PRODUCT INTEGRALS AND MATRICES

THESIS

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CHAPTER I

INTRODUCTION

Two problems are solved in this paper by use of product integrals. First, it is proven that any square, non-singular matrix has n distinct nth roots, for each positive integer n. Secondly, conditions are established under which $\sum_{i=1}^{\infty} A_i$ converges if, and only if, $\prod_{i=1}^{\infty} (1+A_i)$ converges, where $\{A_i\}_{i=1}^{\infty}$ is a sequence of commutative square matrices.

CHAPTER II

DEFINITIONS, NOTATIONS, AND FUNDAMENTAL THEOREMS

Proofs of most of the theorems stated may be found in the references cited in the bibliography.

Notation 2.1. The symbol R denotes the set of nxn matrices of complex numbers, where n is a positive integer.

Definition 2.2. A ring N is complete means for each sequence $\{x_n\}_{n=1}^{\infty}$ of elements in N such that $\lim_{m,n\to\infty}|x_m-x_n|=0$, there exists an x_0 belonging to N such that $\lim_{m\to\infty} x_m = x_0$.

Notation 2.3. $|\circ|$ is a norm for R such that R is complete with respect to $|\circ|$, $|1| = 1$, $|0| = 0$, and $|a\circ b| \leq |a|\circ|b|$ for elements a and b in R.

Notation 2.4. The symbol S denotes the set of real numbers and SxS denotes the set of ordered pairs of real numbers.

Notation 2.5. The symbol [a,b] denotes a closed real number interval, where b > a.

Definition 2.6. D is a subdivision of [a,b] means D is a finite subset $\{x_i\}_{i=0}^{n}$ of $[a,b]$ such that

$$a = x_0 \leq x_1 \leq \cdots \leq x_{i-1} \leq x_i \leq \cdots \leq x_n = b.$$

Definition 2.7. H is a refinement of a subdivision D of $[a,b]$ means H is a subdivision of $[a,b]$ and D is a subset of H.

Notation 2.8. If g is a function from S to R and $\{x_i\}_{i=0}^{n}$ is a subdivision of an interval $[a,b]$, then Δg_i denotes $g(x_i) - g(x_{i-1})$.

Definition 2.9. If F is a function from SxS to R, the statement that F belongs to OB on $[a,b]$ means there is a positive number M such that, if $\{x_i\}_{i=0}^{n}$ is a subdivision of $[a,b]$, then $\sum_{i=1}^{n} |F(x_{i-1}, x_i)| < M$.

Definition 2.10. If G is a function from SxS to R and a and b belong to S, the statement that $\int_a^b G$ exists means there is an element A belonging to R such that, if $\varepsilon > 0$, there is a subdivision D of $[a,b]$ such that, if $\{x_i\}_{i=0}^{n}$ is a refinement of D, then $\left| \sum_{i=1}^{n} G(x_{i-1}, x_i) - A \right| < \varepsilon$.

Definition 2.11. If G is a function from SxS to R and a and b belong to S, the statement that ${}_a\Pi^b G$ exists means there is an element A belonging to R such that, if $\varepsilon > 0$, there is a subdivision D of [a,b] such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then $\left|\prod_{i=1}^n G(x_{i-1}, x_i) - A\right| < \varepsilon$.

Notation 2.12. If f and g are functions from S to R, then the function fdg is the function G from SxS to R such that $G(x,y) = f(x)[g(y)-g(x)]$ for (x,y) belonging to SxS.

Definition 2.13. If $\{A_i\}_{i=1}^\infty$ is a sequence of elements of R, the statement that $\prod_{i=1}^\infty (1+A_i)$ converges means there is an element A of R such that $\lim_{n\to\infty} \prod_{i=1}^n (1+A_i) = A$ and A^{-1} exists.

Notation 2.14. If A belongs to R, then det A denotes the determinant of A.

Theorem 2.15. If n is a positive integer, the complex number (1,0) has n distinct nth roots.

Theorem 2.16. If n is a positive integer and $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are subsets of R, then

$$\prod_{i=1}^n x_i - \prod_{i=1}^n y_i = \sum_{i=1}^n \left[\prod_{j=1}^{i-1} y_j\right][x_i - y_i]\left[\prod_{j=i+1}^n x_j\right].$$

<u>Theorem 2.17.</u> Suppose a and b are real numbers and F and G are functions from SxS to R such that $\int_a^b |F \circ G| = 0$, $\int_a^b |G^2| = 0$, F and G belong to OB, ${}_a\Pi^b(1+F)$ and ${}_a\Pi^b(1+G)$ exist, and all elements commute.

Conclusion:

(1) ${}_a\Pi^b[1+(F+G)]$ exists and is ${}_a\Pi^b(1+F) \circ {}_a\Pi^b(1+G)$.

(2) If n is a positive integer, then

$$[{}_a\Pi^b(1+G)]^{\frac{1}{n}} = {}_a\Pi^b(1+\tfrac{1}{n}G).$$

(3) If ${}_a\Pi^b(1-G)$ exists, then

$$[{}_a\Pi^b(1+G)]^{-1} = {}_a\Pi^b(1-G).$$

<u>Theorem 2.18.</u> If f is a function from S to R, a and b belong to S, and $[f(x)]^{-1}$ exists for each x belonging to [a,b], then

$$f(x) = f(a) \circ {}_a\Pi^x(1+f^{-1}df)$$

for each x belonging to [a,b].

<u>Theorem 2.19.</u> If n is a positive integer and $\{a_i\}_{i=1}^n$ is a set of non-negative real numbers, then

$$\prod_{i=1}^{n}(1+a_i) \leq \exp\left(\sum_{i=1}^{n} a_i\right).$$

<u>Theorem 2.20</u>. If G is a function from SxS to R, a and b are real numbers, and ${}_a\Pi^b(1+G)$ and ${}_a\Pi^b(1+|G|)$ exists, then

$$\left|{}_a\Pi^b(1+G)\right| \le {}_a\Pi^b(1+|G|).$$

<u>Theorem 2.21</u>. If a, x, and y are real numbers and $y > x$, then ${}_x\Pi^y(1+adt)$ exists and is $\exp a(y-x)$.

<u>Theorem 2.22</u>. If f is a continuous function from S to R such that $[f(x)]^{-1}$ exists for each x in S, then f^{-1} is continuous.

<u>Theorem 2.23</u>. If f and g are functions from S to R such that df and dg belong to OB, then ${}_x\Pi^y(1+fdg)$ exists for each (x,y) belonging to SxS.

<u>Theorem 2.24</u>. If f is a function from S to R, a belongs to S, and α belongs to R, then the following two statements are equivalent:

(1) If x belongs to S, then $\int_a^x f\alpha dt$ exists and

$$f(x) = 1 + \int_a^x f\alpha dt.$$

(2) If x belongs to S, then ${}_a\Pi^x(1+\alpha dt)$ exists and

$$f(x) = f(a) \cdot {}_{a}\prod^{x} (1+\alpha dt).$$

CHAPTER III

THE ROOTS OF A SQUARE, NON-SINGULAR MATRIX

Suppose A is a square, non-singular matrix of complex numbers and n is a positive integer greater than one. In this chapter it will be proven that A has n distinct nth roots.

First, a function g will be defined from the real interval [0,1] to the complex numbers, with the properties:

(1) if x belongs to [0,1], $\det [1+(A-1)g(x)] \neq 0$,

(2) g is continuous and has bounded variation on [0,1], and

(3) $g(0) = (0,0)$ and $g(1) = (1,0)$.

For each x belonging to [0,1], $\det [1+(A-1)g(x)]$ is a polynomial of the form

$$b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n, \tag{3.1}$$

where $z = g(x)$. The polynomial (3.1) has only a finite number of roots (x,y) such that $|(x,y)| \leq 1$; let B denote this set. Since $\det [1+(A-1)] = \det A \neq 0$ and since $\det 1 \neq 0$, then (0,0) and (1,0) are not elements of B. If B is empty, or if there exists no (x,y) in B such that y=0, then let $g(x) = (x,0)$ for x in [0,1]. If, on the other hand, y=0 for every (x,y) in B, then let the graph of g be the straight line extending from $(0,0)$ to $(\frac{1}{2},\frac{1}{2})$ plus the straight line extending from $(\frac{1}{2},\frac{1}{2})$ to $(1,0)$. Finally, if there exists an (x,y) in B such that $|y| > 0$, then there exists an (s,t) in B such that $|y| \geq |t| > 0$ for all (x,y) in B. In this case let the graph of g be the straight line extending from $(0,0)$ to

$$\left(\sqrt{1 - \frac{1}{4}|t|^2},\ \frac{1}{2}|t|\right)$$

plus that arc on the unit circle from

$$\left(\sqrt{1 - \frac{1}{4}|t|^2},\ \frac{1}{2}|t|\right)$$

to $(1,0)$. In any of the three above cases, g is a continuous function of bounded variation such that $g(0) = (0,0)$ and $g(1) = (1,0)$. Furthermore, no element of B belongs to the graph of g; therefore, for each x belonging to $[0,1]$, $\det[1+(A-1)g(x)] \neq 0$ and $[1+(A-1)g(x)]^{-1}$ exists.

Define f to be the function from the real interval $[0,1]$ to the ring of matrices such that

$$f(x) = 1 + (A-1)g(x)$$

for x belonging to $[0,1]$. Since g is continuous and of bounded variation, then f has these properties. Furthermore, since $[f(x)]^{-1}$ exists for each x belonging to [0,1] and since f is continuous, then f^{-1} is continuous on [0,1] by Theorem 2.22. Hence, both f and f^{-1} are bounded; therefore $\int_0^1 [f^{-1} df]^2 = 0$.

If r and s are complex numbers, then $[1+(A-1)r]$ and $[1+(A-1)s]^{-1}$ commute because

$$
\begin{aligned}
[1+(A-1)r] \cdot [1+(A-1)s]^{-1} \\
&= (1-r)[1+(A-1)s]^{-1} + Ar[1+(A-1)s]^{-1} \\
&= \{[1+(A-1)s]^{-1}(1-r)\} + r\{[1+(A-1)s]A^{-1}\}^{-1} \\
&= \{\cdot\} + \{A^{-1}+(1-A^{-1})s\}^{-1} r \\
&= \{\cdot\} + \{A^{-1}+A^{-1}(A-1)s\}^{-1} r \\
&= \{\cdot\} + \{A^{-1}[1+(A-1)s]\}^{-1} r \\
&= [1+(A-1)s]^{-1}(1-r) + [1+(A-1)s]^{-1} Ar \\
&= [1+(A-1)s]^{-1}[1+(A-1)r].
\end{aligned}
$$

The above manipulations prove that f and f^{-1} commute. Since the hypothesis of Theorem 2.18 is satisfied and

$f(0) = 1$, then

$$f(1) = 1 \cdot {}_0\Pi^1(1+f^{-1}df).$$

Furthermore, since the hypothesis of Theorem 2.17 is satisfied and since $f(1) = A$, then

$$[(1\circ\alpha_i) \cdot {}_0\Pi^1(1+\tfrac{1}{n}f^{-1}df)]^n = 1 \cdot {}_0\Pi^1(1+f^{-1}df) = A,$$

where α_i, $i=1,2,\cdots,n$, is one of the n distinct nth roots of the complex number $(1,0)$. Therefore, if $1 \le i \le n$, then $(1\circ\alpha_i) \cdot {}_0\Pi^1(1+\frac{1}{n}f^{-1}df)$ is an nth root of A. We now prove that these roots are distinct. Let K denote ${}_0\Pi^1(1+\frac{1}{n}f^{-1}df)$. $K \neq 0$, for, if it were, then $A=K^n=0^n=0$, which is false. Also, if $1 \le i < j \le n$, then $(1\circ\alpha_i)K \neq (1\circ\alpha_j)K$. If this last statement were false, then there exist positive integers i and j such that $1 \le i < j \le n$ and $(1\circ\alpha_i)K = (1\circ\alpha_j)K$, from which it follows that $1\circ(\alpha_i-\alpha_j)K = 0$. Since $1\circ(\alpha_i-\alpha_j)$ is a diagonal matrix with no zeros on the diagonal, it has a multiplicative inverse and it follows that $K=0$, which is false. Therefore, the roots are distinct and we conclude that A has n distinct nth roots.

CHAPTER IV

TWO EQUIVALENT STATEMENTS

Theorems involving product integrals may be used to prove that, if $\{A_i\}_{i=1}^{\infty}$ is a sequence of commutative elements of R and $\sum_{i=1}^{\infty} |A_i|^2$ converges, then the infinite product $\prod_{i=1}^{\infty} (1+A_i)$ converges if, and only if, $\sum_{i=1}^{\infty} A_i$ converges. The following theorems and lemmas are used in the proof of Theorem 4.7.

Theorem 4.1. If A is an element of R and $|A| < 1$, then $(1+A)^{-1}$ exists and is $1+\sum_{i=1}^{\infty} (-A)^i$.

Proof:

Since $|A| < 1$, $\sum_{i=1}^{\infty} |A|^i$ converges. Let $\varepsilon > 0$, then there exists a positive integer N such that $\sum_{i=n}^{m} |A|^i < \varepsilon$ for $m > n > N$. Let $m > n > N$, then

$$\left|\sum_{i=n}^{m} (-A)^i\right| \le \sum_{i=n}^{m} |A|^i < \varepsilon;$$

hence, the series $\sum_{i=1}^{\infty}(-A)^i$ converges by the Cauchy criterion. Since

$$\begin{aligned}(1+A)[1+\sum_{i=1}^{\infty}(-A)^i] &= (1+A)(1-A+A^2-A^3+\cdots) \\ &= (1+A-A^2+A^2+A^3-A^3-A^4+\cdots) \\ &= 1;\end{aligned}$$

then

$$(1+A)^{-1} = 1+\sum_{i=1}^{\infty}(-A)^i.$$

<u>Theorem 4.2</u>. If A is an element of R, then ${}_0\Pi^1(1+Adt)$ exists and ${}_0\Pi^1(1+Adt) = 1+\sum_{n=1}^{\infty}\frac{1}{n!}A^n$.

Proof:

${}_0\Pi^1(1+Adt)$ exists by Theorem 2.23 because the function $F(x,y) = A(y-x)$ belongs to OB. Also, by Theorems 2.20 and 2.21,

$$|{}_0\Pi^x(1+Adt)| \le {}_0\Pi^x(1+|A|dt) = \exp|A|\cdot x \le \exp|A|$$

for $0 \le x \le 1$. Define $f(x) = {}_0\Pi^x(1+Adt)$, then by Theorem 2.24

$$\begin{aligned}{}_0\Pi^1(1+Adt) &= f(1) \\ &= 1+\int_0^1 Afdt\end{aligned}$$

$$= 1 + A\int_o^1 f dt$$

$$= 1 + A\int_o^1 [1+\int_o^t Afdp]dt$$

$$= 1 + A\int_o^1 dt + A^2\int_o^1\int_o^t fdpdt$$

$$= 1 + A + A^2\int_o^1\int_o^t [1 + \int_o^p Afdr]dpdt$$

$$= 1 + A + A^2\int_o^1\int_o^t dpdt + A^3\int_o^1\int_o^t\int_o^p fdrdpdt$$

$$= 1 + A + \frac{1}{2!} A^2 + A^3\int_o^1\int_o^t\int_o^p fdrdpdt.$$

If this process is continued, then for each positive integer n,

$$_o\Pi^1(1+Adt) = 1 + \sum_{k=1}^{n-1} \frac{1}{k!} A^k + r_n,$$

where

$$r_n = A^n\int_o^1\int_o^{x_1} \cdots \int_o^{x_{n-1}} f\, dx_n\, dx_{n-1} \cdots dx_1$$

and

$$|r_n| \leq \frac{|A|^n \exp|A|}{n!} = s_n.$$

Since $\sum_{n=1}^{\infty} s_n$ converges, then $\sum_{n=1}^{\infty} \frac{1}{n!} A^n$ converges and

$$_o\Pi^1(1+Adt) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} A^n.$$

Lemma 4.3. If $\{A_i\}_{i=1}^{\infty}$ is a sequence of commutative elements of R and $\sum_{i=1}^{\infty} A_i$ and $\sum_{i=1}^{\infty} |A_i|^2$ converge, then there is a positive number M such that $|\prod_{i=n}^{m} (1+A_i)| \leq M$ and

$$|\prod_{i=n}^{m} {}_0\Pi^1 (1+A_i dt)| \leq M$$

for positive integers $m > n$.

Proof:

Since $\sum_{i=1}^{\infty} |A_i|^2$ converges, there is a positive integer X such that $\sum_{i=n}^{m} |A_i|^2 < 1$ for $m > n > X$. Also, since $\sum_{i=1}^{\infty} A_i$ converges, there exists a positive integer Y such that $|\sum_{i=n}^{m} A_i| < \frac{1}{2}$ for $m > n > Y$. Choose $N = X + Y$ and let $M = \exp(\sum_{i=1}^{N} |A_i| + 3)$. Let m and n be positive integers such that $m > n$.

If m and n are both less than N, then

$$|\prod_{i=n}^{m} (1+A_i)| \leq \prod_{i=n}^{m} (1+|A_i|)$$

$$\leq \exp\left(\sum_{i=n}^{m} |A_i|\right) \qquad \text{(Th. 2.19)}$$

and

$$|\prod_{i=n}^{m} {}_0\Pi^1 (1+A_i dt)| = |{}_0\Pi^1 (1+\sum_{i=n}^{m} A_i dt)| \qquad \text{(Th. 2.17)}$$

$$\leq {}_{o}\Pi^{1}(1+\sum_{i=n}^{m}|A_i|dt) \qquad \text{(Th. 2.20)}$$

$$= \exp\left(\sum_{i=n}^{m}|A_i|\right), \qquad \text{(Th. 2.21)}$$

but

$$\exp\left(\sum_{i=n}^{m}|A_i|\right) \leq \exp\left(\sum_{i=1}^{N}|A_i|\right) < M;$$

hence,

$$\left|\prod_{i=n}^{m}(1+A_i)\right| < M$$

and

$$\left|\prod_{i=n}^{m} {}_{o}\Pi^{1}(1+A_i dt)\right| < M.$$

If m and n are both greater than N, then for each positive integer $i > N$ define f_i to be the function such that, if t belongs to [0,1], then $f_i(t) = 1 + A_i t$. If $i > N$, then $|A_i| < \frac{1}{2}$; therefore, by Theorem 4.1

$$f_i(t)^{-1} = (1+A_i t)^{-1}$$

$$= 1 + \sum_{n=1}^{\infty}(-A_i t)^n$$

for each t in [0,1]. For each $i > N$ let $B_i = \sum_{n=1}^{\infty}(-A_i t)^n$, then, since $f_i(0) = 1$ and $f_i(1) = 1+A_i$,

$$\left|\prod_{i=n}^{m}(1+A_i)\right| = \left|\prod_{i=n}^{m} {}_{o}\Pi^{1}(1+f_i^{-1}df_i)\right|$$

$$\text{(Th. 2.18)}$$

$$= \left|\prod_{i=n}^{m} {}_{o}\prod^{1}[1 + (1+A_i t)^{-1} A_i dt]\right|$$

$$= \left|{}_{o}\prod^{1}[1 + \sum_{i=n}^{m}(1+A_i t)^{-1} A_i dt]\right| \qquad \text{(Th. 2.17)}$$

$$= \left|{}_{o}\prod^{1}[1 + \sum_{i=n}^{m}(1+B_i)A_i dt]\right|$$

$$\leq \left|{}_{o}\prod^{1}(1 + \sum_{i=n}^{m} A_i dt)\right| \cdot \left|{}_{o}\prod^{1}(1 + \sum_{i=n}^{m} B_i A_i dt)\right|$$

(Th. 2.17).

Furthermore,

$$|B_i A_i| = |A_i B_i| = \left|A_i \sum_{n=1}^{\infty} (-A_i t)^n\right|$$

$$= \left|A_i^2 \sum_{n=1}^{\infty} (-1)^n A_i^{n-1} t^n\right|$$

$$\leq |A_i|^2 \sum_{n=1}^{\infty} |A_i|^{n-1} |t|^n$$

$$< |A_i|^2 \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \qquad (|A_i| < \tfrac{1}{2})$$

$$= 2|A_i|^2.$$

Hence,

$$\left|\prod_{i=n}^{m}(1+A_i)\right| \leq {}_{o}\prod^{1}(1+\left|\sum_{i=n}^{m} A_i\right|dt) \cdot {}_{o}\prod^{1}(1 + \sum_{i=n}^{m} 2|A_i|^2 dt)$$

(Th. 2.20)

$$< {}_{0}\Pi^{1}(1+1dt)\ {}_{0}\Pi^{1}(1+2dt) \qquad (m>n>N)$$

$$= (\exp 1)(\exp 2) \qquad (\text{Th. } 2.21)$$

$$< M.$$

Also,

$$\left|\prod_{i=n}^{m} {}_{0}\Pi^{1}(1+A_i dt)\right| \leq {}_{0}\Pi^{1}\left(1+\left|\sum_{i=n}^{m} A_i\right| dt\right) < M.$$

Finally, if N is between m and n,

$$\left|\prod_{i=n}^{m} (1+A_i)\right| \leq \exp\left(\sum_{i=1}^{N} |A_i|\right)\cdot \exp 3 = M$$

and

$$\left|\prod_{i=n}^{m} {}_{0}\Pi^{1}(1+A_i dt)\right| \leq \exp\left(\sum_{i=1}^{N} |A_i|\right)\cdot \exp 1 < M.$$

<u>Lemma 4.4</u>. Suppose $\{A_i\}_{i=1}^{\infty}$ is a sequence of commutative elements of R, $\prod_{i=1}^{\infty}(1+A_i) = A$, A^{-1} exists, and $(1+A_j)^{-1}$ exists for each positive integer j. Conclusion: There exists a positive number M and a positive integer N such that

$$\left|\left[\prod_{i=1}^{n}(1+A_i)\right]^{-1}\right| < M$$

for n > N.

Proof:

Let $M = 2|A^{-1}|$. Since $\prod_{i=1}^{\infty} (1+A_i) = A$, there exists a positive integer N such that

$$|A - \prod_{i=1}^{n} (1+A_i)| < \frac{1}{M}$$

for $n > N$. Let n be a positive integer such that $n > N$ and let $k = \prod_{i=1}^{n} (1+A_i) - A$; therefore, $|k| < \frac{1}{M}$ and $A + k = \prod_{i=1}^{n} (1+A_i)$.

$$(A+k)[\prod_{i=1}^{n} (1+A_i)]^{-1} = 1;$$

hence,

$$(1+A^{-1}k)[\prod_{i=1}^{n} (1+A_i)]^{-1} = A^{-1}.$$

Since $|A^{-1}k| < |A^{-1}|\frac{1}{M} = \frac{1}{2}$, then by Theorem 4.1

$$(1+A^{-1}k)^{-1} = 1 + \sum_{n=1}^{\infty} (-A^{-1}k)^n;$$

therefore,

$$[\prod_{i=1}^{n} (1+A_i)]^{-1} = (1+A^{-1}k)^{-1}A^{-1}$$

and

$$|[\prod_{i=1}^{n} (1+A_i)]^{-1}| \leq |(1+A^{-1}k)^{-1}|\cdot|A^{-1}|$$

$$= |1 + \sum_{n=1}^{\infty} (-A^{-1}k)^n| \cdot |A^{-1}|$$

$$< 2|A^{-1}|$$

$$= M.$$

<u>Lemma 4.5</u>. Suppose $\{A_i\}_{i=1}^{\infty}$ is a sequence of commutative elements of R, $\prod_{i=1}^{\infty} (1+A_i) = A$, A^{-1} exists, and $(1+A_j)^{-1}$ exists for each positive integer j. Conclusion: There exists a positive integer N such that $|\prod_{i=n}^{m} (1+A_i)| < 2$ for $m > n > N$.

Proof:

By Lemma 4.4 there exist a positive integer X and a positive number Q such that $|[\prod_{i=1}^{n} (1+A_i)]^{-1}| < Q$ for $n > X$. Also, by the Cauchy criterion for products, there exists a positive integer Y such that

$$|\prod_{i=1}^{m} (1+A_i) - \prod_{i=1}^{n} (1+A_i)| < \frac{1}{Q}$$

for $m > n > Y$. Let $N = X + Y$ and let $m > n > N$, then

$$1 = Q \cdot \frac{1}{Q} > |[\prod_{i=1}^{n} (1+A_i)]^{-1}| \cdot |\prod_{i=1}^{m} (1+A_i) - \prod_{i=1}^{n} (1+A_i)|$$

$$\geq \left|\prod_{i=n}^{m} (1+A_i) - 1\right|;$$

therefore,

$$\left|\prod_{i=n}^{m} (1+A_i)\right| < 2.$$

<u>Lemma 4.6</u>. Suppose $\{A_i\}_{i=1}^{\infty}$ is a sequence of commutative elements of R, $\prod_{i=1}^{\infty} (1+A_i)$ exists and is A, A^{-1} exists, $\sum_{i=1}^{\infty} |A_i|^2$ converges, and $(1+A_j)^{-1}$ exists for each positive integer j. Conclusion: There exist a positive number M and a positive integer N such that

$$\left|{}_0\prod^1 (1 + \sum_{i=n}^{m} A_i dt)\right| < M$$

for $m > n > N$.

Proof:

By Lemma 4.5 there exists a positive integer X such that $\left|\prod_{i=n}^{m} (1+A_i)\right| < 2$ for $m > n > X$. Since $\sum_{i=1}^{\infty} |A_i|^2$ converges, there exists a positive integer Y such that $\sum_{i=n}^{m} |A_i|^2 < \frac{1}{2}$ for $m > n > Y$. Let $M = 2(\exp 1)$, $N = X + Y$, and let m and n be integers such that $m > n > N$; then, if t belongs to [0,1] and $i > N$,

$$(1+A_i t)^{-1} = 1 + \sum_{n=1}^{\infty}(-A_i t)^n \qquad \text{(Th. 4.1)}.$$

Define $B_i = \sum_{n=1}^{\infty}(-A_i t)^n$. If we define $f_i(t) = 1 + A_i t$ for each $i > N$ and each t in [0,1] and use manipulations similar to those used in Lemma 4.3, then

$$\left|\prod_{i=n}^{m}(1+A_i)\right| = \left|{}_0\Pi^1\left(1 + \sum_{i=n}^{m} A_i dt\right) \cdot {}_0\Pi^1\left(1 + \sum_{i=n}^{m} A_i B_i dt\right)\right|$$

and $|A_i B_i| < 2|A_i|^2$ for $i > N$; therefore, by Theorem 2.17

$$\left|{}_0\Pi^1\left(1 - \sum_{i=n}^{m} A_i B_i dt\right)\right| \cdot \left|\prod_{i=n}^{m}(1+A_i)\right| \geq \left|{}_0\Pi^1\left(1 + \sum_{i=n}^{m} A_i dt\right)\right|.$$

Also,

$$\begin{aligned} \left|{}_0\Pi^1\left(1 - \sum_{i=n}^{m} A_i B_i dt\right)\right| &\leq {}_0\Pi^1\left(1 + \sum_{i=n}^{m} 2|A_i|^2 dt\right) \\ &< {}_0\Pi^1(1+1dt) \\ &= \exp 1; \end{aligned}$$

therefore, since $\left|\prod_{i=n}^{m}(1+A_i)\right| < 2$, then

$$M = 2(\exp 1) > \left|{}_0\Pi^1\left(1 + \sum_{i=n}^{m} A_i dt\right)\right|.$$

<u>Theorem 4.7.</u> If $\{A_i\}_{i=1}^{\infty}$ is a sequence of commutative

elements of R such that $\sum_{i=1}^{\infty} |A_i|^2$ converges, then $\sum_{i=1}^{\infty} A_i$ converges if, and only if, $\prod_{i=1}^{\infty} (1+A_i)$ converges.

Proof:

Since the convergence of $\sum_{i=1}^{\infty} |A_i|^2$ implies there are only a finite number of terms of the sequence $\{A_i\}_{i=1}^{\infty}$ for which $|A_i| > \frac{1}{2}$ and since a finite number of the terms of a series or the factors of a product may be "discarded" without altering convergence, it will be assumed in the proof of this theorem that $|A_i| < \frac{1}{2}$ for each positive integer i. It follows by Theorem 4.1 that $(1+A_i)^{-1}$ exists for each positive integer i.

First, the Cauchy criterion will be used to prove that, if $\sum_{i=1}^{\infty} A_i$ converges, then $\prod_{i=1}^{\infty} (1+A_i)$ converges. Let $\varepsilon > 0$. From Lemma 4.3 there exists a positive number such that

$$\left| \prod_{i=n}^{m} {}_{o}\prod^{1} (1+A_i dt) \right| < M$$

and

$$\left| \prod_{i=n}^{m} (1+A_i) \right| < M$$

for positive integers $m > n$. Since $\sum_{i=1}^{\infty} |A_i|^2$ converges, there exists a positive integer X such that

$$\sum_{i=n}^{m} |A_i|^2 < \frac{\varepsilon}{2M^3} \tag{4.1}$$

and

$$\sum_{i=n}^{m} |A_i|^2 < 1$$

for m > n > X. Also, since $\sum_{i=1}^{\infty} A_i$ converges, there exists a positive integer Y such that

$$\left|\sum_{i=n}^{m} A_i\right| < \frac{\varepsilon}{4M(\exp 1)} \tag{4.2}$$

and

$$\left|\sum_{i=n}^{m} A_i\right| < 1$$

for m > n > Y. Let N = X + Y and let m and n be integers such that m > n > N. Since ${}_0\Pi^1(1 + \sum_{i=n+1}^{m} A_i dt)$ exists there is a subdivision $\{t_i\}_{i=0}^{r}$ of [0,1] such that

(4.3)
$$\left|\prod_{j=1}^{r}(1 + \sum_{i=n+1}^{m} A_i \Delta t_j) - {}_0\Pi^1(1 + \sum_{i=n+1}^{m} A_i dt)\right| < \frac{\varepsilon}{4M}.$$

Therefore,

$$\left|\prod_{i=1}^{m}(1+A_i) - \prod_{i=1}^{n}(1+A_i)\right|$$

$$\leq \left|\prod_{i=1}^{n}(1+A_i)\right| \cdot \left|\prod_{i=n+1}^{m}(1+A_i) - 1\right|$$

$$< M\left|\prod_{i=n+1}^{n}(1+A_i) - \prod_{i=n+1}^{m} {}_o\prod^1(1+A_i dt)\right|$$

$$+ M\left|\prod_{i=n+1}^{m} {}_o\prod^1(1+A_i dt) - 1\right|.$$

Now,

$$M\left|\prod_{i=n+1}^{m}(1+A_i) - \prod_{i=n+1}^{m} {}_o\prod^1(1+A_i dt)\right|$$

$$= M\left|\sum_{i=n+1}^{m}\left[\prod_{j=n+1}^{i-1} {}_o\prod^1(1+A_j dt)\right]\left[(1+A_i)\right.\right.$$

$$\left.\left. - {}_o\prod^1(1+A_i dt)\right]\left[\prod_{j=i+1}^{m}(1+A_j)\right]\right| \qquad \text{(Th. 2.16)}$$

$$\leq M^3 \sum_{i=n+1}^{m}\left|(1+A_i) - {}_o\prod^1(1+A_i dt)\right|$$

$$= M^3 \sum_{i=n+1}^{m}\left|(1+A_i) - (1+A_i+\sum_{n=2}^{\infty}\frac{1}{n!}A_i^n)\right|$$

$$\text{(Th. 4.2)}$$

$$\leq M^3 \sum_{i=n+1}^{m}|A_i|^2 \cdot \left[\sum_{n=2}^{\infty}\frac{1}{n!}|A_i|^{n-2}\right]$$

$$< M^3 \sum_{i=n+1}^{m}|A_i|^2 \sum_{n=1}^{\infty}\frac{1}{2^n} \qquad (|A_i|<1)$$

$$= M^3 \sum_{i=n+1}^{m}|A_i|^2$$

$$< \frac{\varepsilon}{2} \qquad \text{(Eq. 4.2).}$$

Also,

$$M\left|\prod_{i=n+1}^{m} {}_o\prod^1(1+A_i dt)-1\right|$$

$$= M\left|\,{}_{o}\prod^{1}\left(1+\sum_{i=n+1}^{m} A_i\,dt\right) - 1\right| \qquad \text{(Th. 2.17)}$$

$$\leq M\left|\,{}_{o}\prod^{1}\left(1+\sum_{i=n+1}^{m} A_i\,dt\right) - \prod_{j=1}^{r}\left(1+\sum_{i=n+1}^{m} A_i\Delta t_j\right)\right|$$

$$+ M\left|\prod_{j=1}^{r}\left(1+\sum_{i=n+1}^{m} A_i\Delta t_j\right) - 1\right|$$

$$< M \cdot \frac{\varepsilon}{4M} + M\left|\sum_{j=1}^{r}\left[\sum_{i=n+1}^{m} A_i\Delta t_j\right]\cdot\left[\prod_{k=j+1}^{r}\left(1+\sum_{i=n+1}^{m} A_i\Delta t_k\right)\right]\right|$$

(Eq. 4.3 and Th. 2.16)

$$\leq \frac{\varepsilon}{4} + M\sum_{j=1}^{r}\left|\sum_{i=n+1}^{m} A_i\right| \cdot \Delta t_j \cdot \prod_{k=j+1}^{r}\left(1+\left|\sum_{i=n+1}^{m} A_i\right|\Delta t_k\right)$$

$$< \frac{\varepsilon}{4} + M\sum_{j=1}^{r} \frac{\varepsilon}{4M(\exp 1)} \cdot \Delta t_j \cdot \exp\sum_{k=j+1}^{r}\left|\sum_{i=n+1}^{m} A_i\right|\Delta t_k$$

(Eq. 4.2)

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4(\exp 1)}\sum_{j=1}^{r} \Delta t_j(\exp 1) \qquad \left(\left|\sum_{i=n+1}^{m} A_i\right| < 1\right)$$

$$= \frac{\varepsilon}{2}.$$

When the two preceding inequalities are combined, we obtain

$$\left|\prod_{i=1}^{m}(1+A_i) - \prod_{i=1}^{n}(1+A_i)\right| < \varepsilon;$$

hence, $\prod_{i=1}^{\infty}(1+A_i)$ exists.

It remains to show the existence of $\left[\prod_{i=1}^{\infty}(1+A_i)\right]^{-1}$. To do this, define $f_i(t) = 1+A_i t$ for each positive integer i and for t belonging to $[0,1]$; then,

$$\prod_{i=1}^{n}(1+A_i) = {}_0\prod^1[1+\sum_{i=1}^{n}(1+A_i t)^{-1}A_i dt]$$

as in Lemma 4.3. Since $|A_i| < \frac{1}{2}$, then

(1) $\sum_{i=1}^{\infty}(1+A_i t)^{-1}A_i$ converges uniformly on $[0,1]$,

(2) ${}_0\prod^1[1+\sum_{i=1}^{\infty}(1+A_i t)^{-1}A_i dt] = \prod_{i=1}^{\infty}(1+A_i)$

and

(3) ${}_0\prod^1[1-\sum_{i=1}^{\infty}(1+A_i t)^{-1}A_i dt]$ exists and is the multiplicative inverse of

$${}_0\prod^1[1+\sum_{i=1}^{\infty}(1+A_i t)^{-1}A_i dt].$$

(Ths. 2.23, 2.17)

Therefore $[\prod_{i=1}^{\infty}(1+A_i)]^{-1}$ exists and $\prod_{i=1}^{\infty}(1+A_i)$ converges by Definition 2.13.

Now an indirect proof will be used to prove that, if $\prod_{i=1}^{\infty}(1+A_i)$ converges, then $\sum_{i=1}^{\infty} A_i$ converges. If $\sum_{i=1}^{\infty} A_i$ diverges, then by the Cauchy criterion there exists a positive number ε_0 such that, if N is a positive integer, there exist integers $m > n > N$ such that $|\sum_{i=n}^{m} A_i| > \varepsilon_0$. Define

$\varepsilon > 0$ such that $\varepsilon < \frac{1}{8}$ and $\varepsilon < \varepsilon_o$.

By Lemma 4.4 there exist a positive number P and a positive integer X such that

$$(4.4) \qquad \left|\left[\prod_{i=1}^{n} (1+A_i)\right]^{-1}\right| < P$$

for $n > X$. Since $\prod_{i=1}^{\infty} (1+A_i)$ converges, there exists a positive integer Y such that

$$\left|\prod_{i=1}^{n} (1+A_i) - \prod_{i=1}^{m} (1+A_i)\right| < \frac{\varepsilon}{4P}$$

for $m > n > Y$. Also, by Lemmas 4.5 and 4.6 there exist a positive integer Y and a positive number Q such that

$$\left|\prod_{i=n}^{m} (1+A_i)\right| < 2$$

and

$$(4.5) \qquad \left|{}_o\prod^1(1+\sum_{i=n}^{m} A_i dt)\right| < Q$$

for $m > n > Y$. Therefore, since $\sum_{i=1}^{\infty} |A_i|^2$ converges, there is a positive integer Z such that $\sum_{i=n}^{m} |A_i|^2 < \frac{\varepsilon}{8Q}$ and $|A_n| < \varepsilon < \frac{1}{8}$ for $m > n > Z$.

Let $N = X + Y + Z$; then, from the denial, there exist integers $m > n > N$ such that $\left|\sum_{i=n}^{m} A_i\right| > \varepsilon_o > \varepsilon$. Denote the set of positive integers $\{n,n+1,\cdots,m-1,m\}$ by A and let B be the subset of A such that $B = \{j: |\sum_{i=n}^{j} A_i| > \varepsilon\}$. B is non-empty because m belongs to B and B is bounded below by n because $|A_n| < \varepsilon$; thus, B has a least element r. Now, if $r > p \geq n$, then $|\sum_{i=n}^{p} A_i| \leq \varepsilon$ since r is the least element in B and $|\sum_{i=n}^{r} A_i| > \varepsilon$ since r belongs to B. Furthermore, $|\sum_{i=n}^{r} A_i| < 2\varepsilon < \frac{1}{4}$ since

$$2\varepsilon > |\sum_{i=n}^{r-1} A_i| + |A_i| \geq |\sum_{i=n}^{r} A_i|.$$

Now,

$$\frac{\varepsilon}{4P} > |\prod_{i=1}^{r}(1+A_i) - \prod_{i=1}^{n-1}(1+A_i)|;$$

therefore,

$$\frac{\varepsilon}{4} > |[\prod_{i=1}^{n-1}(1+A_i)]^{-1}| \cdot |\prod_{i=1}^{r}(1+A_i) - \prod_{i=1}^{n-1}(1+A_i)| \qquad \text{(Eq. 4.4)}$$

$$\geq |\prod_{i=n}^{r}(1+A_i) - 1|$$

$$\geq |\prod_{i=n}^{r} {}_0\prod^1(1+A_i dt) - 1|$$

$$- |\prod_{i=n}^{r} {}_0\prod^1(1+A_i dt) - \prod_{i=n}^{r}(1+A_i)|$$

$$\geq |{}_0\prod^1(1+\sum_{i=n}^{r} A_i dt) - 1|$$

$$- \sum_{i=n}^{r} |\prod_{j=n}^{i-1}(1+A_j)| \cdot |{}_0\prod^1(1+A_i dt) - (1+A_i)|.$$

$$\cdot\left|\prod_{j=i+1}^{r} {}_0\prod^{1}(1+A_j dt)\right| \qquad \text{(Ths. 2.17, 2.16)}$$

$$\geq |\cdot| - 2Q\sum_{i=n}^{r}\left|1+A_i+\sum_{n=2}^{\infty}\frac{1}{n!}A_i^n - (1+A_i)\right|$$

(Th. 4.2 and Eq. 4.5)

$$\geq |\cdot| - 2Q\sum_{i=n}^{r}|A_i|^2\sum_{n=2}^{\infty}\frac{1}{n!}|A_i|^{n-2}$$

$$> |\cdot| - 2Q\sum_{i=n}^{r}|A_i|^2\sum_{n=1}^{\infty}\frac{1}{2^n} \qquad (|A_i|<1)$$

$$= |\cdot| - 2Q\sum_{i=n}^{r}|A_i|^2$$

$$> \left|{}_0\prod^{1}(1+\sum_{i=n}^{r}A_i dt) - 1\right| - \frac{\varepsilon}{4} \qquad \left(\sum_{i=n}^{r}|A_i|^2 < \frac{\varepsilon}{8Q}\right).$$

Therefore,

$$\frac{\varepsilon}{2} > \left|{}_0\prod^{1}(1 + \sum_{i=n}^{r} A_i dt) - 1\right|$$

$$= \left|\left[1 + \sum_{i=n}^{r} A_i + \sum_{n=2}^{\infty}\frac{1}{n!}\left(\sum_{i=n}^{r} A_i\right)^n\right] - 1\right|$$

(Th. 4.2)

$$\geq \left|\sum_{i=n}^{r} A_i\right| - \sum_{n=2}^{\infty}\frac{1}{n!}\left|\sum_{i=n}^{r} A_i\right|^n$$

$$\geq \left|\sum_{i=n}^{r} A_i\right| - \left|\sum_{i=n}^{r} A_i\right|^2 \qquad \left(\left|\sum_{i=n}^{r} A_i\right| < \frac{1}{4}\right).$$

Since $\frac{1}{4} > 2\varepsilon > \left|\sum_{i=n}^{r} A_i\right| > \varepsilon$, the above inequality becomes

$\frac{\varepsilon}{2} > \frac{\varepsilon}{2}$; hence, our assumption was false and $\sum_{i=1}^{\infty} A_i$ converges. Hence, both parts of the theorem are proven.

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