Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 103, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMMETRIC CRITICAL FRACTIONAL p-LAPLACIAN PROBLEMS

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Communicated by Marco Squassina

Abstract. We consider the asymmetric critical fractional p-Laplacian problem

$$(-\Delta)_p^s u = \lambda |u|^{p-2}u + u_+^{p_s^*-1}, \quad \text{in } \Omega;$$
$$u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega;$$

where $\lambda > 0$ is a constant, $p_s^* = Np/(N - sp)$ is the fractional critical Sobolev exponent, and $u_+(x) = \max\{u(x), 0\}$. This extends a result in the literature for the local case $s = 1$. We prove the theorem based on the concentration compactness principle of the fractional p-Laplacian and a linking theorem based on the $\mathbb{Z}_2$-cohomological index.

1. Introduction

Beginning with the seminal paper of Ambrosetti and Prodi [2], elliptic boundary value problems with asymmetric nonlinearities have been extensively studied (see, e.g., Berger and Podolak [5], Kazdan and Warner [17], Dancer [8], Amann and Hess [1], and the references therein). In particular, Deng [9], De Figueiredo and Yang [11], Aubin and Wang [3], Calanchi and Ruf [7], and Zhang et al. [32] have obtained existence and multiplicity results for semilinear Ambrosetti-Prodi type problems with critical nonlinearities using variational methods. And the results for the quasilinear Ambrosetti-Prodi type problems can be found in Perera et al. [29].

Recently, a lot of attention has been given to the study of the elliptic equations involving the fractional p-Laplacian, which is the nonlinear nonlocal operator defined on smooth functions by

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} dy,$$

where $p \in (1, +\infty)$, $s \in (0, 1)$ and $N > sp$. Some motivation that have led to the study of this kind of operator can be found in Caffarelli [6]. The operator $(-\Delta)_p^s$ leads naturally to the quasilinear problem

$$(-\Delta)_p^s u = f(x, u), \quad \text{in } \Omega;$$

2010 *Mathematics Subject Classification.* 35B33, 35J92, 35J20.
Key words and phrases. Fractional p-Laplacian; critical nonlinearity; asymmetric nonlinearity; linking; $\mathbb{Z}_2$-cohomological index.

Submitted November 9, 2016. Published April 18, 2017.

$$u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega;$$

where Ω is a domain in $\mathbb{R}^N$. There is currently a rapidly growing literature on this problem when Ω is bounded with Lipschitz boundary. In particular, fractional p-eigenvalue problems have been studied in [12, 16, 18, 26], global Hölder regularity in [15, 22], existence theory in the critical case in [27, 19, 20, 21, 22].

Motivated by [29], in this article, we consider the asymmetric critical fractional p-Laplacian problem

$$\begin{gathered} (-\Delta)_p^s u = \lambda |u|^{p-2}u + u_+^{p_s^*-1}, \quad \text{in } \Omega; \\ u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega; \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $\lambda > 0$ is a constant, $p_s^* = Np/(N - sp)$ is the fractional critical Sobolev exponent, and $u_+(x) = \max\{u(x), 0\}$.

We call that $\lambda \in \mathbb{R}$ is a Dirichlet eigenvalue of $(-\Delta)_p^s$ in Ω if the problem

$$\begin{gathered} (-\Delta)_p^s u = \lambda |u|^{p-2}u, \quad \text{in } \Omega; \\ u = 0, \text{in } \mathbb{R}^N \setminus \Omega; \end{gathered} \tag{1.2}$$

has a nontrivial weak solution. The first eigenvalue λ_1 is positive, simple, and has an associated eigenfunction φ_1 that is positive in Ω. And if $\lambda \geq \lambda_2$ is an eigenvalue, u is a λ-eigenfunction, then u changes sign in Ω. For problem (1.1) when $\lambda = \lambda_1$, $t\varphi_1$ is clearly a negative solution for any $t < 0$. So here we focus on the case λ is not an eigenvalue of $(-\Delta)_p^s$, and our result is the following.

Theorem 1.1. *Let $1 < p < \infty$, $s \in (0, 1)$, $N > sp$, and $\lambda > 0$. Then problem* (1.1) *has a nontrivial weak solution in the following cases*

(i) *$N = sp^2$ and $0 < \lambda < \lambda_1$;*
(ii) *$N > sp^2$ and λ is not an eigenvalue of $(-\Delta)_p^s$.*

2. Preliminaries and some known results

Let

$$[u]_{s,p} = \Big(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy \Big)^{1/p}$$

be the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, and let

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = \big(|u|_p^p + [u]_{s,p}^p \big)^{1/p},$$

where $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$. We work in the closed linear subspace

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$, which is a uniformly convex Banach space. The imbedding $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r \in [1, p_s^*]$ and compact for $r \in [1, p_s^*)$. Weak solutions of problem (1.1) coincide with critical points of the C^1-functional

$$I_\lambda(u) = \frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy - \frac{\lambda}{p} \int_\Omega |u|^p \, dx - \frac{1}{p_s^*} \int_\Omega u_+^{p_s^*} \, dx,$$

for $u \in W_0^{s,p}(\Omega)$.

We recall that I_λ satisfies the Cerami compactness condition at the level $c \in \mathbb{R}$, or the $(C)_c$ condition for short, if every sequence $\{u_j\} \subset W_0^{s,p}(\Omega)$ such that $I_\lambda(u_j) \to c$ and $(1 + \|u_j\|) I_\lambda'(u_j) \to 0$, called a $(C)_c$ sequence, has a convergent subsequence.

Let

$$S = \inf_{u \in W_0^{s,p}(\Omega) \setminus \{0\}} \frac{\|u\|^p}{|u|_{p_s^*}^p}$$

be the best constant in the Sobolev inequality. From [4], we know that for $1 < p < \infty$, $0 < s < 1$, $N > ps$, there exists a minimizer for S, and for every minimizer U, there exist $x_0 \in \mathbb{R}^N$ and a constant sign monotone function $u : \mathbb{R} \to \mathbb{R}$ such that $U(x) = u(|x - x_0|)$. In the following, we shall fix a radially symmetric nonnegative decreasing minimizer $U = U(r)$ for S. Multiplying U by a positive constant if necessary, we may assume that

$$(-\Delta)_p^s U = U^{p_s^* - 1}. \tag{2.1}$$

For any $\varepsilon > 0$, the function

$$U_\varepsilon(x) = \frac{1}{\varepsilon^{(N-sp)/p}} \, U\Big(\frac{|x|}{\varepsilon}\Big)$$

is also a minimizer for S satisfying (2.1). In [20, Lemma 2.2], the following asymptotic estimates for U were provided.

Lemma 2.1 ([20, Lemma 2.2])**.** *There exist constants $c_1, c_2 > 0$ and $\theta > 1$ such that for all $r \geq 1$,*

$$\frac{c_1}{r^{(N-sp)/(p-1)}} \leq U(r) \leq \frac{c_2}{r^{(N-sp)/(p-1)}},$$

and

$$\frac{U(\theta\, r)}{U(r)} \leq \frac{1}{2}.$$

Assume, without loss of generality, that $0 \in \Omega$. For $\varepsilon, \delta > 0$, let

$$m_{\varepsilon,\delta} = \frac{U_\varepsilon(\delta)}{U_\varepsilon(\delta) - U_\varepsilon(\theta\delta)},$$

let

$$g_{\varepsilon,\delta}(t) = \begin{cases} 0, & 0 \leq t \leq U_\varepsilon(\theta\delta), \\ m_{\varepsilon,\delta}^p \, (t - U_\varepsilon(\theta\delta)), & U_\varepsilon(\theta\delta) \leq t \leq U_\varepsilon(\delta), \\ t + U_\varepsilon(\delta) \, (m_{\varepsilon,\delta}^{p-1} - 1), & t \geq U_\varepsilon(\delta), \end{cases}$$

and let

$$G_{\varepsilon,\delta}(t) = \int_0^t g_{\varepsilon,\delta}'(\tau)^{1/p} \, d\tau = \begin{cases} 0, & 0 \leq t \leq U_\varepsilon(\theta\delta), \\ m_{\varepsilon,\delta} \, (t - U_\varepsilon(\theta\delta)), & U_\varepsilon(\theta\delta) \leq t \leq U_\varepsilon(\delta), \\ t, & t \geq U_\varepsilon(\delta). \end{cases}$$

The functions $g_{\varepsilon,\delta}$ and $G_{\varepsilon,\delta}$ are nondecreasing and absolutely continuous. Consider the radially symmetric non-increasing function

$$u_{\varepsilon,\delta}(r) = G_{\varepsilon,\delta}(U_\varepsilon(r)),$$

which satisfies

$$u_{\varepsilon,\delta}(r) = \begin{cases} U_\varepsilon(r), & r \leq \delta, \\ 0, & r \geq \theta\delta. \end{cases}$$

We have the following estimates for $u_{\varepsilon,\delta}$ which were proved in [20, Lemma 2.7].

Lemma 2.2 ([20, Lemma 2.7]). *There exists a constant $C = C(N,p,s) > 0$ such that for any $\varepsilon \le \delta/2$,*

$$\|u_{\varepsilon,\delta}\|^p \le S^{N/sp} + C(\frac{\varepsilon}{\delta})^{(N-sp)/(p-1)},$$

$$|u_{\varepsilon,\delta}|_p^p \ge \begin{cases} \frac{1}{C}\varepsilon^{sp}\log(\frac{\delta}{\varepsilon}), & \text{if } N = sp^2, \\ \frac{1}{C}\varepsilon^{sp}, & \text{if } N > sp^2, \end{cases} \tag{2.2}$$

$$|u_{\varepsilon,\delta}|_{p_s^*}^{p_s^*} \ge S^{N/sp} - C(\frac{\varepsilon}{\delta})^{N/(p-1)},$$

$$\frac{\|u_{\varepsilon,\delta}\|^p - \lambda|u_{\varepsilon,\delta}|^p}{|u_{\varepsilon,\delta}|_{p_s^*}^p} \le \begin{cases} S - \frac{\lambda}{C}\,\varepsilon^{sp}\,\log\Big(\frac{\delta}{\varepsilon}\Big) + C\Big(\frac{\varepsilon}{\delta}\Big)^{sp}, & N = sp^2, \\ S - \frac{\lambda}{C}\,\varepsilon^{sp} + C\Big(\frac{\varepsilon}{\delta}\Big)^{(N-sp)/(p-1)}, & N > sp^2. \end{cases} \tag{2.3}$$

For $p > 1$, and the eigenvalues of problem (1.2), we define a non-decreasing sequence λ_k by means of the cohomological index. This type of construction was introduced for the p-Laplacian by Perera [23]. (see also Perera and Szulkin [25]), and it is slightly different from the traditional one, based on the Krasnoselskii genus (which does not give the additional Morse-theoretical information that we need here).

We briefly recall the definition of Z_2-cohomological index by Fadell and Rabinowitz [10]. Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2 2$ be the quotient space of A with each u and $-u$ identified, let $f : \overline{A} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of A is defined by

$$i(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ \sup\{m \ge 1 : f^*(\omega^{m-1}) \ne 0\}, & \text{if } A \ne \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2 2[\omega]$. See Perera et al. [24] for details.

So the eigenvalues of problem (1.2), coincide with critical values of the functional

$$\Psi(u) = \frac{1}{|u|_p^p}, \quad u \in \mathcal{M} = \{u \in W_0^{s,p}(\Omega) : \|u\| = 1\}.$$

Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$, and set

$$\lambda_k := \inf_{M\in\mathcal{F},\, i(M)\ge k}\ \sup_{u\in M}\ \Psi(u), \quad k \in \mathbb{N}.$$

Then $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots \to +\infty$ is a sequence of eigenvalues of problem (1.2), and

$$\lambda_k < \lambda_{k+1} \implies i(\Psi^{\lambda_k}) = i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}}) = k,$$

where

$$\Psi^a = \{u \in \mathcal{M} : \Psi(u) \le a\}, \quad \Psi_a = \{u \in \mathcal{M} : \Psi(u) \ge a\}, \quad a \in \mathbb{R}.$$

From [20, Proposition 3.1], the sublevel set Ψ^{λ_k} has a compact symmetric subset $E(\lambda_k)$ of index k that is bounded in $L^\infty(\Omega)$. We may assume without loss of generality that $0 \in \Omega$. Let $\delta_0 = (0, \partial\Omega)$, take a smooth function $\eta : [0,\infty) \to [0,1]$ such that $\eta(s) = 0$ for $s \le 3/4$ and $\eta(s) = 1$ for $s \ge 1$, set

$$v_\delta(x) = \eta\big(\frac{|x|}{\delta}\big)v(x), \quad v \in E(\lambda_k),\, 0 < \delta \le \frac{\delta_0}{2},$$

and let $E_\delta = \{\pi(v_\delta) : v \in E(\lambda_k)\}$, where $\pi : W_0^{s,p}(\Omega) \setminus \{0\} \to \mathcal{M}$, $u \mapsto u/\|u\|$ is the radial projection onto $\mathcal{M}$.

Lemma 2.3 ([20, Proposition 3.2])**.** *There exists a constant $C = C(N, \Omega, p, s, k) > 0$ such that for all sufficiently small $\delta > 0$,*

- (i) $\frac{1}{C} \le |\omega|_q \le C$, *for all* $\omega \in E_\delta, 1 \le q \le \infty$,
- (ii) $\sup_{\omega \in E_\delta} I_\lambda(\omega) \le \lambda_k + C\delta^{N-sp}$,
- (iii) $E_\delta \cap \Psi_{\lambda_{k+1}} = \emptyset, i(E_\delta) = k$,
- (iv) $\operatorname{supp}\omega \cap \operatorname{supp}\pi(u_{\varepsilon,\delta}) = \emptyset$ *for all* $\omega \in E_\delta$,
- (v) $\pi(u_{\varepsilon,\delta}) \notin E_\delta$.

We need the following two lemmas for the fractional p-Laplacian.

Lemma 2.4 ([14, P.161])**.** *If $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ is such that $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$, and*

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))((u_n - u)(x) - (u_n - u)(y))}{|x-y|^{N+ps}}\, dx\, dy \to 0,$$

as $n \to \infty$, then $u_n \to u$ in $W_0^{s,p}(\Omega)$ as $n \to \infty$.

Lemma 2.5 ([19, Theorem 2.5])**.** *Let $\{u_n\}$ be a bounded sequence in $W_0^{s,p}(\Omega)$, let $|D^s u_n|^p(x) := \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\, dy$, for a.e.$x \in \mathbb{R}^N$. Then, up to a subsequence, there exists $u \in W_0^{s,p}(\Omega)$, two Borel regular measures μ and ν, Λ denumerable, $x_j \in \Omega, \nu_j \ge 0, \mu_j \ge 0$ with $\mu_j + \nu_j > 0$, such that*

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{s,p}(\Omega), \quad \text{and } u_n \to u \quad \text{strongly in} \quad L^p(\Omega),$$

$$|D^s u_n|^p \xrightarrow{w^*} d\mu, \quad |u_n|^{p_s^*} \xrightarrow{w^*} d\nu,$$

$$d\mu \ge |D^s u|^p + \sum_{j\in\Lambda} \mu_j \delta_{x_j}, \quad \mu_j := \mu(\{x_j\}),$$

$$d\nu = |u|^{p_s^*} + \sum_{j\in\Lambda} \nu_j \delta_{x_j}, \quad \nu_j := \nu(\{x_j\}),$$

$$\mu_j \ge S\nu_j^{p/p_s^*}.$$

We will prove Theorems 1.1 using the following abstract critical point theorem proved in Yang and Perera [31, Theorem 2.2], which was also used successfully in [28, 29, 20], and generalizes the well-known linking theorem of Rabinowitz [30].

Lemma 2.6 ([31, Theorem 2.2])**.** *Let W be a Banach space, let $S = \{u \in W : \|u\| = 1\}$ be the unit sphere in W, and let $\pi : W \setminus \{0\} \to S, u \mapsto u/\|u\|$ be the radial projection onto S. Let I be a C^1-function on W and let A_0 and B_0 be disjoint nonempty closed symmetric subsets of S such that*

$$i(A_0) = i(S \setminus B_0) < \infty.$$

Assume that there exist $R > r > 0$ and $v \in S \setminus A_0$ such that

$$\sup I(A) \le \inf I(B), \quad \sup I(X) < \infty,$$

where

$$A = \{tu : u \in A_0, 0 \le t \le R\} \cup \{R\pi((1-t)u + tv) : u \in A_0, 0 \le t \le 1\},$$
$$B = \{ru : u \in B_0\}, \quad X = \{tu : u \in A, \|u\| = R, 0 \le t \le 1\}.$$

Let $\Gamma = \{\gamma \in C(X, W) : \gamma(X)$ *is closed and* $\gamma|_A = id_A\}$, *and set*

$$c := \inf_{\gamma\in\Gamma} \sup_{u\in\gamma(X)} I(u).$$

Then

$$\inf I(B) \le c \le \sup I(X),$$

in particular, c is finite. If, in addition, I satisfies the $(C)_c$ condition, then c is a critical value of I.

3. Proof of Theorem 1.1

First, we will give our main lemma.

Lemma 3.1. *If $\lambda \ne \lambda_1$, then I_λ satisfies the $(C)_c$ condition for all $c < \frac{s}{N} S^{N/sp}$.*

Proof. Let $c < \frac{s}{N} S^{N/sp}$, and let $\{u_j\}$ be a $(C)_c$ sequence. First we show that $\{u_j\}$ is bounded. We have

$$\frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^p}{|x-y|^{N+ps}}\, dx\, dy - \frac{\lambda}{p} \int_\Omega |u_j|^p\, dx - \frac{1}{p_s^*} \int_\Omega u_{j+}^{p_s^*}\, dx = c + o(1), \tag{3.1}$$

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))(v(x) - v(y))}{|x-y|^{N+ps}}\, dx\, dy \\ &- \lambda \int_\Omega |u_j|^{p-2} u_j v\, dx - \int_\Omega u_{j+}^{p_s^*-1} v\, dx \\ &= \frac{o(1)\|v\|}{1 + \|u_j\|}. \end{aligned} \tag{3.2}$$

Taking $v = u_j$ in (3.2) and combing with (3.1) gives

$$\int_\Omega u_{j+}^{p_s^*}\, dx = \frac{N}{s} c + o(1). \tag{3.3}$$

Taking $v = u_{j+}$ in (3.2), and using the equality

$$|u_+(x) - u_+(y)|^p \le |u(x) - u(y)|^{p-2}(u(x) - u(y))(u_+(x) - u_+(y)), \tag{3.4}$$

gives

$$\iint_{\mathbb{R}^{2N}} \frac{|u_{j+}(x) - u_{j+}(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \le \lambda \int_\Omega u_{j+}^p\, dx + \int_\Omega u_{j+}^{p_s^*}\, dx + o(1).$$

So $\{u_{j+}\}$ is bounded in $W_0^{s,p}(\Omega)$. Suppose $\rho_j := \|u_j\| = (\iint_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^p}{|x-y|^{N+ps}})^{1/p} \to \infty$ for a renamed subsequence. Then $\tilde{u}_j = \frac{u_j}{\|u_j\|}$ converges to some $\tilde{u}$ weakly in $W_0^{s,p}(\Omega)$, strongly in $L^q(\Omega)$ for $1 \le q < p_s^*$, and a.e. in Ω for a further subsequence. Since the sequence $\{u_{j+}\}$ is bounded, dividing (3.1) by ρ_j^p and (3.2) by ρ_j^{p-1} and passing to the limit then gives

$$\lambda \int_\Omega |\tilde{u}|^p\, dx = 1,$$

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}(x)-\tilde{u}(y)|^{p-2}(\tilde{u}(x)-\tilde{u}(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&= \lambda \int_\Omega |\tilde{u}|^{p-2}\tilde{u}v\,dx, \quad \forall v \in W_0^{s,p}(\Omega),
\end{aligned}$$

respectively. Moreover, since $\tilde{u}_{j+} = u_{j+}/\rho_j \to 0$, $\tilde{u} \le 0$ a.e. Hence $\tilde{u} = t\varphi_1$ for some $t < 0$ and $\lambda = \lambda_1$, this is a contradiction with assumption. So $\{u_j\}$ is bounded, and for a renamed subsequence, it converges to some u weakly in $W_0^{s,p}(\Omega)$ and $L^{p_s^*}(\Omega)$. Since $\{u_{j+}\}$ is bounded, according to Lemma 2.5, a renamed subsequence of which then converges to some $v \ge 0$ weakly in $W_0^{s,p}(\Omega)$, strongly in $L^q(\Omega)$ for $1 \le q < p_s^*$ and a.e. in Ω, and

$$|D^s u_{j+}|^p \xrightarrow{w^*} d\mu, \quad |u_{j+}|^{p_s^*} \xrightarrow{w^*} d\nu, \tag{3.5}$$

then there exists an at most countable index set Λ and points $x_i \in \Omega$, $i \in \Lambda$, such that

$$\begin{aligned}
d\mu &\ge |D^s v|^p + \sum_{i\in\Lambda} \mu_i \delta_{x_i}, \quad \mu_i := \mu(\{x_i\}), \\
d\nu &= |v|^{p_s^*} + \sum_{i\in\Lambda} \nu_i \delta_{x_i}, \quad \nu_i := \nu(\{x_i\}),
\end{aligned} \tag{3.6}$$

where μ_i, $\nu_i \ge 0$, $\mu_i + \nu_i > 0$, and $\mu_i \ge S\nu_i^{p/p_s^*}$.

Now for any $\rho > 0$, let $\varphi_{i,\rho} \in C_c^\infty(B_{2\rho}(x_i))$ satisfy

$$0 \le \varphi_{i,\rho}, \quad \varphi_{i,\rho}|_{B_\rho} = 1, \quad |\varphi_{i,\rho}|_\infty \le 1, \quad |\nabla\varphi_{i,\rho}|_\infty \le C/\rho.$$

From [19, (2.14)], for all $w \in L^{p_s^*}(\mathbb{R}^N)$,

$$\lim_{\rho \searrow 0} \int_{\mathbb{R}^N} |w|^p |D^s \varphi_{i,\rho}|^p dx = 0. \tag{3.7}$$

Testing equation (3.2) with $\varphi_{i,\rho}u_{j+}$, which is also bounded in $W_0^{s,p}(\Omega)$, from (3.4), we obtain

$$\begin{aligned}
&o(1) \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))(\varphi_{i,\rho}(x)u_{j+}(x)-\varphi_{i,\rho}(y)u_{j+}(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&\quad - \lambda\int_\Omega |u_j|^{p-2}u_j\varphi_{i,\rho}u_{j+}\,dx - \int_\Omega u_{j+}^{p_s^*-1}\varphi_{i,\rho}u_{j+}\,dx \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))(u_{j+}(x)-u_{j+}(y))}{|x-y|^{N+ps}}\varphi_{i,\rho}(x)\,dx\,dy \\
&\quad - \int_\Omega u_{j+}^{p_s^*}\varphi_{i,\rho}\,dx \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))u_{j+}(y)(\varphi_{i,\rho}(x)-\varphi_{i,\rho}(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&\quad - \lambda\int_\Omega |u_j|^{p-2}u_j\varphi_{i,\rho}u_{j+}\,dx \\
&\ge \iint_{\mathbb{R}^{2N}} \frac{|u_{j+}(x)-u_{j+}(y)|^p}{|x-y|^{N+ps}}\varphi_{i,\rho}(x)\,dx\,dy - \int_\Omega u_{j+}^{p_s^*}\varphi_{i,\rho}\,dx \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))u_{j+}(y)(\varphi_{i,\rho}(x)-\varphi_{i,\rho}(y))}{|x-y|^{N+ps}}\,dx\,dy
\end{aligned} \tag{3.8}$$

$$- \lambda \int_{\Omega} |u_j|^{p-2} u_j \varphi_{i,\rho} u_{j+}\, dx. \tag{3.9}$$

By (3.5), we have

$$\iint_{\mathbb{R}^{2N}} \frac{|u_{j+}(x) - u_{j+}(y)|^p}{|x+y|^{N+sp}} \varphi_{i,\rho}(x) dx\, dy \to \int_{\mathbb{R}^N} \varphi_{i,\rho} d\mu,$$
$$\int_{\Omega} u_{j+}^{p_s^*} \varphi_{i,\rho} dx \to \int_{\Omega} \varphi_{i,\rho} d\nu,$$
$$\int_{\Omega} u_{j+}^{p} \varphi_{i,\rho} dx \to \int_{\Omega} v^p \varphi_{i,\rho} dx.$$

Moreover, by Hölder's inequality, we obtain

$$\begin{aligned}
&| \iint_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y)) u_{j+}(y)(\varphi_{i,\rho}(x) - \varphi_{i,\rho}(y))}{|x-y|^{N+ps}} \, dx\, dy| \\
&\le \iint_{\mathbb{R}^{2N}} |\frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y)) u_{j+}(y)(\varphi_{i,\rho}(x) - \varphi_{i,\rho}(y))}{|x-y|^{N+ps}}| \, dx\, dy \\
&\le \Big(\iint_{\mathbb{R}^{2N}} \Big| \frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))}{|x-y|^{\frac{(p-1)(N+ps)}{p}}} \Big|^{p/(p-1)} \, dx\, dy \Big)^{(p-1)/p} \\
&\quad \times \Big(\int_{\mathbb{R}^N} |u_{j+}|^p |D^s \varphi_{i,\rho}|^p \, dy \Big)^{1/p}.
\end{aligned} \tag{3.10}$$

Notice that $|D^s \varphi_{i,\rho}|^p \in L^\infty(\mathbb{R}^N)$, since

$$\int_{\mathbb{R}^N} \frac{|\varphi_{i,\rho}(x) - \varphi_{i,\rho}(y)|^p}{|x-y|^{N+ps}} \, dy \le \frac{C}{\rho^p} \int_{\mathbb{R}^N} \frac{\min\{1, |x-y|^p\}}{|x-y|^{N+ps}} \, dy \le \frac{C}{\rho^p}, \tag{3.11}$$

then

$$\limsup_{j\to+\infty} \int_{\mathbb{R}^N} |u_{j+}|^p |D^s \varphi_{i,\rho}|^p \, dy = \int_{\mathbb{R}^N} v^p |D^s \varphi_{i,\rho}|^p \, dy,$$

passing to the limit in (3.9) gives,

$$\int_{\mathbb{R}^N} \varphi_{i,\rho} \, d\mu \le \int_{\Omega} \varphi_{i,\rho} \, d\nu + C(\int_{\mathbb{R}^N} v^p |D^s \varphi_{i,\rho}|^p \, dy)^{1/p} + \lambda \int_{\Omega} v^p \varphi_{i,\rho} dx.$$

Letting $\rho \searrow 0$ and using (3.7), gives $\nu_i \ge \mu_i$, which together with $\mu_i \ge S\nu_i^{p/p_s^*}$, then give $\nu_i = 0$ or $\nu_i \ge S^{N/sp}$.

We claim that $\nu_i \ge S^{N/sp}$ is not possible to hold. Indeed, passing to the limit in (3.3) and by (3.5) and (3.6), then $\nu_i \le \frac{N}{s} c < S^{N/sp}$. So $\nu_i = 0$, Λ is empty, and

$$\int_{\Omega} u_{j+}^{p_s^*} \, dx \to \int_{\Omega} v^{p_s^*} \, dx,$$

then $u_{j+} \to v$ strongly in $L^{p_s^*}(\Omega)$ by uniform convexity. Combining the fact that u_j converges to u weakly in $L^{p_s^*}(\Omega)$,

$$\int_{\Omega} u_{j+}^{p_s^*-1} (u_j - u) \, dx \to 0.$$

Now we have

$$\begin{aligned}
&\langle I_\lambda'(u_j), (u_j - u) \rangle \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))((u_j - u)(x) - (u_j - u)(y))}{|x-y|^{N+ps}} \, dx\, dy
\end{aligned}$$

$$- \lambda \int_\Omega |u_j|^{p-2} u_j (u_j - u)\, dx - \int_\Omega u_{j+}^{p_s^*-1} (u_j - u)\, dx \to 0.$$

Therefore

$$\iint_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))((u_j - u)(x) - (u_j - u)(y))}{|x-y|^{N+ps}}\, dx\, dy \to 0.$$

By Lemma 2.4, we obtain $u_j \to u$ in $W_0^{s,p}(\Omega)$. □

Proof of Theorem 1.1. We now give the proof for the case when $\lambda > \lambda_1$ is not one of the eigenvalues. For $0 < \lambda < \lambda_1$, the proof is similar and simpler. Fix λ' such that $\lambda_k < \lambda' < \lambda < \lambda_{k+1}$, and let $\delta > 0$ be so small such that $\lambda_k + C\delta^{N-sp} < \lambda'$, in particular,

$$\Psi(\omega) < \lambda', \quad \forall \omega \in E_\delta. \tag{3.12}$$

Then take $A_0 = E_\delta$ and $B_0 = \Psi_{\lambda_{k+1}}$, and note that A_0 and B_0 are disjoint nonempty closed symmetric subsets of $\mathcal{M}$ such that

$$i(A_0) = i(\mathcal{M} \backslash B_0) = k.$$

Now, let $0 < \varepsilon \le \delta/2$, let $R > r > 0$, and let $v_0 = \pi(u_{\varepsilon,\delta}) \in \mathcal{M} \backslash E_\delta$, and let A, B, and X be as in Lemma 2.6.

For $u \in \Psi_{\lambda_{k+1}}$,

$$I_\lambda(ru) \ge \frac{1}{p}(1 - \frac{\lambda}{\lambda_{k+1}}) r^p - \frac{1}{p_s^* S^{p_s^*/p}} r^{p_s^*}.$$

Since $\lambda < \lambda_{k+1}$, it follows that $\inf I_\lambda(B) > 0$ if r is sufficiently small. Next we show $I_\lambda \le 0$ on A if R is sufficiently large. For $\omega \in E_\delta$ and $t \ge 0$,

$$\begin{aligned} I_\lambda(t\omega) &= \frac{1}{p}\|t\omega\|^p - \frac{\lambda}{p}|t\omega|_p^p - \frac{1}{p_s^*}|t\omega_+|_{p_s^*}^{p_s^*} \\ &\le \frac{t^p}{p}(1 - \frac{\lambda}{\Psi(\omega)}) \le 0, \end{aligned}$$

by (3.12). Now let $\omega \in E_\delta$, $0 \le t \le 1$, and set $u = \pi((1-t)\omega + tv_0)$. Clearly, $\|(1-t)\omega + tv_0\| \le 1$, and since the supports of ω and v_0 are disjoint by Lemma 2.3(iv),

$$\begin{gathered} |(1-t)\omega + tv_0|_{p_s^*}^{p_s^*} = (1-t)^{p_s^*}|\omega|_{p_s^*}^{p_s^*} + t^{p_s^*}|v_0|_{p_s^*}^{p_s^*}, \\ |u|_p^p = \frac{|(1-t)\omega + tv_0|_p^p}{\|(1-t)\omega + tv_0\|^p} \ge \frac{(1-t)^p}{\Psi(\omega)} \ge \frac{(1-t)^p}{\lambda'}. \end{gathered}$$

Since

$$|v_0|_{p_s^*}^{p_s^*} = \frac{|u_{\varepsilon,\delta}|_{p_s^*}^{p_s^*}}{\|u_{\varepsilon,\delta}\|^{p_s^*}} \ge \frac{1}{S^{N/(N-sp)}} + O(\varepsilon^{(N-sp)/(p-1)}), \tag{3.13}$$

it follows that

$$\begin{aligned} |u_+|_{p_s^*}^{p_s^*} &= \frac{|[(1-t)\omega + tv_0]_+|_{p_s^*}^{p_s^*}}{\|(1-t)\omega + tv_0\|^{p_s^*}} \\ &\ge (1-t)^{p_s^*}|\omega_+|_{p_s^*}^{p_s^*} + t^{p_s^*}|v_0|_{p_s^*}^{p_s^*} \\ &\ge t^{p_s^*}|v_0|_{p_s^*}^{p_s^*} \ge \frac{t^{p_s^*}}{C}, \end{aligned} \tag{3.14}$$

if ε is sufficiently small, where $C = C(N, \Omega, p, s, k) > 0$. Then

$$\begin{aligned} I_\lambda(Ru) &= \frac{R^p}{p}\|u\|^p - \frac{\lambda R^p}{p}|u|_p^p - \frac{R^{p_s^*}}{p_s^*}|u_+|_{p_s^*}^{p_s^*} \\ &\le -\frac{R^p}{p}\big[\frac{\lambda}{\lambda'}(1-t)^p - 1\big] - \frac{t^{p_s^*}}{p_s^* C}R^{p_s^*}. \end{aligned} \tag{3.15}$$

The above expression is clearly non-positive if $t \le 1 - (\lambda'/\lambda)^{1/p} =: t_0$. For $t > t_0$, it is non-positive if R is sufficiently large.

Now, it only remains to show that

$$\sup I_\lambda(X) < \frac{s}{N} S^{N/sp}, \tag{3.16}$$

if ϵ is sufficiently small, where

$$X = \{\rho\pi((1-t)\omega + tv_0) : \omega \in E_\delta, 0 \le t \le 1, 0 \le \rho \le R\}.$$

Set again $u = \pi((1-t)\omega + tv_0)$. From (3.15), $I_\lambda(\rho u) \le 0$, for all $0 \le \rho \le R$, if $0 \le t \le t_0$. So we only need to consider the case that $1 \ge t \ge t_0$. Then

$$\begin{aligned} \sup_{0\le\rho\le R} I_\lambda(\rho u) &\le \sup_{\rho\ge 0}\Big[\frac{\rho^p}{p}(1 - \lambda|u|_p^p) - \frac{\rho^{p_s^*}}{p_s^*}|u_+|_{p_s^*}^{p_s^*}\Big] \\ &= \frac{s}{N}\Big[\frac{(1-\lambda|u|_p^p)_+}{|u_+|_{p_s^*}^p}\Big]^{N/sp} \\ &= \frac{s}{N}\Big[\frac{(\|(1-t)\omega + tv_0\|^p - \lambda|(1-t)\omega + tv_0|_p^p)_+}{|[(1-t)\omega + tv_0]_+|_{p_s^*}^p}\Big]^{N/sp}. \end{aligned} \tag{3.17}$$

From the arguments in [20, pp.17-18 (3.15)-(3.17)],

$$\|(1-t)\,\omega + tv_0\|^p \le \frac{\lambda}{\lambda'}\,(1-t)^p + t^p + C\varepsilon^{N-(N-sp)\,q/p}, \tag{3.18}$$

where $q \in (N(p-1)/(N-sp), p)$,

$$\begin{aligned} |(1-t)\,\omega + tv_0|_p^p &= (1-t)^p|\omega|_p^p + t^p|v_0|_p^p, \\ |[(1-t)\,\omega + tv_0]_+|_{p_s^*}^{p_s^*} &\ge (1-t)^{p_s^*}|\omega_+|_{p_s^*}^{p_s^*} + t^{p_s^*}|v_0|_{p_s^*}^{p_s^*}. \end{aligned} \tag{3.19}$$

By (3.13), $|v_0|_{p_s^*}$ is bounded away from zero, if ε is sufficiently small, so the last expression in (3.19) is bounded away from a certain number for $1 \ge t \ge t_0$. It follows from (3.18), (3.19) and $|\omega|_p \ge \frac{1}{\lambda'}$ by (3.12), that

$$\begin{aligned} &\frac{\|(1-t)\,\omega + tv_0\|^p - \lambda|(1-t)\omega + tv_0|_p^p}{|[(1-t)\omega + tv_0]_+|_{p_s^*}^p} \\ &\le \frac{1-\lambda|v_0|_p^p}{|v_0|_{p_s^*}^p} + C\varepsilon^{N-(N-sp)q/p} \\ &\le \frac{\|u_{\varepsilon,\delta}\|^p - \lambda|u_{\varepsilon,\delta}|^p}{|u_{\varepsilon,\delta}|_{p_s^*}^p} + C\varepsilon^{N-(N-sp)\,q/p} \\ &\le S - (\frac{\lambda}{C} - C\varepsilon^{(N-sp^2)/(p-1)} - C\varepsilon^{(N-sp)(1-q/p)})\varepsilon^{sp}, \end{aligned}$$

by $v_0 = u_{\varepsilon,\delta}/\|u_{\varepsilon,\delta}\|$, and (2.3). Since $N > sp^2$ and $q < p$, it follows from this that the last expression in (3.17) is strictly less than $\frac{s}{N} S^{N/sp}$ if ε is sufficiently small.

So $0 < c < \frac{s}{N} S^{N/sp}$. Then I_λ satisfies the $(C)_c$ condition by Lemma 3.1, and hence c is a critical value of I_λ by Lemma 2.6. $\square$

Acknowledgments. Yang Yang was supported by the NSFC, projects 11501252 and 11571176.

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