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Uniqueness theorem for p-biharmonic equations *

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Abstract

The goal of this paper is to prove existence and uniqueness of a solution of the initial value problem for the equation

$$(|u''|^{p-2}u'')'' = \lambda |u|^{q-2}u$$

where $\lambda \in \mathbb{R}$ and $p, q > 1$. We prove the existence for $p \geq q$ only, and give a counterexample which shows that for $p < q$ there need not exist a global solution (blow-up of the solution can occur). On the other hand, we prove the uniqueness for $p \leq q$, and show that for $p > q$ the uniqueness does not hold true (we give a corresponding counterexample again). Moreover, we deal with continuous dependence of the solution on the initial conditions and parameters.

1 Introduction

In 2000, Drábek and Ôtani proved [3] that the initial value problem

$$\begin{gathered}
\big(|u''(t)|^{p-2}u''(t)\big)'' = \lambda |u(t)|^{p-2}u(t), \quad t \in [t_0, t_0+\varepsilon], \\
u(t_0) = \alpha, \quad u'(t_0) = \beta, \\
|u''(t_0)|^{p-2}u''(t_0) = \gamma, \quad \big(|u''(t)|^{p-2}u''(t)\big)'\Big|_{t=t_0} = \delta
\end{gathered} \tag{1.1}$$

where $\lambda > 0$ and $p > 1$, has a unique locally defined solution (for some $\varepsilon > 0$). The equation in (1.1) is a generalization of the one-dimensional version of the well-known linear clamped plate equation, which we obtain choosing $p = 2$ in (1.1).

It should be mentioned that the existence and uniqueness problem for (1.1) cannot be inferred from the classical theory. Indeed, let us denote $v := |u''|^{p-2}u''$ and rewrite (1.1) as the equivalent problem

$$\begin{aligned}
u''(t) &= |v(t)|^{\frac{2-p}{p-1}}v(t), \qquad u(t_0) = \alpha, \quad u'(t_0) = \beta, \\
v''(t) &= \lambda |u(t)|^{p-2}u(t), \qquad v(t_0) = \gamma, \quad v'(t_0) = \delta,
\end{aligned} \quad t \in [t_0, t_0+\varepsilon]. \tag{1.2}$$

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Whenever $p \neq 2$, at least one of the right-hand sides in (1.2) satisfies neither Lipschitz (see, e.g., [2]) nor any other general condition that guarantees existence or uniqueness of a solution. For example, the very general Kamke's Theorem (or its corollaries—Nagumo's (Rosenblatt's), Osgood's, Tonelli's Criterion, see, e.g., [4, pp. 31–35]) cannot be used to prove the uniqueness here.

In what follows, we show how the situation gets more complicated as we carry forward to more general problems than (1.1), namely to problems with

- different growth of the nonlinearity depending on u'' and on u (non-homogeneous equation),
- jumping nonlinearity,
- non-constant coefficients.

Let us consider the problem with a non-homogeneous equation

$$\begin{gathered} \big(|u''(t)|^{p-2}u''(t)\big)'' = \lambda |u(t)|^{q-2}u(t), \quad t \in \mathcal{I}, \\ u(t_0) = \alpha, \quad u'(t_0) = \beta, \\ |u''(t_0)|^{p-2}u''(t_0) = \gamma, \quad \big(|u''(t)|^{p-2}u''(t)\big)'\Big|_{t=t_0} = \delta \end{gathered} \tag{1.3}$$

where $\lambda \in \mathbb{R}$, $p, q > 1$ and $\mathcal{I} = [t_0, t_1]$, $t_0 < t_1$, or $\mathcal{I} = [t_0, \infty)$. Taking $p = q$ in (1.3) we obtain (1.1), but for $p \neq q$ the situation is more complex: for $p < q$ we lose the existence of a globally defined solution (we call this a "global existence"), and for $p > q$ we lose the uniqueness of a locally defined solution (we call this a "local uniqueness"). In Sections 3 and 4 we introduce the corresponding counterexamples.

We can further generalize (1.3) adding the jumping nonlinearity to the right-hand side:

$$\begin{gathered} \big(|u''(t)|^{p-2}u''(t)\big)'' = \mu |u(t)|^{q_1-2}u^+(t) - \nu |u(t)|^{q_2-2}u^-(t), \quad t \in \mathcal{I}, \\ u(t_0) = \alpha, \quad u'(t_0) = \beta, \\ |u''(t_0)|^{p-2}u''(t_0) = \gamma, \quad \big(|u''(t)|^{p-2}u''(t)\big)'\Big|_{t=t_0} = \delta \end{gathered} \tag{1.4}$$

where $p, q_1, q_2 > 1$, $\mu, \nu \in \mathbb{R}$, $u^+ = \max\{u, 0\}$ (positive part of u) and $u^- = \max\{-u, 0\}$ (negative part of u). Putting $q := q_1 = q_2$ and $\lambda := \mu = \nu$ into (1.4) we arrive at (1.3). Now the situation is analogous to the previous case (1.3): to prove the global existence we have to assume $p \geq \max\{q_1, q_2\}$, and to prove the local uniqueness we must have $p \leq \min\{q_1, q_2\}$.

Taking into account non-constant coefficients in (1.3) we obtain:

$$\begin{gathered} \big(|a(t)u''(t)|^{p-2}u''(t)\big)'' = b(t)|u(t)|^{q-2}u(t), \quad t \in \mathcal{I}, \\ u(t_0) = \alpha, \quad u'(t_0) = \beta, \\ |a(t_0)u''(t_0)|^{p-2}u''(t_0) = \gamma, \quad \big(a(t)|u''(t)|^{p-2}u''(t)\big)'\Big|_{t=t_0} = \delta \end{gathered} \tag{1.5}$$

where $a, b \in C(\mathcal{I})$ and $a > 0$. When $p > 2$, it is not enough to assume $p \leq q$ for proving the local uniqueness. We have to add a condition on b. It suffices to assume $b \ge 0$ or $b \le 0$ on the whole interval $\mathcal{I}$, i.e., that b does not change its sign on $\mathcal{I}$. Less restrictive is to assume that b have the property $\mathcal{P}$ (stated below) on $\mathcal{I}$.

Definiton 1.1 We say that a function f has a property $\mathcal{P}$ on the interval $\mathcal{I} = [t_0, t_1]$, or $\mathcal{I} = [t_0, \infty]$, if

$$\forall \tilde{t} \in \mathcal{I}^* \ \exists \xi > 0, \quad f(t) \ge 0 \ \forall t \in [\tilde{t}, \tilde{t} + \xi] \quad \text{or} \quad f(t) \le 0 \ \forall t \in [\tilde{t}, \tilde{t} + \xi]$$

where $\mathcal{I}^* = [t_0, t_1)$, or $\mathcal{I}^* = [t_0, \infty)$, respectively. In other words, for every point $\tilde{t}$ of $\mathcal{I}$ (except a contingent right boundary point) there exists some right closed neighborhood of $\tilde{t}$ in which f does not change its sign.

Note that a continuous function that does not have the property $\mathcal{P}$ is, e.g.,

$$f(t) = (t - t_0) \sin \frac{1}{t - t_0} \quad \text{for } t > t_0, \quad f(t_0) = 0.$$

It is clear that any constant function has the property $\mathcal{P}$.

We prove the (both local and global) existence and uniqueness for the most general non-homogeneous problem including the jumping nonlinearity and non-constant coefficients as well:

$$\begin{gathered}
\big(a(t)|u''(t)|^{p-2}u''(t)\big)'' = b_1(t)|u(t)|^{q_1-2}u^+(t) - b_2(t)|u(t)|^{q_2-2}u^-(t), \quad t \in \mathcal{I}, \\
u(t_0) = \alpha, \quad u'(t_0) = \beta, \\
a(t_0)|u''(t_0)|^{p-2}u''(t_0) = \gamma, \quad \big(a(t)|u''(t)|^{p-2}u''(t)\big)'\Big|_{t=t_0} = \delta
\end{gathered} \tag{1.6}$$

where $b_1, b_2 \in C(\mathcal{I})$ (the other parameters are as above).

Denoting $u_1 := u$ and $u_3 := a|u''|^{p-2}u''$ we can rewrite (1.6) as the equivalent initial value problem for a system of four equations of the first order

$$\begin{aligned}
&u_1'(t) = u_2(t), && u_1(t_0) = \alpha, \\
&u_2'(t) = a^{-\frac{1}{p-1}}(t)|u_3(t)|^{\frac{2-p}{p-1}}u_3(t), && u_2(t_0) = \beta, \qquad t \in \mathcal{I}. \\
&u_3'(t) = u_4(t), && u_3(t_0) = \gamma, \\
&u_4'(t) = b_1(t)|u_1(t)|^{q_1-2}u_1^+(t) - b_2(t)|u_1(t)|^{q_2-2}u_1^-(t), && u_4(t_0) = \delta,
\end{aligned} \tag{1.7}$$

The main results of this paper are the following.

Proposition 1.2 (local existence) *There exists $\varepsilon > 0$ such that* (1.6) *has a solution on $\mathcal{I} = [t_0, t_0 + \varepsilon]$.*

Theorem 1.3 (global existence) *Let $p \ge \max\{q_1, q_2\}$. Then* (1.6) *has a solution on $\mathcal{I} = [t_0, \infty)$.*

Corollary 1.4 *If $p \ge \max\{q_1, q_2\}$, then* (1.4) *has a solution on $\mathcal{I} = [t_0, \infty)$.*
If $p \ge q$, then (1.5) *and* (1.3) *have a solution on $\mathcal{I} = [t_0, \infty)$.*

Proposition 1.5 (local uniqueness) *Let one of these conditions hold true:*

- $|\alpha| + |\beta| + |\gamma| + |\delta| > 0$ *or*
- $p \le \min\{q_1, q_2\}$.

Moreover, let at least one of the following conditions hold true:

- $p \le 2$ *or*
- $\alpha = \beta = 0$ *or*
- $|\gamma| + |\delta| > 0$ *or*
- *there exists some right closed neighborhood of* t_0 *in which neither* b_1 *nor* b_2 *changes its sign.*

Then there exists $\varepsilon > 0$ *such that* (1.6) *has at most one solution on* $\mathcal{I} = [t_0, t_0 + \varepsilon]$.

Remark 1.6 For the special cases (1.3) and (1.4) of (1.6) the last condition of the latter four is trivially satisfied, and so it remains to satisfy only one of the former two conditions.

Theorem 1.7 (global uniqueness) *Let* $p \le \min\{q_1, q_2\}$. *Further, let* $p \le 2$ *or functions* b_1, b_2 *have the property* $\mathcal{P}$ *on* $\mathcal{I}$ *(see Definition* 1.1*). Then* (1.6) *has at most one solution.*

Corollary 1.8 *If* $p \le \min\{q_1, q_2\}$ *and, furthermore,* $p \le 2$ *or neither* b_1 *nor* b_2 *changes its sign on* $\mathcal{I}$*, then* (1.6) *has at most one solution.*

If $p \le q$ *and, furthermore,* $p \le 2$ *or* b *has the property* $\mathcal{P}$ *on* $\mathcal{I}$*, then* (1.5) *has at most one solution.*

If $p \le q$ *and, furthermore,* $p \le 2$ *or* b *does not change its sign on* $\mathcal{I}$*, then* (1.5) *has at most one solution.*

If $p \le \min\{q_1, q_2\}$*, then* (1.4) *has at most one solution.*

If $p \le q$*, then* (1.3) *has at most one solution.*

The paper is organized as follows. In Section 2 we define the solution of (1.6). In Section 3 we prove Proposition 1.2 and Theorem 1.3. Section 4 contains a proof of Proposition 1.5 and Theorem 1.7. In Section 5 we introduce some open problems related to this paper.

Tables 1 and 2 summarize the cases when the global existence, and the local uniqueness, respectively, of a solution of (1.3) is guaranteed or foreclosed (there exists a counterexample).

The following two corollaries are consequences of the global existence guaranteed by Theorem 1.3 and the global uniqueness guaranteed by Theorem 1.7. The reader is invited to accomplish their proofs following, e.g., that of [2, Th. 4.1, p. 59].

<table>
<tr><td>$p \geq q$</td><td colspan="3">YES (Corollary 1.4)</td></tr>
<tr><td rowspan="5">$p < q$</td><td rowspan="3">$\lambda > 0$</td><td>$\alpha, \beta, \gamma, \delta \geq 0$, $\alpha + \beta + \gamma + \delta > 0$
or
$\alpha, \beta, \gamma, \delta \leq 0$, $\alpha + \beta + \gamma + \delta < 0$</td><td>NO (Example 3.1, Remark 3.2)—blow-up to ∞ or $-\infty$</td></tr>
<tr><td>$\alpha = \beta = \gamma = \delta = 0$</td><td>YES (trivial)</td></tr>
<tr><td>$\exists \kappa_1, \kappa_2 \in \{\alpha, \beta, \gamma, \delta\} : \kappa_1 \kappa_2 < 0$</td><td>?</td></tr>
<tr><td>$\lambda = 0$</td><td colspan="2">YES (trivial)</td></tr>
<tr><td>$\lambda < 0$</td><td colspan="2">?</td></tr>
</table>

Table 1: Existence of a solution of (1.3) on $\mathcal{I} = [t_0, \infty)$.

<table>
<tr><td>$|\alpha| + |\beta| + |\gamma| + |\delta| > 0$</td><td colspan="3">YES (Proposition 1.5, Remark 1.6)</td></tr>
<tr><td rowspan="4">$\alpha = \beta = \gamma = \delta = 0$</td><td>$p \leq q$</td><td colspan="2">YES (Proposition 1.5, Remark 1.6)</td></tr>
<tr><td rowspan="3">$p > q$</td><td>$\lambda > 0$</td><td>NO (Example 4.5)</td></tr>
<tr><td>$\lambda = 0$</td><td>YES (trivial)</td></tr>
<tr><td>$\lambda < 0$</td><td>?</td></tr>
</table>

Table 2: Uniqueness of a solution of (1.3) on $\mathcal{I} = [t_0, t_0 + \varepsilon]$ for some $\varepsilon > 0$.

Corollary 1.9 *Let $\tilde{p} \leq \min\{\tilde{q}_1, \tilde{q}_2\}$. Further, let $\tilde{p} \leq 2$ or $\tilde{b}_1$, $\tilde{b}_2$ have the property $\mathcal{P}$ on $[a, b]$. Let $\tilde{\boldsymbol{u}}$ be a solution of* (1.7) *with $p = \tilde{p}$, $q_1 = \tilde{q}_1$, $q_2 = \tilde{q}_2$, $a = \tilde{a} > 0$, $b_1 = \tilde{b}_1$, $b_2 = \tilde{b}_2$, $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$, $\delta = \tilde{\delta}$, $t_0 = \tilde{\tau}$ and $\mathcal{I} = [a, b]$, $a < \tilde{\tau} < b$.*

Then there exists $\varepsilon > 0$ such that for any $p, q_1, q_2, \alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}$ and $a, b_1, b_2 \in C(\mathcal{I})$ satisfying

$$|p - \tilde{p}| + |q_1 - \tilde{q}_1| + |q_2 - \tilde{q}_2| + \|a - \tilde{a}\|_{C(\mathcal{I})} + \|b_1 - \tilde{b}_1\|_{C(\mathcal{I})} + \|b_2 - \tilde{b}_2\|_{C(\mathcal{I})} + \\ + |\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| + |\gamma - \tilde{\gamma}| + |\delta - \tilde{\delta}| + |\tau - \tilde{\tau}| < \varepsilon$$

all solutions $\boldsymbol{u} = \boldsymbol{u}(t, p, q_1, q_2, a, b_1, b_2, \alpha, \beta, \gamma, \delta, \tau)$ of (1.7) *with $t_0 = \tau$ exist over $\mathcal{I}$, and, as $(p, q_1, q_2, a, b_1, b_2, \alpha, \beta, \gamma, \delta, \tau) \to (\tilde{p}, \tilde{q}_1, \tilde{q}_2, \tilde{a}, \tilde{b}_1, \tilde{b}_2, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\tau})$,*

$$\boldsymbol{u}(t, p, q_1, q_2, a, b_1, b_2, \alpha, \beta, \gamma, \delta, \tau) \to \tilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t, \tilde{p}, \tilde{q}_1, \tilde{q}_2, \tilde{a}, \tilde{b}_1, \tilde{b}_2, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\tau})$$

uniformly over $[a, b]$.

Corollary 1.10 *Let $\tilde{p} = \tilde{q}_1 = \tilde{q}_2$. Further, let $\tilde{p} \leq 2$ or $\tilde{b}_1$, $\tilde{b}_2$ have the property $\mathcal{P}$ on $[a, b]$. Let $\tilde{p}, \tilde{q}_1, \tilde{q}_2, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\tau} \in \mathbb{R}$, $a < \tilde{\tau} < b$, and $\tilde{a}, \tilde{b}_1, \tilde{b}_2 \in C(\mathcal{I})$, $a > 0$, be fixed. Then there exists a solution $\tilde{\boldsymbol{u}}$ of* (1.7) *with $p = \tilde{p}$, $q_1 = \tilde{q}_1$, $q_2 = \tilde{q}_2$, $a = \tilde{a}$, $b_1 = \tilde{b}_1$, $b_2 = \tilde{b}_2$, $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$, $\delta = \tilde{\delta}$, $t_0 = \tilde{\tau}$ and $\mathcal{I} = [a, b]$, and the conclusion of Corollary* 1.9 *holds true.*

This paper is a brief version of the first chapter of the author's diploma thesis [1] which is available in Czech only.

2 Preliminaries

Let us define a function $\psi_p\colon \mathbb{R} \to \mathbb{R}$, $p > 1$, by $\psi_p(s) = |s|^{p-2}s$ for $s \neq 0$, and $\psi_p(0) = 0$. Now we can rewrite (1.6) as

$$\begin{gathered} \big(a(t)\psi_p(u''(t))\big)'' = b_1(t)\psi_{q_1}(u^+(t)) - b_2(t)\psi_{q_2}(u^-(t)), \quad t \in \mathcal{I}, \\ u(t_0) = \alpha, \quad u'(t_0) = \beta, \\ a(t_0)\psi_p(u''(t_0)) = \gamma, \quad \big(a(t)\psi_p(u''(t))\big)'\Big|_{t=t_0} = \delta. \end{gathered} \tag{2.1}$$

We denote $p' = \frac{p}{p-1}$. One can simply show that ψ_p and $\psi_{p'}$ are inverse functions. The problem (1.7) then takes the form

$$\begin{aligned} u_1'(t) &= u_2(t), & u_1(t_0) &= \alpha, \\ u_2'(t) &= c(t)\psi_{p'}(u_3(t)), & u_2(t_0) &= \beta, \\ u_3'(t) &= u_4(t), & u_3(t_0) &= \gamma, \\ u_4'(t) &= b_1(t)\psi_{q_1}(u_1^+(t)) - b_2(t)\psi_{q_2}(u_1^-(t)), & u_4(t_0) &= \delta, \end{aligned} \quad t \in \mathcal{I} \tag{2.2}$$

where $c(t) = \psi_{p'}\big(\frac{1}{a(t)}\big)$ ($c \in C(\mathcal{I})$, $c > 0$).

Definiton 2.1 By a solution of (2.2) we understand a vector function $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ of the class $(C^1(\mathcal{I}))^4$ which satisfy the equations in (2.2) at every point of $\mathcal{I}$, and fulfill the initial conditions in (2.2).

By a solution of the problem (2.1) we understand a function u of the class $C^2(\mathcal{I})$, such that $\big(u, u', a\psi_p(u''), (a\psi_p(u''))'\big)$ is a solution of the corresponding problem (2.2).

Remark 2.2 We transferred the problem of existence and uniqueness of a solution of (2.1) (i.e. (1.6)) to the equivalent problem for (2.2).

3 Existence

Proof of Proposition 1.2 By integration of the equations in (2.2) we obtain that $\boldsymbol{u}$ is a solution of (2.2) if and only if (u_1, u_3) is a fixed point of the operator $T\colon C(\mathcal{I}) \times C(\mathcal{I}) \to C(\mathcal{I}) \times C(\mathcal{I})$ defined by

$$\begin{aligned} T(u,v) \quad = \quad & \Big(\alpha + \beta t + \int_0^t (t-\tau)c(\tau)\psi_{p'}(v(\tau))\,\mathrm{d}\tau, \\ & \gamma + \delta t + \int_0^t (t-\tau)\Big(b_1(\tau)\psi_{q_1}(u^+(\tau)) - b_2(\tau)\psi_{q_2}(u^-(\tau))\Big)\,\mathrm{d}\tau\Big). \end{aligned}$$

The reader is invited to prove that there exists $\varepsilon > 0$ such that the Schauder Fixed Point Theorem guarantees the existence of at least one fixed point of T. It completes the proof of Proposition 1.2 (see Remark 2.2). $\square$

Now we want to prove that the local solution can be extended to ∞, i.e., that there exists a solution of (2.2) on $\mathcal{I} = [t_0, \infty)$. We find that it is not always possible, and we must add some conditions on the parameters in (2.2). We begin with the example which shows that it is necessary.

Example 3.1 Let in (1.3) $p < q$ and $\lambda > 0$. Let $H > t_0$ be arbitrary (fixed). Then one can compute that the function $u(t) = K(H - t)^r$ where

$$r = \frac{2p}{p-q} \quad \text{and} \quad K = \left(\frac{2^p (p-1) q \big(p(p+q)\big)^{p-1} (2pq - p - q)}{\lambda (q-p)^{2p}} \right)^{1/(q-p)}$$

is a solution of (1.3) with

$$\begin{gathered} \alpha = K(H-t_0)^r, \quad \beta = -Kr(H-t_0)^{r-1}, \\ \gamma = \big(Kr(r-1)\big)^{p-1} (H-t_0)^{(r-2)(p-1)}, \\ \delta = -\big(Kr(r-1)\big)^{p-1} (r-2)(p-1)(H-t_0)^{(r-2)(p-1)-1} \end{gathered}$$

on $\mathcal{I} = [t_0, t_1]$ for any $t_1 \in (t_0, H)$. However, this solution cannot be extended to $\mathcal{I} = [t_0, H]$ because $u(t) \to \infty$ as $t \to H$. This situation is called a blow-up of the solution.

Remark 3.2 Using Example 3.1 one can prove that each of the conditions

- $\alpha, \beta, \gamma, \delta \geq 0$, $\alpha + \beta + \gamma + \delta > 0$, $p < q_1$ and $c, b_1 \geq C > 0$ on $[t_0, \infty)$, and
- $\alpha, \beta, \gamma, \delta \leq 0$, $\alpha + \beta + \gamma + \delta < 0$, $p < q_2$ and $c, b_2 \geq C > 0$ on $[t_0, \infty)$

is sufficient for existence of $H > t_0$ such that there is no solution of (1.6) on $\mathcal{I} = [t_0, H]$. The idea of the proof is based on comparison of solutions of the initial value problem (1.6).

Remark 3.3 Example 3.1 can be generalized for the initial value problem of the $(2n)^{\text{th}}$-order ($n \in \mathbb{N}$)

$$\begin{gathered} (-1)^n \big(\psi_p(u^{(n)}(t))\big)^{(n)} = \lambda \psi_q(u(t)), \quad t \in \mathcal{I}, \\ u^{(i)}(t_0) = \alpha_i, \quad \big(\psi_p(u^{(n)}(t))\big)^{(i)}\Big|_{t=t_0} = \beta_i, \quad i = 0, \dots, n-1 \end{gathered} \tag{3.1}$$

where $p < q$ and $(-1)^n \lambda > 0$. The solution is defined similarly as for (2.1). Let $H > t_0$ be arbitrary (fixed). The reader is invited to justify that the function $u(t) = K(H-t)^r$ where

$$r = \frac{np}{p-q} \quad \text{and}$$

$$K = \left(\frac{\left(\prod_{k=0}^{n-1} \big((n-k)p + kq\big) \right)^{p-1} \prod_{k=0}^{n-1} \big(npq - kp - (n-k)q\big)}{(-1)^n \lambda (q-p)^{np}} \right)^{1/(q-p)}$$

is a solution of (3.1) (with some initial conditions) on $\mathcal{I} = [t_0, t_1]$ for any $t_1 \in (t_0, H)$. As in Example 3.1, this solution cannot be extended to $\mathcal{I} = [t_0, H]$ because $u(t) \to \infty$ as $t \to H$.

Proof of Theorem 1.3 Now we begin the proof of the existence of a solution of (1.6) on $\mathcal{I} = [t_0, \infty)$ assuming $p \geq \max\{q_1, q_2\}$. It suffices to prove that there exists at least one solution of (2.2) (see Remark 2.2) on $\mathcal{I} = [t_0, t_1]$ for any t_1 satisfying $t_0 < t_1$.

Let us have the auxiliary problem

$$\begin{aligned} \hat{u}_1'(t) &= \hat{u}_2(t), & \hat{u}_1(t_0) &= \hat{\alpha}, \\ \hat{u}_2'(t) &= C\psi_{p'}(\hat{u}_3(t)), & \hat{u}_2(t_0) &= \hat{\beta}, \\ \hat{u}_3'(t) &= \hat{u}_4(t), & \hat{u}_3(t_0) &= \hat{\gamma}, \\ \hat{u}_4'(t) &= B\psi_p(\hat{u}_1(t)), & \hat{u}_4(t_0) &= \hat{\delta}, \end{aligned} \quad t \in [t_0, t_1] \tag{3.2}$$

where $|b_1(t)| \leq B$, $|b_2(t)| \leq B$ and $|c(t)| \leq C$ on $[t_0, t_1]$. The vector function $\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4)$ where

$$\hat{u}_1(t) = K\mathrm{e}^{r(t-t_0)}, \quad \hat{u}_2(t) = Kr\mathrm{e}^{r(t-t_0)},$$
$$\hat{u}_3(t) = \Big(\frac{Kr^2}{C}\Big)^{p-1} \mathrm{e}^{r(p-1)(t-t_0)}, \quad \hat{u}_4(t) = \Big(\frac{Kr^2}{C}\Big)^{p-1} r(p-1)\mathrm{e}^{r(p-1)(t-t_0)},$$

$K > 0$ is arbitrary and

$$r = \Big(\frac{(p-1)^2}{BC^{p-1}}\Big)^{1/(2p)},$$

is a solution of (3.2) with

$$\hat{\alpha} = K, \quad \hat{\beta} = Kr, \quad \hat{\gamma} = \Big(\frac{Kr^2}{C}\Big)^{p-1}, \quad \hat{\delta} = \Big(\frac{Kr^2}{C}\Big)^{p-1} r(p-1).$$

We choose K big enough to have $|\alpha| < \hat{\alpha}$, $|\beta| < \hat{\beta}$, $|\gamma| < \hat{\gamma}$ and $|\delta| < \hat{\delta}$. We shall prove that for any solution $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ of (2.2) on $\mathcal{I} = [t_0, t_1]$

$$|u_1(t)| \leq \hat{u}_1(t), \quad |u_2(t)| \leq \hat{u}_2(t), \quad |u_3(t)| \leq \hat{u}_3(t), \quad |u_4(t)| \leq \hat{u}_4(t) \tag{3.3}$$

for every $t \in [t_0, t_1]$. We have $|u_1(t_0)| < \hat{u}_1(t_0)$, and so the set

$$T = \{\tilde{t} \in [t_0, t_1] : |u_1(\tilde{t})| \leq \hat{u}_1(\tilde{t})\}$$

is non-empty and closed, and there exists

$$t_m = \max\{\tilde{t} \in [t_0, t_1] : [t_0, \tilde{t}] \subseteq T\}.$$

We can assume $K \geq 1$. Then for any $t \in [t_0, t_m]$ we have $\hat{u}_1(t) \geq 1$ and

$$|u_4'(t)| \leq B|u_1(t)|^{q-1} \leq B(\hat{u}_1(t))^{q-1} \leq B(\hat{u}_1(t))^{p-1} = \hat{u}_4'(t)$$

where $q = q_1$ if $u_1(t) \geq 0$, and $q = q_2$ if $u_1(t) < 0$. For any $t \in [t_0, t_m]$ we now have

$$|u_4(t)| \leq |\delta| + \int_{t_0}^{t} |u_4'(\tau)| \,\mathrm{d}\tau \leq \hat{\delta} + \int_{t_0}^{t} \hat{u}_4'(\tau) \,\mathrm{d}\tau = \hat{u}_4(t),$$

$$|u_3(t)| \leq |\gamma| + \int_{t_0}^{t} |u_4(\tau)| \,\mathrm{d}\tau \leq \hat{\gamma} + \int_{t_0}^{t} \hat{u}_4(\tau) \,\mathrm{d}\tau = \hat{u}_3(t).$$

Similarly we can show that for $t \in [t_0, t_m]$

$$|u_2'(t)| \leq \hat{u}_2'(t) \quad \text{and} \quad |u_2(t)| \leq \hat{u}_2(t).$$

Thus

$$|u_1(t_m)| \leq |\alpha| + \int_{t_0}^{t_m} |u_1'(\tau)| \,\mathrm{d}\tau < \hat{\alpha} + \int_{t_0}^{t_m} \hat{u}_1'(\tau) \,\mathrm{d}\tau = \hat{u}_1(t_m).$$

This inequality would for $t_m < t_1$ contradict with the maximality of t_m, and so $t_m = t_1$ and (3.3) is proved. Using the standard continuation arguments, the proof of Theorem 1.3 is completed. $\square$

4 Uniqueness

In this section we prove the local uniqueness (Proposition 1.5). We distinguish the cases (a) $|\alpha| + |\beta| + |\gamma| + |\delta| > 0$ and (b) $\alpha = \beta = \gamma = \delta = 0$.
(a) Here u_1 does not change its sign on some right neighborhood of t_0. Hence in this case it suffices to prove the uniqueness for the problem without the jumping nonlinearity, i.e. (1.5). The proof is divided into four parts: Lemma 4.1 (for $p \leq 2$, $q \geq 2$), Lemma 4.2 (for $p \leq 2$, $q < 2$), Lemma 4.3 (for $p > 2$, $q \geq 2$) and Lemma 4.4 (for $p > 2$, $q < 2$).
(b) We assume $p \leq \min\{q_1, q_2\}$ here (see Proposition 1.5). Lemma 4.7 deals with this case. Before Lemma 4.7 we introduce the example of non-uniqueness of a solution of (1.3) for $\alpha = \beta = \gamma = \delta = 0$ and $p > q$.

In all proofs in this section we denote by $A, B, C > 0$ such constants that $|a(t)| \leq A$ (i.e. $|c(t)| \geq A^{1-p'}$), $|b_1(t)| \leq B$, $|b_2(t)| \leq B$ (i.e. $|b(t)| \leq B$ for b from (1.5)) and $|c(t)| \leq C$ (i.e. $|a(t)| \geq C^{1-p}$) for every $t \in \mathcal{I}$. We can also assume $t_0 = 0$. According to Remark 2.2, we prove the assertions for (2.2). For Lemmata 4.1–4.4, formulated for (1.5), the fourth equation in (2.2) takes the form $u_4'(t) = b(t)\psi_q(u(t))$.

Lemma 4.1 *Let $|\alpha| + |\beta| + |\gamma| + |\delta| > 0$, $p \leq 2$ and $q \geq 2$. Then there exists $\varepsilon > 0$ such that* (1.5) *has at most one solution on $\mathcal{I} = [t_0, t_0 + \varepsilon]$.*

Proof Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be solutions of the special case of (2.2), corresponding to (1.5). From the former two equations we conclude

$$u_3(t) - v_3(t) = a(t)\Big(\psi_p(u_1''(t)) - \psi_p(v_1''(t))\Big),$$

and from the latter two equations we obtain

$$u_3''(t) - v_3''(t) = b(t)\Big(\psi_q(u_1(t)) - \psi_q(v_1(t))\Big),$$

$t \in \mathcal{I}$. Then

$$a(t)\Big(\psi_p(u_1''(t)) - \psi_p(v_1''(t))\Big) = \int_0^t (t-\tau)b(\tau)\Big(\psi_q(u_1(\tau)) - \psi_q(v_1(\tau))\Big)\,\mathrm{d}\tau. \quad (4.1)$$

There exists a constant $K_1 > 0$ such that $|u_1''(t)| \le K_1$ and $|v_1''(t)| \le K_1$ on $\mathcal{I}$. Since $p \le 2$, $\psi_p'(\tau) \ge \psi_p'(K_1)$ for $|\tau| \le K_1$, and so

$$\begin{aligned} \Big|a(t)\Big(\psi_p(u_1''(t)) - \psi_p(v_1''(t))\Big)\Big| &\ge C^{1-p}\Big|\int_{v_1''(t)}^{u_1''(t)} \psi_p'(\tau)\,\mathrm{d}\tau\Big| \ge \\ &\ge (p-1)K_1^{p-2}C^{1-p}|u_1''(t) - v_1''(t)|. \end{aligned} \quad (4.2)$$

There exists a constant $K_2 > 0$ such that $|u_1(\tau)| \le K_2$ and $|v_1(\tau)| \le K_2$ on $\mathcal{I}$. Since $q \ge 2$, $\psi_q'(\sigma) \le \psi_q'(K_2)$ for $|\sigma| \le K_2$. For $\tau \in \mathcal{I}$ it yields

$$|\psi_q(u_1(\tau)) - \psi_q(v_1(\tau))| = \Big|\int_{v_1(\tau)}^{u_1(\tau)} \psi_q'(\sigma)\,\mathrm{d}\sigma\Big| \le (q-1)K_2^{q-2}|u_1(\tau) - v_1(\tau)|. \quad (4.3)$$

Using the estimate

$$|u_1(\tau) - v_1(\tau)| = \Big|\int_0^\tau (\tau-\sigma)(u_1''(\sigma) - v_1''(\sigma))\,\mathrm{d}\sigma\Big| \le \tau^2\|u_1'' - v_1''\|_{C(\mathcal{I})} \quad (4.4)$$

we conclude

$$\Big|\int_0^t (t-\tau)b(\tau)\Big(\psi_q(u_1(\tau)) - \psi_q(v_1(\tau))\Big)\,\mathrm{d}\tau\Big| \le t^4(q-1)K_2^{q-2}B\|u_1'' - v_1''\|_{C(\mathcal{I})}. \quad (4.5)$$

We combine (4.1), (4.2) and (4.5), take the maximum over $t \in \mathcal{I}$, and get

$$\|u_1'' - v_1''\|_{C(\mathcal{I})} \le \varepsilon^4 \frac{(q-1)K_2^{q-2}B}{(p-1)K_1^{p-2}C^{1-p}}\|u_1'' - v_1''\|_{C(\mathcal{I})}. \quad (4.6)$$

For $\varepsilon > 0$ small enough this implies $u_1'' = v_1''$, and so $u_3 = v_3$. Since $u_1(0) = v_1(0)$ and $u_1'(0) = v_1'(0)$, it is then $u_1 = v_1$. Thus $\boldsymbol{u} = \boldsymbol{v}$. $\square$

Lemma 4.2 *Let $|\alpha| + |\beta| + |\gamma| + |\delta| > 0$, $p \le 2$ and $q < 2$. Then there exists $\varepsilon > 0$ such that* (1.5) *has at most one solution on $\mathcal{I} = [t_0, t_0 + \varepsilon]$.*

Proof We distinguish the cases (i) $\alpha \ne 0$, (ii) $\alpha = 0$, $\beta \ne 0$, (iii) $\alpha = \beta = 0$, $\gamma \ne 0$ and (iv) $\alpha = \beta = \gamma = 0$, $\delta \ne 0$.
(i) We proceed as in the proof of Lemma 4.1. The assumption $u_1(0) = v_1(0) = \alpha \ne 0$ guarantees the existence of a constant $K_2 > 0$ such that $|u_1(\tau)| \ge K_2$

and $|v_1(\tau)| \ge K_2$ in $[0,\varepsilon]$ for $\varepsilon > 0$ small enough. Since $q < 2$, $\psi_q'(\sigma) \le \psi_q'(K_2)$ for $|\sigma| \ge K_2$. Hence (4.3) still holds true for all $\tau \in \mathcal{I}$, and we arrive again at (4.6).

(ii) We modify again the proof of Lemma 4.1. Due to the assumptions ($\alpha = 0$, $\beta \neq 0$), $\frac{u_1(\tau)}{\tau} \to \beta$ and $\frac{v_1(\tau)}{\tau} \to \beta \neq 0$ as $\tau \to 0_+$. Hence there exists a constant $K_2 > 0$ such that $\left|\frac{u_1(\tau)}{\tau}\right| \ge K_2$ and $\left|\frac{v_1(\tau)}{\tau}\right| \ge K_2$ for all $\tau \in (0,\varepsilon]$ with $\varepsilon > 0$ small enough. Thus

$$\Big|\psi_q\Big(\frac{u_1(\tau)}{\tau}\Big) - \psi_q\Big(\frac{v_1(\tau)}{\tau}\Big)\Big| = \Big|\int_{\frac{v_1(\tau)}{\tau}}^{\frac{u_1(\tau)}{\tau}} \psi_q'(\sigma)\,\mathrm{d}\sigma\Big| \le \frac{(q-1)K_2^{q-2}}{\tau}|u_1(\tau) - v_1(\tau)|. \tag{4.7}$$

Using (4.7) instead of (4.3) we get

$$\|u_1'' - v_1''\|_{C(\mathcal{I})} \le \varepsilon^{q+2}\frac{(q-1)K_2^{q-2}B}{(p-1)K_1^{p-2}C^{1-p}}\|u_1'' - v_1''\|_{C(\mathcal{I})}. \tag{4.8}$$

(iii) We follow again the proof of Lemma 4.1. By the assumptions ($\alpha = \beta = 0$, $\gamma \neq 0$), $\frac{u_1(\tau)}{\tau^2} \to \frac{1}{2}c(0)\psi_{p'}(\gamma)$ and $\frac{v_1(\tau)}{\tau^2} \to \frac{1}{2}c(0)\psi_{p'}(\gamma) \neq 0$ as $\tau \to 0_+$. Thus, there exists a constant $K_2 > 0$ such that $\left|\frac{u_1(\tau)}{\tau^2}\right| \ge K_2$ and $\left|\frac{v_1(\tau)}{\tau^2}\right| \ge K_2$ for every $\tau \in (0,\varepsilon]$ with $\varepsilon > 0$ small enough. Then

$$\Big|\psi_q\Big(\frac{u_1(\tau)}{\tau^2}\Big) - \psi_q\Big(\frac{v_1(\tau)}{\tau^2}\Big)\Big| = \Big|\int_{\frac{v_1(\tau)}{\tau^2}}^{\frac{u_1(\tau)}{\tau^2}} \psi_q'(\sigma)\,\mathrm{d}\sigma\Big| \ge \frac{(q-1)K_2^{q-2}}{\tau^2}|u_1(\tau) - v_1(\tau)|. \tag{4.9}$$

Using (4.9) instead of (4.3) we get

$$\|u_1'' - v_1''\|_{C(\mathcal{I})} \le \varepsilon^{2q}\frac{(q-1)K_2^{q-2}B}{(p-1)K_1^{p-2}C^{1-p}}\|u_1'' - v_1''\|_{C(\mathcal{I})}. \tag{4.10}$$

(iv) Here we cannot follow the proof of Lemma 4.1. Like (4.1), we can derive that for every $t \in \mathcal{I}$

$$\big|b(t)(u_1(t) - v_1(t))\big| \le |b(t)|\int_0^t (t-\tau)|c(\tau)|\Big|\psi_{p'}(u_3(\tau)) - \psi_{p'}(v_3(\tau))\Big|\,\mathrm{d}\tau. \tag{4.11}$$

By the assumptions ($\gamma = 0$, $\delta \neq 0$), $\frac{u_1''(\tau)}{\tau^{p'-1}} \to c(0)\psi_{p'}(\delta) \neq 0$ as $\tau \to 0_+$. So there exists a constant $K_1 > 0$ such that $\left|\frac{u_1''(\tau)}{\tau^{p'-1}}\right| \ge K_1$ for any $\tau \in (0,\varepsilon]$ with $\varepsilon > 0$ small enough. Thus, for every $t \in \mathcal{I}$

$$|u_1(t)| = \int_0^t (t-\tau)|u_1''(\tau)|\,\mathrm{d}\tau \ge \int_0^t (t-\tau)K_1\tau^{p'-1}\,\mathrm{d}\tau = \frac{K_1 t^{p'+1}}{p'(p'+1)},$$

i.e. $\left|\psi_q\big(\frac{u_1(t)}{t^{p'+1}}\big)\right| \ge \tilde{K}_1 := \psi_q\big(\frac{K_1}{p'(p'+1)}\big) > 0$, and analogously $\left|\psi_q\big(\frac{v_1(t)}{t^{p'+1}}\big)\right| \ge \tilde{K}_1$ for all $t \in (0,\varepsilon]$. Since $q < 2$, $\psi_{q'}'(\sigma) \ge \psi_{q'}'(\tilde{K}_1)$ for $|\sigma| \ge \tilde{K}_1$. Thus, for all $t \in (0,\varepsilon]$,

$$\big|b(t)(u_1(t) - v_1(t))\big| = |b(t)|t^{p'+1}\Big|\psi_{q'}\Big(\psi_q\Big(\frac{u_1(t)}{t^{p'+1}}\Big)\Big) - \psi_{q'}\Big(\psi_q\Big(\frac{v_1(t)}{t^{p'+1}}\Big)\Big)\Big| =$$

$$= |b(t)|t^{p'+1}\Big|\int_{\psi_q\left(\frac{v_1(t)}{t^{p'+1}}\right)}^{\psi_q\left(\frac{u_1(t)}{t^{p'+1}}\right)} \psi'_{q'}(\sigma)\,\mathrm{d}\sigma\Big| \geq t^{(2-q)(p'+1)}(q'-1)\tilde{K}_1^{q'-2}|u_3''(t)-v_3''(t)|. \tag{4.12}$$

Since $\frac{u_3(\tau)}{\tau} \to \delta$ and $\frac{v_3(\tau)}{\tau} \to \delta$ as $\tau \to 0_+$, there exists $K_2 > 0$ such that $\left|\frac{u_3(\tau)}{\tau}\right| \leq K_2$ and $\left|\frac{v_3(\tau)}{\tau}\right| \leq K_2$ for any $\tau \in (0,\varepsilon]$. Since $p \leq 2$, $\psi'_{p'}(\sigma) \leq \psi'_{p'}(K_2)$ for $|\sigma| \leq K_2$, and so

$$\Big|\psi_{p'}\Big(\frac{u_3(\tau)}{\tau}\Big)-\psi_{p'}\Big(\frac{v_3(\tau)}{\tau}\Big)\Big| = \Big|\int_{\frac{v_3(\tau)}{\tau}}^{\frac{u_3(\tau)}{\tau}} \psi'_{p'}(\sigma)\,\mathrm{d}\sigma\Big| \leq \frac{(p'-1)K_2^{p'-2}}{\tau}|u_3(\tau)-v_3(\tau)|. \tag{4.13}$$

Since, analogously to (4.4),

$$|u_3(\tau)-v_3(\tau)| \leq \tau^2\|u_3''-v_3''\|_{C(\mathcal{I})} \qquad \forall \tau \in \mathcal{I}, \tag{4.14}$$

we have the following estimate for all $t \in \mathcal{I}$:

$$\begin{aligned} |b(t)|\int_0^t (t-\tau)|c(\tau)|\psi_{p'}(\tau)\Big|\psi_{p'}\Big(\frac{u_3(\tau)}{\tau}\Big)-\psi_{p'}\Big(\frac{v_3(\tau)}{\tau}\Big)\Big|\,\mathrm{d}\tau \leq \\ \leq t^{p'+2}(p'-1)K_2^{p'-2}BC\|u_3''-v_3''\|_{C(\mathcal{I})}. \end{aligned} \tag{4.15}$$

Putting (4.12), (4.11) and (4.15) together and passing to the maximum over $\mathcal{I}$ (for $t=0$ the inequality is trivially satisfied) we obtain

$$\|u_3''-v_3''\|_{C(\mathcal{I})} \leq \varepsilon^{(q-1)(p'+1)+1}\frac{p'-1}{q'-1}\tilde{K}_1^{2-q'}K_2^{p'-2}BC\|u_3''-v_3''\|_{C(\mathcal{I})}. \tag{4.16}$$

Since for any $p,q>1$ we have $(q-1)(p'+1)+1>0$, the proof is complete. ☐

Lemma 4.3 *Let $|\alpha|+|\beta|+|\gamma|+|\delta|>0$, $p>2$ and $q\geq 2$. Moreover, let $|\gamma|+|\delta|>0$ or b not change its sign on some right closed neighborhood of t_0. Then there exists $\varepsilon>0$ such that* (1.5) *has at most one solution on $\mathcal{I} = [t_0, t_0+\varepsilon]$.*

Proof We distinguish the cases (i) $\gamma \neq 0$, (ii) $\gamma = 0$, $\delta \neq 0$, (iii) $\gamma=\delta=0$, $\alpha \neq 0$ and (iv) $\gamma=\delta=\alpha=0$, $\beta\neq 0$. Let again $\boldsymbol{u}$ a $\boldsymbol{v}$ be solutions of the special case (2.2), corresponding to (1.5).
(i) Since $u_1''(0) = v_1''(0) = c(0)\psi_{p'}(\gamma) \neq 0$, there exists a constant $K_1>0$ such that $|u_1''(t)| \geq K_1$ and $|v_1''(t)| \geq K_1$ for all $t \in [0,\varepsilon]$ with $\varepsilon > 0$ small enough. We have $p>2$, and so $\psi'_p(\tau) \geq \psi'_p(K_1)$ for $|\tau| \geq K_1$. Hence (4.2) holds true, and we get (4.6).
(ii) As in the part (iv) of the proof of Lemma 4.2, there exists a constant $K_1 > 0$ such that $\left|\frac{u_1''(t)}{t^{p'-1}}\right| \geq K_1$ and $\left|\frac{v_1''(t)}{t^{p'-1}}\right| \geq K_1$ for all $t \in (0,\varepsilon]$ with $\varepsilon>0$ small enough. Hence

$$\begin{aligned} \Big|a(t)\Big(\psi_p(u_1''(t))-\psi_p(v_1''(t))\Big)\Big| = |a(t)|t\Big|\psi_p\Big(\tfrac{u_1''(t)}{t^{p'-1}}\Big)-\psi_p\Big(\tfrac{v_1''(t)}{t^{p'-1}}\Big)\Big| \geq \\ \geq C^{1-p}t\Big|\int_{\frac{v_1''(t)}{t^{p'-1}}}^{\frac{u_1''(t)}{t^{p'-1}}}\psi'_p(\sigma)\,\mathrm{d}\sigma\Big| \geq t^{2-p'}(p-1)K_1^{p-2}C^{1-p}|u_1''(t)-v_1''(t)|. \end{aligned} \tag{4.17}$$

Using (4.17) instead of (4.2) we obtain

$$\|u_1'' - v_1''\|_{C(\mathcal{I})} \le \varepsilon^{p'+2} \frac{(q-1)K_2^{q-2}B}{(p-1)K_1^{p-2}C^{1-p}} \|u_1'' - v_1''\|_{C(\mathcal{I})}. \tag{4.18}$$

(iii) We can assume

$$f(t) := \int_0^t (t-\tau)|b(\tau)|\,\mathrm{d}\tau > 0 \quad \forall t \in (0,\varepsilon]$$

(otherwise $b(\tau) = 0$ for all $\tau \in [0, t_0]$ with some $t_0 > 0$, and the uniqueness is then trivial). Since $u_1(0) = \alpha \ne 0$, there exists a constant $K_1 > 0$ such that $|u_1(\tau)| \ge K_1$, and so $|u_3''(\tau)| \ge K_1^{q-1}|b(\tau)|$ for all $\tau \in [0,\varepsilon]$ with $\varepsilon > 0$ small enough. We suppose that b and u_3'' does not change its sign on $\mathcal{I}$. Hence for any $t \in \mathcal{I}$

$$|u_3(t)| = \int_0^t (t-\tau)|u_3''(\tau)|\,\mathrm{d}\tau \ge K_1^{q-1}\int_0^t (t-\tau)|b(\tau)|\,\mathrm{d}\tau = K_1^{q-1}f(t).$$

Thus

$$|u_1''(t)| = |c(t)\psi_{p'}(u_3(t))| \ge K_1^{(q-1)(p'-1)}A^{1-p'}f^{p'-1}(t),$$

and the same estimate holds for $|v_1''(t)|$, $t \in \mathcal{I}$. For $t \in (0,\varepsilon]$ we can write

$$\begin{aligned}\Big|a(t)\Big(\psi_p(u_1''(t)) - \psi_p(v_1''(t))\Big)\Big| &\ge C^{1-p}f(t)\Big|\psi_p\Big(\tfrac{u_1''(t)}{f^{p'-1}(t)}\Big) - \psi_p\Big(\tfrac{v_1''(t)}{f^{p'-1}(t)}\Big)\Big| = \\ &= C^{1-p}f(t)\Big|\int_{\frac{v_1''(t)}{f^{p'-1}(t)}}^{\frac{u_1''(t)}{f^{p'-1}(t)}} \psi_p'(\tau)\,\mathrm{d}\tau\Big| \ge \\ &\ge (p-1)K_1^{\frac{(q-1)(p-2)}{p-1}}A^{-\frac{p-2}{p-1}}C^{1-p}f^{2-p'}(t)|u_1''(t) - v_1''(t)|.\end{aligned} \tag{4.19}$$

Using (4.3) and (4.4) we get for $t \in \mathcal{I}$

$$\Big|\int_0^t (t-\tau)b(\tau)\Big(\psi_q(u_1(\tau)) - \psi_q(v_1(\tau))\Big)\,\mathrm{d}\tau\Big| \le t^2(q-1)K_2^{q-2}f(t)\|u_1'' - v_1''\|_{C(\mathcal{I})}. \tag{4.20}$$

Obviously $f(t) \le t^2B$. Putting (4.19), (4.1) and (4.20) together and passing to the maximum over $\mathcal{I}$ we arrive at

$$\|u_1'' - v_1''\|_{C(\mathcal{I})} \le \varepsilon^{2p'} \frac{(q-1)K_2^{q-2}B^{p'-1}}{(p-1)K_1^{\frac{(q-1)(p-2)}{p-1}}A^{-\frac{p-2}{p-1}}C^{1-p}} \|u_1'' - v_1''\|_{C(\mathcal{I})}.$$

(iv) We proceed as analogically to (iii). We can assume

$$f(t) := \int_0^t (t-\tau)\tau^{q-1}|b(\tau)|\,\mathrm{d}\tau > 0 \quad \forall t \in (0,\varepsilon].$$

Since $\frac{u_1(\tau)}{\tau} \to \beta \neq 0$, there exists a constant $K_1 > 0$ such that $\left|\frac{u_1(\tau)}{\tau}\right| \geq K_1$, and also $|u_3''(\tau)| \geq (K_1\tau)^{q-1}|b(\tau)|$ for $\tau \in [0,\varepsilon]$ with $\varepsilon > 0$ small enough. We suppose that b and u_3'' does not change sign on $\mathcal{I}$. For any $t \in \mathcal{I}$ we have

$$|u_3(t)| = \int_0^t (t-\tau)|u_3''(\tau)|\,\mathrm{d}\tau \geq K_1^{q-1}\int_0^t (t-\tau)\tau^{q-1}|b(\tau)|\,\mathrm{d}\tau = K_1^{q-1}f(t).$$

Analogously as (4.19) we can now for $t \in (0,\varepsilon]$ derive

$$\begin{aligned} &\Big|a(t)\Big(\psi_p(u_1''(t)) - \psi_p(v_1''(t))\Big)\Big| \geq \\ &\geq (p-1)K_1^{\frac{(q-1)(p-2)}{p-1}}A^{-\frac{p-2}{p-1}}C^{1-p}f^{2-p'}(t)|u_1''(t) - v_1''(t)|. \end{aligned} \tag{4.21}$$

There exists a constant $K_2 > 0$ such that $\left|\frac{u_1(\tau)}{\tau}\right| \leq K_2$ and $\left|\frac{v_1(\tau)}{\tau}\right| \leq K_2$ for all $\tau \in (0,\varepsilon]$. Since $q \geq 2$, $\psi_q'(\sigma) \leq \psi_q'(K_2)$ for $|\sigma| \leq K_2$. Thus, (4.7) holds true for all $t \in (0,\varepsilon]$. Together with (4.4) it yields for $t \in \mathcal{I}$

$$\Big|\int_0^t (t-\tau)b(\tau)\Big(\psi_q(u_1(\tau)) - \psi_q(v_1(\tau))\Big)\,\mathrm{d}\tau\Big| \leq t(q-1)K_2^{q-2}f(t)\|u_1'' - v_1''\|_{C(\mathcal{I})}. \tag{4.22}$$

Obviously $f(t) \leq t^{q+1}B$. Using (4.21), (4.1), (4.22) we obtain for the maximum over $\mathcal{I}$

$$\|u_1'' - v_1''\|_{C(\mathcal{I})} \leq \varepsilon^{(q+1)(p'-1)+1}\frac{(q-1)K_2^{q-2}B^{p'-1}}{(p-1)K_1^{\frac{(q-1)(p-2)}{p-1}}A^{-\frac{p-2}{p-1}}C^{1-p}}\|u_1'' - v_1''\|_{C(\mathcal{I})}.$$

Since for $p,q > 1$ it is $(q+1)(p'-1)+1 > 0$, we proved the assertion of this lemma. $\square$

Lemma 4.4 *Let $|\alpha| + |\beta| + |\gamma| + |\delta| > 0$, $p > 2$ and $q < 2$. Moreover, let $|\gamma| + |\delta| > 0$ or b not change its sign on some right closed neighborhood of t_0. Then there exists $\varepsilon > 0$ such that* (1.5) *has at most one solution on $\mathcal{I} = [t_0, t_0+\varepsilon]$.*

Proof We combine the previous techniques. Consequently, we distinguish the cases (i) $\alpha \neq 0$, $\gamma \neq 0$, (ii) $\alpha \neq 0$, $\gamma = 0$, $\delta \neq 0$, (iii) $\alpha \neq 0$, $\gamma = \delta = 0$, (iv) $\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$, (v) $\alpha = 0$, $\beta \neq 0$, $\gamma = 0$, $\delta \neq 0$, (vi) $\alpha = 0$, $\beta \neq 0$, $\gamma = \delta = 0$, (vii) $\alpha = \beta = 0$, $\gamma \neq 0$ and (viii) $\alpha = \beta = \gamma = 0$, $\delta \neq 0$.
(i) As in the part (i) of the proof of Lemma 4.3, we can use (4.2), and, as in the part (i) of the proof of Lemma 4.2, we can use (4.3). Using (4.1) we arrive at (4.6).
(ii) From (4.17), (4.1) and (4.3) we derive (4.18).
(iii) As (4.3) holds true, we follow the part (iii) of the proof of Lemma 4.3.
(iv) From (4.2), (4.1) and (4.7) we get (4.8).
(v) Here (4.17), (4.1) and (4.7) yield

$$\|u_1'' - v_1''\|_{C(\mathcal{I})} \leq \varepsilon^{p'+q}\frac{(q-1)K_2^{q-2}B}{(p-1)K_1^{p-2}C^{1-p}}\|u_1'' - v_1''\|_{C(\mathcal{I})}.$$

(vi) Since (4.7) holds true for some $K_2 > 0$, we can proceed as in the part (iv) of the proof of Lemma 4.3.
(vii) From (4.2), (4.1) and (4.9) we conclude (4.10).
(viii) We follow the part (iv) of the proof of Lemma 4.2. Due to the assumptions ($\gamma = 0$, $\delta \neq 0$), there exists a constant $K_2 > 0$ such that $\left|\frac{u_3(\tau)}{\tau}\right| \geq K_2$ and $\left|\frac{u_3(\tau)}{\tau}\right| \geq K_2$ for all $\tau \in (0, \varepsilon]$ with $\varepsilon > 0$ small enough. Since $p > 2$, $\psi'_{p'}(\sigma) \leq \psi'_{p'}(K_2)$ for $|\sigma| \geq K_2$, and so (4.13) holds true. Now we put (4.12), (4.11) and (4.13) together to obtain (4.16). $\square$

As we promised, we show now that if $p > q$, $\alpha = \beta = \gamma = \delta = 0$ and $\lambda > 0$, then (1.3) has a non-trivial solution besides of the trivial one, and so the uniqueness is broken.

Example 4.5 Let in (1.3) $p > q$, $\alpha = \beta = \gamma = \delta = 0$, $\lambda > 0$ and $\mathcal{I} = [0, \varepsilon]$ with some $\varepsilon > 0$. Then one can compute that $u(t) = 0$ and $u(t) = K(t - t_0)^r$ where

$$r = \frac{2p}{p-q} \quad \text{and} \quad K = \left(\frac{2^p(p-1)q\big(p(p+q)\big)^{p-1}(2pq - p - q)}{\lambda(p-q)^{2p}}\right)^{1/(q-p)}$$

are solutions of (1.3).

Remark 4.6 Example 4.5 can be generalized (cf. Example 3.1) for the initial value problem (3.1) of the $(2n)^{\text{th}}$-order, $n \in \mathbb{N}$, where $p > q$, $(-1)^n\lambda > 0$ and $\alpha_i = \beta_i = 0$, $i = 0, \dots, n-1$. The reader is invited to justify that $u(t) = 0$ and $u(t) = K(H - t)^r$ where

$$r = \frac{np}{p-q} \quad \text{and}$$

$$K = \left(\frac{\left(\prod_{k=0}^{n-1}\big((n-k)p + kq\big)\right)^{p-1}\prod_{k=0}^{n-1}\big(npq - kp - (n-k)q\big)}{(-1)^n\lambda(p-q)^{np}}\right)^{1/(q-p)}$$

are solutions of (3.1).

Lemma 4.7 *Let $\alpha = \beta = \gamma = \delta = 0$ and $p \leq \min\{q_1, q_2\}$. Then there exists $\varepsilon > 0$ such that* (1.6) *has at most one solution on $\mathcal{I} = [t_0, t_0 + \varepsilon]$.*

Proof Let $\boldsymbol{u}$ be a solution of (1.6). We prove that $u_1 = u_2 = u_3 = u_4 = 0$ on $[0, \varepsilon]$ for some $\varepsilon > 0$. For $t \in \mathcal{I}$

$$|a(t)\psi_p(u_1''(t))| \leq \int_0^t (t-\tau)\Big(|b_1(\tau)|\psi_{q_1}(|u_1^+(\tau)|) + |b_2(\tau)|\psi_{q_2}(|u_1^-(\tau)|)\Big)\,\mathrm{d}\tau \tag{4.23}$$

Since for $\tau \in \mathcal{I}$ obviously $|u(\tau)| \le \tau^2 \|u''\|_{C(\mathcal{I})}$, we have

$$\begin{aligned} |b_1(\tau)|\psi_{q_1}(|u_1^+(\tau)|) + |b_2(\tau)|\psi_{q_2}(|u_1^-(\tau)|) &\le \\ \le B\Big(\tau^{2q_1-2}\|u_1''\|_{C(\mathcal{I})}^{q_1-1} + \tau^{2q_2-2}\|u_1''\|_{C(\mathcal{I})}^{q_2-1}\Big) &\le \\ \le B\Big(\tau^{2q_1-2}\|u_1''\|_{C(\mathcal{I}_0)}^{q_1-p} + \tau^{2q_2-2}\|u_1''\|_{C(\mathcal{I}_0)}^{q_2-p}\Big)\|u_1''\|_{C(\mathcal{I})}^{p-1} & \end{aligned} \tag{4.24}$$

where $\mathcal{I}_0 = [0, \varepsilon_0]$ with $\varepsilon_0 > 0$ arbitrary, but fixed, and $\varepsilon \le \varepsilon_0$. We used the assumption $p \le \min\{q_1, q_2\}$ which implies that $\|u_1''\|_{C[0,\varepsilon]}^{q_i-p}$, $i = 1, 2$, are increasing functions of ε. Using the estimate $|a(t)\psi_p(u_1''(t))| \ge C^{1-p}|u_1''(t)|^{p-1}$ we can infer from (4.23) and (4.24) that for every $t \in \mathcal{I}$

$$C^{1-p}|u_1''(t)|^{p-1} \le B\Big(t^{2q_1}\|u_1''\|_{C(\mathcal{I}_0)}^{q_1-p} + t^{2q_2}\|u_1''\|_{C(\mathcal{I}_0)}^{q_2-p}\Big)\|u_1''\|_{C(\mathcal{I})}^{p-1}.$$

Now we pass to the maximum for $t \in \mathcal{I}$. If we suppose that $\varepsilon \le 1$, we obtain

$$\|u_1''\|_{C(\mathcal{I})}^{p-1} \le \varepsilon^{2\min\{q_1,q_2\}} BC^{p-1}\Big(\|u''\|_{C(\mathcal{I}_0)}^{q_1-p} + \|u''\|_{C(\mathcal{I}_0)}^{q_2-p}\Big)\|u_1''\|_{C(\mathcal{I})}^{p-1}.$$

For $\varepsilon > 0$ small enough this inequality guarantees that $\|u''\|_{C(\mathcal{I})}^{p-1} = 0$, and so $u_1'' = 0$, $u_1 = 0$, and also $u_2 = u_3 = u_4 = 0$ on $\mathcal{I} = [0, \varepsilon]$. $\square$

Now that we completed the proof of Proposition 1.5. Theorem 1.7 is a direct consequence of this proposition.

5 Open Problems

The main problems we leave open are:

1. Does the conclusion of Proposition 1.5 (local uniqueness) hold true even without the latter four conditions, i.e. for $p > 2$, $\gamma = \delta = 0$, α or β nonzero and b_1 or b_2 changing its sign on arbitrarily small right neighborhood of t_0? If it did, then the sufficient condition for the global uniqueness (see Theorem 1.7) would be $p \le \min\{q_1, q_2\}$ only (we showed that this assumption cannot be left out).

   We can simplify this problem: Can there exist two different solutions of

$$\begin{array}{lll} u''(t) = \psi_{p'}(v(t)), & u(t_0) = \alpha, \quad u'(t_0) = \beta, & \\ v''(t) = b(t)\psi_p(u(t)), & v(t_0) = 0, \quad v'(t_0) = 0, & t \in \mathcal{I} \end{array} \tag{5.1}$$

   with $\mathcal{I} = [t_0, t_1]$, $b \in C(\mathcal{I})$ arbitrary, $p > 2$ and $|\alpha| + |\beta| > 0$? Note that the system of equations in (5.1) is homogeneous!

2. We gave Example 3.1 which showed that for $p < q$ and some initial conditions the solution of (1.3) did not have to exist on $[t_0, \infty)$. In this Example we assumed $\lambda > 0$, for $\lambda = 0$ the global existence is trivial, but for $\lambda < 0$ we leave the question of global existence open (see Table 1).

3. Analogously to the previous open problem, for $\lambda < 0$, $\alpha = \beta = \gamma = \delta = 0$ and $p > q$ we gave neither the proof of the local uniqueness of the solution of (1.3) nor a counterexample (see Table 2), and so we leave it as an open question, too.

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It was brought to my knowledge by a colleague that in Remark 4.6, fourth line (page 15) there should be

$$u(t) = K(t - t_0)^r$$

instead of

$$u(t) = K(H - t)^r \,.$$

Even though I think that this mistake is not misleading for the reader, I want to correct it.