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ASYMPTOTIC SHAPE OF SOLUTIONS TO NONLINEAR EIGENVALUE PROBLEMS

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ABSTRACT. We consider the nonlinear eigenvalue problem

$$-u''(t) = f(\lambda, u(t)), \quad u > 0, \quad u(0) = u(1) = 0,$$

where $\lambda > 0$ is a parameter. It is known that under some conditions on $f(\lambda, u)$, the shape of the solutions associated with λ is almost 'box' when $\lambda \gg 1$. The purpose of this paper is to study precisely the asymptotic shape of the solutions as $\lambda \to \infty$ from a standpoint of L^1-framework. To do this, we establish the asymptotic formulas for L^1-norm of the solutions as $\lambda \to \infty$.

1. INTRODUCTION

We consider the nonlinear eigenvalue problem

$$-u''(t) = f(\lambda, u(t)), \quad t \in I := (0,1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(0) = u(1) = 0, \tag{1.3}$$

where $\lambda > 0$ is a parameter. The nonlinearities considered here are as follows:

$$f(\lambda, u) = \lambda \sin u, \tag{1.4}$$

$$f(\lambda, u) = \lambda \sin u - g(u), \tag{1.5}$$

$$f(\lambda, u) = \lambda(u - u^3). \tag{1.6}$$

Equation (1.1)–(1.3) with the nonlinearities (1.4) and (1.5) are called the simple pendulum type equations (SPE), and that with (1.6) is derived from the logistic equation of population dynamics (LEPD). Throughout this paper, in (1.5), we assume that $g(u)$ satisfies the following conditions:

- (A1) $g \in C^1(\mathbf{R})$ and $g(u) > 0$ for $u > 0$.
- (A2) $g(0) = g'(0) = 0$.
- (A3) $g(u)/u$ is strictly increasing for $0 \le u \le \pi$.

Nonlinear eigenvalue problems and singularly perturbed problems are intensively investigated by many authors. We refer to [5, 9] and the references therein. One of the most interesting problems to study in these fields is to clarify the asymptotic shapes of the solutions. We know (cf. [2, 3]) that for a given $\lambda > \pi^2$, there exists a

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unique solution $u \in C^2(I)$ to (1.1)–(1.3). We denote by $u_{0,\lambda}, u_\lambda$ and v_λ the solutions u of (1.1)–(1.3) with (1.4), (1.5) and (1.6), respectively. Let $m_\lambda := \min\{m > 0 : f(\lambda, m) = 0\}$. Then it is well known (cf. [2, 3] and Appendix) that $\|u_{0,\lambda}\|_\infty < m_\lambda$ (resp. $\|u_\lambda\|_\infty < m_\lambda$, $\|v_\lambda\|_\infty < m_\lambda$). Clearly, $m_\lambda = \pi$ and $m_\lambda = 1$ for (1.4) and (1.6), respectively. By (A1), we see that $0 < m_\lambda < \pi$ for (1.5). Furthermore, it is known (cf. [2, 6]) that

$$u_{0,\lambda} \to \pi, \quad u_\lambda \to \pi, \quad v_\lambda \to 1 \tag{1.7}$$

uniformly on any compact interval in I as $\lambda \to \infty$. In other words, the asymptotic shape of these solutions are *almost boxes*. Therefore, a natural question we have to ask here is "*how close to the boxes are the shape of the solutions* $u_{0,\lambda}, u_\lambda$ *and* v_λ *asymptotically?*

The purpose of this paper is to answer this question from a viewpoint of L^1-framework. More precisely, we restrict our attention to the typical nonlinearities (1.4)–(1.6), and establish the precise asymptotic formulas for L^1-norm of $\|u_{0,\lambda}\|_1, \|u_\lambda\|_1$ and $\|v_\lambda\|_1$ as $\lambda \to \infty$. By these formulas, we understand well

(i) The difference of the shape between $u_{0,\lambda}$ and u_λ from a *non-local* point of view, and
(ii) The difference between $\|v_\lambda\|_1$ and $\|v_\lambda\|_2$ when $\lambda \gg 1$.

The first approach to the study of the asymptotic shape of the solutions of (SPE) is to investigate the asymptotic behavior of the L^∞-norm of the solutions as $\lambda \to \infty$ and the following results have been obtained in [6, 8]:

$$\|u_{0,\lambda}\|_\infty = \pi - 8e^{-\sqrt{\lambda}/2} + o(e^{-\sqrt{\lambda}/2}), \tag{1.8}$$

$$\|u_\lambda\|_\infty = \pi - \frac{g(\pi)}{\lambda} + \frac{g(\pi)g'(\pi)}{\lambda^2} + o\big(\frac{1}{\lambda^2}\big). \tag{1.9}$$

By (1.8) and (1.9), we understand the difference between the *pointwise (local)* behavior of $u_{0,\lambda}$ and u_λ. However, we do not know the difference between the *total mass* of $u_{0,\lambda}$ and u_λ, which gives us the important information about the *non-local* property of $u_{0,\lambda}$ and u_λ. Therefore, it seems meaningful for us to establish the asymptotic formulas for $\|u_{0,\lambda}\|_1$ and $\|u_\lambda\|_1$, which give us the better understanding of the difference between the original (SPE) and the perturbed (SPE).

We now state the results for (SPE).

Theorem 1.1. *As* $\lambda \to \infty$

$$\|u_{0,\lambda}\|_1 = \pi - C_1\frac{1}{\sqrt{\lambda}} + \frac{8}{\sqrt{\lambda}}e^{-\sqrt{\lambda}/2} + o\big(\frac{1}{\sqrt{\lambda}}e^{-\sqrt{\lambda}/2}\big), \tag{1.10}$$

$$\|u_\lambda\|_1 = \pi - C_1\frac{1}{\sqrt{\lambda}} - \frac{g(\pi)}{\lambda} + O\big(\frac{\log\lambda}{\lambda\sqrt{\lambda}}\big), \tag{1.11}$$

where

$$C_1 = 8\int_0^{\pi/4} \log(\cot\theta)d\theta.$$

Roughly speaking, the second terms in (1.10) and (1.11) are derived from the width of the boundary layers of $u_{0,\lambda}$ and u_λ, while the third terms come directly from the second terms in (1.8) and (1.9).

We next consider (LEPD). The motivation to consider the asymptotic behavior of $\|v_\lambda\|_1$ as $\lambda \to \infty$ is as follows. Recently, from a viewpoint of L^2-bifurcation theory and nonlinear eigenvalue problems, the following formula for $\|v_\lambda\|_2$ as $\lambda \to \infty$ has been obtained.

$$\|v_\lambda\|_2 = 1 - \sqrt{\frac{2}{\lambda}} - \frac{1}{\lambda} + o\big(\frac{1}{\lambda}\big). \tag{1.12}$$

This has been obtained by using [7, Theorem 1]. On the other hand, since (LEPD) is a model equation of population density for some species, $\|v_\lambda\|_1$ stands for the total population of the species with λ, which describes the reciprocal number of its diffusion rate. Motivated by this biological background, it is also important to investigate the asymptotic behavior of $\|v_\lambda\|_1$ as $\lambda \to \infty$. Furthermore, since (1.6) is a special nonlinearlity, we also obtain the asymptotic formula for $\|v_\lambda\|_\infty$ better than (1.8) as $\lambda \to \infty$.

Now we state our second results.

Theorem 1.2. *As* $\lambda \to \infty$

$$\|v_\lambda\|_1 = 1 - \frac{2\sqrt{2}\log 2}{\sqrt{\lambda}} - 12e^{-\sqrt{2\lambda}} + o(e^{-\sqrt{2\lambda}}), \tag{1.13}$$

$$\|v_\lambda\|_\infty = 1 - 4e^{-\sqrt{\lambda}/\sqrt{2}} - 8e^{-2\sqrt{\lambda}/\sqrt{2}} - 24\sqrt{2}\sqrt{\lambda}e^{-3\sqrt{\lambda}/\sqrt{2}} + o(\sqrt{\lambda}e^{-3\sqrt{\lambda}/\sqrt{2}}). \tag{1.14}$$

We see from (1.12) and (1.13) that the third term of $\|v_\lambda\|_1$ and $\|v_\lambda\|_2$ are totally different each other. The further direction of this study will be to treat more general nonlinear term $f(\lambda, u)$ and extend our results to PDE cases.

The remainder of this paper is organized as follows. In Sections 2 and 3, we prove (1.11) and (1.10) in Theorem 1.1, respectively. By using the properties of complete elliptic integral, we prove Theorem 1.2 in Section 4.

2. Proof of (1.11) in Theorem 1.1

In this section, we consider (1.1)–(1.3) with (1.5) and prove the formula (1.11). In what follows, the character C denotes various positive constants independent of $\lambda \gg 1$. We know that

$$\begin{gathered} u_\lambda(t) = u_\lambda(1-t) \quad \text{for } t \in \bar{I}, \\ u_\lambda'(t) > 0 \quad \text{for } 0 \le t < 1/2, \\ \|u_\lambda\|_\infty = u_\lambda(1/2). \end{gathered} \tag{2.1}$$

We begin with the fundamental equality. Multiply (1.1) by u_λ'. Then

$$\{u_\lambda''(t) + \lambda \sin u_\lambda(t) - g(u_\lambda(t))\}u_\lambda'(t) = 0.$$

This implies that for $t \in \bar{I}$

$$\frac{d}{dt}\big\{\frac{1}{2}u_\lambda'(t)^2 - \lambda\cos u_\lambda(t) - G(u_\lambda(t))\big\} \equiv 0,$$

where $G(u) = \int_0^u g(s)ds$. By this and (2.1), for $0 \le t \le 1$,

$$\frac{1}{2}u_\lambda'(t)^2 - \lambda\cos u_\lambda(t) - G(u_\lambda(t)) \equiv \text{constant} = -\lambda\cos\|u_\lambda\|_\infty - G(\|u_\lambda\|_\infty). \tag{2.2}$$

By this and (2.1), for $0 \le t \le 1/2$,

$$u_\lambda'(t) = \sqrt{2\lambda(\cos u_\lambda(t) - \cos\|u_\lambda\|_\infty) + 2(G(u_\lambda(t)) - G(\|u_\lambda\|_\infty))} \tag{2.3}$$

We know from (1.7) that as $\lambda \to \infty$

$$r_1(\lambda) := \|u_\lambda\|_\infty - \|u_\lambda\|_1 \to 0. \tag{2.4}$$

Since we have (1.9), we establish an asymptotic formula for $r_1(\lambda)$ as $\lambda \to \infty$ to obtain (1.11). By (2.1) and (2.3), for $\lambda \gg 1$

$$\begin{aligned} r_1(\lambda) &= 2\int_0^{1/2} (\|u_\lambda\|_\infty - u_\lambda(t))dt \\ &= 2\int_0^{1/2} \frac{(\|u_\lambda\|_\infty - u_\lambda(t))u_\lambda'(t)dt}{\sqrt{2\lambda(\cos u_\lambda(t) - \cos\|u_\lambda\|_\infty) + 2(G(u_\lambda(t)) - G(\|u_\lambda\|_\infty))}} \\ &= 2\int_0^{\|u_\lambda\|_\infty} \frac{(\|u_\lambda\|_\infty - \theta)d\theta}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty) + 2(G(\theta) - G(\|u_\lambda\|_\infty))}} \\ &= K_1(\lambda) + K_2(\lambda), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} K_1(\lambda) &:= 2\int_{\|u_\lambda\|_\infty - 1/\lambda}^{\|u_\lambda\|_\infty} \frac{(\|u_\lambda\|_\infty - \theta)d\theta}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty) + 2(G(\theta) - G(\|u_\lambda\|_\infty))}}, \\ K_2(\lambda) &:= 2\int_0^{\|u_\lambda\|_\infty - 1/\lambda} \frac{(\|u_\lambda\|_\infty - \theta)d\theta}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty) + 2(G(\theta) - G(\|u_\lambda\|_\infty))}}. \end{aligned}$$

Lemma 2.1. *$K_1(\lambda) = O(\lambda^{-3/2})$ for $\lambda \gg 1$.*

Proof. For $j = 1, 2, \dots$, let

$$I_j := \big[\|u_\lambda\|_\infty - \frac{1}{j\lambda}, \|u_\lambda\|_\infty - \frac{1}{(j+1)\lambda}\big].$$

We put

$$J_j := 2\int_{I_j} \frac{(\|u_\lambda\|_\infty - \theta)d\theta}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty) + 2(G(\theta) - G(\|u_\lambda\|_\infty))}}. \tag{2.6}$$

We know by [2] that

$$\lambda \sin\|u_\lambda\|_\infty > g(\|u_\lambda\|_\infty). \tag{2.7}$$

Let an arbitrary $0 < \epsilon \ll 1$ be fixed. Let $\eta_{\lambda,\epsilon} := \min_{\|u_\lambda\|_\infty - 2\epsilon \le u \le \|u_\lambda\|_\infty} g'(u)$. Then by (A3), we see that $\eta_{\lambda,\epsilon} > 0$. Then for $\theta \in [\|u_\lambda\|_\infty - 2\epsilon, \|u_\lambda\|_\infty]$, by (1.7), (2.7) and Taylor expansion, we have

$$\begin{aligned} &2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty) + 2(G(\theta) - G(\|u_\lambda\|_\infty)) \\ &\ge 2\left(\lambda\sin\|u_\lambda\|_\infty - g(\|u_\lambda\|_\infty)\right)(\|u_\lambda\|_\infty - \theta) \\ &\quad + 2\left(-\frac{\lambda}{2}\cos(\|u_\lambda\|_\infty - 2\epsilon) + \frac{1}{2}\eta_{\lambda,\epsilon}\right)(\|u_\lambda\|_\infty - \theta)^2 \\ &\ge C\lambda(\|u_\lambda\|_\infty - \theta)^2. \end{aligned} \tag{2.8}$$

By this and (2.6), for $\lambda > 1/\epsilon$,

$$\begin{aligned} J_j &\le \int_{I_j} \frac{\|u_\lambda\|_\infty - \theta}{\sqrt{C\lambda(\|u_\lambda\|_\infty - \theta)^2}} d\theta \\ &= \frac{1}{\sqrt{C\lambda}}\frac{1}{\lambda}\big(\frac{1}{j} - \frac{1}{j+1}\big) \le \frac{C}{\lambda^{3/2}}\frac{1}{j(j+1)}. \end{aligned}$$

By this,

$$K_1(\lambda) = \sum_{j=1}^\infty J_j \le \sum_{j=1}^\infty \frac{C}{\lambda^{3/2}}\frac{1}{j(j+1)} \le \frac{C}{\lambda^{3/2}}.$$

Thus the proof is complete. □

The formula (1.11) follows from (1.9), Lemma 2.1 and the following Proposition.

Proposition 2.2. *For $\lambda \gg 1$*

$$K_2(\lambda) = \frac{C_1}{\sqrt{\lambda}} + O\left(\frac{\log\lambda}{\lambda^{3/2}}\right). \tag{2.9}$$

We prove this proposition using Lemmas 2.3–2.7 below. We put

$$K_2(\lambda) := K_{2,1}(\lambda) + K_{2,2}(\lambda),$$

where

$$K_{2,1}(\lambda) := 2\int_0^{\|u_\lambda\|_\infty - 1/\lambda} \frac{(\|u_\lambda\|_\infty - \theta)d\theta}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}},$$
$$K_{2,2}(\lambda) := K_2(\lambda) - K_{2,1}(\lambda).$$

Furthermore, we put

$$K_{2,1}(\lambda) := L_1(\lambda) + L_2(\lambda),$$

where

$$L_1(\lambda) := 2\int_0^{\pi - 1/\lambda} \frac{\pi - \theta}{\sqrt{2\lambda(\cos\theta + 1)}} d\theta.$$

Lemma 2.3. *For $\lambda \gg 1$*

$$L_1(\lambda) = \frac{C_1}{\sqrt{\lambda}} + o\Big(\frac{\log\lambda}{\lambda^{3/2}}\Big).$$

Proof. For $\lambda \gg 1$

$$\begin{aligned}
L_1(\lambda) &= \frac{1}{\sqrt{\lambda}}\int_0^{\pi-1/\lambda} \frac{\pi-\theta}{\cos(\theta/2)}d\theta \\
&= \frac{1}{\sqrt{\lambda}}\int_{1/\lambda}^{\pi} \frac{t}{\sin(t/2)}dt = \frac{4}{\sqrt{\lambda}}\int_{1/(2\lambda)}^{\pi/2} \frac{t}{\sin t}dt \quad (\text{put } \theta = \tan(t/2)) \\
&= \frac{8}{\sqrt{\lambda}}\int_{\tan(1/(4\lambda))}^{1} \frac{\tan^{-1}\theta}{\theta}d\theta \\
&= \frac{8}{\sqrt{\lambda}}\big\{[\log\theta\tan^{-1}\theta]_{\tan(1/(4\lambda))}^{1} - \int_{\tan(1/(4\lambda))}^{1}\frac{\log\theta}{1+\theta^2}d\theta\big\} \quad (\text{put } \theta = \tan t) \\
&= \frac{2+o(1)}{\lambda^{3/2}}\log\lambda - \frac{8}{\sqrt{\lambda}}\int_0^{\pi/4}\log(\tan t)dt + \frac{8}{\sqrt{\lambda}}\int_0^{1/(4\lambda)}\log(\tan t)dt \\
&= \frac{2+o(1)}{\lambda^{3/2}}\log\lambda + \frac{8}{\sqrt{\lambda}}\int_0^{\pi/4}\log(\cot t)dt + \frac{8}{\sqrt{\lambda}}(1+o(1))\int_0^{1/(4\lambda)}\log t dt \\
&= \frac{2+o(1)}{\lambda^{3/2}}\log\lambda + \frac{8}{\sqrt{\lambda}}\int_0^{\pi/4}\log(\cot t)dt + \frac{8}{\sqrt{\lambda}}(1+o(1))\Big(\frac{1}{4\lambda}\log\frac{1}{4\lambda} - \frac{1}{4\lambda}\Big) \\
&= \frac{8}{\sqrt{\lambda}}\int_0^{\pi/4}\log(\cot t)dt + o\Big(\frac{\log\lambda}{\lambda^{3/2}}\Big).
\end{aligned}$$

Thus the proof is complete. □

Next, we calculate $L_2(\lambda)$. To do this, we put

$$L_2(\lambda) = K_{2,1}(\lambda) - L_1(\lambda) := D_1(\lambda) + D_2(\lambda) + D_3(\lambda), \tag{2.10}$$

where

$$D_1(\lambda) := 2\int_0^{\|u_\lambda\|_\infty - 1/\lambda} \frac{\|u_\lambda\|_\infty - \pi}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}} d\theta,$$

$$D_2(\lambda) := 2\int_0^{\|u_\lambda\|_\infty - 1/\lambda} \Big(\frac{\pi - \theta}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}} - \frac{\pi - \theta}{\sqrt{2\lambda(\cos\theta + 1)}}\Big) d\theta,$$

$$D_3(\lambda) := -2\int_{\|u_\lambda\|_\infty - 1/\lambda}^{\pi - 1/\lambda} \frac{\pi - \theta}{\sqrt{2\lambda(\cos\theta + 1)}} d\theta.$$

Lemma 2.4. *$D_1(\lambda) = O(\lambda^{-3/2})$ as $\lambda \to \infty$.*

Proof. Let an arbitrary $0 < \epsilon \ll 1$ be fixed. Then for $\lambda \gg 1$

$$\begin{aligned} D_1(\lambda) &= D_{1,1}(\lambda) + D_{1,2}(\lambda) \\ &:= 2\int_{\|u_\lambda\|_\infty - \epsilon}^{\|u_\lambda\|_\infty - 1/\lambda} \frac{\|u_\lambda\|_\infty - \pi}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}} d\theta \\ &\quad + 2\int_0^{\|u_\lambda\|_\infty - \epsilon} \frac{\|u_\lambda\|_\infty - \pi}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}} d\theta. \end{aligned}$$

For $0 \le \theta \le \|u_\lambda\|_\infty - \epsilon$, there exists a constant $C_\epsilon > 0$ such that for $\lambda \gg 1$

$$C_\epsilon \le \cos\theta - \cos\|u_\lambda\|_\infty. \tag{2.11}$$

By this and (1.9), for $\lambda \gg 1$,

$$|D_{1,2}(\lambda)| \le \frac{2g(\pi)}{\lambda}(1 + o(1))\frac{1}{\sqrt{2C_\epsilon\lambda}}\pi \le C(\lambda^{-3/2}). \tag{2.12}$$

We next estimate $D_{1,1}(\lambda)$. For a given $\lambda \gg 1$, there exists $k_\lambda \in N$ satisfying

$$\|u_\lambda\|_\infty - 2\epsilon \le \|u_\lambda\|_\infty - \frac{k_\lambda + 1}{\lambda} \le \|u_\lambda\|_\infty - \epsilon \le \|u_\lambda\|_\infty - \frac{k_\lambda}{\lambda}. \tag{2.13}$$

For $j = 1, 2, \dots, k_\lambda$, we define an interval

$$M_j = \big[\|u_\lambda\|_\infty - \frac{j+1}{\lambda}, \|u_\lambda\|_\infty - \frac{j}{\lambda}\big]. \tag{2.14}$$

By (2.13), we see that $k_\lambda \le \epsilon\lambda$. By this, (1.9) and (2.8), we obtain

$$\begin{aligned} |D_{1,1}(\lambda)| &\le \frac{g(\pi)}{\lambda}(1 + o(1))\sum_{j=1}^{k_\lambda}\int_{M_j} \frac{1}{\sqrt{2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty)}} d\theta \\ &\le \frac{g(\pi)}{\sqrt{2C}\lambda^{3/2}}(1 + o(1))\sum_{j=1}^{k_\lambda}\int_{M_j} \frac{1}{\|u_\lambda\|_\infty - \theta} d\theta \\ &\le C\lambda^{-3/2}\sum_{j=1}^{k_\lambda}(\log(j+1) - \log j) \\ &= C\lambda^{-3/2}\log(k_\lambda + 1) \le C\lambda^{-3/2}\log\lambda. \end{aligned}$$

By this and (2.12), we obtain our conclusion. Thus the proof is complete. $\square$

Lemma 2.5. *$D_2(\lambda) = O(\lambda^{-3/2})$ for $\lambda \gg 1$.*

Proof. We put

$$A_\lambda(\theta) := 2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty), \quad B_\lambda(\theta) := 2\lambda(\cos\theta + 1). \tag{2.15}$$

By (1.9) and Taylor expansion, for $\lambda \gg 1$, we have

$$1 + \cos\|u_\lambda\|_\infty = \frac{g(\pi)^2}{2\lambda^2}(1 + o(1)). \tag{2.16}$$

Note that $A_\lambda(\theta) \le B_\lambda(\theta)$. By this, (2.16) and Taylor expansion, for a fixed $0 < \epsilon \ll 1$

$$\begin{aligned} D_2(\lambda) &= 2\int_0^{\|u_\lambda\|_\infty - 1/\lambda} \frac{2\lambda(\pi-\theta)(1+\cos\|u_\lambda\|_\infty)}{\sqrt{A_\lambda(\theta)}\sqrt{B_\lambda(\theta)}(\sqrt{A_\lambda(\theta)} + \sqrt{B_\lambda(\theta)})} d\theta \\ &\le \frac{2g(\pi)^2}{\lambda}(1+o(1))\Big[\int_{\|u_\lambda\|_\infty - \epsilon}^{\|u_\lambda\|_\infty - 1/\lambda} \frac{\pi - \theta}{(2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty))^{3/2}} d\theta \\ &\quad + \int_0^{\|u_\lambda\|_\infty - \epsilon} \frac{\pi-\theta}{(2\lambda(\cos\theta - \cos\|u_\lambda\|_\infty))^{3/2}} d\theta\Big] \\ &:= D_{2,1}(\lambda) + D_{2,2}(\lambda). \end{aligned} \tag{2.17}$$

We know that $2\theta/\pi \le \sin\theta$ for $0 \le \theta \le \pi/2$. By this, (1.9) and mean value theorem, for $\theta \in M_j$ defined by (2.14), we have

$$\begin{aligned} \cos\theta - \cos\|u_\lambda\|_\infty &\ge \sin\Big(\|u_\lambda\|_\infty - \frac{j}{\lambda}\Big)(\|u_\lambda\|_\infty - \theta) \\ &= \sin\Big(\pi - \frac{g(\pi)}{\lambda}(1+o(1)) - \frac{j}{\lambda}\Big)(\|u_\lambda\|_\infty - \theta) \\ &= \sin\Big(\frac{g(\pi)}{\lambda}(1+o(1)) + \frac{j}{\lambda}\Big)(\|u_\lambda\|_\infty - \theta) \\ &\ge \frac{2}{\pi}\Big(\frac{g(\pi)(1+o(1)) + j}{\lambda}\Big)(\|u_\lambda\|_\infty - \theta). \end{aligned}$$

By this, (1.9) and (2.17),

$$\begin{aligned} D_{2,1}(\lambda) &\le \frac{C}{\lambda}\sum_{j=1}^{k_\lambda}\int_{M_j} \frac{(j+1+g(\pi)+o(1))/\lambda}{\{\frac{4}{\pi}(j+g(\pi)+o(1))(\|u_\lambda\|_\infty - \theta)\}^{3/2}} d\theta \\ &\le \frac{C}{\lambda^2}\sum_{j=1}^{k_\lambda}(j+g(\pi)+1+o(1))(j+g(\pi)+o(1))^{-3/2}\int_{M_j}(\|u_\lambda\|_\infty - \theta)^{-3/2} d\theta \\ &\le \frac{C}{\lambda^{3/2}}\sum_{j=1}^{k_\lambda} j^{-1/2}\Big(\frac{1}{\sqrt{j}} - \frac{1}{\sqrt{j+1}}\Big) \\ &= \frac{C}{\lambda^{3/2}}\sum_{j=1}^{k_\lambda}\frac{1}{\sqrt{j}\sqrt{j+1}(\sqrt{j}+\sqrt{j+1})} j^{-1/2} \\ &\le \frac{C}{\lambda^{3/2}}\sum_{j=1}^{k_\lambda}\frac{1}{j^2} \le \frac{C}{\lambda^{3/2}}. \end{aligned} \tag{2.18}$$

By (2.11), we have

$$D_{2,2}(\lambda) \le \frac{2g(\pi)^2 + o(1)}{\lambda} \frac{1}{(2C_\epsilon \lambda)^{3/2}} \pi^2 \le C\lambda^{-5/2}.$$

This along with (2.17) and (2.18) implies our conclusion. □

Lemma 2.6. *$D_3(\lambda) = O(\lambda^{-3/2})$ for $\lambda \gg 1$.*

Proof. By (1.9), for $\lambda \gg 1$

$$\begin{aligned}
|D_3(\lambda)| &= \int_{\|u_\lambda\|_\infty - 1/\lambda}^{\pi - 1/\lambda} \frac{\pi - \theta}{\sqrt{\lambda}\cos(\theta/2)} d\theta \\
&= \frac{1}{\sqrt{\lambda}} \int_{\pi - \|u_\lambda\|_\infty + 1/\lambda}^{1/\lambda} \frac{-t}{\sin(t/2)} dt \\
&= \frac{4}{\sqrt{\lambda}} \int_{1/(2\lambda)}^{(\pi - \|u_\lambda\|_\infty + 1/\lambda)/2} \frac{\theta}{\sin\theta} d\theta \\
&= \frac{4}{\sqrt{\lambda}} (1 + o(1)) \frac{\pi - \|u_\lambda\|_\infty}{2} = \frac{2g(\pi)}{\lambda^{3/2}} (1 + o(1)).
\end{aligned}$$

Thus the proof is complete. □

By Lemmas 2.3–2.6, we see that

$$K_{2,1}(\lambda) = \frac{C_1}{\sqrt{\lambda}} + O\left(\frac{\log\lambda}{\lambda^{3/2}}\right). \tag{2.19}$$

Now we estimate $K_{2,2}(\lambda)$.

Lemma 2.7. *$K_{2,2}(\lambda) = O(\lambda^{-3/2}\log\lambda)$ for $\lambda \gg 1$.*

Proof. We put $E_\lambda(\theta) := 2(G(\theta) - G(\|u_\lambda\|_\infty))$. We recall $A_\lambda(\theta)$ defined in (2.15). Let an arbitrary $0 < \epsilon \ll 1$ be fixed. For $\lambda \gg 1$

$$\begin{aligned}
&K_{2,2}(\lambda) \\
&= 2\int_0^{\|u_\lambda\|_\infty - 1/\lambda} \Big(\frac{\|u_\lambda\|_\infty - \theta}{\sqrt{A_\lambda(\theta) + E_\lambda(\theta)}} - \frac{\|u_\lambda\|_\infty - \theta}{\sqrt{A_\lambda(\theta)}} \Big) d\theta \\
&= 2\int_0^{\|u_\lambda\|_\infty - \epsilon} \frac{(\|u_\lambda\|_\infty - \theta)(G(\|u_\lambda\|_\infty) - G(\theta))}{\sqrt{A_\lambda(\theta) + E_\lambda(\theta)}\sqrt{A_\lambda(\theta)}(\sqrt{A_\lambda(\theta) + E_\lambda(\theta)} + \sqrt{A_\lambda(\theta)})} d\theta \\
&\quad + 2\int_{\|u_\lambda\|_\infty - \epsilon}^{\|u_\lambda\|_\infty - 1/\lambda} \frac{(\|u_\lambda\|_\infty - \theta)(G(\|u_\lambda\|_\infty) - G(\theta))}{\sqrt{A_\lambda(\theta) + E_\lambda(\theta)}\sqrt{A_\lambda(\theta)}(\sqrt{A_\lambda(\theta) + E_\lambda(\theta)} + \sqrt{A_\lambda(\theta)})} d\theta \\
&= H_1(\lambda) + H_2(\lambda).
\end{aligned}$$

By (2.11) we see that for $\lambda \gg 1$,

$$H_1(\lambda) \le 2g(\pi)\pi^3(2C_\epsilon\lambda)^{-3/2}.$$

Note that $A_\lambda(\theta) + E_\lambda(\theta) \le A_\lambda(\theta)$ for $\|u_\lambda\|_\infty - \epsilon \le \theta \le \|u_\lambda\|_\infty - 1/\lambda$. Then by (2.8),

$$
\begin{aligned}
H_2(\lambda) &\le 2\int_{\|u_\lambda\|_\infty-\epsilon}^{\|u_\lambda\|_\infty-1/\lambda} \frac{g(\pi)(\|u_\lambda\|_\infty-\theta)^2}{(A_\lambda+E_\lambda)^{3/2}}d\theta \\
&\le \frac{C}{\lambda^{3/2}}\int_{\|u_\lambda\|_\infty-\epsilon}^{\|u_\lambda\|_\infty-1/\lambda} \frac{1}{\|u_\lambda\|_\infty-\theta}d\theta \\
&= \frac{C}{\lambda^{3/2}}(\log\epsilon - \log(1/\lambda)) \le \frac{C}{\lambda^{3/2}}\log\lambda.
\end{aligned}
$$

Thus the proof is complete. $\square$

By (2.19) and Lemma 2.7, we obtain Proposition 2.2. Now (1.11) follows from (1.9), (2.4), (2.5), Lemma 2.1 and Proposition 2.2. Thus the proof is complete.

3. Proof of (1.10) in Theorem 1.1

To prove (1.10), we put

$$
Q(\lambda) := \pi - \|u_{0,\lambda}\|_1. \tag{3.1}
$$

By the similar calculation to that in (2.5), we have

$$
Q(\lambda) = 2\int_0^{\|u_{0,\lambda}\|_\infty} \frac{\pi-\theta}{\sqrt{2\lambda(\cos\theta - \cos\|u_{0,\lambda}\|_\infty)}}d\theta = Q_1(\lambda) + Q_2(\lambda), \tag{3.2}
$$

$$
Q_2(\lambda) := Q(\lambda) - Q_1(\lambda), \tag{3.3}
$$

where

$$
Q_1(\lambda) := 2\int_0^{\pi} \frac{\pi-\theta}{\sqrt{2\lambda(\cos\theta+1)}}d\theta. \tag{3.4}
$$

Lemma 3.1. $Q_1(\lambda) = C_1\lambda^{-1/2}$.

Proof.

$$
\begin{aligned}
Q_1(\lambda) &= \frac{1}{\sqrt{\lambda}}\int_0^{\pi} \frac{\pi-\theta}{\cos(\theta/2)}d\theta \\
&= \frac{1}{\sqrt{\lambda}}\int_0^{\pi} \frac{t}{\sin(t/2)}dt = \frac{4}{\sqrt{\lambda}}\int_0^{\pi/2} \frac{t}{\sin t}dt \quad (\text{put } \theta = \tan(t/2)) \\
&= \frac{8}{\sqrt{\lambda}}\int_0^{1} \frac{\tan^{-1}\theta}{\theta}d\theta \\
&= \frac{8}{\sqrt{\lambda}}\Big\{[\log\theta\tan^{-1}\theta]_0^1 - \int_0^1 \frac{\log\theta}{1+\theta^2}d\theta\Big\} \quad (\text{put } \theta = \tan t) \\
&= -\frac{8}{\sqrt{\lambda}}\int_0^{\pi/4}\log(\tan t)dt = \frac{8}{\sqrt{\lambda}}\int_0^{\pi/4}\log(\cot t)dt
\end{aligned}
$$

Thus the proof is complete. $\square$

Lemma 3.2. $Q_2(\lambda) = -\frac{8}{\sqrt{\lambda}}(1+o(1))e^{-\sqrt{\lambda}/2}$ *as* $\lambda \to \infty$.

Proof. We put

$$Q_2(\lambda) := R(\lambda) + S(\lambda), \tag{3.5}$$

where

$$R(\lambda) = \sqrt{\frac{2}{\lambda}} \int_0^{\|u_{0,\lambda}\|_\infty} \Big(\frac{\pi-\theta}{\sqrt{\cos\theta - \cos\|u_{0,\lambda}\|_\infty}} - \frac{\pi-\theta}{\sqrt{\cos\theta+1}}\Big) d\theta$$

$$S(\lambda) = -\sqrt{\frac{2}{\lambda}} \int_{\|u_{0,\lambda}\|_\infty}^{\pi} \frac{\pi-\theta}{\sqrt{\cos\theta+1}} d\theta.$$

For $\lambda \gg 1$

$$\begin{aligned}
R(\lambda) &= \sqrt{\frac{2}{\lambda}} \int_0^{\|u_{0,\lambda}\|_\infty} \frac{(\pi-\theta)(\sqrt{\cos\theta+1} - \sqrt{\cos\theta - \cos\|u_{0,\lambda}\|_\infty})}{\sqrt{\cos\theta+1}\sqrt{\cos\theta-\cos\|u_{0,\lambda}\|_\infty}} d\theta \\
&= \sqrt{\frac{2}{\lambda}}(1+o(1)) \int_0^{\|u_{0,\lambda}\|_\infty} \frac{(\pi-\theta)(\cos\|u_{0,\lambda}\|_\infty + 1)}{(\cos\theta+1)(\sqrt{\cos\theta-\cos\|u_{0,\lambda}\|_\infty} + \sqrt{\cos\theta+1})} d\theta \\
&= \sqrt{\frac{2}{\lambda}}(\cos\|u_{0,\lambda}\|_\infty + 1)(1+o(1)) \int_0^{\|u_{0,\lambda}\|_\infty} \frac{\pi-\theta}{2(\cos\theta+1)^{3/2}} d\theta \\
&= \sqrt{\frac{2}{\lambda}}(\cos\|u_{0,\lambda}\|_\infty + 1)(1+o(1)) \int_0^{\|u_{0,\lambda}\|_\infty} \frac{\pi-\theta}{4\sqrt{2}\cos^3(\theta/2)} d\theta \\
&= \sqrt{\frac{2}{\lambda}}(\cos\|u_{0,\lambda}\|_\infty + 1)(1+o(1)) \int_\pi^{\pi-\|u_{0,\lambda}\|_\infty} \frac{-\theta}{4\sqrt{2}\sin^3(\theta/2)} d\theta \\
&= \sqrt{\frac{1}{\lambda}}(\cos\|u_{0,\lambda}\|_\infty + 1)(1+o(1)) \int_{(\pi-\|u_{0,\lambda}\|_\infty)/2}^{\pi/2} \frac{\theta}{\sin^3\theta} d\theta \\
&= \sqrt{\frac{1}{\lambda}}(\cos\|u_{0,\lambda}\|_\infty + 1)(1+o(1)) \int_{(\pi-\|u_{0,\lambda}\|_\infty)/2}^{\pi/2} \frac{1}{\sin^2\theta} d\theta \\
&= \sqrt{\frac{1}{\lambda}}(\cos\|u_{0,\lambda}\|_\infty + 1)(1+o(1)) [-\cot\theta]_{(\pi-\|u_{0,\lambda}\|_\infty)/2}^{\pi/2} \\
&= \sqrt{\frac{1}{\lambda}} \frac{\cos((\pi-\|u_{0,\lambda}\|_\infty)/2)}{\sin((\pi-\|u_{0,\lambda}\|_\infty)/2)} (\cos\|u_{0,\lambda}\|_\infty + 1)(1+o(1)) \\
&= \sqrt{\frac{1}{\lambda}} \frac{2}{\pi-\|u_{0,\lambda}\|_\infty} (\cos\|u_{0,\lambda}\|_\infty + 1)(1+o(1)).
\end{aligned} \tag{3.6}$$

By (1.8) and Taylor expansion, for $\lambda \gg 1$

$$\begin{aligned}
\cos\|u_{0,\lambda}\|_\infty &= \cos\big(\pi - 8(1+o(1))e^{-\sqrt{\lambda}/2}\big) \\
&= -\cos\big(8(1+o(1))e^{-\sqrt{\lambda}/2}\big) \\
&= -1 + 32(1+o(1))e^{-\sqrt{\lambda}}.
\end{aligned} \tag{3.7}$$

By this, (1.8) and (3.6), for $\lambda \gg 1$

$$R(\lambda) = \frac{8}{\sqrt{\lambda}}(1+o(1))e^{-\sqrt{\lambda}/2}. \tag{3.8}$$

Next, we calculate $S(\lambda)$. By (1.8), for $\lambda \gg 1$

$$\begin{aligned} S(\lambda) &= -\sqrt{\frac{2}{\lambda}} \int_{\pi - \|u_{0,\lambda}\|_\infty}^{0} \frac{-t}{\sqrt{2}\cos((\pi - t)/2)} dt \\ &= -\sqrt{\frac{1}{\lambda}} \int_{0}^{\pi - \|u_{0,\lambda}\|_\infty} \frac{t}{\sin(t/2)} dt \\ &= -\frac{4}{\sqrt{\lambda}} \int_{0}^{(\pi - \|u_{0,\lambda}\|_\infty)/2} \frac{\theta}{\sin\theta} d\theta \\ &= -\frac{4}{\sqrt{\lambda}} \frac{\pi - \|u_{0,\lambda}\|_\infty}{2}(1 + o(1)) \\ &= -\frac{16}{\sqrt{\lambda}}(1 + o(1))e^{-\sqrt{\lambda}/2}. \end{aligned}$$

By this, (3.5) and (3.8), we obtain our conclusion. Thus the proof is complete. $\square$

Now (1.10) follows from (1.8), (3.1), (3.2) and Lemmas 3.1 and 3.2. Thus the proof is complete.

4. Proof of Theorem 1.2

In this section, we consider (1.1)–(1.3) with (1.6). By (1.1), we have

$$(v_\lambda''(t) + \lambda(v_\lambda(t) - v_\lambda^3(t)))v_\lambda'(t) = 0.$$

This implies that for $t \in \bar{I}$

$$\frac{d}{dt}\big(\frac{1}{2}v_\lambda'(t)^2 + \frac{1}{2}\lambda v_\lambda^2(t) - \frac{1}{4}\lambda v_\lambda^4(t)\big) = 0.$$

This implies that for $t \in \bar{I}$,

$$\frac{1}{2}v_\lambda'(t)^2 + \frac{1}{2}\lambda v_\lambda^2(t) - \frac{1}{4}\lambda v_\lambda^4(t) \equiv \text{constant} = \frac{1}{2}\lambda\|v_\lambda\|_\infty^2 - \frac{1}{4}\lambda\|v_\lambda\|_\infty^4. \tag{4.1}$$

We know that

$$v_\lambda'(t) \ge 0, \quad 0 \le t \le 1/2, \quad v_\lambda(t) = v_\lambda(1 - t), \quad t \in I. \tag{4.2}$$

Therefore, by (4.1) and (4.2), for $0 \le t \le 1/2$,

$$v_\lambda'(t) = \sqrt{\lambda\{(\|v_\lambda\|_\infty^2 - v_\lambda(t)^2) - \frac{1}{2}(\|v_\lambda\|_\infty^4 - v_\lambda(t)^4)\}}. \tag{4.3}$$

The following Lemma 4.1 implies (1.14) in Theorem 1.2.

Lemma 4.1. *As $\lambda \to \infty$*

$$\|v_\lambda\|_\infty = 1 - 4e^{-\sqrt{\lambda}/\sqrt{2}} - 8e^{-2\sqrt{\lambda}/\sqrt{2}} - 24\sqrt{2}\sqrt{\lambda}e^{-3\sqrt{\lambda}/\sqrt{2}} + o(\sqrt{\lambda}e^{-3\sqrt{\lambda}/\sqrt{2}}). \tag{4.4}$$

Proof. By (4.3),

$$
\begin{aligned}
\frac{1}{2} &= \int_0^{1/2} dt = \int_0^{1/2} \frac{v_\lambda'(t)}{\sqrt{\lambda\{(\|v_\lambda\|_\infty^2 - v_\lambda(t)^2) - \frac{1}{2}(\|v_\lambda\|_\infty^4 - v_\lambda(t)^4)\}}} dt \\
&= \frac{1}{\sqrt{\lambda}} \int_0^{\|v_\lambda\|_\infty} \frac{1}{\sqrt{(\|v_\lambda\|_\infty^2 - \theta^2) - \frac{1}{2}(\|v_\lambda\|_\infty^4 - \theta^4)}} d\theta \quad (\text{put } \theta = \|v_\lambda\|_\infty s) \\
&= \frac{1}{\sqrt{\lambda}} \frac{\sqrt{2}}{\sqrt{2 - \|v_\lambda\|_\infty^2}} \int_0^1 \frac{1}{\sqrt{(1-s^2)(1-k^2 s^2)}} ds \\
&= \frac{1}{\sqrt{\lambda}} \frac{\sqrt{2}}{\sqrt{2 - \|v_\lambda\|_\infty^2}} K(k),
\end{aligned} \tag{4.5}
$$

where $k = \|v_\lambda\|_\infty / \sqrt{2 - \|v_\lambda\|_\infty^2}$ and

$$
K(k) := \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta.
$$

It is known (cf. [4, p.909, 8.113]) that as $k \to 1$

$$
\begin{aligned}
K(k) = &-\frac{1}{2}\log(1-k^2) + 2\log 2 - \frac{1-k^2}{8}\log(1-k^2) \\
&+ \big(\frac{1}{2}\log 2 - \frac{1}{4}\big)(1-k^2) - \frac{9}{128}(1-k^2)^2\log(1-k^2) \\
&+ o\big((1-k^2)^2\log(1-k^2)\big).
\end{aligned} \tag{4.6}
$$

We put $\xi_\lambda := 1 - \|v_\lambda\|_\infty^2$. Then $\xi_\lambda > 0$ and $\xi_\lambda \to 0$ as $\lambda \to \infty$ by (1.7). Then

$$
1 - k^2 = \frac{2(1 - \|v_\lambda\|_\infty^2)}{2 - \|v_\lambda\|_\infty^2} = \frac{2\xi_\lambda}{1 + \xi_\lambda}. \tag{4.7}
$$

By this, the Taylor expansion, and (4.6),

$$
\begin{aligned}
K(k) &= -\frac{1}{2}\big(\log 2 + \log\xi_\lambda - \log(1+\xi_\lambda)\big) \\
&\quad + 2\log 2 - \frac{\xi_\lambda}{4(1+\xi_\lambda)}\big(\log 2 + \log\xi_\lambda - \log(1+\xi_\lambda)\big) \\
&\quad + \big(\frac{1}{2}\log 2 - \frac{1}{4}\big)\frac{2\xi_\lambda}{1+\xi_\lambda} - \frac{9}{32}\frac{\xi_\lambda^2}{(1+\xi_\lambda)^2}\big(\log 2 + \log\xi_\lambda - \log(1+\xi_\lambda)\big) \\
&\quad + o(\xi_\lambda^2\log\xi_\lambda) \\
&= -\frac{1}{2}\big(\log 2 + \log\xi_\lambda - \xi_\lambda + O(\xi_\lambda^2)\big) + 2\log 2 \\
&\quad - \frac{1}{4}\xi_\lambda(1 - \xi_\lambda + O(\xi_\lambda^2))\big(\log 2 + \log\xi_\lambda - \xi_\lambda + O(\xi_\lambda^2)\big) \\
&\quad + \big(\log 2 - \frac{1}{2}\big)\xi_\lambda(1 - \xi_\lambda + O(\xi_\lambda^2)) - \frac{9}{32}\xi_\lambda^2\log\xi_\lambda + o(\xi_\lambda^2\log\xi_\lambda) \\
&= -\frac{1}{2}\log\xi_\lambda + \frac{3}{2}\log 2 - \frac{1}{4}\xi_\lambda\log\xi_\lambda + \frac{3\log 2}{4}\xi_\lambda - \frac{1}{32}\xi_\lambda^2\log\xi_\lambda + o(\xi_\lambda^2\log\xi_\lambda).
\end{aligned} \tag{4.8}
$$

Furthermore, by Taylor expansion, for $\lambda \gg 1$,

$$\frac{1}{\sqrt{2-\|v_\lambda\|_\infty^2}} = (1+\xi_\lambda)^{-1/2} = 1 - \frac{1}{2}\xi_\lambda + \frac{3}{8}\xi_\lambda^2 + o(\xi_\lambda^2).$$

This along with (4.5) and (4.8) implies that

$$\frac{\sqrt{\lambda}}{2\sqrt{2}} = -\frac{1}{2}\log\xi_\lambda + \frac{3}{2}\log 2 - \frac{3}{32}\xi_\lambda^2\log\xi_\lambda + o(\xi_\lambda^2\log\xi_\lambda). \tag{4.9}$$

By this, for $\lambda \gg 1$, we have

$$\frac{\sqrt{\lambda}}{\sqrt{2}} = -\log\xi_\lambda + \log 8 + \log(1+o(1)) = \log\frac{8(1+o(1))}{\xi_\lambda}.$$

This implies that for $\lambda \gg 1$, $\xi_\lambda = 8(1+o(1))e^{-\sqrt{\lambda}/\sqrt{2}}$. Then for $\lambda \gg 1$,

$$-\frac{3}{32}\xi_\lambda^2\log\xi_\lambda = 3\sqrt{2}(1+o(1))\sqrt{\lambda}e^{-\sqrt{2\lambda}}.$$

By this, (4.9) and Taylor expansion, for $\lambda \gg 1$

$$\begin{aligned}
\xi_\lambda &= 8e^{-\sqrt{\lambda}/\sqrt{2}} \cdot e^{6\sqrt{2}(1+o(1))\sqrt{\lambda}e^{-\sqrt{2\lambda}}} \\
&= 8e^{-\sqrt{\lambda}/\sqrt{2}}(1+6\sqrt{2}(1+o(1))\sqrt{\lambda}e^{-\sqrt{2\lambda}})) \\
&= 8e^{-\sqrt{\lambda}/\sqrt{2}} + 48\sqrt{2}(1+o(1))\sqrt{\lambda}e^{-3\sqrt{\lambda}/\sqrt{2}}.
\end{aligned} \tag{4.10}$$

By this and Taylor expansion, for $\lambda \gg 1$,

$$\begin{aligned}
\|v_\lambda\|_\infty &= \sqrt{1-\xi_\lambda} \\
&= \Big(1 - 8e^{-\sqrt{\lambda}/\sqrt{2}} - 48\sqrt{2}(1+o(1))\sqrt{\lambda}e^{-3\sqrt{\lambda}/\sqrt{2}})\Big)^{1/2} \\
&= 1 + \frac{1}{2}\Big(- 8e^{-\sqrt{\lambda}/\sqrt{2}} - 48\sqrt{2}(1+o(1))\sqrt{\lambda}e^{-3\sqrt{\lambda}/\sqrt{2}})\Big) \\
&\quad - \frac{1}{8}(-8e^{-\sqrt{\lambda}/\sqrt{2}} - 48\sqrt{2}(1+o(1))\sqrt{\lambda}e^{-3\sqrt{\lambda}/\sqrt{2}}))^2 + O(e^{-3\sqrt{\lambda}/\sqrt{2}}).
\end{aligned}$$

By this, we obtain (4.4). $\square$

The following implies (1.13) in Theorem 1.2.

Lemma 4.2. *As* $\lambda \to \infty$

$$\|v_\lambda\|_1 = 1 - \frac{2\sqrt{2}\log 2}{\sqrt{\lambda}} - 12e^{-\sqrt{2\lambda}} + o(e^{-\sqrt{2\lambda}}). \tag{4.11}$$

Proof. By (4.3), for $\lambda \gg 1$

$$
\begin{aligned}
\|v_\lambda\|_1 &= 2\int_0^{1/2} v_\lambda(t)\,dt \\
&= 2\int_0^{1/2} \frac{v_\lambda(t)v_\lambda'(t)}{\sqrt{\lambda\{(\|v_\lambda\|_\infty^2 - v_\lambda(t)^2) - \frac{1}{2}(\|v_\lambda\|_\infty^4 - v_\lambda(t)^4)\}}}\,dt \\
&= \frac{2}{\sqrt{\lambda}}\int_0^{\|v_\lambda\|_\infty} \frac{\theta}{\sqrt{\|v_\lambda\|_\infty^2 - \theta^2 - \frac{1}{2}(\|v_\lambda\|_\infty^4 - \theta^4)}}\,d\theta \\
&= \frac{2\|v_\lambda\|_\infty}{\sqrt{\lambda}}\int_0^1 \frac{s}{\sqrt{(1-s^2) - \frac{1}{2}\|v_\lambda\|_\infty^2(1-s^4)}}\,ds \\
&= \frac{\|v_\lambda\|_\infty}{\sqrt{\lambda}}\frac{\sqrt{2}}{\sqrt{2-\|v_\lambda\|_\infty^2}}\int_0^1 \frac{1}{(\sqrt{1-t})(1-k^2t)}\,dt \\
&= \sqrt{\frac{2}{\lambda}}k\int_0^1 \frac{1}{\sqrt{(1-t)(1-k^2t)}}\,dt.
\end{aligned}
\tag{4.12}
$$

By putting $s = \sqrt{(1-k^2t)/(1-t)}$, we obtain easily

$$
\int_0^1 \frac{1}{\sqrt{(1-t)(1-k^2t)}}dt = \frac{1}{k}\log\big(\frac{1+k}{1-k}\big).
$$

This along with (4.12) implies that

$$
\|v_\lambda\|_1 = \sqrt{\frac{2}{\lambda}}\log\frac{1+k}{1-k} = \sqrt{\frac{2}{\lambda}}\log\frac{(1+k)^2}{1-k^2} = \sqrt{\frac{2}{\lambda}}(2\log(1+k) - \log(1-k^2)). \tag{4.13}
$$

By (4.7), (4.10) and Taylor expansion, for $\lambda \gg 1$

$$
\begin{aligned}
\log(1-k^2) &= \log\frac{2\xi_\lambda}{1+\xi_\lambda} \\
&= \log 2 + \log\xi_\lambda - \log(1+\xi_\lambda) \\
&= 4\log 2 - \sqrt{\frac{\lambda}{2}} + 6\sqrt{2}\sqrt{\lambda}e^{-\sqrt{2\lambda}} - \big(\xi_\lambda + O(\xi_\lambda^2)\big) \\
&= 4\log 2 - \sqrt{\frac{\lambda}{2}} - 8e^{-\sqrt{\lambda/2}} + 6\sqrt{2}\sqrt{\lambda}e^{-\sqrt{2\lambda}} + o(\sqrt{\lambda}e^{-\sqrt{2\lambda}}).
\end{aligned}
\tag{4.14}
$$

By Lemma 4.1, (4.10) and Taylor expansion, for $\lambda \gg 1$,

$$
\begin{aligned}
k &= \frac{\|v_\lambda\|_\infty}{\sqrt{2-\|v_\lambda\|_\infty}} = \|v_\lambda\|_\infty(1+\xi_\lambda)^{-1/2} \\
&= \|v_\lambda\|_\infty\Big(1 - \frac{1}{2}\xi_\lambda + \frac{3}{8}\xi_\lambda^2(1+o(1))\Big) \\
&= (1 - 4e^{-\sqrt{\lambda/2}} - 8e^{-2\sqrt{\lambda/2}}(1+o(1)))(1 - 4e^{-\sqrt{\lambda/2}} + 24e^{-2\sqrt{\lambda/2}}(1+o(1))) \\
&= 1 - 8e^{-\sqrt{\lambda/2}} + 32e^{-2\sqrt{\lambda/2}}(1+o(1)).
\end{aligned}
$$

By this and Taylor expansion, for $\lambda \gg 1$,

$$\begin{aligned}\log(1+k) &= \log(2-(1-k)) = \log 2 - \frac{1}{2}(1-k) - \frac{1}{8}(1-k)^2 + o((1-k)^2)\\ &= \log 2 - 4e^{-\sqrt{\lambda/2}} + 8(1+o(1))e^{-2\sqrt{\lambda/2}}.\end{aligned}$$

By this and (4.14), we obtain

$$\log\Big(\frac{1+k}{1-k}\Big) = -2\log 2 + \sqrt{\frac{\lambda}{2}} - 6\sqrt{2}(1+o(1))\sqrt{\lambda}e^{-\sqrt{2\lambda}}.$$

By this and (4.13) and (4.14), we obtain (4.11). Thus the proof is complete. □

5. Appendix

We show that $\|u_\lambda\|_\infty < \pi$ for completeness. By (2.1),

$$-u_\lambda''(1/2) = \lambda \sin \|u_\lambda\|_\infty - g(\|u_\lambda\|_\infty) \ge 0.$$

This along with (A1) implies that there exists a non-negative integer k such that

$$2k\pi < \|u_\lambda\|_\infty < (2k+1)\pi. \tag{5.1}$$

Assume that $k \ge 1$. Then by (2.1), there exists a unique $t_\lambda \in (0,1/2)$ such that $u_\lambda(t_\lambda) = \|u_\lambda\|_\infty - 2k\pi$. Then by (2.2),

$$\begin{aligned}&\frac{1}{2}u_\lambda'(t_\lambda)^2 - \lambda\cos u_\lambda(t_\lambda) - G(u_\lambda(t_\lambda))\\ &= \frac{1}{2}u_\lambda'(t_\lambda)^2 - \lambda\cos\|u_\lambda\|_\infty - G(\|u_\lambda\|_\infty - 2k\pi)\\ &= -\lambda\cos\|u_\lambda\|_\infty - G(\|u_\lambda\|_\infty).\end{aligned} \tag{5.2}$$

Since $G(u)$ is strictly increasing for $u \ge 0$ by (A1), by (5.2), we obtain

$$\frac{1}{2}u_\lambda'(t_\lambda)^2 = G(\|u_\lambda\|_\infty - 2k\pi) - G(\|u_\lambda\|_\infty) < 0.$$

This is a contradiction. Thus $k = 0$ in (5.1) and we get our assertion.

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