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FENG'S FIRST-INTEGRAL METHOD TO TRAVELING WAVE SOLUTIONS OF THE OSTROVSKY SYSTEM

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Abstract. In this paper, we apply Feng's first-integral method to study traveling wave solutions to a two-component generalization of the Ostrovsky system. We convert the two-component generalization of the Ostrovsky system to an equivalent autonomous system. Then we use the Divisor Theorem of two variables in the complex domain to seek the polynomial first-integral to this autonomous system. Through analyzing the derived first-integral, we obtain traveling wave solutions to the two-component generalization of the Ostrovsky system under certain parametric conditions.

1. Introduction

The Ostrovsky equation [16] describes weakly nonlinear waves in continuous media with two kinds of dispersion: small scale dispersion typically for the Koretweg-de Vries (KdV) equation and large-scale dispersion typically for electromagnetic or optic waves in wave guides, or because of the effect of background rotation for internal and surface waves [11]. If we neglect small scale dispersion, then the Ostrovsky equation becomes the nonlinear evolution equation

$$(u_t + \mu u u_x)_x = \gamma u, \tag{1.1}$$

where μ is the nonlinear coefficient and γ is the dispersion coefficient. This equation is the reduced Ostrovsky equation [17, 19] (also known variously as the Ostrovsky-Hunter equation [1] or the Vakhnenko equation [18, 21]). Equation (1.1) has been previously studied numerically and theoretically, see [1, 5, 10, 15, 19, 20, 21].

In this article, we consider traveling wave solutions for a more general Ostrovsky system in the form

$$\begin{gathered}(u_t + ru + uu_x)_x = 3u + c(1-\rho),\\ \rho_t + (\rho u)_x = 0,\end{gathered} \tag{1.2}$$

where r is a real constant, u and ρ are functions of x and t. If $r = 0$ then (1.2) reduces to a generalized two-component of the Ostrovsky system [6]

$$\begin{gathered}(u_t + uu_x)_x = 3u + c(1-\rho),\\ \rho_t + (\rho u)_x = 0.\end{gathered} \tag{1.3}$$

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System (1.3) is integrable because of the existence of Lax pairs and has multi-soliton solutions. In this study, we apply Feng's first integral method [12, 22] to find loop soliton solutions, kink-profile wave solutions, and cusped solitary wave solutions under certain parametric conditions. The key idea of Feng's first integral method [7, 8, 9] is to utilize the Division Theorem based on ring theory of commutative algebras to derive the first integrals [3]. Then making use of this first integral, we reduce the second-order nonlinear differential equation to a first-order integrable differential equation.

The rest of this article is organized as follow. In Section 2, in order to study the dynamical behaviors and obtain different kinds of exact traveling solutions of system (1.2), we transform (1.2) into an equivalent two-dimensional autonomous system. By applying Feng's first-integral methods we explore the first integral of this autonomous system. In Section 3, Some traveling wave solutions of system (1.2) are presented explicitly.

2. Traveling wave system

Assume that system (1.2) has the traveling wave solution of the form

$$u(x,t)=u(\xi),\quad \rho(x,t)=\rho(\xi),\quad \xi=x-\omega t, \tag{2.1}$$

where ω ($\omega\neq 0$) is the wave speed. Substituting (2.1) into (1.2) gives the ordinary differential equation

$$\begin{gathered}\big(-\omega u'+ru+uu'\big)'=3u+c(1-\rho),\\ -\omega\rho'+(\rho u)'=0,\end{gathered} \tag{2.2}$$

where the prime denotes differentiation with respect to ξ. Integrating the second equation of (2.2) once with respect to ξ gives

$$\rho=\frac{g}{u-\omega}, \tag{2.3}$$

where g ($g\neq 0$) is the arbitrary constant of integration. Substituting (2.3) into the first equation of (2.2) yields

$$(u-\omega)^2u''+(u-\omega)(u')^2-(u-\omega)(3u+c)+r(u-\omega)u'+cg=0. \tag{2.4}$$

Let $\frac{du}{d\xi}=y$. Equation (2.4) can be changed into an equivalent planar system

$$\begin{gathered}\frac{du}{d\xi}=y,\\ \frac{dy}{d\xi}=\frac{(u-\omega)(3u+c)-(u-\omega)y^2-r(u-\omega)y-cg}{(u-\omega)^2}.\end{gathered} \tag{2.5}$$

Since this system possesses a singularity at $u=\omega$, we can remove it by introducing the parameter τ such that

$$\frac{d\tau}{d\xi}=\frac{1}{(u-\omega)^2}\implies \tau(\xi)=\int_0^\xi\frac{ds}{(u-s)^2}\,. \tag{2.6}$$

Except at $u=\omega$ where $d\tau/d\xi$ is not defined, and $d\tau/d\xi>0$. Thus τ has an inverse τ^{-1} which in principle can be derived from (2.6). We then have a topologically equivalent system

$$\begin{aligned} \frac{du}{d\tau} &= (u-\omega)^2 y, \\ \frac{dy}{d\tau} &= 3u^2 - (3\omega - c)u - (u-\omega)y^2 - r(u-\omega)y - c(\omega+g). \end{aligned} \tag{2.7}$$

System (2.7) does not possess any singularity. So we can apply Feng's first-integral method to study the first integral of autonomous system (2.7). This innovative method was initially proposed in [7, 8, 9], which is based on the theory of commutative algebras and Division Theorem for two variables in the complex domain $\mathbb{C}$ to seek the first-integral for two-dimensional systems. In order to present our results in a straightforward manner, let us first recall the Division Theorem for two variables in the complex domain $\mathbb{C}$.

Lemma 2.1 (Divisor Theorem)**.** *Suppose that $P_0(\omega, z)$ and $Q_0(\omega, z)$ are polynomials in $\mathbb{C}[\omega, z]$, and $P_0(\omega, z)$ is irreducible in $\mathbb{C}[\omega, z]$. If $Q_0(\omega, z)$ vanishes at all zero points of $P_0(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $\mathbb{C}[\omega, z]$ such that*

$$Q_0(\omega, z) = P_0(\omega, z) \cdot G(\omega, z).$$

Suppose $u = u(\tau)$ and $y = y(\tau)$ are nontrivial solutions of system (2.7), and $P(u, y) = \sum_{i=0}^{m} a_i(u)y^i$ is an irreducible polynomial in $\mathbb{C}[u, y]$ such that

$$P[u(\tau), y(\tau)] = \sum_{i=0}^{m} a_i(u)y^i = 0, \tag{2.8}$$

where $a_i(u)$ $(i = 0, 1, \dots, m)$ are polynomials of u and all relatively prime in $\mathbb{C}[u, y]$, and $a_m(u) \neq 0$. The equation (2.8) is called the first-integral of the polynomial form to system (2.7). We start our discussion by assuming $m = 2$ in (2.8). Note that $\frac{dp}{d\tau}$ is a polynomial in u and y, and $p[u(\tau), y(\tau)] = 0$ always implies $\frac{dp}{d\tau} = 0$. By the Division Theorem, there exists a polynomial $H(u, y) = \alpha(u) + \beta(u)y$ in $\mathbb{C}[u, y]$ such that

$$\begin{aligned} \frac{dp}{d\tau} &= \frac{\partial p}{\partial y}\frac{dy}{d\tau} + \frac{\partial p}{\partial u}\frac{du}{d\tau} \\ &= \sum_{i=0}^{2} i a_i(u) y^{i-1}[3u^2 - (3\omega - c)u - (u-\omega)y^2 - r(u-\omega)y - c(\omega+g)] \\ &\quad + \sum_{i=0}^{2} a_i'(u)(u-\omega)^2 y^{i+1} \\ &= [\alpha(u) + \beta(u)y]\Big[\sum_{i=0}^{2} a_i(u)y^i\Big]. \end{aligned} \tag{2.9}$$

By equating the coefficients of y^i $(i = 3, 2, 1, 0)$ on both sides of (2.9), we have

$$\begin{aligned} a_2'(u)(u-\omega)^2 &= a_2(u)[2(u-\omega) + \beta(u)], \\ a_1'(u)(u-\omega)^2 &= a_2(u)[\alpha(u) + 2r(u-\omega)] + a_1(u)[\beta(u) + (u-\omega)], \\ a_0'(u)(u-\omega)^2 &= -2a_2(u)[3u^2 - (3\omega - c)u - c(\omega+g)] \\ &\quad + a_1(u)[\alpha(u) + r(u-\omega)] + a_0(u)\beta(u), \\ a_0(u)\alpha(u) &= a_1(u)[3u^2 - (3\omega - c)u - c(\omega+g)]. \end{aligned} \tag{2.10}$$

For simplicity, we denote equation (2.10) by

$$\mathbf{a}'(u) = \mathbf{A}(u)\mathbf{a}(u),$$
$$a_0(u)\alpha(u) = a_1(u)[3u^2 - (3\omega - c)u - c(\omega + g)],$$

where $\mathbf{a}(u) = (a_2(u), a_1(u), a_0(u))^T$, and

$$\mathbf{A}(u) = \begin{pmatrix} \frac{\beta(u)+2(u-\omega)}{(u-\omega)^2} & 0 & 0 \\ \frac{\alpha(u)+2r(u-\omega)}{(u-\omega)^2} & \frac{\beta(u)+(u-\omega)}{(u-\omega)^2} & 0 \\ \frac{-2[3u^2-(3\omega-c)u-c(\omega+g)]}{(u-\omega)^2} & \frac{\alpha(u)+r(u-\omega)}{(u-\omega)^2} & \frac{\beta(u)}{(u-\omega)^2} \end{pmatrix}.$$

Setting $\beta(u) = l_1u + l_0$. Solving the first equation of (2.10) directly, we have

$$a_2(u) = D(u - \omega)^{l_1+2} e^{-\frac{l_1\omega+l_0}{u-\omega}}, \tag{2.11}$$

where D is an arbitrary constant. Since $a_2(u)$ is a polynomial and ω is the wave speed, by (2.11), we deduce that $l_1 = l_0 = 0$, namely, $\beta(u) = 0$. For simplicity, we take the constant of integration $D = 1$; then

$$a_2(u) = (u - \omega)^2.$$

Substituting $a_2(u)$ and $\beta(u)$ into (2.10) yields

$$\begin{gathered} a_1'(u) = \alpha(u) + 2r(u - \omega) + \frac{a_1(u)}{u - \omega}, \\ a_0'(u) = -2[3u^2 - (3\omega - c)u - c(\omega + g)] + \frac{ra_1(u)}{u - \omega} + \frac{a_1(u)\alpha(u)}{(u - \omega)^2}, \\ a_0(u)\alpha(u) = a_1(u)[3u^2 - (3\omega - c)u - c(\omega + g)]. \end{gathered} \tag{2.12}$$

We suppose that $\deg a_1(u) = m$, $\deg a_0(u) = n$ and $\deg \alpha(u) = k$, where m, n, k are non-negative integers. Since $a_1(u)$ is a polynomial, by the first and the third equations of (2.12), we conclude that $m \geq 1$, $a_0(u) \neq 0$ and $\alpha(u) \neq 0$.

Step 1. We show that $k \geq 1$. In fact, if $k = 0$, namely, $\alpha(u) = d_1 \neq 0$. By the second equation of (2.12), we have that $m = \deg a_1(u) \geq 2$. Consequently, by the third equation of (2.12), we conclude that $\deg a_0(u) = n = m + 2 \geq 4$ and $\deg a_0'(u) \geq 3$. Since

$$\deg \frac{ra_1(u)}{u - \omega} > \deg \frac{a_1(u)\alpha(u)}{(u - \omega)^2},$$

by the second equation of (2.12) again, we have $n - 1 = m - 1$, namely, $n = m$. This is a contradiction with $n = m + 2$. Consequently, we have $k \geq 1$.

Step 2. We claim that $m \geq 2$. Actually, by the first equation of (2.12), we claim that $m - 1 \geq k$, namely, $m \geq k + 1 \geq 2$. Otherwise, if $m - 1 < k$, then we have $\alpha(u) + 2r(u - \omega) = 0$ and $0 \leq m - 1 < k = \deg \alpha(u) = 1$. That is, $m = k = 1$. By the second equation of (2.12), it follows that $n = 3$. Substituting $m = k = 1$ and $n = 3$ into the third equation of (2.12) yields that $3 + 1 = 1 + 2$. This is a contradiction.

Step 3. We prove that $n \geq 1$. If $n = 0$, without loss of generality, we suppose $a_0(u) = d_2 \neq 0$. By the first equation and the third equation of (2.12), it leads to $k \leq m - 1$ and $k = m + 2$. Apparently, this is a contradiction.

Step 4. By the third equation of (2.12), we have that $n + k = m + 2$, namely, $n = m - k + 2 \geq 1 + 2 = 3$. If $n = m + k - 1 > 3$, by the last two equations of (2.12), we obtain that $n - 1 = m + k - 2$ and $n + k = m + 2$, which implies that

$k=3/2$. This yields a contradiction with $k\in\mathbb{Z}^+$. Consequently, only possibility is $n=3$. By the last two equations of (2.12) again, we have that $0\le m+k-2\le 2$ and $3+k=m+2$, which implies that $m=2$ and $k=1$.

Thus, we now have that $m=2$, $n=3$ and $k=1$. By the first two equations of (2.12), we assume that $a_1(u)=A_1(u-\omega)^2+A_0(u-\omega)$ and $\alpha(u)=B_1(u-\omega)$. Substituting $a_1(u)$ and $\alpha(u)$ into (2.12), we have that $A_1=B_1+2r$ and

$$\begin{aligned} a_0(u)&=-2u^3+\big[(3\omega-c)+\frac{A_1(r+B_1)}{2}\big]u^2\\ &\quad+[2c(\omega+g)+(r+B_1)(A_0-A_1\omega)]u+h,\\ a_0(u)B_1&=3A_1u^3+[3(A_0-A_1\omega)-A_1(3\omega-c)]u^2\\ &\quad-[(A_0-A_1\omega)(3\omega-c)+A_1c(\omega+g)]u-c(A_0-A_1\omega)(\omega+g), \end{aligned}\tag{2.13}$$

where h is an arbitrary constant. Substituting the first equation of(2.13) into the second equation of (2.13) and equating the coefficients of u^i $(i=3,2,1,0)$, we have

$$\begin{gathered} -2B_1=3A_1,\\ [(3\omega-c)+\frac{A_1(r+B_1)}{2}]B_1=3(A_0-A_1\omega)-A_1(3\omega-c),\\ [2c(\omega+g)+(r+B_1)(A_0-A_1\omega)]B_1=-[(A_0-A_1\omega)(3\omega-c)+A_1c(\omega+g)],\\ -c(A_0-A_1\omega)(\omega+g)=hB_1. \end{gathered}$$

Solving this equatiobn, we find that the solutions only exist under the parametric restriction

$$cg=\frac{3r^4}{625}-\frac{(3\omega+c)^2}{12},\tag{2.14}$$

and they are

$$B_1=\frac{-6r}{5},\quad A_1=\frac{4r}{5},\quad A_0=\frac{4r^3}{125}+\frac{2r(3\omega+c)}{15},\quad h=c(\omega+g)(\frac{2r^2}{75}-\frac{3\omega-c}{9}).$$

Thus, we obtain that $a_2(u)=(u-\omega)^2$, $a_1(u)=\frac{4r}{5}(u-\omega)[u+\frac{r^2}{25}-\frac{3\omega-c}{6}]$ and

$$\begin{aligned} a_0&=-2u^3-[\frac{2r^2}{25}-(3\omega-c)]u^2+[\frac{2r^4}{625}+\frac{2r^2(3\omega-c)}{75}-\frac{(3\omega-c)^2}{6}]u\\ &\quad+c(\omega+g)\Big(\frac{2r^2}{75}-\frac{3\omega-c}{9}\Big). \end{aligned}$$

Substituting $a_i(u)$ $(i=0,1,2)$ into (2.8), we derive a first integral of the polynomial form for system (2.7),

$$\begin{aligned} P(u,y)&=(u-\omega)^2y^2+\frac{4r}{5}(u-\omega)\big[u+\frac{r^2}{25}-\frac{3\omega-c}{6}\big]y-2u^3-\big[\frac{2r^2}{25}-(3\omega-c)\big]u^2\\ &\quad+\big[\frac{2r^4}{625}+\frac{2r^2(3\omega-c)}{75}-\frac{(3\omega-c)^2}{6}\big]u+c(\omega+g)\Big(\frac{2r^2}{75}-\frac{3\omega-c}{9}\Big). \end{aligned}$$

Changing to the original variables, we obtain a first-integral of the polynomial form to equation (2.4) as follows

$$\begin{aligned} &(u-\omega)^2(u')^2+\frac{4r}{5}(u-\omega)\big[u+\frac{r^2}{25}-\frac{3\omega-c}{6}\big]u'-2u^3-\big[\frac{2r^2}{25}-(3\omega-c)\big]u^2\\ &+\big[\frac{2r^4}{625}+\frac{2r^2(3\omega-c)}{75}-\frac{(3\omega-c)^2}{6}\big]u+c(\omega+g)\Big(\frac{2r^2}{75}-\frac{3\omega-c}{9}\Big)=0. \end{aligned}\tag{2.15}$$

3. Traveling wave solutions

In this section, we consider traveling wave solutions of (2.7) through the first integral (2.15) in different cases in terms of values of the parameter r and the constant of integration g.

3.1. First we consider $r \neq 0$, and consider two parts according to the values of the constant of integration g.

3.1.1. $g = 0$. In (2.3), $g = 0$ implies $\rho = 0$, then equation (1.2) can be reduced to

$$(u_t + ru + uu_x)_x = 3u + c. \tag{3.1}$$

By using the transformation of traveling wave, equation (2.2) can be converted into

$$(u - \omega)u'' = 3u - (u')^2 - ru' + c. \tag{3.2}$$

So system (2.7) becomes

$$\begin{aligned} \frac{du}{d\tau} &= (u - \omega)y, \\ \frac{dy}{d\tau} &= 3u^2 + c - y^2 - ry. \end{aligned} \tag{3.3}$$

As in the preceding section, we suppose that $P[u(\tau), y(\tau)] = \sum_{i=0}^{2} a_i(u)y^i{=}0$ is the first-integral of the polynomial form to system (3.3). Then, by the Division Theorem, there exists $H(u, y) = \alpha(u) + \beta(u)y$ such that

$$\frac{dp}{d\tau} = \frac{dp}{du}\frac{du}{d\tau} + \frac{dp}{dy}\frac{dy}{d\tau} = H(u, y)P(u, y). \tag{3.4}$$

By a straightforward computation, we derive that $\alpha(u) = -\frac{6r}{5}$, $\beta(u) = 0$ and

$$\begin{aligned} &P(u, y) \\ &= (u - \omega)^2 y^2 + \frac{4r}{5}(u - \omega)\big[u + \frac{r^2}{25} - \frac{3\omega - c}{6}\big]y - 2u^3 - \big[\frac{2r^2}{25} - (3\omega - c)\big]u^2 \\ &\quad + \big[\frac{2r^4}{625} + \frac{2r^2(3\omega - c)}{75} - \frac{(3\omega - c)^2}{6}\big]u + c\omega\big[\frac{2r^2}{75} - \frac{3\omega - c}{9}\big] = 0, \end{aligned} \tag{3.5}$$

with the parametric restriction

$$\frac{r^4}{625} = \frac{(3\omega + c)^2}{36}.$$

Case 1. When $\frac{r^2}{25} = \frac{3\omega + c}{6}$, equation (3.5) can be reduced to

$$(u - \omega)^2 y^2 + \frac{4r}{5}(u - \omega)(u + \frac{c}{3})y - 2u^3 + 2(\omega - \frac{2c}{3})u^2 + \frac{2c(6\omega - c)}{9}u + \frac{2c^2\omega}{9} = 0.$$

Solving this equation gives

$$y = \frac{du}{d\xi} = \frac{-6r(3u + c) \pm 5(3u + c)\sqrt{6(3u + c)}}{45(u - \omega)}.$$

Thus, we obtain two traveling wave solutions to (3.1) as follows

$$\sqrt{\frac{2(3u + c)}{3}} + [\frac{2r}{5} \mp \frac{2r}{5}] \ln\big|\sqrt{3u + c} \mp \frac{\sqrt{6}r}{5}\big| \pm \frac{r}{5}\ln|3u + c| = \pm\xi + D, \tag{3.6}$$

where $\xi = x - \omega t$ and D is an arbitrary constant.

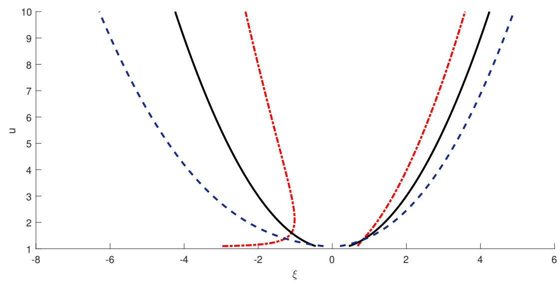

Figure 1. Traveling wave solution of (3.6), when $c = -3$. Red dotted line: $r = -1$; black solid line: $r = 0$; and blue dashed line: $r = 1$.

Case 2. When $\frac{r^2}{25} = -\frac{3\omega+c}{6}$, equation (3.5) can be reduced to

$$(u-\omega)^2y^2 + \frac{4r}{5}(u-\omega)^2y - 2u^3 + 2(2\omega - \frac{c}{3})u^2 - \frac{2\omega(3\omega-2c)}{3}u - \frac{2c\omega^2}{3} = 0.$$

Solving this equation, we obtain

$$y = \frac{du}{d\xi} = -\frac{2r}{5} \pm \sqrt{2(u-\omega)};$$

thus we obtain two traveling wave solutions to (3.1) as:

$$\sqrt{2(u-\omega)} \pm \frac{2r}{5}\ln\big|\sqrt{2(u-\omega)} \mp \frac{2r}{5}\big| = \pm\xi + D, \tag{3.7}$$

where $\xi = x - \omega t$ and D is an arbitrary constant.

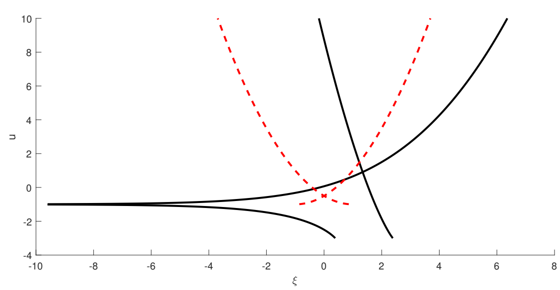

Figure 2. Traveling wave solution of (3.7), when $c = 3$ and $D = 1$. Black solid line: $r = 5$, and red dashed line: $r = 0$.

Remark 3.1. Let us take a closer look at (3.4). Since $H(u, y) = -\frac{6}{5}$, we have that

$$\frac{dp}{d\tau} = -\frac{6r}{5}p. \tag{3.8}$$

Solving equation (3.8), we get two new quasi-polynomial first integrals of explicit form for equation (3.2) as:

$$
\begin{aligned}
&(u-\omega)^2y^2+\frac{4r}{5}(u-\omega)\big[u+\frac{r^2}{25}-\frac{3\omega-c}{6}\big]y-2u^3-\big[\frac{2r^2}{25}-(3\omega-c)\big]u^2\\
&+\big[\frac{2r^4}{625}+\frac{2r^2(3\omega-c)}{75}-\frac{(3\omega-c)^2}{6}\big]u+c\omega\big[\frac{2r^2}{75}-\frac{3\omega-c}{9}\big]\\
&=Ie^{-6r/5},
\end{aligned}
\tag{3.9}
$$

with the parametric restriction

$$
\frac{r^4}{625}=\frac{(3\omega+c)^2}{36}.
$$

In the case of $\frac{r^2}{25}=\frac{(3\omega+c)}{6}$, equation (3.9) can be reduced to

$$
\begin{aligned}
&(u-\omega)^2y^2+\frac{4r}{5}(u-\omega)(u+\frac{c}{3})y-2u^3+2(\omega-\frac{2c}{3})u^2+\frac{2c(6\omega-c)}{9}u+\frac{2c^2\omega}{9}\\
&=Ie^{-6r/5}.
\end{aligned}
$$

In the case of $\frac{r^2}{25}=-\frac{(3\omega+c)}{6}$, equation (3.9) can be reduced to

$$
(u-\omega)^2y^2+\frac{4r}{5}(u-\omega)^2y-2u^3+2(2\omega-\frac{c}{3})u^2-\frac{2\omega(3\omega-2c)}{3}u-\frac{2c\omega^2}{3}=Ie^{-6r/5}.
$$

3.1.2. Now we consider $g=-\omega$. By (2.14), we have $\frac{r^2}{25}=\pm\frac{3\omega-c}{6}$.

Case 1. When $\frac{r^2}{25}=\frac{3\omega-c}{6}$, equation (2.15) can be reduced to

$$
P(u,\,u')=(u-\omega)^2(u')^2+\frac{4r}{5}(u-\omega)uu'-2u^3+\frac{2}{3}(3\omega-c)u^2=0,
$$

Solving this equation, we obtain

$$
u'=-\frac{\frac{2ru}{5}\pm u\sqrt{2u+4[\frac{r^2}{25}-\frac{3\omega-c}{6}]}}{u-\omega}=-\frac{\frac{2ru}{5}\pm u\sqrt{2u}}{u-\omega}. \tag{3.10}
$$

Integrating this equation directly, we obtain two exact traveling wave solutions to equation (1.2) as

$$
\pm\sqrt{2u}-\frac{5\omega}{2r}\ln u+\big(\frac{5\omega}{r}-\frac{2r}{5}\big)\ln\big(5\sqrt{2u}\pm 2r\big)+\xi=D. \tag{3.11}
$$

where $\xi=x-\omega t$ and D is an arbitrary constant.

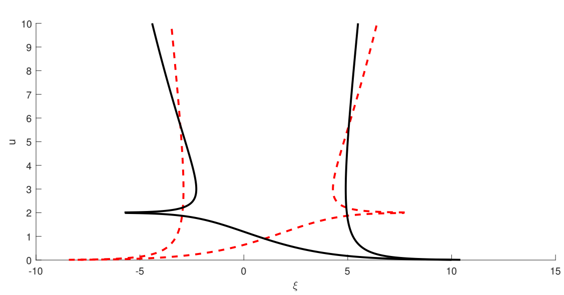

FIGURE 3. Traveling wave solution of (3.11), when $c=3$ and $D=1$. Black solid line: $r=-5$, and red dashed line: $r=5$.

Case 2. When $\frac{r^2}{25} = -\frac{3\omega - c}{6}$, equation (2.15) can be reduced to

$$P(u,\, u') = (u-\omega)^2(u')^2 + \frac{4r(u-\omega)}{5}\big(u + \frac{2r^2}{25}\big)u' - 2u^3 - \frac{8r^2}{25}u^2 - \frac{8r^4}{625}u = 0.$$

Solving this equation gives

$$u' = \frac{-\frac{2r}{5}(u + \frac{2r^2}{25}) \pm (u + \frac{2r^2}{25})\sqrt{2(u + \frac{2r^2}{25})}}{u - \omega}. \tag{3.12}$$

By using a transformation $v = \sqrt{2(u + \frac{2r^2}{25})}$, from (3.12) we obtain

$$d\xi = \frac{\big(\frac{v^2}{2} - \frac{2r^2}{25} - \omega\big)dv}{\pm\frac{v^2}{2} - \frac{rv}{5}}.$$

Integrating directly, we obtain another two traveling wave solutions to equation (1.2) as

$$\pm\sqrt{2(u + \frac{2r^2}{25})} - \frac{5\omega}{r}\ln\big(5\sqrt{2(u + \frac{2r^2}{25})} \mp 2r\big) + \big(\frac{r}{5} + \frac{5\omega}{2r}\big)\ln\big(u + \frac{2r^2}{25}\big) = \xi + D, \tag{3.13}$$

where $\xi = x - \omega t$ and D is an arbitrary constant.

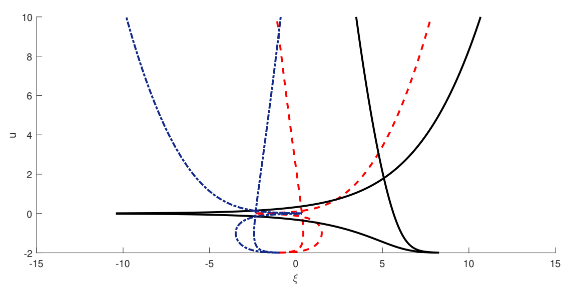

FIGURE 4. Traveling wave solution of (3.13), when $D = 1$. Black solid line: $c = -3$ and $r = 5$. Red dashed line: $c = 3$ and $r = 5$. Blue dashed line: $c = 3$ and $r = -5$.

3.1.3. Now we consider g as an arbitrary constant. We are looking for general traveling wave solutions to equation(1.2) through (2.15). Denote equation (2.15) by

$$ay^2 + by + e = 0,$$

where

$$a = (u-\omega)^2, \quad y = u';$$

$$b = \frac{4r}{5}(u-\omega)[u + \frac{r^2}{25} - \frac{3\omega - c}{6}];$$

$$\begin{aligned} e = -2u^3 - [\frac{2r^2}{25} - (3\omega - c)]u^2 + [\frac{2r^4}{625} + \frac{2r^2(3\omega - c)}{75} - \frac{(3\omega - c)^2}{6}]u \\ + c(\omega + g)[\frac{2r^2}{75} - \frac{3\omega - c}{9}]. \end{aligned}$$

Combining this with (2.14), we have

$$\Delta = b^2 - 4ae = \frac{(u-\omega)^2(6r^2+25c-75\omega+150u)^3}{421875} = 8(u-\omega)^2\Big(u+\frac{r^2}{25}-\frac{3\omega-c}{6}\Big)^3.$$

If $(u+\frac{r^2}{25}-\frac{3\omega-c}{6}) \geq 0$, then $y = u' = \frac{du}{d\xi}$ can be expressed in terms of u as

$$\frac{du}{d\xi} = -\frac{\frac{2r}{5}(u+\frac{r^2}{25}-\frac{3\omega-c}{6}) \pm (u+\frac{r^2}{25}-\frac{3\omega-c}{6})\sqrt{2(u+\frac{r^2}{25}-\frac{3\omega-c}{6})}}{(u-\omega)}.$$

Using the transformation $v = \sqrt{2(u+\frac{r^2}{25}-\frac{3\omega-c}{6})}$ gives us

$$\frac{dv}{d\xi} = -\frac{\frac{rv}{5} \pm \frac{v^2}{2}}{\frac{v^2}{2}-\frac{r^2}{25}-\frac{3\omega+c}{6}}.$$

Solving this equation yields

$$v \pm [\frac{5(3\omega+c)}{6r}-\frac{r}{5}]\ln(5v\pm 2r) \mp [\frac{5(3\omega+c)}{6r}+\frac{r}{5}]\ln v \pm \xi = D, \tag{3.14}$$

where D is an arbitrary constant. Substituting $v = \sqrt{2(u+\frac{r^2}{25}-\frac{3\omega-c}{6})}$ and $\xi = x-\omega t$ into equation (3.14), we obtain two exact traveling wave solutions to (1.2) as

$$\begin{aligned} &\sqrt{2(u+\frac{r^2}{25}-\frac{3\omega-c}{6})} \pm [\frac{5(3\omega+c)}{6r}-\frac{r}{5}]\ln\Big|5\sqrt{2(u+\frac{r^2}{25}-\frac{3\omega-c}{6})}\pm 2r\Big| \\ &\mp [\frac{5(3\omega+c)}{12r}+\frac{r}{10}]\ln\Big|u+\frac{r^2}{25}-\frac{3\omega-c}{6}\Big| \pm \xi = D. \end{aligned} \tag{3.15}$$

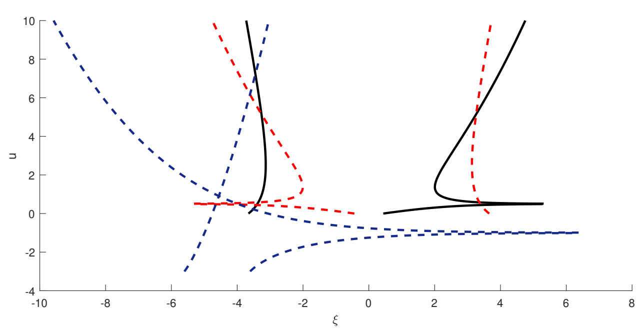

FIGURE 5. Traveling wave solution of (3.15), when $D = 1$ and $c = 3$. Black solid line: $r = 5$ and $\omega = 3$. Red dashed line: $r = -5$ and $\omega = 3$. Blue dashed line: $r = 5$ and $\omega = -3$.

3.2. Now we consider $r = 0$. For the sake of simplicity, we assume that $c > 0$ in this subsection. If $r = 0$, then equation (1.2) can be reduced to

$$\begin{gathered} (u_t + uu_x)_x = 3u + c(1-\rho), \\ \rho_t + (\rho u)_x = 0. \end{gathered} \tag{3.16}$$

At the same time, equation (2.4) and system (2.7) can also be reduced to

$$(u-\omega)^2u'' + (u-\omega)(u')^2 - (u-\omega)(3u+c) + cg = 0,$$

with the associated Hamiltonian system

$$\begin{aligned} \frac{du}{d\tau} &= (u-\omega)^2 y, \\ \frac{dy}{d\tau} &= 3u^2 - (3\omega - c)u - (u-\omega)y^2 - c(\omega + g). \end{aligned} \tag{3.17}$$

By the characteristics of the Hamiltonian system, we can obtain the first integral of polynomial form of system (3.17) as

$$P(u,y) = (u-\omega)^2 y^2 - [2u^3 - (3\omega - c)u^2 - 2c(\omega+g)u] = p, \tag{3.18}$$

where p is an arbitrary constant.

Let $\Delta_1 = (3\omega + c)^2 + 12cg$. Obviously, system (3.17) has two equilibrium points at $(u_1, 0)$ and $(u_2, 0)$ in the u-axis when $\Delta_1 > 0$, has one equilibrium point at $(u_{12}, 0)$ in the u-axis when $\Delta_1 = 0$, has no equilibrium point in the u-axis when $\Delta_1 < 0$, and has no equilibrium point in the straight line $u = \omega$, where

$$u_{1,2} = \frac{3\omega - c \pm \sqrt{\Delta_1}}{6}, \quad u_{12} = \frac{3\omega - c}{6}.$$

From (3.18), we have

$$\begin{aligned} P(\omega, 0) &= \omega(\omega^2 + c\omega + 2cg) \equiv p_s, \\ P(u_{12}, 0) &= -\frac{1}{108}(27\omega^3 - 27c\omega^2 + 9c^2\omega - c^3) \equiv p_{12}, \\ P(u_1, 0) &= \frac{1}{54}(\Delta_1\sqrt{\Delta_1} + 27\omega^3 + 27c\omega^2 + 9c(6g - c)\omega - c^3 - 18c^2 g) \equiv p_1, \\ P(u_2, 0) &= -\frac{1}{54}(\Delta_1\sqrt{\Delta_1} - 27\omega^3 - 27c\omega^2 - 9c(6g - c)\omega + c^3 + 18c^2 g) \equiv p_2. \end{aligned}$$

For a fixed c, by using the properties of the equilibrium points and dynamical systems [14], we obtain the following bifurcation curves of system (3.17):

$$\begin{aligned} C_1 : &\quad g = 0, \\ C_2 : &\quad g = -\frac{1}{16}(9\omega^2 + 6c\omega + c^2) := g^\star, \\ C_3 : &\quad g = -\frac{1}{12}(9\omega^2 + 6c\omega + c^2) := g^*. \end{aligned}$$

For $c > 0$, we get $g^\star > g^*$.

3.2.1. *Loop soliton solutions.* For a given $p = p_1$ in (3.18), we know that there are three open curves defined by $P(u,y) = p_1$: two of them approaching the straight line $u = \omega$, and a curve $u = u_s(y)$ defined by $P(u,y) = p_s$, respectively. One curve passes through the point $(u_m, 0)$ and connects with the saddle point $(u_1, 0)$, and another curve connects with the saddle point $(u_1, 0)$, if and only if one of the following conditions holds:

(i1) $\omega \neq 0$, $g > 0$,
(i2) $\omega < c/3$ $g^\star < g < 0$,

where $u_m = \frac{1}{6}(3\omega - c - 2\sqrt{\Delta_1})$. In the (u,y)-plane, their formula expressions are

$$y = \pm\frac{(u_1 - u)\sqrt{2(u - u_m)}}{u - \omega}, \tag{3.19}$$

for $u \in [u_m, u_s(y)) \cup (\omega, u_1)$ or $u \in [u_m, \omega) \cup (u_s(y), u_1)$, and

$$y = \pm\frac{(u-u_1)\sqrt{2(u-u_m)}}{u-\omega}, \quad u_1 < u < +\infty.$$

Corresponding to (3.19), we can obtain one loop soliton solution of (3.16) as follows

$$\begin{gathered} u(\xi) = u_m + (u_1 - u_m)\tanh^2(\alpha_1 s), \\ \xi = (u_1 - \omega)s - \frac{u_1 - u_m}{\alpha_1}\tanh(\alpha_1 s), \end{gathered} \tag{3.20}$$

where $\alpha_1 = \sqrt{\frac{1}{2}(u_1 - u_m)}$, and s is a new variable parameter.

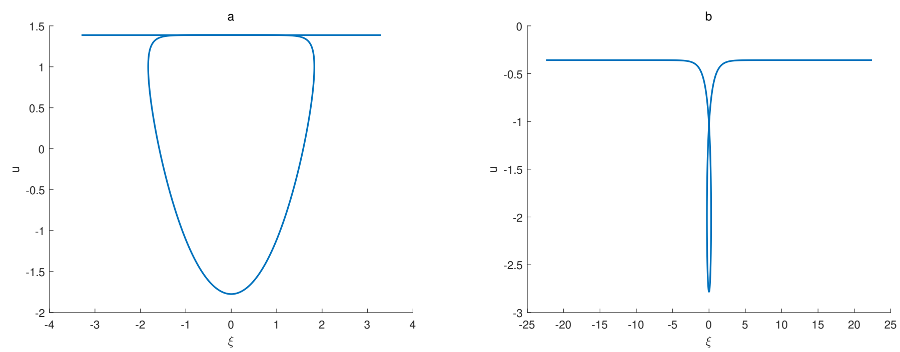

FIGURE 6. Loop soliton solution of (3.20). a: Condition (i1) $c = 1$, $\omega = 1$, $g = 2$. b: Condition (i2) $c = 1$, $\omega = -2$, $g^\star = -25/16$, $g = -1/8$.

3.2.2. *Smooth solitary wave solutions.* For a given $p = p_1$, we know that there is a homoclinic orbit defined by $P(u, y) = p_1$ which connects with the saddle point $(u_1, 0)$ and passes through the point $(u_m, 0)$, if and only if one of the following conditions holds:

(ii1) $\omega < -c/3$, $g^* < g < g^\star$,
(ii2) $\omega > -c/3$, $g^* < g < 0$.

In the (u, y)-plane, its expression is

$$y = \pm\frac{(u_1-u)\sqrt{2(u-u_m)}}{u-\omega}, \quad u_m \le u < u_1. \tag{3.21}$$

Corresponding to (3.21), we can get one smooth solitary wave solution of (3.16) as (3.20).

3.2.3. *Kink-like wave solutions.* For a given $p = p_1$, we know that there is one open curve defined by $P(u, y) = p_1$ connecting with the saddle point $(u_1, 0)$ and approaching the straight line $u = \omega$ when $\omega > -c/3$ and $g^* < g < 0$. In the (u, y)-plane, its expression is

$$y = \pm\frac{(u-u_1)\sqrt{2(u-u_m)}}{u-\omega}, \quad u_1 < u < \omega. \tag{3.22}$$

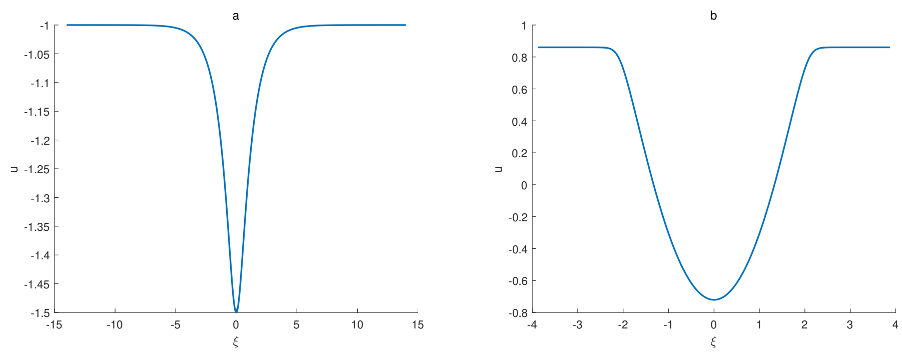

FIGURE 7. Smooth solitary solution of (3.20). a: Condition (ii1) $c = 1$, $\omega = -2$, $g^\star = -25/16$, $g^* = -25/12$, $g = -2$. b: Condition (ii2) $c = 1$, $\omega = 1$, $g^* = -4/3$, $g = -1/2$.

Substituting $\frac{du}{d\xi} = y$ into (3.22) and integrating it along the curve, we have

$$\int_{u_0}^{u} \frac{(s-\omega)ds}{(s-u_1)\sqrt{s-u_m}} = \pm\sqrt{2}\int_0^{\xi} ds, \tag{3.23}$$

where $u_1 < u_0 < \omega$. Letting $\xi \to \xi_1$ in (3.23), we obtain

$$\int_{\omega}^{u_0} \frac{(s-\omega)ds}{(s-u_1)\sqrt{s-u_m}} = \pm\sqrt{2}\int_0^{\xi_1} ds. \tag{3.24}$$

From (3.23) and (3.24), we get the kink-like wave solutions of equation (3.16) as

$$\begin{gathered} u(\xi) = \frac{\kappa\sinh(\alpha_2 s) + u_1(u_1-2u_m)\sinh^2(\frac{\alpha_2}{2}s) + u_0u_1\cosh^2(\frac{\alpha_2}{2}s) - u_0u_m}{\left(\sqrt{u_0-u_m}\sinh(\frac{\alpha_2}{2}s) + \sqrt{u_1-u_m}\cosh(\frac{\alpha_2}{2}s)\right)^2}, \\ \xi = \sqrt{2}\Big(\frac{1}{2}s + \sqrt{u_0-u_m} - \sqrt{u(\xi)-u_m}\Big), \quad -\infty < \xi < \xi_1, \end{gathered}$$

and

$$\begin{gathered} u(\xi) = \frac{\kappa\sinh(\alpha_2 s) + u_1(u_1-2u_m)\sinh^2(\frac{\alpha_2}{2}s) + u_0u_1\cosh^2(\frac{\alpha_2}{2}s) - u_0u_m}{\left(\sqrt{u_0-u_m}\sinh(\frac{\alpha_2}{2}s) + \sqrt{u_1-u_m}\cosh(\frac{\alpha_2}{2}s)\right)^2}, \\ \xi = -\sqrt{2}\Big(\frac{1}{2}s + \sqrt{u_0-u_m} - \sqrt{u(\xi)-u_m}\Big), \quad -\xi_1 < \xi < +\infty, \end{gathered}$$

where

$$\begin{gathered} \kappa = u_1\sqrt{(u_0-u_m)(u_1-u_m)}, \quad \alpha_2 = \frac{\sqrt{u_1-u_m}}{u_1-\omega}, \\ \xi_1 = \sqrt{2}\Big(\sqrt{u_0-u_m} - \sqrt{\omega-u_m} + \frac{\omega-u_1}{2\sqrt{u_1-u_m}}\ln|\sigma|\Big), \\ \sigma = \frac{(\sqrt{\omega-u_m}-\sqrt{u_1-u_m})(\sqrt{u_0-u_m}+\sqrt{u_1-u_m})}{(\sqrt{\omega-u_m}+\sqrt{u_1-u_m})(\sqrt{u_0-u_m}-\sqrt{u_1-u_m})}, \end{gathered}$$

and s is a new variable parameter.

For a given $p = p_{12}$, we know that there is one open curve defined by $P(u, y) = p_{12}$ connecting with the cusp $(u_{12}, 0)$ and approaching the the straight line $u = \omega$

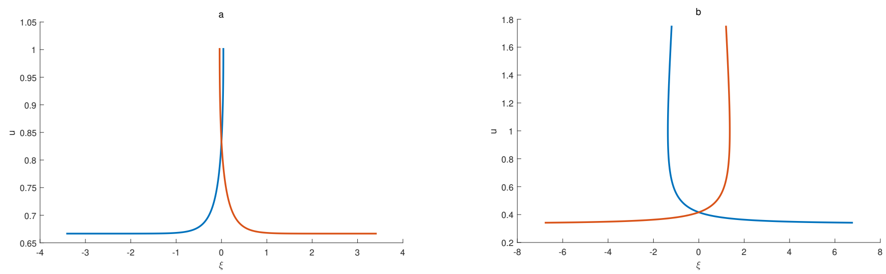

FIGURE 8. Graphs of kink-like wave solutions: a: $p = p_1$, $c = 1$, $\omega = 1$, $g = g^\star = -1$. b: $p = p_{12}$, $c = 1$, $\omega = 1$, $g = g^* = -4/3$.

when $\omega > -c/3$ and $g = g^*$. In the (u, y)-plane, its expression is

$$y = \pm\frac{(u - u_{12})\sqrt{2(u - u_{12})}}{u - \omega}, \quad u_{12} < u < \omega. \tag{3.25}$$

Substituting (3.25) into $\frac{du}{d\xi} = y$ and integrating it along the open curve, we have

$$\int_{u_0}^{u} \frac{(\omega - s)ds}{(s - u_{12})\sqrt{s - u_{12}}} = \pm\sqrt{2}\int_0^{\xi} ds, \tag{3.26}$$

where $u_{12} < u_0 < \omega$. Letting $\xi \to \xi_2$ in (3.26), we obtain

$$\int_{u_0}^{\omega} \frac{(\omega - s)ds}{(s - u_{12})\sqrt{s - u_{12}}} = \sqrt{2}\int_0^{\xi_2} ds. \tag{3.27}$$

From (3.26) and (3.27), we get the kink-like solutions of equation (3.16) as

$$u(\xi) = u_{12} + \Big(\frac{1}{\frac{1}{\sqrt{u_0 - u_{12}}} - \frac{s}{2(\omega - u_{12})}}\Big)^2,$$
$$\xi = \sqrt{2}(\frac{1}{2}s + \sqrt{u_0 - u_{12}} - \frac{1}{\frac{1}{\sqrt{u_0 - u_{12}}} - \frac{s}{2(\omega - u_{12})}}), \quad -\infty < \xi < \xi_2,$$

and

$$u(\xi) = u_{12} + \Big(\frac{1}{\frac{1}{\sqrt{u_0 - u_{12}}} - \frac{s}{2(\omega - u_{12})}}\Big)^2,$$
$$\xi = -\sqrt{2}\Big(\frac{1}{2}s + \sqrt{u_0 - u_{12}} - \frac{1}{\frac{1}{\sqrt{u_0 - u_{12}}} - \frac{s}{2(\omega - u_{12})}}\Big), \quad -\xi_2 < \xi < +\infty,$$

where

$$\xi_2 = \sqrt{2}\Big(\sqrt{u_0 - u_{12}} - 2\sqrt{\omega - u_{12}} + \frac{\omega - u_{12}}{\sqrt{u_0 - u_{12}}}\Big).$$

3.2.4. *Cusped solitary wave solutions.* For a given $p = p_s$, we know that there is one open curve defined by $P(u, y) = p_s$ connecting with the saddle point $(u_*, 0)$ and infinity approaching the straight line $u = \omega$ when $\omega < -c/3$ and $g = g^\star$, where $u_* = \frac{1}{4}(\omega - c)$. In the (u, y)-plane, its expression is

$$y = \pm\frac{(u_* - u)\sqrt{2(u - \omega)}}{u - \omega}, \quad \omega \le u < u_*. \tag{3.28}$$

Corresponding to (3.28), we can obtain one cusped solitary wave solution of (3.16) as

$$
\begin{gathered}
u(\xi)=\omega+(u_*-\omega)\tanh^2(\alpha_3 s),\\
\xi=(u_*-\omega)s-\frac{u_*-\omega}{\alpha_3}\tanh(\alpha_3 s),
\end{gathered}
$$

where $\Omega_3=\sqrt{\frac{1}{2}(u_*-\omega)}$.

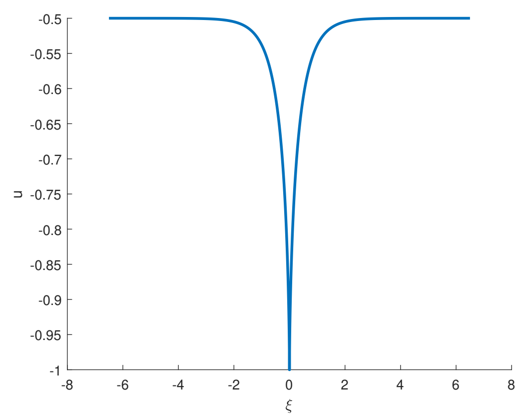

FIGURE 9. Cusped solitary wave solutions: $c=1$, $\omega=-1$, $g=g^\star=-1/4$.

3.2.5. *Periodic loop soliton solutions.* For a given $p\in(p_s,\,p_1)$, we know that there are three open curves defined by $P(u,y)=p$, $p\in(p_s,\,p_1)$: two of them approaching the straight line $u=\omega$ and a curve $u=u_s(y)$ defined by $P(u,y)=p_s$, respectively. One curve passes through the points $(\gamma_3,0)$ and $(\gamma_2,0)$ and another passes through the point $(\gamma_1,0)$, if and only if one of the following conditions holds:

(iv1) $\omega\neq 0$, $g>0$,
(iv2) $\omega<-c/3$, $g^\star<g<0$,

where γ_1, γ_2 and γ_3 ($\gamma_3<\gamma_2<\gamma_1$) are real roots of

$$
2X^3+(c-3\omega)X^2-2c(\omega+g)X+p=0,\quad p\in(p_s,p_1).
$$

In the (u,y)-plane, their expressions are

$$
y=\pm\frac{\sqrt{2(\gamma_1-u)(\gamma_2-u)(u-\gamma_3)}}{u-\omega},\tag{3.29}
$$

for $u\in[\gamma_3,\omega)\cup(u_s(y),\gamma_2)$ or $u\in[\gamma_3,u_s(y))\cup(\omega,\gamma_2]$, and

$$
y=\pm\frac{\sqrt{2(u-\gamma_1)(u-\gamma_2)(u-\gamma_3)}}{u-\omega},\quad \gamma_1\le u<+\infty,
$$

respectively. Corresponding to (3.29) , we can get periodic loop soliton solutions of (3.16) as

$$
\begin{gathered}
u(\xi)=\gamma_3+(\gamma_2-\gamma_3)\,\mathrm{sn}^2(\alpha_4 s,k_1),\\
\xi=(\gamma_1-\omega)s-\frac{\gamma_2-\gamma_3}{\alpha_4 k_1^2}\,\mathrm{E}(\mathrm{am}(\alpha_4 s,k_1),k_1),
\end{gathered}\tag{3.30}
$$

where

$$\alpha_4 = \sqrt{\frac{1}{2}(\gamma_1 - \gamma_3)}, \quad k_1 = \sqrt{\frac{\gamma_2 - \gamma_3}{\gamma_1 - \gamma_3}},$$

$\mathrm{sn}(\cdot,\cdot)$ is the Jacobian elliptic function, $\mathrm{E}(\cdot,\cdot)$ is the elliptic integral of the second kind, and $\mathrm{am}(v_1,\cdot)$ is the amplitude v_1 (see [2] and [4]).

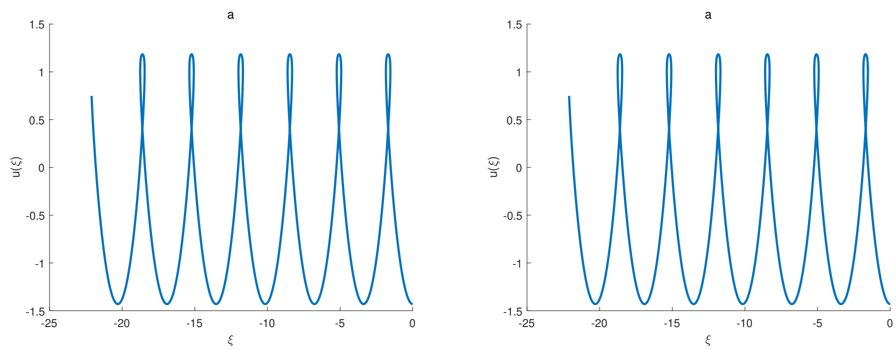

FIGURE 10. Graphs of periodic loop soliton solution of (3.30). a: Condition (iv1) $c = 1$, $\omega = 1$, $g = 1$, $p_s = 4$, $p_1 = \frac{28\sqrt{7}+40}{27}$ and $p = 4.22$. b: Condition (iv2) $c = 1$, $\omega = -2$, $g^\star = -25/16$, $g = -1$, $p_s = 0$, $p_1 = \frac{13\sqrt{13}+35}{54}$ and $p = 1.5$.

3.2.6. *Smooth periodic wave solutions.* For a given $p \in (p_2, p_s)$ or $p \in (p_2, p_1)$, or $p \in (p_s, p_1)$, we know that there is one periodic orbit defined by $P(u, y) = p$ passing through the points $(\gamma_3, 0)$ and $(\gamma_2, 0)$, if and only if one of the following conditions holds:

- (v1) $\omega \neq 0$, $g > 0$, $p \in (p_2, p_s)$,
- (v2) $\omega > -c/3$, $g^\star < g < 0$, $p \in (p_2, p_s)$,
- (v3) $\omega < -c/3$, $g^\star \le g < 0$, $p \in (p_2, p_s)$,
- (v4) $\omega > -c/3$, $g^* < g \le g^\star$, $p \in (p_2, p_1)$,
- (v5) $\omega < -c/3$, $g^* < g < g^\star$, $p \in (p_2, p_1)$,
- (v6) $\omega > -c/3$, $g^\star < g < 0$, $p \in (p_s, p_1)$.

In the (u, y)-plane, its expression is

$$y = \pm \frac{\sqrt{2(\gamma_1 - u)(\gamma_2 - u)(u - \gamma_3)}}{u - \omega}, \quad \gamma_3 \le u \le \gamma_2, \tag{3.31}$$

where γ_1, γ_2 and γ_3 (with $\gamma_3 < \gamma_2 < \gamma_1$) are real roots of

$$2X^3 + (c - 3\omega)X^2 - 2c(\omega + g)X + p = 0.$$

Corresponding to (3.31), we can get the smooth periodic wave solutions of (3.16) as (3.30).

For a given $p = p_s$, we know that there is one periodic orbit defined by $P(u, y) = p_s$ passing through the points $(u_n, 0)$ and $(u_N, 0)$, if and only if the following condition holds:

- (vi) $\omega > -c/3$, $g^\star < g < 0$,

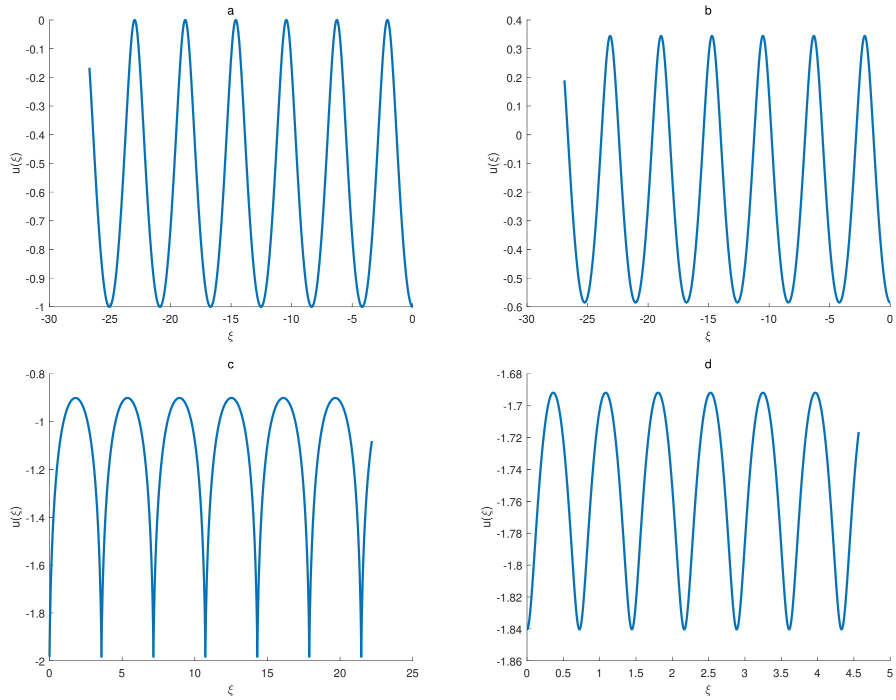

FIGURE 11. Graphs of smooth periodic wave solutions of (3.30). a: Condition (v1) $c = 1$, $\omega = 1$, $g = 1$, $p_s = 4$, $p_2 = \frac{-28\sqrt{7}+40}{27}$, $p = 0$. b: Condition (v2) $c = 1$, $\omega = 1$, $g^\star = -1$, $g = \frac{1}{2}$, $p_s = 1$, $p_2 = \frac{-5\sqrt{10}+13}{27}$, $p = \frac{1}{2}$. c: Condition (v3) $c = 1$, $\omega = -2$, $g = g^\star = -\frac{25}{16}$, $p_s = \frac{9}{4}$, $p_2 = \frac{361}{216}$, $p = 2.2$. d: Condition (v3) $c = 1$, $\omega = -2$, $g^\star = -\frac{25}{16}$, $g = -1$, $p_s = 0$, $p_2 = \frac{-13\sqrt{13}+35}{27}$, $p = \frac{1}{5}$.

where $u_{N,n} = \frac{1}{4}(\omega - c \pm \sqrt{\Delta_2})$ and $\Delta_2 = c^2 + 6c\omega + 9\omega^2 + 16cg$. In the (u, y)-plane, its expression is

$$y = \pm\frac{\sqrt{2(\omega - u)(u_N - u_n)(u - u_n)}}{u - \omega}, \quad u_n \le u \le u_N. \tag{3.32}$$

Corresponding to (3.32), we can obtain the smooth periodic wave solutions of (3.16) as

$$\begin{aligned} u(\xi) &= u_n + (u_N - u_n)\,\mathrm{sn}^2(\alpha_5 s, k_2), \\ \xi &= \frac{u_n - u_N}{\alpha_5 k_2^2}\mathrm{E}(\mathrm{am}(\alpha_5 s, k_2), k_2), \end{aligned} \tag{3.33}$$

where $\alpha_5 = \sqrt{\frac{1}{2}(\omega - u_n)}$ and $k_2 = \sqrt{\frac{u_N - u_n}{\omega - u_n}}$.

Conclusions

In this work, we have applied Feng's first-integral method to the two-component generalization of the reduced Ostrovsky equation, and found some new traveling wave solutions, loop soliton, kink-like wave cusped solitary wave, periodic loop

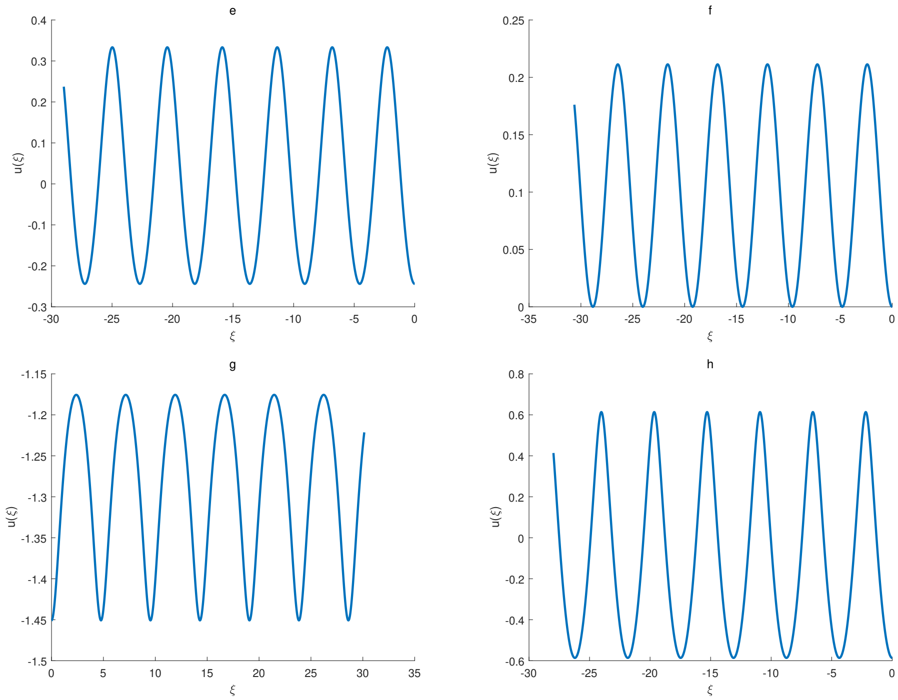

FIGURE 12. Graphs of smooth periodic wave solutions of (3.30). e: Condition (v4) $c = 1$, $\omega = 1$, $g = g^\star = -1$, $p_1 = 8/27$, $p_2 = 0$, $p = 4/27$. f: Condition (v4) $c = 1$, $\omega = 1$, $g^\star = -1$, $g^* = -4/3$, $g = -7/6$, $p_1 = \frac{\sqrt{2}+1}{27}$, $p_2 = \frac{-\sqrt{2}+1}{27}$, $p = 0$. g: Condition (v5) $c = 1$, $\omega = -2$, $g^\star = -25/16$, $g^* = -25/12$, $g = -2$, $p_1 = 3$, $p_2 = 80/27$, $p = 2.98$. h: Condition (v6) $c = 1$, $\omega = 1$, $g^\star = -1$, $g = -2/3$, $p_1 = \frac{8\sqrt{2}+10}{27}$, $p_2 = 2/3$, $p = 0.7$.

soliton, and periodic wave for the two-component generalization of the reduced Ostrovsky equation. Bifurcations of phase portrait of traveling waves were also provide and discussed under various parametric conditions. It is worthy to mention that we can apply this powerful method to solve traveling wave solutions for nonlinear partial differential equations described in [13, 17, 18, 19, 23]. We believe that this method is advantageous for a rather diverse group of scientists.

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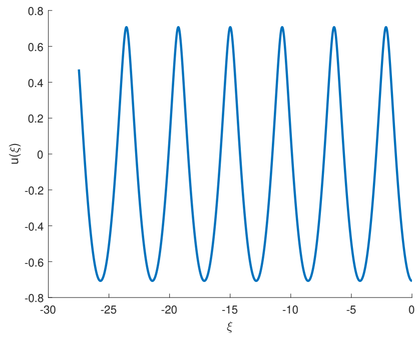

Figure 13. Graph of smooth periodic wave solution of (3.33). Condition (vi) $c = 1, \omega = 1, g^{\star} = -1, g = -1/2$.

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