \times V \to V'$ defined by

$$B(u, v) = (u \cdot \nabla)v,$$

$B_1 : V \times H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$B_1(u, \omega) = (u \cdot \nabla)\omega,$$

and $\mathcal{K} : V_4 \to V_4'$ defined by

$$\mathcal{K}u = -\nabla \cdot [2M(|e(u)|^2)e(u)].$$

Definition 1.1. Let $f \in L^2(\tau, T; H)$, $g \in L^2(\tau, T; L^2(\Omega))$, $u_\tau \in H$ and $w_\tau \in L^2(\Omega)$. A weak solution of problem (1.1)-(1.2) is a pair of functions (u, w) such that for each $T > \tau$,

$$u \in L^\infty(\tau, T; H) \cap L^4(\tau, T; V_4),$$
$$w \in L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)),$$

with $u' \in L^{4/3}(\tau, T; V_4')$ and $w' \in L^2(\tau, T; H^{-1}(\Omega))$ such that $u(x, \tau) = u_\tau(x)$, $w(x, \tau) = w_\tau(x)$ and satisfying the following identities for all $\varphi \in V_4$ and $\phi \in H_0^1(\Omega)$,

$$\begin{aligned} &\frac{d}{dt}(u(t), \varphi) + (\nu + \nu_r)a(u(t), \varphi) + (\mathcal{K}u, \varphi) + (B(u(t), u(t)), \varphi) \\ &= 2\nu_r(\operatorname{rot} w(t), \varphi) + (f(t), \varphi) \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} &\frac{d}{dt}(w(t), \phi) + \nu_1(\nabla w(t), \nabla \phi) + (B_1(u(t), w(t)), \phi) + 4\nu_r(w(t), \phi) \\ &= 2\nu_r(\operatorname{rot} u(t), \phi) + (g(t), \phi) \end{aligned} \tag{1.7}$$

in the sense of scalar distributions on (τ, ∞).

System (1.1) was investigated in [1] on a bounded smooth domain of $\mathbb{R}^d$, the authors proved the existence of weak solution for $d \leqslant 3$ and uniqueness for $d = 2$ under these same conditions in M; namely, the authors proved the existence and uniqueness of solution of problem (1.1)-(1.2) in the sense of Definition 1.1. Moreover, for each $t > \tau$ the map $(u_\tau, w_\tau) \mapsto (u(t), w(t))$ is continuous as a map defined in $H \times L^2(\Omega)$.

Now we recall the definition of nonlinear evolution process (or non-autonomous dynamical systems) and pullback attractors, for more details we refer the reader to [5, 6, 16] and references therein.

Definition 1.2. An evolution process in a Banach space X is a family of continuous maps $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ from X into itself with the following properties:

(i) $S(t, t)x = x$, for all $t \in \mathbb{R}$ and $x \in X$;
(ii) $S(t, \tau) = S(t, s)S(s, \tau)$, for all $t \geqslant s \geqslant \tau$;
(iii) $(t, \tau) \mapsto S(t, \tau)x$ is continuous for $t \geq \tau$, $x \in X$.

Let $\mathcal{D}$ be a nonempty class of parameterised sets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

Definition 1.3. An evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ in a Banach space X is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, and any sequences $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ the set $\{S(t, \tau_n)x_n\}_{n \in \mathbb{N}}$ is precompact in X.

Definition 1.4. Let $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ be an evolution process in a Banach space X. The family $\widehat{B}$ is pullback $\mathcal{D}$-absorbing for the process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(\widehat{B}, t) \leqslant t$ such that

$$S(t, \tau)D(\tau) \subset B(t) \quad \text{for any } \tau \leq \tau_0(\widehat{B}, t).$$

Observe that in the above definition the set $\widehat{B}$ does not necessarily belong to the class $\mathcal{D}$. In the sequel we introduce the concept of pullback $\mathcal{D}$-attractor.

Definition 1.5. Let $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be an evolution process in a Banach space X. A family $\widehat{A} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ of subsets of X is said to be the pullback $\mathcal{D}$-attractor for the evolution process $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ if the following conditions are satisfied

- (i) $A(t)$ is compact for all $t \in \mathbb{R}$;
- (ii) $\widehat{A}$ invariant, i.e., $S(t,\tau)A(\tau) = A(t)$ for all $t \geq \tau$;
- (iii) $\widehat{A}$ pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{\tau \to -\infty} \operatorname{dist}(S(t,\tau)D(\tau), A(t)) = 0, \quad \text{for all } \widehat{D} \in \mathcal{D} \text{ and } t \in \mathbb{R}.$$

- (iv) $\widehat{A}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets which is pullback $\mathcal{D}$-attracting, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Theorem 1.6. *Let $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be an evolution process in a Banach space X. Suppose that the process $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is pullback $\mathcal{D}$-asymptotically compact and that $\widehat{B} \in \mathcal{D}$ is a family pullback $\mathcal{D}$-absorbing. Then $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ has a unique pullback $\mathcal{D}$-attractor $\widehat{A}$ given by*

$$A(t) = \Lambda(\widehat{B}, t) = \cap_{s \leq t} \overline{\cup_{\tau \leq s} S(t,\tau)B(\tau)}.$$

Let us investigate the existence of pullback attractor for the problem (1.1)-(1.2). For this, let us consider the class of all families tempered in $\mathcal{H} = H \times L^2(\Omega)$, equipped with the usual norm, as the attraction universe $\mathcal{D}$, i.e.,

$$\mathcal{D} = \big\{\widehat{D} : \widehat{D} = \{D(t) : t \in \mathbb{R}\},\ \lim_{\tau \to -\infty} e^{\varepsilon\tau}\|D(\tau)\| = 0,\ \forall \varepsilon > 0\big\},$$

where $\|D(\tau)\| := \sup_{(u,v) \in D(\tau)} \|(u,v)\|_{\mathcal{H}}$ for $\tau \in \mathbb{R}$. The main result of the paper is as follows.

Theorem 1.7. *Let $\mathcal{H} = H \times L^2(\Omega)$ equipped with the usual norm. Assume that*

$$\int_{-\infty}^{t} e^{\alpha_2 s}(|f(s)|^2 + |g(s)|^2)ds < \infty, \quad \text{for all } t \in \mathbb{R},$$

where $\alpha_2 > 0$ is constant. Then

- (i) *The evolution process generated by problem* (1.1)-(1.2) *possesses a unique pullback $\mathcal{D}$-attractor $\widehat{A} = \{A(t) : t \in \mathbb{R}\}$ in $\mathcal{H}$;*
- (ii) *For each $\nu_r \in [0,1]$, the family of pullback $\mathcal{D}$-attractor $\widehat{A}_{\nu_r} = \{A_{\nu_r}(t) : t \in \mathbb{R}\}$ in $\mathcal{H}$ is upper semicontinuity at $\nu_r = 0$ in the sense of Hausdorff semidistance in $\mathcal{H}$, that is, for each $t \in \mathbb{R}$,*

$$\lim_{\nu_r \to 0} \operatorname{dist}(A_{\nu_r}(t), A_0(t)) := \lim_{\nu_r \to 0} \sup_{x \in A_{\nu_r}(t)} \inf_{y \in A_0(t)} \|x - y\|_{\mathcal{H}} = 0.$$

This article is organized as follows. In Section 2 we prove the existence of pullback attractor for the evolution process generated by the problem (1.1)-(1.2) in $\mathcal{H}$. In section 3 we prove that the family of pullback attractors indexed by ν_r converge upper semicontinuously to the pullback attractor associated with (3.3)-(3.4) as $\nu_r \to 0$.

2. Existence of pullback attractor

In this section we prove the Theorem 1.7(i) via Theorem 1.6.

Lemma 2.1. *For each $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$, there exists $\tau_0(\widehat{D}, t) < t$ such that the solution (u, w) of the problem* (1.1)-(1.2) *satisfy the following estimate*

$$\begin{aligned} |u(t,\tau,u_\tau)|^2 + |w(t,\tau,w_\tau)|^2 &\le 1 + \alpha_3 \int_{-\infty}^{t} e^{-\alpha_2(t-s)}(|f(s)|^2 + |g(s)|^2)\,ds \\ &:= R_1(t) < \infty. \end{aligned} \tag{2.1}$$

uniformly in $(u_\tau, w_\tau) \in D(\tau)$ and $\tau \le \tau_0(\widehat{D}, t)$, where α_2 and α_3 are positive constants.

Proof. From (1.6) and (1.7) with $\varphi = u$ and $\psi = w$ we see that

$$\frac{1}{2}\frac{d}{dt}|u|^2 + (\nu+\nu_r)\|u\|^2 + 2\sum_{i,j=1}^{2}\int_\Omega M(|e(u)|^2)|e_{ij}(u)|^2\,dx = 2\nu_r(\operatorname{rot} w, u) + (f, u), \tag{2.2}$$

and

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu_1\|w\|^2 + 4\nu_r|w|^2 = 2\nu_r(\operatorname{rot} u, w) + (g, w). \tag{2.3}$$

By Schwarz's inequality, we also deduce that

$$2\nu_r(\operatorname{rot} w, u) = 2\nu_r(w, \operatorname{rot} u) \leqslant 2\nu_r|w|^2 + \frac{\nu_r}{2}\|u\|^2.$$

and using Korn's inequality (see e.g. [17]) and (1.4) we have

$$\int_\Omega M(|e(u)|^2)|e_{ij}(u)|^2\,dx \ge c_1\int_\Omega |e(u)|^4\,dx \ge c_1K_4^4\|u\|^4_{(W^{1,4}(\Omega))^2}, \tag{2.4}$$

and by Poincaré inequality

$$(f, u) \leqslant |f||u| \leqslant \frac{1}{\sqrt{\lambda_1}}\|u\||f| \leqslant \frac{\nu}{2}\|u\|^2 + \frac{1}{2\nu\lambda_1}|f|^2, \tag{2.5}$$

where $\lambda_1 > 0$ is the first eigenvalue of the Stokes operator A. Thus, from (2.2), (2.4) and (2.5) we obtain that

$$\frac{d}{dt}|u|^2 + (\nu+\nu_r)\|u\|^2 + 4c_1K_4^4\|u\|^4_{(W^{1,4}(\Omega))^2} \leqslant 4\nu_r|w|^2 + \frac{1}{\nu\lambda_1}|f|^2. \tag{2.6}$$

On other hand, we have

$$2\nu_r(\operatorname{rot} u, w) \leqslant 2\nu_r\|u\||w| \leqslant 2\nu_r|w|^2 + \frac{\nu_r}{2}\|u\|^2.$$

and again by Poincaré inequality

$$(g, w) \leqslant |g||w| \leqslant \frac{1}{\sqrt{\lambda}}|g|\|w\| \leqslant \frac{\nu_1}{2}\|w\|^2 + \frac{1}{2\nu_1\lambda}|g|^2,$$

where $\lambda > 0$ denotes the first eigenvalues of the negative Laplacian operator $-\Delta$ in $L^2(\Omega)$ with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$.

Using above estimates in (2.3) we also deduce that

$$\frac{d}{dt}|w|^2 + \nu_1\|w\|^2 + 4\nu_r|w|^2 \leqslant \nu_r\|u\|^2 + \frac{1}{\nu_1\lambda}|g|^2. \tag{2.7}$$

From (2.6) and (2.7) we obtain that

$$\frac{d}{dt}(|u|^2+|w|^2)+\nu\|u\|^2+\nu_1\|w\|^2+4c_1K_4^4\|u\|^4_{(W^{1,4}(\Omega))^2}\leqslant\frac{1}{\nu\lambda_1}|f|^2+\frac{1}{\nu_1\lambda}|g|^2. \tag{2.8}$$

Setting

$$\alpha_1=\min\{\nu,\nu_1\},\quad \alpha_2=\alpha_1\min(\lambda_1,\lambda),\quad \alpha_3=\max\Big(\frac{1}{\nu\lambda_1},\frac{1}{\nu_1\lambda}\Big). \tag{2.9}$$

Then from (2.8) we have

$$\frac{d}{dt}(|u|^2+|w|^2)+\alpha_1(\|u\|^2+\|w\|^2)+4c_1K_4^4\|u\|^4_{(W^{1,4}(\Omega))^2}\leqslant\alpha_3(|f|^2+|g|^2), \tag{2.10}$$

and by Poincaré inequality

$$\frac{d}{dt}(|u|^2+|w|^2)+\alpha_2(|u|^2+|w|^2)\leqslant\alpha_3(|f|^2+|g|^2). \tag{2.11}$$

Now multiplying (2.11) by $e^{\alpha_2 t}$, we get

$$\frac{d}{dt}\left\{e^{\alpha_2 t}(|u|^2+|w|^2)\right\}\le\alpha_3e^{\alpha_2 t}(|f(t)|^2+|g(t)|^2) \tag{2.12}$$

Integrating (2.12) from τ to t, we obtain

$$\begin{aligned}
&|u(t,\tau,u_\tau)|^2+|w(t,\tau,w_\tau)|^2\\
&\le e^{-\alpha_2(t-\tau)}(|u_\tau|^2+|w_\tau|^2)+\alpha_3\int_\tau^t e^{-\alpha_2(t-s)}(|f(s)|^2+|g(s)|^2)\,ds\\
&\le e^{-\alpha_2(t-\tau)}(|u_\tau|^2+|w_\tau|^2)+\alpha_3\int_{-\infty}^t e^{-\alpha_2(t-s)}(|f(s)|^2+|g(s)|^2)\,ds.
\end{aligned}$$

Since $(u_\tau,w_\tau)\in D(\tau)$ and $\widehat{D}\in\mathcal{D}$ it follows that there exists a $\tau_0(\widehat{D},t)\le t$ such that

$$|u(t,\tau,u_\tau)|^2+|w(t,\tau,w_\tau)|^2\le 1+\alpha_3\int_{-\infty}^t e^{-\alpha_2(t-s)}(|f(s)|^2+|g(s)|^2)\,ds,$$

for all $\tau\le\tau_0(\widehat{D},t)$. The proof is complete. $\square$

Lemma 2.2. *For each $t\in\mathbb{R}$ and $\widehat{D}\in\mathcal{D}$, there exists $\tau_0(\widehat{D},t)\le t$ given by Lemma 2.1 such that the solution (u,w) of the problem* (1.1)-(1.2) *satisfy the estimate*

$$\int_t^{t+1}\big(\|u(s)\|^2+\|w(s)\|^2+\|u(s)\|^4_{(W^{1,4}(\Omega))^2}\big)\,ds\le R_2(t)<\infty \tag{2.13}$$

uniformly in $(u_\tau,w_\tau)\in D(\tau)$ and $\tau\le\tau_0(\widehat{D},t)$, where $R_2(t)$ is given by (2.14).

Proof. Integrating (2.10) from τ to t, we see that

$$\begin{aligned}
&\alpha_1\int_t^{t+1}(\|u(s)\|^2+\|w(s)\|^2)\,ds+4c_1K_4^4\int_t^{t+1}\|u(s)\|^4_{(W^{1,4}(\Omega))^2}\,ds\\
&\leqslant(|u(t)|^2+|w(t)|^2)+\alpha_3\int_t^{t+1}(|f(s)|^2+|g(s)|^2)\,ds\\
&\le(|u(t)|^2+|w(t)|^2)+\alpha_3\int_{-\infty}^{t+1}e^{-\alpha_2(t-s)}(|f(s)|^2+|g(s)|^2)\,ds.
\end{aligned}$$

Taking $\mu = \min(\alpha_1, 4c_1K_4^4)$, applying the Lemma 2.1, we get that for any $\tau \leq \tau_0(\widehat{D}, t)$,

$$\int_t^{t+1} \big(\|u(s)\|^2 + \|w(s)\|^2 + \|u(s)\|^4_{(W^{1,4}(\Omega))^2}\big)\, ds \leq R_2(t),$$

where

$$R_2(t) := \frac{1}{\mu}\big\{1 + 2\alpha_3 \int_{-\infty}^{t+1} e^{-\alpha_2(t-s)}(|f(s)|^2 + |g(s)|^2)\, ds\big\}. \tag{2.14}$$

The proof is complete. □

Lemma 2.3. *For any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$, there exists $\tau_0(\widehat{D}, t) \leq t$ given by Lemma 2.1 such that the solution (u, w) of the problem* (1.1)-(1.2) *satisfy the estimate*

$$\|u(t, \tau, u_\tau)\|^2 + \|w(t, \tau, w_\tau)\|^2 \leq R_3(t) < \infty.$$

uniformly in $(u_\tau, w_\tau) \in D(\tau)$ and $\tau \leq \tau_0(\widehat{D}, t)$.

Proof. Using $\partial_t u$ as a test function in (1.6) we get

$$\begin{aligned} &|\partial_t u|^2 + \frac{\nu + \nu_r}{2}\frac{d}{dt}\|u\|^2 + \sum_{i,j=1}^2 \int_\Omega \tau_{ij}(e(u))e_{ij}(\partial_t u)dx \\ &= -\sum_{i,j=1}^2 \int_\Omega u_i(\partial_{x_i} u_j)\partial_t u_j\, dx + 2\nu_r(\operatorname{rot} w, \partial_t u) + (f, \partial_t u). \end{aligned} \tag{2.15}$$

where τ_{ij} is a tensor given by $\tau_{ij}(e) = 2M(|e|^2)e_{ij}$.

Using the definition of the potential

$$\Phi(e) = \int_0^{|e|^2} M(t)\, dt, \tag{2.16}$$

tt follows that

$$\frac{d}{dt}\int_\Omega \Phi(e(u))\, dx = \int_\Omega \tau_{ij}(e(u))e_{ij}(\partial_t u)\, dx. \tag{2.17}$$

Using Young's inequality we have

$$2\nu_r(\operatorname{rot} w, \partial_t u) \leqslant 4\nu_r^2\|w\|^2 + \frac{1}{4}|\partial_t u|^2,$$

and

$$(f, \partial_t u) \leqslant |f|^2 + \frac{1}{4}|\partial_t u|^2. \tag{2.18}$$

By (2.15), (2.17) and (2.18) we get

$$\begin{aligned} &\frac{1}{2}|\partial_t u|^2 + \frac{d}{dt}\Big(\frac{\nu + \nu_r}{2}\|u\|^2 + \int_\Omega \Phi(e(u))\, dx\Big) \\ &\leqslant \sum_{i,j=1}^2 \int_\Omega |u_i||\partial_{x_i} u_j||\partial_t u_j|\, dx + 4\nu_r^2\|w\|^2 + |f|^2. \end{aligned} \tag{2.19}$$

Note that

$$\sum_{i,j=1}^2 \int_\Omega |u_i||\partial_{x_i} u_j||\partial_t u_j|\, dx \leqslant \|u\|^2_{(L^\infty(\Omega))^2}\|u\|^2 + \frac{1}{4}|\partial_t u|^2.$$

By embedding $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $q > 2$, there exists $c > 0$ such that

$$\sum_{i,j=1}^{2} \int_\Omega |u_i||\partial_{x_i} u_j||\partial_t u_j|\,dx \leqslant c\|u\|^2_{(W^{1,4}(\Omega))^2}\|u\|^2 + \frac{1}{4}|\partial_t u|^2, \tag{2.20}$$

and from (2.19) and (2.20) it follows that

$$\begin{aligned}&\frac{1}{4}|\partial_t u|^2 + \frac{d}{dt}\Big(\frac{\nu+\nu_r}{2}\|u\|^2 + \int_\Omega \Phi(e(u))\,dx\Big)\\ &\leqslant c\|u\|^2_{(W^{1,4}(\Omega))^2}\|u\|^2 + 4\nu_r^2\|w\|^2 + |f|^2.\end{aligned} \tag{2.21}$$

By Korn's inequality, (1.4) and (2.16) we have

$$\int_\Omega \Phi(e(u))\,dx \geq c_1 \int_\Omega |e(u)|^2\,dx \geq c_1 K_2^2\|u\|^2. \tag{2.22}$$

Thus,

$$\|u\|^2 \leqslant k_0\Big(\frac{\nu+\nu_r}{2}\|u\|^2 + \int_\Omega \Phi(e(u))\,dx\Big), \tag{2.23}$$

where $k_0 = \frac{1}{c_1K_2^2}$. Employing (2.23) in (2.21) we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{\nu+\nu_r}{2}\|u\|^2 + \int_\Omega \Phi(e(u))dx\Big)\\ &\leqslant ck_0\|u\|^2_{(W^{1,4}(\Omega))^2}\Big(\frac{\nu+\nu_r}{2}\|u\|^2 + \int_\Omega \Phi(e(u))\,dx\Big) + 4\nu_r^2\|w\|^2 + |f|^2.\end{aligned} \tag{2.24}$$

If we denote

$$y(t) = \frac{\nu+\nu_r}{2}\|u\|^2 + \int_\Omega \Phi(e(u))\,dx, \quad g(t) = ck_0\|u\|^2_{(W^{1,4}(\Omega))^2},$$
$$h(t) = 4\nu_r^2\|w\|^2 + |f|^2,$$

then

$$\frac{dy(t)}{dt} \leqslant g(t)y(t) + h(t). \tag{2.25}$$

Using (1.4) we obtain

$$\begin{aligned}\Phi(e(u)) &\leqslant c_2\int_0^{|e(u)|^2}(1+\sqrt{t})^2\,dt\\ &\leqslant 2c_2\int_0^{|e(u)|}(1+t)^3\,dt\\ &= \frac{c_2}{2}[(1+|e(u)|)^4 - 1]\\ &\leqslant \frac{c_2}{2}(1+|e(u)|)^4.\end{aligned}$$

Integrating the above estimate over Ω, we have

$$\begin{aligned}\int_\Omega \Phi(e(u))\,dx &\leqslant \frac{c_2}{2}\int_\Omega (1+|e(u)|)^4\,dx\\ &\leqslant \frac{c_2k_1}{2}\int_\Omega (1+|e(u)|^4)\,dx\\ &\leqslant \frac{c_2k_1}{2}|\Omega| + \frac{c_2k_1}{2}\|u\|^4_{(W^{1,4}(\Omega))^2},\end{aligned} \tag{2.26}$$

where $|\Omega|$ is the Lebesgue measure of the set Ω.

By Lemma 2.2 and (2.26) we have

$$\int_t^{t+1} y(s)\,ds \leqslant \frac{1}{2}[(\nu+\nu_r+c_2k_1)R_2(t)+c_2k_1|\Omega|] \equiv \xi_3(t)<\infty,$$

for all $\tau \le \tau_0(\widehat{D},t)$,

$$\int_t^{t+1} g(s)\,ds \leqslant ck_0\left(\int_t^{t+1}\|u\|^4_{(W^{1,4}(\Omega))^2}\,ds\right)^{1/2} \leqslant ck_0\sqrt{R_2(t)} \equiv \xi_1(t)<\infty,$$

for all $\tau \le \tau_0(\widehat{D},t)$, and

$$\begin{aligned}\int_t^{t+1} h(s)\,ds &\leqslant 4\nu_r^2R_2(t)+\int_t^{t+1}|f(s)|^2\,ds\\ &\le 4\nu_r^2R_2(t)+\int_{-\infty}^{t+1}e^{-\alpha_2(t-s)}|f(s)|^2\,ds \ \equiv \xi_2(t)<\infty,\end{aligned}$$

for all $\tau \le \tau_0(\widehat{D},t)$.

From uniform Gronwall Lemma and the above considerations we conclude that

$$\begin{aligned}&\frac{\nu+\nu_r}{2}\|u(t+1)\|^2+\int_\Omega \Phi(e(u(t+1)))\,dx\\ &\leqslant (\xi_3(t)+\xi_2(t))\,e^{\xi_1(t)}<\infty, \quad \text{for all } \tau\le\tau_0(\widehat{D},t).\end{aligned} \tag{2.27}$$

Since, by (2.22) we have

$$\|u\|^2 \leqslant k_0\int_\Omega \Phi(e(u))\,dx,$$

by (2.27) we conclude that

$$\|u(t+1,\tau,u_\tau)\|^2 \leqslant k_0\,(\xi_3(t)+\xi_2(t))\,e^{\xi_1(t)} \equiv \tilde{R}(t)<\infty, \tag{2.28}$$

for all $\tau \le \tau_0(\widehat{D},t)$.

Now, let us use $\partial_t w$ as a test function in (1.7), we obtain

$$\begin{aligned}&|\partial_t w|^2+\frac{\nu_1}{2}\frac{d}{dt}\|w\|^2+2\nu_r\frac{d}{dt}|w|^2\\ &=-\sum_{i,j=1}^2\int_\Omega u_i\partial_{x_i}w\partial_t w\,dx+2\nu_r(\operatorname{rot}u,\partial_t w)+(g,\partial_t w).\end{aligned} \tag{2.29}$$

By Young's inequality we have

$$2\nu_r(\operatorname{rot}u,\partial_t w) \leqslant 4\nu_r^2\|u\|^2+\frac{1}{4}|\partial_t w|^2, \tag{2.30}$$

$$(g,\partial_t w) \leqslant |g|^2+\frac{1}{4}|\partial_t w|^2,$$

$$\begin{aligned}\sum_{i,j=1}^2\int_\Omega |u_i||\partial_{x_i}w||\partial_t w|\,dx &\leqslant \|u\|^2_{(L^\infty(\Omega))^2}\|w\|^2+\frac{1}{4}|\partial_t w|^2\\ &\leqslant c\|u\|^2_{(W^{1,4}(\Omega))^2}\|w\|^2+\frac{1}{4}|\partial_t w|^2.\end{aligned} \tag{2.31}$$

Employing (2.30) and (2.31) in (2.29) we have

$$\begin{aligned}&\frac{1}{4}|\partial_t w|^2+\frac{d}{dt}\left(\frac{\nu_1}{2}\|w\|^2+2\nu_r|w|^2\right)\\ &\leqslant c\|u\|^2_{(W^{1,4}(\Omega))^2}\|w\|^2+4\nu_r^2\|u\|^2+|g|^2\end{aligned}$$

$$
\begin{aligned}
&\leqslant c\|u\|^2_{(W^{1,4}(\Omega))^2}\|w\|^2 + \frac{4c\nu_r}{\nu_1}\|u\|^2_{(W^{1,4}(\Omega))^2}|w|^2 + 4\nu_r^2\|u\|^2 + |g|^2 \\
&= \frac{2c}{\nu_1}\|u\|^2_{(W^{1,4}(\Omega))^2}\Big(\frac{\nu_1}{2}\|w\|^2 + 2\nu_r|w|^2\Big) + 4\nu_r^2\|u\|^2 + |g|^2.
\end{aligned}
$$

Hence

$$
\frac{d}{dt}\Psi(t) \leqslant \Psi(t)\mathcal{G}(t) + \mathcal{N}(t),
$$

where

$$
\Psi(t) = \frac{\nu_1}{2}\|w\|^2 + 2\nu_r|w|^2, \quad \mathcal{G}(t) = \frac{2c}{\nu_1}\|u\|^2_{(W^{1,4}(\Omega))^2}, \quad \mathcal{N}(t) = 4\nu_r^2\|u\|^2 + |g|^2.
$$

By Lemmas 2.1 and 2.2 we have

$$
\begin{gathered}
\int_t^{t+1}\Psi(s)\,ds \leqslant \frac{\nu_1 R_2(t)}{2} + 2\nu_r R_1(t) \equiv \zeta_3(t) < \infty, \quad \text{for all } \tau \le \tau_0(\widehat{D}, t), \\
\int_t^{t+1}\mathcal{G}(s)\,ds \leqslant \frac{2c}{\nu_1}\Big(\int_t^{t+1}\|u(s)\|^4_{(W^{1,4}(\Omega))^2}\Big)^{1/2} \leqslant \frac{2c}{\nu_1}\sqrt{R_2(t)} \equiv \zeta_1(t) < \infty, \\
\text{for all } \tau \le \tau_0(\widehat{D}, t), \\
\begin{aligned}
\int_t^{t+1}\mathcal{N}(s)\,ds &\leqslant 4\nu_r^2 R_2(t) + \int_t^{t+1}|g(s)|^2\,ds \\
&\leqslant 4\nu_r^2 R_2(t) + \int_{-\infty}^{t+1} e^{-\alpha_2(t-s)}|g(s)|^2\,ds \equiv \zeta_2(t) < \infty,
\end{aligned} \\
\text{for all } \tau \le \tau_0(\widehat{D}, t).
\end{gathered}
$$

Thus by uniform Gronwall Lemma we deduce that

$$
\|w(t+1,\tau,w_\tau)\|^2 \leqslant \frac{2}{\nu_1}\left(\zeta_3(t) + \zeta_2(t)\right)e^{\zeta_1(t)} \equiv \hat{R}(t) < \infty \tag{2.32}
$$

for all $\tau \le \tau_0(\widehat{D}, t)$; by (2.28) and (2.32) we conclude the proof. □

Let us finally verify that the evolution process is pullback $\mathcal{D}$-asymptotically compact in $\mathcal{H}$ to conclude the proof of the Theorem 1.7(i).

Theorem 1.7(i). Let $\{S(t,\tau) : t \ge \tau \in \mathbb{R}\}$ be the evolution process generated by the problem (1.1)-(1.2) in $\mathcal{H}$, and let $\widehat{B} = \{B(t) : t \in \mathbb{R}\}$ and $\widehat{K} = \{K(t) : t \in \mathbb{R}\}$ be families of sets given by

$$
\begin{aligned}
B(t) &= \{(u,v) \in \mathcal{H} : |u|^2 + |v|^2 \le R_1(t)\}, \\
K(t) &= \{(u,v) \in \mathcal{H} : |u|^2 + |v|^2 \le R_3(t)\}
\end{aligned}
$$

where $R_1(t)$ is given by Lemma 2.1 and $R_3(t)$ is given by Lemma 2.3. By Lemma 2.1 $\widehat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for $\{S(t,\tau) : t \ge \tau \in \mathbb{R}\}$ in $\mathcal{H}$ and Lemma 2.3 it follows that $\widehat{K}$ is pullback $\mathcal{D}$-absorbing for $\{S(t,\tau) : t \ge \tau \in \mathbb{R}\}$ in $\mathcal{V} = V \times H_0^1(\Omega)$, equipped with the usual norm, and $\{S(t,\tau) : t \ge \tau \in \mathbb{R}\}$ is pullback $\mathcal{D}$-asymptotically compact in $\mathcal{H}$, thus the proof is complete by Theorem 1.6. □

Remark 2.4. In face of previous results and following the same arguments of [22] it is possible establish that there exists a unique family of Borel invariant probability measures on the pullback attractor.

3. Upper semicontinuity of pullback attractors

In this section, we investigate the upper semicontinuity of pullback attractors as $\nu_r \to 0$. To indicate the dependence of solutions on ν_r, we write the solution of problem (1.1)-(1.2) as (u_{ν_r}, w_{ν_r}) and the corresponding evolution process as $\{S_{\nu_r}(t,\tau) : t \ge \tau \in \mathbb{R}\}$. Thus (u_{ν_r}, w_{ν_r}) satisfies

$$\begin{gathered}
\partial_t u_{\nu_r} - \nabla\cdot\tau(e(u_{\nu_r})) + (u_{\nu_r}\cdot\nabla)u_{\nu_r} + \nabla p = 2\nu_r \operatorname{rot} w_{\nu_r} + f(x,t), \\
x\in\Omega,\ t>\tau, \\
\nabla\cdot u_{\nu_r} = 0, \quad x\in\Omega,\ t>\tau, \\
\partial_t w_{\nu_r} - \nu_1\Delta w_{\nu_r} + (u_{\nu_r}\cdot\nabla)w_{\nu_r} + 4\nu_r w_{\nu_r} = 2\nu_r \operatorname{rot} u_{\nu_r} + g(x,t), \\
x\in\Omega,\ t>\tau,
\end{gathered} \tag{3.1}$$

with the corresponding initial-boundary condition

$$\begin{gathered}
u_{\nu_r}(x,\tau) = u_\tau(x),\ w_{\nu_r}(x,\tau) = w_\tau(x), \quad x\in\Omega, \\
u_{\nu_r}(x,t) = 0,\ w_{\nu_r}(x,t) = 0, \quad x\in\partial\Omega,\ t\geqslant\tau.
\end{gathered} \tag{3.2}$$

For $\nu_r = 0$ the problem (1.1)-(1.2) reduces to

$$\begin{gathered}
\partial_t u - \nabla\cdot\{2(\nu + M(|e(u)|^2))\} + (u\cdot\nabla)u + \nabla p = f(x,t), \quad x\in\Omega,\ t>\tau, \\
\nabla\cdot u = 0, \quad x\in\Omega,\ t>\tau, \\
\partial_t w - \nu_1\Delta w + (u\cdot\nabla)w = g(x,t), \quad x\in\Omega,\ t>\tau,
\end{gathered} \tag{3.3}$$

with corresponding initial-boundary condition

$$\begin{gathered}
u(x,\tau) = u_\tau(x), \quad w(x,\tau) = w_\tau(x), \quad x\in\Omega, \\
u(x,t) = 0, \quad w(x,t) = 0, \quad x\in\partial\Omega, \\
;t\geqslant\tau.
\end{gathered} \tag{3.4}$$

Throughout this section, we assume $\nu_r \in [0,1]$. It follows from previous sections that for each $\nu_r > 0$ the process $\{S_{\nu_r}(t,\tau) : t \ge \tau \in \mathbb{R}\}$ has a pullback $\mathcal{D}$-attractor $\widehat{A}_{\nu_r} = \{A_{\nu_r}(t) : t \in \mathbb{R}\}$ in $\mathcal{H}$. It is clear that problem (3.3)-(3.4) generates a process $\{S_0(t,\tau) : t \ge \tau \in \mathbb{R}\}$ and possesses a unique pullback $\mathcal{D}$-attractor $\widehat{A}_0 = \{A_0(t) : t \in \mathbb{R}\}$ in $\mathcal{H}$.

In the this part we assume that there exist a constant $C > 0$ time independent, such that

$$\int_{-\infty}^t e^{-\alpha_2(t-s)}(|f(s)|^2 + |g(s)|^2)\,ds < C, \quad \forall t\in\mathbb{R}. \tag{3.5}$$

Proof of Theorem 1.7(ii). By [6, Proposition 1.20] it is suffices to prove that:

(i) There exist $\delta > 0$ and $t_0 \in \mathbb{R}$ such that

$$\cup_{\nu_r\in(0,\delta)} \cup_{s\le t_0} A_{\nu_r}(s)$$

is bounded.

(ii) For any $t\in\mathbb{R}$, $T \ge 0$ and all bounded set $B \subset \mathcal{H}$,

$$\lim_{\nu_r\to 0} \sup_{\tau\in[T-t,t],z\in B} \|S_{\nu_r}(t,\tau)z - S_0(t,\tau)z\|_{\mathcal{H}} = 0. \tag{3.6}$$

To prove (i), consider the t-dependent term involved in $R_1(t)$ given by (2.1). Using (3.5) we obtain that

$$R_1(t) \le 1 + \alpha_3 C := R.$$

Thus, the family $\widehat{B_0} = \{B_0(t) : t \in \mathbb{R}\}$ of sets given by $B_0(t) = \overline{B}(0, R)$ is pullback $\mathcal{D}$-absorbing for $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ in $\mathcal{H}$. In particular, we have that $\widehat{B_0} \in \mathcal{D}$. By the invariance of $\widehat{A}_{\nu_r} = \{A_{\nu_r}(t) : t \in \mathbb{R}\}$, for any $t \in \mathbb{R}$ and $\nu_r \in [0, 1]$ we have $A_{\nu_r}(t) \subset \overline{B}(0, R)$, we conclude (i).

To prove (ii), for any $z \in B$ and $t \geq \tau$, we write $U_{\nu_r} = u_{\nu_r} - u$ and $W_{\nu_r} = w_{\nu_r} - w$, then

$$
\begin{gathered}
\partial_t U_{\nu_r} - (\nu + \nu_r)\Delta U_{\nu_r} - \nu_r \Delta u + \mathcal{K} u_{\nu_r} - \mathcal{K} u + (u_{\nu_r} \cdot \nabla) u_{\nu_r} - (u \cdot \nabla) u \\
= 2\nu_r \operatorname{rot} w_{\nu_r}, \\
\partial_t W_{\nu_r} - \nu_1 \Delta W_{\nu_r} = 2\nu_r \operatorname{rot} u_{\nu_r} - 4\nu_r w_{\nu_r} + (u \cdot \nabla) w - (u_{\nu_r} \cdot \nabla) w_{\nu_r}, \\
\nabla \cdot U_{\nu_r} = 0.
\end{gathered}
$$

Hence, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|U_{\nu_r}|^2 + (\nu + \nu_r)\|U_{\nu_r}\|^2 + (\mathcal{K} u_{\nu_r} - \mathcal{K} u, U_{\nu_r}) \\
&= 2\nu_r(\operatorname{rot} w_{\nu_r}, U_{\nu_r}) - \nu_r(\nabla u, \nabla U_{\nu_r}) - (B(U_{\nu_r}, u), U_{\nu_r}), \\
&\frac{1}{2}\frac{d}{dt}|W_{\nu_r}|^2 + \nu_1\|W_{\nu_r}\|^2 \\
&= 2\nu_r(\operatorname{rot} u_{\nu_r}, W_{\nu_r}) - 4\nu_r(w_{\nu_r}, W_{\nu_r}) + (B_1(U_{\nu_r}, W_{\nu_r}), w).
\end{aligned}
\tag{3.7}
$$

We estimate the terms on the right-hand side. Note that

$$
\begin{gathered}
2\nu_r(\operatorname{rot} w_{\nu_r}, U_{\nu_r}) = 2\nu_r(w_{\nu_r}, \operatorname{rot} U_{\nu_r}) \leqslant 2\nu_r|w_{\nu_r}|^2 + \frac{\nu_r}{2}\|U_{\nu_r}\|^2, \\
|\nu_r(\nabla u, \nabla U_{\nu_r})| \leqslant \frac{\nu_r}{2}\|u\|^2 + \frac{\nu_r}{2}\|U_{\nu_r}\|^2, \\
|(B(U_{\nu_r}, u), U_{\nu_r})| \leqslant \frac{c}{\nu}\|u\|^2|U_{\nu_r}|^2 + \frac{\nu}{4}\|U_{\nu_r}\|^2, \\
2\nu_r(\operatorname{rot} u_{\nu_r}, W_{\nu_r}) = 2\nu_r(u_{\nu_r}, \operatorname{rot} W_{\nu_r}) \leqslant \frac{8\nu_r^2}{\nu_1}|u_{\nu_r}|^2 + \frac{\nu_1}{8}\|W_{\nu_r}\|^2, \\
|4\nu_r(w_{\nu_r}, W_{\nu_r})| \leqslant 4\nu_r|w_{\nu_r}|\frac{1}{\lambda}\|W_{\nu_r}\| \leqslant \frac{16\nu_r^2}{\lambda\nu_1}|w_{\nu_r}|^2 + \frac{\nu_1}{4}\|W_{\nu_r}\|^2.
\end{gathered}
$$

Finally, by Hölder's inequality and by following inequality

$$
\|\xi\|_4 \leqslant c|\xi|^{1/2}\|\xi\|^{1/2}, \quad \forall \xi \in H_0^1(\Omega) \tag{3.8}
$$

see e.g. [9], we obtain

$$
\begin{aligned}
(B_1(U_{\nu_r}, W_{\nu_r}), w) &\leqslant \|U_{\nu_r}\|_4\|W_{\nu_r}\|\|w\|_4 \\
&\leqslant c|U_{\nu_r}|^{1/2}\|U_{\nu_r}\|^{1/2}\|W_{\nu_r}\||w|^{1/2}\|w\|^{1/2} \\
&\leqslant \frac{\nu_1}{8}\|W_{\nu_r}\|^2 + c_4|U_{\nu_r}||w|\|U_{\nu_r}\|\|w\| \\
&\leq \frac{\nu_1}{8}\|W_{\nu_r}\|^2 + \frac{\nu}{4}\|U_{\nu_r}\|^2 + c|U_{\nu_r}|^2|w|^2\|w\|^2.
\end{aligned}
$$

Using the above estimates in (3.7) we conclude that

$$
\begin{aligned}
&\frac{d}{dt}(|U_{\nu_r}|^2 + |W_{\nu_r}|^2) + \nu\|U_{\nu_r}\|^2 + \nu_1\|U_{\nu_r}\|^2 \\
&\leq c(\|u\|^2 + |w|^2\|w\|^2)|U_{\nu_r}|^2 + c\nu_r(|w_{\nu_r}|^2 + \|u\|^2) + c\nu_r^2(|u_{\nu_r}|^2 + |w_{\nu_r}|^2),
\end{aligned}
$$

where we have used the fact that $(\mathcal{K}u_{\nu_r} - \mathcal{K}u, U_{\nu_r}) \geq 0$. Hence

$$\frac{d}{dt}(|U_{\nu_r}|^2 + |W_{\nu_r}|^2) \leq k(t)(|U_{\nu_r}|^2 + |W_{\nu_r}|^2) + \nu_r h(t),$$

where

$$h(t) = c(|w_{\nu_r}|^2 + \|u\|^2) + c\nu_r(|u_{\nu_r}|^2 + |w_{\nu_r}|^2),$$
$$k(t) = c(\|u\|^2 + |w|^2\|w\|^2).$$

Then, by Gronwall's lemma we have

$$|U_{\nu_r}(t)|^2 + |W_{\nu_r}(t)|^2 \leq \nu_r \int_\tau^t h(s)e^{\int_s^t g(r)\,dr}\,ds,$$

and thus

$$\|S_{\nu_r}(t,\tau)z - S_0(t,\tau)z\|_{\mathcal{H}}^2 \leq \nu_r \int_{t-T}^t h(s)e^{\int_s^t g(r)\,dr}\,ds,$$

for any $\tau \in [t-T, t]$ and $z \in B$. Thus the proof is complete. □

Acknowledgements. The authors would like to thank the anonymous referee for a careful reading of the manuscript and for constructive suggestions which resulted in the present version of the paper.

References

[1] G. M. de Araújo, M. A. F. de Araújo, E. F. L. Lucena; *On a system of equations of a non-Newtonian micropolar fluid*, Journal of Applied Mathematics, (2015), 2015.
[2] Bo-Qing Dong, Zhi-Min Chen; *Global attractors of two-dimensional micropolar fluid flows in some unbounded domains*, Appl. Math. Comp., **182**(2006), 610–620.
[3] Bo-Qing Dong, Z. Zhang; *Global regularity of the 2D micropolar fluid flows with zero angular viscosity*, J. Differential Equations, **249**(2010), 200–213.
[4] C. Calmelet-Eluhu, D. R. Majundar; *Flow of a micropolar fluid trhough a circular cylinder subject to longitudinal and torsional oscilations*, Math. Comput. Modelling, **27** (8) (1998), 69–78.
[5] T. Carabalho and G. Łukaszewicz, *Pullback attractors for asymptotically compact non-autonomous dynamical systems*, Nonlinear Anal. 64 (2006) 484–498.
[6] A. N. Carvalho, J. A. Langa, J. C. Robinson; *Attractors for infinite-dimensional non-autonomous dynamical systems.* Applied Mathematical Sciences 182. New York, Springer-Verlag, 2013.
[7] J. Chen, Z. Chen, B. Dong; *Existence of H^2-global attractors of two-dimensional micropolar fluid flows*, J. Math. Anal. Appl., **322** (2006), 512–522.
[8] A. C. Eringen; *Theory of micropolar fluids*, J. Math. Fluid Mech., **16** (1966), 1–18.
[9] O. A. Ladyzhenskaya; *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, Science Publishers, New York. (1963).
[10] G. Łukaszewicz; *Micropolar Fluids, Theory and applications, Modeling and simulations in Science*, Engineering and Technology, Birkhäuser Boston, Inc., Boston, MA, 1999.
[11] G. Łukaszewicz; *Long time behavior of 2D micropolar fluid flows*, Mathematical and Computer Modelling, **34** (2001), 487–509.
[12] G. Łukaszewicz, *Asymptotic behavior of micropolar fluid flows*, Inter. J. Eng. Sci., **41**(2003), 259–269.
[13] G. Łukaszewicz, W. Sadowski; *Uniform attractor for 2D magneto-micropolar fluid flow in some unbounded domains*, Z. Angew. Math. Phys., **55** (2004), 247–257.
[14] G. Łukaszewicz, A. Tarasińska; *On H1-pullback attractors for nonautonomous micropolar fluid equations in a bounded domain*, Nonlinear Anal.-TMA, **71** (2009), 782–788.
[15] J. Málek, J. Nečas, M. Rokyta, M. Růžička; *Weak and Measure-valued Solutions to Evolutionary PDEs*, Chapman & Hall, First Edition 1996.
[16] P. Marín-Rubio, J. Real; *On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems*, Nonlinear Anal., 71 (2009), 3956–3963.

[17] J. Nečas; *Sur le normes équivalentes dans* $W_p^k(\Omega)$ *et sur la coercivité des formes formellement positives*, Séminaire Equations aux Dérivées Partielles, Montreal (1966), 102–128.
[18] C. Zhao; *Pullback asymptotic behavior of solutions for a non-autonomous non-Newtonian fluid on two-dimensional unbounded domains*, J. Math. Phys., **53** (2012), 122702-1- 12.
[19] C. Zhao, Y. Li, S. Zhou; *Regularity of trajectory attractor and upper semicontinuity of global attractor for a 2D non-Newtonian fluid*, J. Differential Equations, **247**(2009), 2331–2363.
[20] C. Zhao, G. Liu, W. Wang; *Smooth pullback attractors for a non-autonomous 2D non-Newtonian fluid and their tempered behaviors*, J. Math. Fluid Mech., **16** (2014), 243–262.
[21] C. Zhao, W. Sun; *Global well-posedness and pullback attractors for a two-dimensional non-autonomous micropolar fluid flows with infinite delays*, Comm. Math. Sci., **15**(1) (2017), 97–121.
[22] C. Zhao, L. Yang; *Pullback attractor and invariant measure for the globally modified Navier-Stokes equations*, Comm. Math. Sci., **15** (2017), 1565–1580.
[23] C. Zhao, S. Zhou; *Pullback attractors for a non-autonomous incompressible non-Newtonian fluid*, J. Differential Equations, **238** (2007), 394–425.

Geraldo M. de Araujo
Departamento de Matemática, Universidade Federal do Pará, Rua Augusto Corrêa s/n, 66000-000, Belém PA, Brazil
E-mail address: gera@ufpa.br

Marcos A. F. Araujo
Departamento de Matemática, Universidade Federal do Maranhão, 65080-805, São Luís MA, Brazil
E-mail address: marcosttte@gmail.com

Flank D. M. Bezerra
Departamento de Matemática, Universidade Federal da Paraíba, 58051-900 João Pessoa PB, Brazil
E-mail address: flank@mat.ufpb.br

Mirelson M. Freitas
Departamento de Matemática, Universidade Federal do Pará, Rua Augusto Corrêa s/n, 66000-000, Belém PA, Brazil
E-mail address: freitas.ufpa@outlook.com