Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 19, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NODAL SOLUTIONS OF FOURTH-ORDER KIRCHHOFF EQUATIONS WITH CRITICAL GROWTH IN $\mathbb{R}^N$

HONGLING PU, SHIQI LI, SIHUA LIANG, DUŠAN D. REPOVŠ

Abstract. We consider a class of fourth-order elliptic equations of Kirchhoff type with critical growth in $\mathbb{R}^N$. By using constrained minimization in the Nehari manifold, we establish sufficient conditions for the existence of nodal (that is, sign-changing) solutions.

1. Introduction

In this article we studies the existence of nodal solutions to the fourth-order elliptic equations of Kirchhoff type with critical growth in $\mathbb{R}^N$,

$$\Delta^2 u - \Big(1 + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\Big)\Delta u + V(x)u = \lambda f(u) + |u|^{2^{**}-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\Delta^2 u$ is the biharmonic operator, $2^{**} = 2N/(N-4)$ is the critical Sobolev exponent with $5 \le N < 8$, and b and λ are positive parameters. The continuous functions $V(x)$ and $f(u)$ satisfy the following conditions:

(A1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf\limits_{x\in\mathbb{R}^N} V(x) \ge V_0 > 0$, where V_0 is a positive constant. For each $M > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \le M\} < \infty$, where $\text{meas}(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^N$;
(A2) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $f(u) = o(|u|)$, as $u \to 0$;
(A3) There exists $p \in (4, 2^{**})$ such that $\lim_{u\to\infty} f(u)/u^{p-1} = 0$;
(A4) $\lim_{u\to\infty} F(u)/u^4 = +\infty$, where $F(u) = \int_0^u f(t)dt$;
(A5) $f(u)/|u|^3$ is a strictly increasing function for $u \in \mathbb{R} \setminus \{0\}$.

Problem (1.1) originates from the Kirchhoff equation

$$-\Big(a + b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u) \quad \text{in } \Omega, \tag{1.2}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain, $a > 0$, $b \ge 0$, and u satisfies certain boundary conditions. The above equation stems from a typical model proposed by Kirchhoff [11],

$$u_{tt} - \Big(a + b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u), \tag{1.3}$$

2010 *Mathematics Subject Classification.* 35A15, 35J60, 47G20.
Key words and phrases. Fourth-order elliptic equation; Kirchhoff problem; critical exponent; variational methods; nodal solution.

Submitted January 24, 2021. Published March 25, 2021.

which serves as a generalization of the classical D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=f(x,u),$$

by taking into account the effects of changes in the length of strings during vibrations. The nonlocal term thus appears. See for example [6, 24] for more background on such problems. Thanks to the pioneering work of Lions [19] on problem (1.3), a lot of attention has been drawn to these nonlocal problems during the last decade. That was followed by some interesting results on the existence of various solutions to (1.2), including positive solutions, multiple solutions, bound state solutions, multi-bump solutions, and semiclassical state solutions, both on bounded domains and on the entire space. For more results on the Kirchhoff-type equations we refer to [7, 15, 16, 13, 23, 27] and the references therein. Problem (1.2) with critical nonlinearity, however, is seldom covered, mainly because of the challenge - the lack of compactness - presented by the presence of the critical Sobolev exponent. We also refer the interested readers to [9, 18, 25, 34, 33, 35] on the fractional Kirchhoff type problems.

Recently, various approaches have been adopted for considering the fourth-order elliptic equations of the Kirchhoff type,

$$\begin{gathered}\Delta^2 u-\Big(a+b\int_\Omega|\nabla u|^2\,dx\Big)\Delta u=f(x,u),\quad x\in\Omega,\\ u=\Delta u=0,\quad x\in\partial\Omega,\end{gathered}$$

where $\Delta^2 u$ is the biharmonic operator, with different hypotheses on the nonlinearity. For instance, Ma [21] studied the existence and multiplicity of positive solutions to the fourth-order equation with the fixed point theorems in cones of ordered Banach spaces. Wang et al. [30] applied the mountain pass and the truncation methods to get the existence of nontrivial solutions to the fourth-order elliptic equations of the Kirchhoff type with one parameter λ. Liang and Zhang [17] used the variational methods to obtain the existence and multiplicity of solutions to the fourth-order elliptic equations of the Kirchhoff type with critical growth in $\mathbb{R}^N$.

The motivation for this paper comes from [26, 28, 29, 36]. In [26], the existence was proved of one least energy nodal solution u_b to problem (1.2), with its energy strictly larger than the ground state energy. Meanwhile, the asymptotic behavior of u_b, as the parameter $b\searrow 0$, was investigated as well. Later, under some more weak assumptions on f (especially, with the Nehari type monotonicity condition removed), Tang and Cheng [28] improved and generalized some results obtained in [26] with some new analytical skills and the non-Nehari manifold method. In [29], the authors obtained the existence of least energy nodal solutions to the Kirchhoff-type equation with critical growth in bounded domains by using the constraint variational method and the quantitative deformation lemma. In [36], the authors studied the fourth-order elliptic equation of the Kirchhoff-type,

$$\begin{gathered}\Delta^2 u-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,dx\Big)\Delta u+V(x)u=f(u),\quad x\in\mathbb{R}^N,\\ u\in H^2(\mathbb{R}^N),\end{gathered}$$

where $a>0$ and $b\geq 0$ are constants. By the constraint variational method and the quantitative deformation lemma, they proved that the problem possesses one least energy nodal solution. For more results on nodal solutions to the Kirchhoff-type equations, please refer to [8, 12, 20, 37] and the references therein. However, to the best of our knowledge, there are no such results concerning the existence of nodal solutions of the problem (1.1) involving critical nonlinearities in the whole space.

The purpose of this article is to study the existence, energy estimates and convergence properties of the least energy nodal solutions to the fourth-order elliptic equation (1.1). The novelty of this paper is that problem (1.1) concerns the critical case on the entire space. Based on these facts, the problem turns out to be extremely complicated and more difficult than the one without critical nonlinearities in bounded domains. Since problem (1.1) involves critical exponents in the nonlinearity, it is rather difficult to show that the energy functional reaches a lower infimum on the Nehari manifold because of the lack of compactness caused by the critical term. As we will see, this problem prevents us from using the approach in [2, 26, 28, 36]. So we need some new ideas to overcome the above difficulties. Moreover, we use the constraint variational method, the topological degree theory and the quantitative deformation lemma to prove our main results. Thus, our main results generalize papers [2, 26, 28, 36] in several directions.

Before stating our main results, we define

$$H^2(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : |\nabla u|, \Delta u \in L^2(\mathbb{R}^N)\},$$

endowed with the norm

$$\|u\|_{H^2(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} \big((\Delta u)^2 + (\nabla u)^2 + u^2\big)\, dx\Big)^{1/2}.$$

Now, we introduce the space

$$E := \big\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2\, dx < \infty\big\}$$

with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v + V(x)uv)\ dx$$

and the norm

$$\|u\| = \int_{\mathbb{R}^N} \big(|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2\big)\ dx.$$

Under condition (A1), it is known that the embedding $E \hookrightarrow H^2(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in (2, 2^{**})$ is compact, and continuous for $p \in [2, 2^{**}]$ (see [4]), and

$$S_p|u|_p \le \|u\|, \quad \text{for every } u \in E. \tag{1.4}$$

In particular, the best Sobolev constant for the embedding $E \hookrightarrow L^{2^{**}}(\mathbb{R}^N)$ is

$$S = \inf\big\{ \int_{\mathbb{R}^N} |\Delta u|^2 dx : \int_{\mathbb{R}^N} |u|^{2^{**}} dx = 1\big\}.$$

Definition 1.1. We say that $u \in E$ is a weak solution to problem (1.1), if

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\Delta u \Delta \phi + \nabla u \nabla \phi + V(x)u\phi)\ dx + b\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla \phi\, dx\\
&= \lambda \int_{\mathbb{R}^N} f(u)\phi\, dx + \int_{\mathbb{R}^N} |u|^{2^{**}-2}u\phi\, dx,
\end{aligned}$$

for every $\phi \in E$.

For convenience, we will omit the term weak throughout this paper. The corresponding energy functional $I_b^\lambda : E \to \mathbb{R}$ to problem (1.1) is defined by

$$\begin{aligned} I_b^\lambda(u) =& \frac{1}{2}\int_{\mathbb{R}^N} \big(|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2\big)\,dx + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\Big)^2 \\ & - \lambda\int_{\mathbb{R}^N} F(u)\,dx - \frac{1}{2^{**}}\int_{\mathbb{R}^N} |u|^{2^{**}}dx. \end{aligned} \tag{1.5}$$

It is easy to see that I_b^λ belongs to $C^1(E,\mathbb{R})$ and the critical points of I_b^λ are the solutions to (1.1). For every $u \in E$ we can write

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}\,.$$

Then every solution $u \in E$ to problem (1.1) with the property that $u^\pm \neq 0$ is a nodal solution to problem (1.1).

Our objective is to find the least energy nodal solutions to problem (1.1). There exist several interesting studies on the following typical semilinear equation, which is related to problem (1.1) (see [3, 4]),

$$-\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

These methods, however, depend heavily upon the decompositions:

$$J(u) = J(u^+) + J(u^-), \tag{1.7}$$

$$\langle J'(u), u^+\rangle = \langle J'(u^+), u^+\rangle \quad \text{and} \quad \langle J'(u), u^-\rangle = \langle J'(u^-), u^-\rangle, \tag{1.8}$$

where J is the energy functional of (1.6), given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx.$$

However, if $b > 0$, the energy functional I_b^λ cannot be decomposed in the same way as it is done in (1.7) and (1.8). In fact, we have

$$I_b^\lambda(u) = I_b^\lambda(u^+) + I_b^\lambda(u^-) + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u^+|^2 dx \int_{\mathbb{R}^N} |\nabla u^-|^2 dx;$$

if $u^+ \not\equiv 0$, then

$$\langle (I_b^\lambda)'(u), u^+\rangle = \langle (I_b^\lambda)'(u^+), u^+\rangle + b\int_{\mathbb{R}^N} |\nabla u^+|^2 dx \int_{\mathbb{R}^N} |\nabla u^-|^2 dx > \langle (I_b^\lambda)'(u^+), u^+\rangle;$$

if $u^- \not\equiv 0$, then

$$\langle (I_b^\lambda)'(u), u^-\rangle = \langle (I_b^\lambda)'(u^-), u^-\rangle + b\int_{\mathbb{R}^N} |\nabla u^-|^2 dx \int_{\mathbb{R}^N} |\nabla u^+|^2 dx > \langle (I_b^\lambda)'(u^-), u^-\rangle.$$

Therefore, the methods used for obtaining nodal solutions to the local problem (1.6) do not seem applicable to problem (1.1). In this paper, we follow the approach in [5] by defining the constrained set

$$\mathcal{N}_b^\lambda = \{u \in E : u^\pm \neq 0, \langle (I_b^\lambda)'(u), u^\pm\rangle = 0\} \tag{1.9}$$

and considering a minimization problem of I_b^λ on $\mathcal{N}_b^\lambda$. Shuai [26] proved that $\mathcal{N}_b^\lambda \neq \emptyset$, in the absence of the nonlocal term, by applying the parametric method and the implicit theorem. However, it is the nonlocal terms in problem (1.1), the biharmonic operator and the nonlocal term involved, that add to our difficulties. Roughly speaking, compared to the general Kirchhoff type problem (1.2), decompositions (1.7) and (1.8) corresponding to I_b^λ, are much more complicated, which accounts for some technical difficulties during the proof of the nonemptiness of $\mathcal{N}_b^\lambda$. Moreover, the parametric method and implicit theorem are not applicable to problem (1.1) because the complexity of the nonlocal problem there. Hence, inspired by [1], we follow a different path, specifically, we resort to a modified Miranda's theorem (see [22]). It is also feasible to prove that the minimizer of the constrained problem is also a nodal solution via the quantitative deformation lemma and degree theory. We can now present our first main result.

Theorem 1.2. *Assume that* (A1)—(A5) *hold. Then there exists* $\lambda^* > 0$ *such that for all* $\lambda \geq \lambda^*$, *problem* (1.1) *has a least energy nodal solution* $u_b \in \mathcal{N}_b^\lambda$ *such that* $I_b^\lambda(u_b) = \inf_{u \in \mathcal{N}_b^\lambda} I_b^\lambda(u)$.

Another goal of this paper is to establish the so-called energy doubling property (cf. [31]), i.e., the energy of any nodal solution to problem (1.1) is strictly larger than twice the ground state energy. The conclusion is trivial for the semilinear equation problem (1.6). When $b > 0$, a similar result was obtained by Shuai [26] in a bounded domain Ω. We are also interested in whether energy doubling property still holds for problem (1.1). To answer this question, we prove the following result.

Theorem 1.3. *Assume that* (A1)–(A5) *hold. Then there exists* $\lambda^{**} > 0$ *such that for all* $\lambda \geq \lambda^{**}$, $c^* := \inf_{u \in \mathcal{M}_b^\lambda} I_b^\lambda(u) > 0$ *is achieved, and* $I_b^\lambda(u) > 2c^*$, *where* $\mathcal{M}_b^\lambda = \{u \in E \setminus \{0\} : \langle (I_b^\lambda)'(u), u\rangle = 0\}$ *and* u *is the least energy nodal solution obtained in Theorem 1.2. In particular,* $c^* > 0$ *is achieved either by a positive or a negative function.*

It is obvious that the energy of the nodal solution u_b obtained in Theorem 1.2 depends on b. Next, we establish a convergence property of u_b as $b \to 0$, which demonstrates a relationship between $b > 0$ and $b = 0$ for problem (1.1).

Theorem 1.4. *Assume that* (A1)–(A5) *hold. Then for any sequence* $\{b_n\}$ *with* $b_n \to 0$ *as* $n \to \infty$, *there exists a subsequence, still denoted by* $\{b_n\}$, *such that* $\{u_n\}$ *strongly converges to* u_0 *in* E *as* $n \to \infty$, *where* u_0 *is a least energy nodal solution to the problem*

$$\Delta^2 u - \Delta u + V(x) = \lambda f(u) + |u|^{2^{**}-2}u \quad \text{in } \mathbb{R}^N. \tag{1.10}$$

The structure of this article is as follows: Section 2 contains the proof of the achieving the least energy for the constraint problem (1.1). While section 3 is devoted to the proofs of our main theorems.

Throughout this paper, we use standard notation. For simplicity, we use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space, respectively. By C and C_i we denote various positive constants, and by ":=" definitions. To simplify the notation, we denote a subsequence of a sequence $\{u_n\}_n$ also as $\{u_n\}_n$, unless otherwise specified.

2. Some technical lemmas

To begin, fix $u \in E$ with $u^\pm \neq 0$. Consider the function $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and the mapping $W : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^2$, where

$$\varphi(\alpha, \beta) = I_b^\lambda(\alpha u^+ + \beta u^-), \tag{2.1}$$

$$W(\alpha, \beta) = \big(\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+\rangle, \langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^-\rangle\big). \tag{2.2}$$

For brevity, we define the quantities

$$A^+(u)=\int_{\mathbb{R}^N}|\nabla u^+|^2\,dx,\quad A^-(u)=\int_{\mathbb{R}^N}|\nabla u^-|^2\,dx,\quad B(u)=\int_{\mathbb{R}^N}\Delta u^+\Delta u^-\,dx.$$

Lemma 2.1. *Assume that* (A1)–(A5) *hold. Then for any* $u\in E$ *with* $u^\pm\neq 0$, *there is the unique maximum point pair of positive numbers* (α_u,β_u) *such that* $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$.

Proof. Our proof consists in verifying three claims.

Claim 1. There exists a pair of positive numbers (α_u,β_u) such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$, for any $u\in E$ with $u^\pm\neq 0$. Note that

$$\begin{aligned}
&\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-),\alpha u^+\rangle\\
&=\int_{\mathbb{R}^N}\Delta(\alpha u^+ + \beta u^-)\Delta\alpha u^+dx+\int_{\mathbb{R}^N}|\nabla\alpha u^+|^2\,dx+\int_{\mathbb{R}^N}V(x)|\alpha u^+|^2dx\\
&\quad+b\int_{\mathbb{R}^N}|\nabla(\alpha u^+ + \beta u^-)|^2dx\int_{\mathbb{R}^N}|\nabla\alpha u^+|^2dx\\
&\quad-\lambda\int_{\mathbb{R}^N}f(\alpha u^+)\alpha u^+dx-\int_{\mathbb{R}^N}|\alpha u^+|^{2^{**}}dx
\end{aligned}$$

and

$$\begin{aligned}
&\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-),\beta u^-\rangle\\
&=\int_{\mathbb{R}^N}\Delta(\alpha u^+ + \beta u^-)\Delta\beta u^-dx+\int_{\mathbb{R}^N}|\nabla\beta u^-|^2\,dx+\int_{\mathbb{R}^N}V(x)|\beta u^-|^2dx\\
&\quad+b\int_{\mathbb{R}^N}|\nabla(\alpha u^+ + \beta u^-)|^2dx\int_{\mathbb{R}^N}|\nabla\beta u^-|^2dx\\
&\quad-\lambda\int_{\mathbb{R}^N}f(\beta u^-)\beta u^-dx-\int_{\mathbb{R}^N}|\beta u^-|^{2^{**}}dx.
\end{aligned}$$

By a direct computation we obtain that

$$\begin{aligned}
&\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-),\alpha u^+\rangle\\
&=\alpha^2\|u^+\|^2+\alpha^2\beta^2 bA^+(u)A^-(u)+\alpha^4 b\left(A^+(u)\right)^2\\
&\quad+\alpha\beta B(u)-\lambda\int_{\mathbb{R}^N}f(\alpha u^+)\alpha u^+dx-\int_{\mathbb{R}^N}|\alpha u^+|^{2^{**}}dx
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}
&\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-),\beta u^-\rangle\\
&=\beta^2\|u^-\|^2+\alpha^2\beta^2 bA^+(u)A^-(u)+\beta^4 b\left(A^-(u)\right)^2\\
&\quad+\alpha\beta B(u)-\lambda\int_{\mathbb{R}^N}f(\beta u^-)\beta u^-dx-\int_{\mathbb{R}^N}|\beta u^-|^{2^{**}}dx.
\end{aligned}\tag{2.4}$$

By assumptions (A2) and (A3), we have

$$\int_{\mathbb{R}^N}f(\alpha u^+)\alpha u^+dx\le\varepsilon\int_{\mathbb{R}^N}|\alpha u^+|^2\,dx+C_\varepsilon\int_{\mathbb{R}^N}|\alpha u^+|^p\,dx.\tag{2.5}$$

Choose $\varepsilon>0$ small enough such that $(1-\lambda\varepsilon C_\varepsilon)>0$, which together with (2.5) and (2.3), yields

$$\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-),\alpha u^+\rangle\ge(1-\lambda\varepsilon C_\varepsilon)\alpha^2\|u^+\|^2+\alpha^2\beta^2 bA^+(u)A^-(u)$$

$$+ \alpha^4 b \left(A^+(u)\right)^2 - \lambda C_\varepsilon \int_{\mathbb{R}^N} |\alpha u^+|^p \, dx - \int_{\mathbb{R}^N} |\alpha u^+|^{2^{**}} dx.$$

Since $2^{**} > 4$, we have $\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+\rangle > 0$ for a small enough α and for all $\beta \geq 0$.

Similarly, according to (2.5) and (2.4), we get $\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^-\rangle > 0$, for small enough β and all $\alpha \geq 0$. Hence, there exists $r > 0$ such that

$$\langle (I_b^\lambda)'(ru^+ + \beta u^-), ru^+\rangle > 0 \quad \text{and} \quad \langle (I_b^\lambda)'(\alpha u^+ + ru^-), ru^-\rangle > 0, \tag{2.6}$$

for all $\alpha, \beta \geq 0$.

On the other hand, by (A3) and (A4), we have

$$f(t)t > 0, t \neq 0; \quad F(t) \geq 0, \quad t \in \mathbb{R}. \tag{2.7}$$

Now, choose $R > r$. For sufficiently large R, and by (2.3), (2.4), (2.7), we have

$$\langle (I_b^\lambda)'(Ru^+ + \beta u^-), Ru^+\rangle < 0 \quad \text{and} \quad \langle (I_b^\lambda)'(\alpha u^+ + Ru^-), Ru^-\rangle < 0, \tag{2.8}$$

for all $\alpha, \beta \in [r, R]$. Invoking Miranda's theorem [22], together with (2.6) and (2.8), we can conclude that there exists $(\alpha_u, \beta_u) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $W(\alpha_u, \beta_u) = (0, 0)$, i.e., $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$.

Claim 2. The pair (α_u, β_u) is unique.

• Case $u \in \mathcal{N}_b^\lambda$. Then we have

$$\langle (I_b^\lambda)'(u), u^+\rangle = 0 \quad \text{and} \quad \langle (I_b^\lambda)'(u), u^-\rangle = 0,$$

that is,

$$\begin{aligned} &\|u^+\|^2 + B(u) + bA^+(u)\big(A^+(u) + A^-(u)\big) \\ &= \lambda \int_{\mathbb{R}^N} f(u^+)u^+ dx + \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\|u^-\|^2 + B(u) + bA^-(u)\big(A^+(u) + A^-(u)\big) \\ &= \lambda \int_{\mathbb{R}^N} f(u^-)u^- dx + \int_{\mathbb{R}^N} |u^-|^{2^{**}} dx. \end{aligned} \tag{2.10}$$

By Claim 1, we know that there exists at least one positive pair (α_0, β_0) satisfying $\alpha_0 u^+ + \beta_0 u^- \in \mathcal{N}_b^\lambda$.

Next we show that $(\alpha_0, \beta_0) = (1, 1)$ is the unique pair of numbers. Without loss of generality, let us assume that $\alpha_0 \leq \beta_0$. It follows from (2.8) that

$$\begin{aligned} &\alpha_0^2 \left(\|u^+\|^2 + B(x)\right) + \alpha_0^4 bA^+(u) \left(A^+(u) + A^-(u)\right) \\ &= \lambda \int_{\mathbb{R}^N} f(\alpha_0 u^+)\alpha_0 u^+ \, dx + \int_{\mathbb{R}^N} |\alpha_0 u^+|^{2^{**}} dx. \end{aligned} \tag{2.11}$$

If $\alpha_0 < 1$, then from (2.9), (2.11) and (A5), we have

$$\begin{aligned} 0 &< [(\alpha_0)^{-2} - 1] \left(\|u^+\|^2 + B(u)\right) \\ &\leq \lambda \int_{\mathbb{R}^N} \Big(\frac{f(x, \alpha_0 u^+)}{(\alpha_0 u^+)^3} - \frac{f(u^+)}{(u^+)^3}\Big)(u^+)^4 \, dx \\ &\quad + [(\alpha_0)^{2^{**}-4} - 1] \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx < 0, \end{aligned} \tag{2.12}$$

which is a contradiction. Hence, $1 \leq \alpha_0 \leq \beta_0$.

Adopting a similar approach, we can see that $\beta_0 \leq 1$, which implies that $\alpha_0 = \beta_0 = 1$.

• Case $u \notin \mathcal{N}_b^\lambda$. Assume there exist two other pairs of positive numbers (α_1, β_1) and (α_2, β_2) such that

$$\sigma_1 = \alpha_1 u^+ + \beta_1 u^- \in \mathcal{N}_b^\lambda \quad \text{and} \quad \sigma_2 = \alpha_2 u^+ + \beta_2 u^- \in \mathcal{N}_b^\lambda.$$

Then

$$\sigma_2 = \big(\frac{\alpha_2}{\alpha_1}\big)\alpha_1 u^+ + \big(\frac{\beta_2}{\beta_1}\big)\beta_1 u^- = \big(\frac{\alpha_2}{\alpha_1}\big)\sigma_1^+ + \big(\frac{\beta_2}{\beta_1}\big)\sigma_1^- \in \mathcal{N}_b^\lambda.$$

Since $\sigma_1 \in \mathcal{N}_b^\lambda$, it is clear that

$$\frac{\alpha_2}{\alpha_1} = \frac{\beta_2}{\beta_1} = 1,$$

which means that $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$.

Claim 3. The pair (α_u, β_u) is the unique maximum point of the function φ on $\mathbb{R}_+ \times \mathbb{R}_+$. We know from the above that (α_u, β_u) is the unique critical point of φ on $\mathbb{R}_+ \times \mathbb{R}_+$. By definition and (2.5), we have

$$\begin{aligned}
\varphi(\alpha,\beta) &= I_b^\lambda(\alpha u^+ + \beta u^-) \\
&= \frac{\alpha^2}{2}\|u^+\|^2 + \frac{\beta^2}{2}\|u^-\|^2 + \alpha\beta B(u) + \frac{\alpha^4 b}{4}\left(A^+(u)\right)^2 + \frac{\beta^4 b}{4}\left(A^-(u)\right)^2 \\
&\quad + \frac{\alpha^2\beta^2 b}{2} A^+(u)A^-(u) - \lambda\int_{\mathbb{R}^N} F(\alpha u^+)\,dx - \lambda\int_{\mathbb{R}^N} F(\beta u^-)\,dx \\
&\quad - \frac{\alpha^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N} |u^+|^{2^{**}} dx - \frac{\beta^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N} |u^-|^{2^{**}} dx \\
&< \frac{\alpha^2}{2}\|u^+\|^2 + \frac{\beta^2}{2}\|u^-\|^2 + \alpha\beta B(u) + \frac{\alpha^4 b}{4}\left(A^+(u)\right)^2 + \frac{\beta^4 b}{4}\left(A^-(u)\right)^2 \\
&\quad + \frac{\alpha^2\beta^2 b}{2} A^+(u)A^-(u) - \frac{\alpha^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N} |u^+|^{2^{**}} dx - \frac{\beta^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N} |u^-|^{2^{**}} dx,
\end{aligned}$$

as $|(\alpha,\beta)| \to \infty$. This implies that $\lim_{|(\alpha,\beta)|\to\infty} \varphi(\alpha,\beta) = -\infty$, because $2^{**} > 4$. Hence, it suffices to show that the maximum point cannot be achieved on the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$.

We carry out the proof by contradiction. Assuming $(0, \bar\beta)$ is the global maximum point of φ with $\bar\beta \geq 0$, we have

$$\begin{aligned}
\varphi(\alpha,\bar\beta) &= \frac{\alpha^2}{2}\|u^+\|^2 + \frac{\bar\beta^2}{2}\|u^-\|^2 + \alpha\bar\beta B(u) + \frac{\alpha^4 b}{4}\left(A^+(u)\right)^2 + \frac{\bar\beta^4 b}{4}\left(A^-(u)\right)^2 \\
&\quad + \frac{\alpha^2\bar\beta^2 b}{2} A^+(u)A^-(u) - \lambda\int_{\mathbb{R}^N} F(\alpha u^+)\,dx - \lambda\int_{\mathbb{R}^N} F(\bar\beta u^-)\,dx \\
&\quad - \frac{\alpha^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N} |u^+|^{2^{**}} dx - \frac{\bar\beta^{2^{**}}}{2^{**}}\int_{\mathbb{R}^N} |u^-|^{2^{**}} dx.
\end{aligned}$$

Hence, it is clear that

$$\begin{aligned}
\varphi'_\alpha(\alpha,\bar\beta) &= \alpha\|u^+\|^2 + \bar\beta B(u) + \alpha^3 b\left(A^+(u)\right)^2 + \alpha\bar\beta^2 b A^+(u)A^-(u) \\
&\quad - \lambda\int_{\mathbb{R}^N} f(\alpha u^+)u^+ dx - \alpha^{2^{**}-1}\int_{\mathbb{R}^N} |u^+|^{2^{**}} dx > 0,
\end{aligned}$$

for small enough α. This means that φ is an increasing function with respect to α if α is small enough, which is a contradiction. In a similar way, we can deduce that φ cannot achieve its global maximum at $(\alpha, 0)$ with $\alpha \geq 0$. Thus, we have completed the proof. $\square$

Lemma 2.2. *Assume that* (A1) —(A5) *hold. Then for any $u \in E$ with $u^{\pm} \neq 0$ such that $\langle (I_b^{\lambda})'(u), u^{\pm} \rangle \leq 0$, the unique maximum point pair of φ on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfies $0 < \alpha_u, \beta_u \leq 1$.*

Proof. Without loss of generality, we may assume that $\alpha_u \geq \beta_u > 0$. Since $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^{\lambda}$, we have

$$\begin{aligned} &\alpha_u^2 \|u^+\|^2 + \alpha_u \beta_u B(u) + \alpha_u^2 \beta_u^2 b A^+(u) A^-(u) + \alpha_u^4 b \left(A^+(u)\right)^2 \\ &= \lambda \int_{\mathbb{R}^N} f(\alpha_u u^+) \alpha_u u^+ dx + \int_{\mathbb{R}^N} |\alpha_u u^+|^{2^{**}} dx. \end{aligned} \tag{2.13}$$

Furthermore, since $\langle (I_b^{\lambda})'(u), u^+ \rangle \leq 0$, we have

$$\|u^+\|^2 + B(u) + b\left(A^+(u)\right)^2 + bA^+(u)A^-(u) \leq \lambda \int_{\mathbb{R}^N} f(u^+)u^+ dx + \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx.$$

Then by (2.13), we have

$$\begin{aligned} &[(\alpha_u)^{-2} - 1] \left(\|u^+\|^2 + B(u)\right) \\ &\geq \lambda \int_{\mathbb{R}^N} \Big(\frac{f(\alpha_u u^+)}{(\alpha_u u^+)^3} - \frac{f(u^+)}{(u^+)^3} \Big) (u^+)^4 \, dx + [(\alpha_u)^{2^{**}-4} - 1] \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx. \end{aligned} \tag{2.14}$$

Obviously, the left hand side of (2.14) is negative for $\alpha_u > 1$ whereas the right hand side is positive, which is a contradiction. Therefore $0 < \alpha_u, \beta_u \leq 1$. $\square$

Lemma 2.3. *Suppose that $c_b^{\lambda} = \inf_{u \in \mathcal{N}_b^{\lambda}} I_b^{\lambda}(u)$. Then $\lim_{\lambda \to \infty} c_b^{\lambda} = 0$.*

Proof. For every $u \in \mathcal{N}_b^{\lambda}$, we have $\langle (I_b^{\lambda})'(u), u \rangle = 0$, thus

$$\|u^+\|^2 + \|u^-\|^2 + 2B(u) + b\left(A^+(u) + A^-(u)\right)^2 = \lambda \int_{\mathbb{R}^N} f(u)u \, dx + \int_{\mathbb{R}^N} |u|^{2^{**}} dx.$$

Then, by (2.5), we have

$$\begin{aligned} \|u\|^2 &\leq \lambda \int_{\mathbb{R}^N} f(u^{\pm})u^{\pm} \, dx + \int_{\mathbb{R}^N} |u^{\pm}|^{2^{**}} dx \\ &\leq \lambda\varepsilon \int_{\mathbb{R}^N} |u^{\pm}|^2 dx + \lambda C_{\varepsilon} \int_{\mathbb{R}^N} |u^{\pm}|^p \, dx + \int_{\mathbb{R}^N} |u^{\pm}|^{2^{**}} dx. \end{aligned} \tag{2.15}$$

Choose ε small so that $\lambda\varepsilon \int_{\mathbb{R}^N} |u^{\pm}|^2 dx \leq \frac{1}{2}\|u^{\pm}\|^2$. Then we can claim that there exists $\rho > 0$ such that

$$\|u^{\pm}\|^2 \geq \rho \quad \text{for all } u \in \mathcal{N}_b^{\lambda}, \tag{2.16}$$

since $4 < 2^{**}$. Next, by (A5), we have for $t \neq 0$ that

$$\mathcal{F}(t) := tf(t) - 4F(t) \geq 0,$$

and $\mathcal{F}(t)$ is increasing when $t > 0$, and decreasing when $t < 0$. Therefore,

$$\begin{aligned} I_b^{\lambda}(u) &= I_b^{\lambda}(u) - \frac{1}{4}\langle (I_b^{\lambda})'(u), u \rangle \\ &= \frac{1}{4}\|u\|^2 + (\frac{1}{4} - \frac{1}{2^{**}}) \int_{\mathbb{R}^N} |u|^{2^{**}} dx + \frac{\lambda}{4} \int_{\mathbb{R}^N} [f(u)u - 4F(u)] \, dx \\ &\geq \frac{1}{4}\|u\|^2 \geq \frac{\rho}{4} > 0. \end{aligned} \tag{2.17}$$

So we have $I_b^\lambda(u) > 0$ for all $u \in \mathcal{N}_b^\lambda$, which means that $c_b^\lambda = \inf_{u\in\mathcal{N}_b^\lambda} I_b^\lambda(u)$ is well-defined.

Fix $u \in E$ with $u^\pm \neq 0$. According to Lemma 2.1, for each $\lambda > 0$, there exist $\alpha_\lambda, \beta_\lambda > 0$ such that $\alpha_\lambda u^+ + \beta_\lambda u^- \in \mathcal{N}_b^\lambda$. Therefore,

$$
\begin{aligned}
0 \le c_b^\lambda &= \inf_{u\in\mathcal{N}_b^\lambda} I_b^\lambda(u) \\
&\le I_b^\lambda(\alpha_\lambda u^+ + \beta_\lambda u^-) \\
&\le \frac{1}{2}\|\alpha_\lambda u^+ + \beta_\lambda u^-\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla(\alpha_\lambda u^+ + \beta_\lambda u^-)|^2 dx\Big)^2 \\
&= \frac{\alpha_\lambda^2}{2}\|u^+\|^2 + \frac{\beta_\lambda^2}{2}\|u^-\|^2 + \alpha_\lambda\beta_\lambda B(u) + \frac{\alpha_\lambda^4 b}{4}\left(A^+(u)\right)^2 \\
&\quad + \frac{\beta_\lambda^4 b}{4}\left(A^-(u)\right)^2 + \frac{\alpha_\lambda^2\beta_\lambda^2 b}{2} A^+(u)A^-(u).
\end{aligned}
$$

It suffices to prove that $\alpha_\lambda \to 0$ and $\beta_\lambda \to 0$, as $\lambda \to \infty$. Let

$$
\mathcal{T} = \{(\alpha_\lambda, \beta_\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ : W(\alpha_\lambda, \beta_\lambda) = (0,0), \lambda > 0\},
$$

where W is defined as in (2.2). Then

$$
\begin{aligned}
&\alpha_\lambda^{2^{**}} \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx + \beta_\lambda^{2^{**}} \int_{\mathbb{R}^N} |u^-|^{2^{**}} dx \\
&\le \alpha_\lambda^{2^{**}} \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx + \beta_\lambda^{2^{**}} \int_{\mathbb{R}^N} |u^-|^{2^{**}} dx \\
&\quad + \lambda \int_{\mathbb{R}^N} f(\alpha_\lambda u^+)\alpha_\lambda u^+ dx + \lambda \int_{\mathbb{R}^N} f(\beta_\lambda u^-)\beta_\lambda u^- dx \\
&= \|\alpha_\lambda u^+ + \beta_\lambda u^-\|^2 + b\left(\alpha_\lambda^2 A^+(u) + \beta_\lambda^2 A^-(u)\right)^2.
\end{aligned}
$$

Therefore, $\mathcal{T}$ is bounded, since $4 < 2^{**}$. Let $\{\lambda_n\} \subset (0,\infty)$ be such that $\lambda_n \to \infty$, as $n \to \infty$. Then there exist α_0 and β_0 such that $(\alpha_{\lambda_n}, \beta_{\lambda_n}) \to (\alpha_0, \beta_0)$, as $n \to \infty$.

Now we claim that $\alpha_0 = \beta_0 = 0$. Assume, to the contrary, that $\alpha_0 > 0$ or $\beta_0 > 0$. Since $\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^- \in \mathcal{N}_b^{\lambda_n}$, then for any $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|^2 + b\left(\alpha_{\lambda_n}^2 A^+(u) + \beta_{\lambda_n}^2 A^-(u)\right)^2 \\
&= \lambda_n \int_{\mathbb{R}^N} f(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)\, dx \\
&\quad + \int_{\mathbb{R}^N} |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^{2^{**}} dx.
\end{aligned} \tag{2.18}
$$

Then, invoking $\alpha_{\lambda_n} u^+ \to \alpha_0 u^+, \beta_{\lambda_n} u^- \to \beta_0 u^-$ in E and the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)\, dx \\
&\to \int_{\mathbb{R}^N} f(\alpha_0 u^+ + \beta_0 u^-)(\alpha_0 u^+ + \beta_0 u^-)\, dx > 0,
\end{aligned}
$$

as $n \to \infty$. This contradicts (2.18), given that $\lambda_n \to \infty$, as $n \to \infty$ and that $\{\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\}$ is bounded in E. Therefore, $\alpha_0 = \beta_0 = 0$, which implies $\lim_{\lambda\to\infty} c_b^\lambda = 0$. $\square$

Lemma 2.4. *There exists $\lambda^* > 0$ such that the infimum c_b^λ is achieved for all $\lambda \geq \lambda^*$.*

Proof. According to the definition of c_b^λ, there exists a sequence $\{u_n\} \subset \mathcal{N}_b^\lambda$ such that $\lim_{n\to\infty} I_b^\lambda(u_n) = c_b^\lambda$. Clearly, $\{u_n\}$ is bounded in E. By Lemma 2.1 and the properties of L^p space, up to a subsequence, we have

$$\begin{gathered}
u_n^\pm \rightharpoonup u^\pm \quad \text{in } E, \\
u_n^\pm \to u^\pm \quad \text{in } L^p(\mathbb{R}^N) \text{ for } p \in [2, 2^{**}), \\
u_n^\pm \to u^\pm \quad \text{a.e. in } \mathbb{R}^N.
\end{gathered}$$

In view of Lemma 2.1, we also have

$$I_b^\lambda(\alpha u_n^+ + \beta u_n^-) \leq I_b^\lambda(u_n),$$

for all $\alpha, \beta \geq 0$. So, by the Brézis-Lieb lemma, Fatou's lemma and the weak lower semicontinuity of norm, we can conclude that

$$\begin{aligned}
&\liminf_{n\to\infty} I_b^\lambda(\alpha u_n^+ + \beta u_n^-) \\
&\geq \frac{\alpha^2}{2} \lim_{n\to\infty} (\|u_n^+ - u^+\|^2 + \|u^+\|^2) + \frac{\beta^2}{2} \lim_{n\to\infty} (\|u_n^- - u^-\|^2 + \|u^-\|^2) \\
&\quad + \frac{\alpha^4 b}{4} \Big[\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n^+ - \nabla u^+|^2 \,dx + \int_{\mathbb{R}^N} |\nabla u^+|^2 \,dx \Big]^2 \\
&\quad + \frac{\beta^4 b}{4} \Big[\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n^- - \nabla u^-|^2 \,dx + \int_{\mathbb{R}^N} |\nabla u^-|^2 \,dx \Big]^2 \\
&\quad - \frac{\alpha^{2^{**}}}{2^{**}} \Big[\lim_{u} ndersetn \to \infty \int_{\mathbb{R}^N} |u_n^+ - u^+|^{2^{**}} dx + \lim_{n\to\infty} \int_{\mathbb{R}^N} |u^+|^{2^{**}} dx \Big] \\
&\quad - \frac{\beta^{2^{**}}}{2^{**}} \Big[\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n^- - u^-|^{2^{**}} dx + \lim_{n\to\infty} \int_{\mathbb{R}^N} |u^-|^{2^{**}} dx \Big] \\
&\quad - \lambda \int_{\mathbb{R}^N} F(\alpha u^+) \,dx - \lambda \int_{\mathbb{R}^N} F(\beta u^-) \,dx \\
&\quad + \frac{\alpha^2 \beta^2 b}{2} \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n^+|^2 dx \int_{\mathbb{R}^N} |\nabla u_n^-|^2 dx \\
&\geq I_b^\lambda(\alpha u^+ + \beta u^-) + \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3^2 + \frac{\alpha^4 b}{2} A_3 A^+(u) - \frac{\alpha^{2^{**}}}{2^{**}} B_1 \\
&\quad + \frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^2 + \frac{\beta^4 b}{2} A_4 A^-(u) - \frac{\beta^{2^{**}}}{2^{**}} B_2,
\end{aligned}$$

where

$$\begin{gathered}
A_1 = \lim_{n\to\infty} \|u_n^+ - u^+\|^2, \quad A_2 = \lim_{n\to\infty} \|u_n^- - u^-\|^2, \\
A_3 = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n^+ - \nabla u^+|^2 dx, \quad A_4 = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n^- - \nabla u^-|^2 dx, \\
B_1 = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n^+ - u^+|^{2^{**}} dx, \quad B_2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n^- - u^-|^{2^{**}} dx.
\end{gathered}$$

That is,

$$\begin{aligned} c_b^\lambda \geq I_b^\lambda(\alpha u^+ + \beta u^-) &+ \frac{\alpha^2}{2}A_1 + \frac{\alpha^4 b}{4}A_3^2 + \frac{\alpha^4 b}{2}A_3 A^+(u) - \frac{\alpha^{2^{**}}}{2^{**}}B_1 \\ &+ \frac{\beta^2}{2}A_2 + \frac{\beta^4 b}{4}A_4^2 + \frac{\beta^4 b}{2}A_4 A^-(u) - \frac{\beta^{2^{**}}}{2^{**}}B_2, \end{aligned} \tag{2.19}$$

for all $\alpha, \beta \geq 0$.

Step 1: $u^\pm \neq 0$. We carry out our proof by contradiction. Assume that $u^+ = 0$. Lettong $\beta = 0$ in (2.19) we have

$$c_b^\lambda \geq \frac{\alpha^2}{2}A_1 + \frac{\alpha^4 b}{4}A_3^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1 := \phi(\alpha), \tag{2.20}$$

for all $\alpha \geq 0$.

Case 1: $B_1 = 0$. If $A_1 = 0$, then $u_n^+ \to u^+$ in E. By (2.15), we obtain $\|u^\pm\| > 0$, which contradicts our assumption. If $A_1 > 0$, then by (2.20), we have $c_b^\lambda \geq \frac{\alpha^2}{2}A_1$ for all $\alpha \geq 0$, which contradicts Lemma 2.3.

Case 2: $B_1 > 0$. From the definition of S and Lemma 2.3, there exists $\lambda^* > 0$ such that

$$c_b^\lambda < \frac{2}{N}S^{-2/N} \tag{2.21}$$

for all $\lambda \geq \lambda^*$. According to the Sobolev embedding and the fact that $B_1 > 0$, we obtain $A_1 > 0$. By (2.20), we have

$$\begin{aligned} \frac{2}{N}S^{-2/N} &\leq \frac{2}{N}\Big[\frac{A_1^{\frac{2^{**}}{2}}}{B_1}\Big]^{\frac{2}{2^{**}-2}} \\ &\leq \max_{\alpha \geq 0}\big\{\frac{\alpha^2}{2}A_1 - \frac{\alpha^{2^{**}}}{2^{**}}B_1\big\} \\ &\leq \max_{\alpha \geq 0}\big\{\frac{\alpha^2}{2}A_1 + \frac{\alpha^4 b}{4}A_3^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1\big\} \leq c_b^\lambda, \end{aligned}$$

which is a contradiction. Hence, we can conclude that $u^+ \neq 0$. Similarly, we get that $u^- \neq 0$.

Step 2: $B_1 = B_2 = 0$. Given that the proof of $B_2 = 0$ is analogous, we just prove $B_1 = 0$. By contradiction, assume $B_1 > 0$.

Case 1: $B_2 > 0$. Since $B_1, B_2 > 0$, we get $A_1, A_2 > 0$. Clearly, $\phi(\alpha) > 0$ for α small enough, where $\phi(\alpha)$ is given by (2.20), and $\phi(\alpha) < 0$ for α sufficiently large. Therefore, by continuity of $\phi(\alpha)$, there exists $\bar{\alpha} > 0$ such that

$$\frac{\bar{\alpha}^2}{2}A_1 + \frac{\bar{\alpha}^4 b}{4}A_3^2 - \frac{\bar{\alpha}^{2^{**}}}{2^{**}}B_1 = \max_{\alpha \geq 0}\big\{\frac{\alpha^2}{2}A_1 + \frac{\alpha^4 b}{4}A_3^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1\big\}.$$

Similarly, there exists $\bar{\beta} > 0$ such that

$$\frac{\bar{\beta}^2}{2}A_2 + \frac{\bar{\beta}^4 b}{4}A_4^2 - \frac{\bar{\beta}^{2^{**}}}{2^{**}}B_2 = \max_{\beta \geq 0}\big\{\frac{\beta^2}{2}A_2 + \frac{\beta^4 b}{4}A_4^2 - \frac{\beta^{2^{**}}}{2^{**}}B_2\big\}.$$

In view of the compactness of $[0, \bar{\alpha}] \times [0, \bar{\beta}]$ and the continuity of ϕ, there exists $(\alpha_u, \beta_u) \in [0, \bar{\alpha}] \times [0, \bar{\beta}]$ such that

$$\varphi(\alpha_u, \beta_u) = \max_{(\alpha,\beta) \in [0,\bar{\alpha}] \times [0,\bar{\beta}]} \varphi(\alpha, \beta),$$

where φ is defined as in Lemma 2.1.

Now we prove that $(\alpha_u,\beta_u)\in(0,\bar\alpha)\times(0,\bar\beta)$. Note that if β is small enough, then we have

$$\varphi(\alpha,0)=I_b^\lambda(\alpha u^+)<I_b^\lambda(\alpha u^+)+I_b^\lambda(\beta u^-)\le I_b^\lambda(\alpha u^++\beta u^-)=\varphi(\alpha,\beta),$$

for all $\alpha\in[0,\bar\alpha]$. Thus, there exists $\beta_0\in[0,\bar\beta]$ such that $\varphi(\alpha,0)\le\varphi(\alpha,\beta_0)$, for all $\alpha\in[0,\bar\alpha]$. That is, $(\alpha_u,\beta_u)\notin[0,\bar\alpha]\times\{0\}$. With a similar method, we can show that $(\alpha_u,\beta_u)\notin\{0\}\times[0,\bar\beta]$.

It is obvious that

$$\frac{\alpha^2}{2}A_1+\frac{\alpha^4 b}{4}A_3^2+\frac{\alpha^4 b}{2}A_3A^+(u)-\frac{\alpha^{2^{**}}}{2^{**}}B_1>0,\quad \alpha\in(0,\bar\alpha] \tag{2.22}$$

and

$$\frac{\beta^2}{2}A_2+\frac{\beta^4 b}{4}A_4^2+\frac{\beta^4 b}{2}A_4A^-(u)-\frac{\beta^{2^{**}}}{2^{**}}B_2>0,\quad \beta\in(0,\bar\beta]. \tag{2.23}$$

Thus we obtain

$$\begin{aligned}\frac{2}{N}S^{-2/N}\le{}&\frac{\bar\alpha^2}{2}A_1+\frac{\bar\alpha^4 b}{4}A_3^2-\frac{\bar\alpha^{2^{**}}}{2^{**}}B_1+\frac{\bar\alpha^4 b}{2}A_3A^+(u)\\&+\frac{\beta^2}{2}A_2+\frac{\beta^4 b}{4}A_4^2+\frac{\beta^4 b}{2}A_4A^-(u)-\frac{\beta^{2^{**}}}{2^{**}}B_2\end{aligned}$$

and

$$\begin{aligned}\frac{2}{N}S^{-2/N}\le{}&\frac{\bar\beta^2}{2}A_2+\frac{\bar\beta^4 b}{4}A_4^2-\frac{\bar\beta^{2^{**}}}{2^{**}}B_2+\frac{\bar\beta^4 b}{2}A_4A^-(u)\\&+\frac{\alpha^2}{2}A_1+\frac{\alpha^4 b}{4}A_3^2+\frac{\alpha^4 b}{2}A_3A^+(u)-\frac{\alpha^{2^{**}}}{2^{**}}B_1,\end{aligned}$$

for all $\alpha\in[0,\bar\alpha]$, $\beta\in[0,\bar\beta]$.

From the these inequalities and (2.19), we obtain $\varphi(\bar\alpha,\beta)\le0$, $\varphi(\alpha,\bar\beta)\le0$ for all $\alpha\in[0,\bar\alpha]$, $\beta\in[0,\bar\beta]$. Therefore, $(\alpha_u,\beta_u)\notin\{\bar\alpha\}\times[0,\bar\beta]$ and $(\alpha_u,\beta_u)\notin[0,\bar\alpha]\times\{\bar\beta\}$, which means $(\alpha_u,\beta_u)\in(0,\bar\alpha)\times(0,\bar\beta)$. It follows that (α_u,β_u) is a critical point of φ.

So, $\alpha_uu^++\beta_uu^-\in\mathcal{N}_b^\lambda$. By (2.19), we have

$$\begin{aligned}c_b^\lambda\ge{}&I_b^\lambda(\alpha_uu^++\beta_uu^-)+\frac{\alpha_u^2}{2}A_1+\frac{\alpha_u^4 b}{4}A_3^2+\frac{\alpha_u^4 b}{2}A_3A^+(u)-\frac{\alpha_u^{2^{**}}}{2^{**}}B_1\\&+\frac{\beta_u^2}{2}A_2+\frac{\beta_u^4 b}{4}A_4^2+\frac{\beta_u^4 b}{2}A_4A^-(u)-\frac{\beta_u^{2^{**}}}{2^{**}}B_2>I_b^\lambda(\alpha_uu^++\beta_uu^-)\ge c_b^\lambda,\end{aligned}$$

which is a contradiction. Therefore $B_1=0$.

Case 2: $B_2=0$**.** In this case, we can maximize in $[0,\bar\alpha]\times[0,\infty)$. It is possible to show that there exists $\beta_0\in[0,\infty)$ satisfying

$$I_b^\lambda(\alpha_uu^++\beta_uu^-)\le0\quad\text{for all }(\alpha,\beta)\in[0,\bar\alpha]\times[\beta_0,\infty).$$

Then there is $(\alpha_u,\beta_u)\in[0,\bar\alpha]\times[0,\infty)$ such that

$$\varphi(\alpha_u,\beta_u)=\max_{(\alpha,\beta)\in[0,\bar\alpha]\times[0,\infty)}\varphi(\alpha,\beta).$$

We claim that $(\alpha_u,\beta_u)\in(0,\bar\alpha)\times(0,\infty)$. Indeed, $\varphi(\alpha,0)<\varphi(\alpha,\beta)$ for $\alpha\in[0,\bar\alpha]$ and β small enough, while $\varphi(0,\beta)<\varphi(\alpha,\beta)$ for $\beta\in[0,\infty)$ and α sufficiently small, which implies $(\alpha_u,\beta_u)\notin[0,\bar\alpha]\times\{0\}$ and $(\alpha_u,\beta_u)\notin\{0\}\times[0,\infty)$.

Note that

$$\frac{2}{N}S^{-2/N} \le \frac{\bar\alpha^2}{2}A_1+\frac{\bar\alpha^4 b}{4}A_3^2-\frac{\bar\alpha^{2^{**}}}{2^{**}}B_1+\frac{\bar\alpha^4 b}{2}A_3A^+(u)+\frac{\beta^2}{2}A_2+\frac{\beta^4 b}{4}A_4^2+\frac{\beta^4 b}{2}A_4A^-(u),$$

for every $\beta\in[0,\infty)$. Therefore, we have $\varphi(\bar\alpha,\beta)\le 0$ for all $\beta\in[0,\infty)$, which means $(\alpha_u,\beta_u)\notin\{\bar\alpha\}\times[0,\infty)$. Based on the above, we get $(\alpha_u,\beta_u)\in(0,\bar\alpha)\times(0,\infty)$, that is, (α_u,β_u) is an inner maximizer of φ in $[0,\bar\alpha]\times[0,\infty)$. Therefore, $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$. In that case, by (2.22), we have

$$\begin{aligned}
c_b^\lambda &\ge I_b^\lambda(\alpha_u u^+ + \beta_u u^-) + \frac{\bar\alpha^2}{2}A_1 + \frac{\bar\alpha^4 b}{4}A_3^2 - \frac{\bar\alpha^{2^{**}}}{2^{**}}B_1\\
&\quad + \frac{\bar\alpha^4 b}{2}A_3A^+(u) + \frac{\beta^2}{2}A_2 + \frac{\beta^4 b}{4}A_4^2 + \frac{\beta^4 b}{2}A_4A^-(u)\\
&> I_b^\lambda(\alpha_u u^+ + \beta_u u^-) \ge c_b^\lambda,
\end{aligned}$$

which is a contradiction. Hence, we have $B_1 = B_2 = 0$.

Step 3: c_b^λ is achieved. Given $u^\pm \ne 0$, according to Lemma 2.1, there exists $\alpha_u,\beta_u > 0$ such that $\hat u := \alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$. Moreover, $\langle (I_b^\lambda)'(u), u^\pm\rangle \le 0$. By Lemma 2.2, we have $0<\alpha_u,\beta_u\le 1$.

Combining $u_n\in\mathcal{N}_b^\lambda$ and Lemma 2.1, we obtain

$$I_b^\lambda(\alpha_u u_n^+ + \beta_u u_n^-) \le I_b^\lambda(u_n^+ + u_n^-) = I_b^\lambda(u_n).$$

Taking into consideration $B_1 = B_2 = 0$ and the semicontinuity of the norm, we obtain

$$\begin{aligned}
c_b^\lambda &\le I_b^\lambda(\hat u)\\
&= I_b^\lambda(\hat u) - \frac14\langle (I_b^\lambda)'(\hat u), \hat u\rangle\\
&= \frac14\|\hat u\|^2 + (\frac14 - \frac{1}{2^{**}})\int_{\mathbb{R}^N}|\hat u|^{2^{**}}dx + \frac{\lambda}{4}\int_{\mathbb{R}^N}[f(\hat u)\hat u - 4F(\hat u)]\,dx\\
&\le \frac14\|u\|^2 + (\frac14 - \frac{1}{2^{**}})\int_{\mathbb{R}^N}|u|^{2^{**}}dx + \frac{\lambda}{4}\int_{\mathbb{R}^N}[f(u)u - 4F(u)]\,dx\\
&\le \liminf_{n\to\infty}\Big[I_b^\lambda(u_n) - \frac14\langle (I_b^\lambda)'(u_n), u_n\rangle\Big] \le c_b^\lambda.
\end{aligned}$$

Hence, we can conclude that $\alpha_u = \beta_u = 1$, and c_b^λ is achieved by $u_b := u^+ + u^- \in \mathcal{N}_b^\lambda$. □

3. Proofs of main results

Proof of Theorem 1.2. Thanks to Lemma 2.4, we only need to prove that the minimizer u_b for c_b^λ is indeed a nodal solution to problem (1.1).

Because $u_b\in\mathcal{N}_b^\lambda$, we have $\langle (I_b^\lambda)'(u_b), u_b^+\rangle = \langle (I_b^\lambda)'(u_b), u_b^-\rangle = 0$. In view of Lemma 2.1, for $(\alpha,\beta)\in(\mathbb{R}_+\times\mathbb{R}_+)\backslash(1,1)$, we have

$$I_b^\lambda(\alpha u_b^+ + \beta u_b^-) < I_b^\lambda(u_b^+ + u_b^-) = c_b^\lambda. \tag{3.1}$$

Now we proceed by contradiction. Suppose $(I_b^\lambda)'(u_b)\ne 0$, then there exist $\delta>0$ and $\theta>0$ such that

$$\|(I_b^\lambda)'(v)\| \ge \theta \quad \text{for all } \|v-u_b\|\le 3\delta.$$

Choose $\tau \in (0, \min\{1/2, \frac{\delta}{\sqrt{2}\|u_b\|}\})$, and define

$$D := (1-\tau, 1+\tau) \times (1-\tau, 1+\tau),$$
$$g(\alpha,\beta) := \alpha u_b^+ + \beta u_b^- \quad \text{for all } (\alpha,\beta) \in D.$$

By (3.1), we have

$$\bar{c}_\lambda := \max_{\partial D}(I_b^\lambda \circ g) < c_b^\lambda. \tag{3.2}$$

Let $\varepsilon := \min\{(c_b^\lambda - \bar{c}_\lambda)/2, \theta\delta/8\}$ and $S_\delta := B(u_b, \delta)$. By [32, Lemma 2.3], there exists a deformation $\eta \in C([0,1] \times D, D)$ such that

(a) $\eta(1,v) = v$ if $v \notin (I_b^\lambda)^{-1}([c_b^\lambda - 2\varepsilon, c_b^\lambda + 2\varepsilon] \cap S_{2\delta})$,
(b) $\eta(1, (I_b^\lambda)^{c_b^\lambda + \varepsilon} \cap S_\delta) \subset (I_b^\lambda)^{c_b^\lambda - \varepsilon}$,
(c) $I_b^\lambda(\eta(1,v)) \le I_b^\lambda(v)$ for all $v \in E$.

Clearly,

$$\max_{(\alpha,\beta)\in\bar{D}} I_b^\lambda(\eta(1, g(\alpha,\beta))) < c_b^\lambda. \tag{3.3}$$

Therefore we claim that $\eta(1, g(D)) \cap \mathcal{N}_b^\lambda \neq \emptyset$, which contradicts the definition of c_b^λ.

We define $h(\alpha,\beta) := \eta(1, g(\alpha,\beta))$,

$$\begin{aligned}\Phi_0(\alpha,\beta) :&= (\langle (I_b^\lambda)'(g(\alpha,\beta)), u_b^+\rangle, \langle (I_b^\lambda)'(g(\alpha,\beta)), u_b^-\rangle) \\ &= (\langle (I_b^\lambda)'(\alpha u_b^+ + \beta u_b^-), u_b^+\rangle, \langle (I_b^\lambda)'(\alpha u_b^+ + \beta u_b^-), u_b^-\rangle)\end{aligned}$$

and

$$\Phi_1(\alpha,\beta) := \Big(\frac{1}{\alpha}\langle (I_b^\lambda)'(h(\alpha,\beta)),\, (h(\alpha,\beta))^+\rangle, \frac{1}{\beta}\langle (I_b^\lambda)'(h(\alpha,\beta)), (h(\alpha,\beta))^-\rangle\Big).$$

With an approach similar to [14], we use degree theory to obtain $\deg(\Phi_0, D, 0) = 1$. Then by (3.2), we obtain

$$g(\alpha,\beta) = h(\alpha,\beta) \quad \text{on } \partial D,$$

as a result of which, we have $\deg(\Phi_1, D, 0) = \deg(\Phi_0, D, 0) = 1$. Hence, $\Phi_1(\alpha_0, \beta_0) = 0$ for some $(\alpha_0, \beta_0) \in D$ so that

$$\eta(1, g(\alpha_0,\beta_0)) = h(\alpha_0,\beta_0) \in \mathcal{N}_b^\lambda,$$

which contradicts (3.3). Hence, $(I_b^\lambda)'(u_b) = 0$, which implies u_b is a critical point of I_b^λ. Thus, we can deduce that u_b is a nodal solution to problem (1.1). □

By Theorem 1.2, we obtain a least energy nodal solution u_b to problem (1.1), contributing to the establishment of Theorem 1.3, where we shall prove that the energy of u_b is strictly larger than twice the ground state energy.

Proof of Theorem 1.3. As in the proof of Lemma 2.3, there exists $\lambda_1^* > 0$ such that for all $\lambda \ge \lambda_1^*$, and for each $b > 0$, there exists $v_b \in \mathcal{M}_b^\lambda$ such that $I_b^\lambda(v_b) = c^* > 0$. By standard arguments (see [10, Corollary 2.13]), the critical points of the functional I_b^λ on $\mathcal{M}_b^\lambda$ are critical points of I_b^λ in E, so we obtain $(I_b^\lambda)'(v_b) = 0$. That is, v_b is a ground state solution to problem (1.1).

As stated in Theorem 1.2, u_b is known as a least energy nodal solution to problem (1.1), which changes sign only once when $\lambda \ge \lambda^*$.

Let $\lambda^{**} = \max\{\lambda^*, \lambda_1^*\}$ and assume $u_b = u_b^+ + u_b^-$. Adopting the same approach as in Lemma 2.1, we claim there exist $\alpha_{u_b^+} > 0$ and $\beta_{u_b^-} > 0$ such that $\alpha_{u_b^+} u_b^+ \in \mathcal{M}_b^\lambda$

and $\beta_{u_b^-}u_b^- \in \mathcal{M}_b^\lambda$. Then, by Lemma 2.2, we obtain $\alpha_{u_b^+}, \beta_{u_b^-} \in (0,1)$. Hence, thanks to Lemma 2.1, we have

$$2c^* \le I_b^\lambda(\alpha_{u_b^+}u_b^+) + I_b^\lambda(\beta_{u_b^-}u_b^-) \le I_b^\lambda(\alpha_{u_b^+}u_b^+ + \beta_{u_b^-}u_b^-) < I_b^\lambda(u_b^+ + u_b^-) = c_b^\lambda.$$

It follows that $c^* > 0$ cannot be achieved by a nodal function. □

We complete this section with the proof of Theorem 1.4. In the sequel, we regard $b > 0$ as a parameter in problem (1.1).

Proof of Theorem 1.4. In 3 steps, we analyze the convergence property of u_b as $b \to 0$, where u_b is the least energy nodal solution obtained in Theorem 1.2.

Step 1. For any sequence $\{b_n\}$, we prove that $\{u_{b_n}\}$ is bounded in E, if $b_n \searrow 0$. Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be a nonzero function with $\chi^\pm \neq 0$ fixed. Analogous to the argument in Lemma 2.1, for any $b \in [0,1]$, there exists a pair of positive numbers (λ_1, λ_2) independent of b, such that

$$\langle (I_b^\lambda)'(\lambda_1\chi^+ + \lambda_2\chi^-), \lambda_1\chi^+\rangle < 0 \quad \text{and} \quad \langle (I_b^\lambda)'(\lambda_1\chi^+ + \lambda_2\chi^-), \lambda_2\chi^-\rangle < 0.$$

Then according to Lemma 2.2, for any $b \in [0,1]$, there exists a unique pair $(\alpha_\chi(b), \beta_\chi(b)) \in (0,1] \times (0,1]$ such that $\overline{\chi} := \alpha_\chi(b)\lambda_1\chi^+ + \beta_\chi(b)\lambda_2\chi^- \in \mathcal{N}_b^\lambda$. Therefore, by (2.5), it follows that, for any $b \in [0,1]$,

$$\begin{aligned}
I_b^\lambda(u_b) \le I_b^\lambda(\overline{\chi}) &= I_b^\lambda(\overline{\chi}) - \frac{1}{4}\langle (I_b^\lambda)'(\overline{\chi}), \overline{\chi}\rangle \\
&= \frac{1}{4}\|\overline{\chi}\|^2 + (\frac{1}{4} - \frac{1}{2^{**}})\int_{\mathbb{R}^N} |\overline{\chi}|^{2^{**}}dx + \frac{\lambda}{4}\int_{\mathbb{R}^N} [f(\overline{\chi})\overline{\chi} - 4F(\overline{\chi})]\,dx \\
&\le \frac{1}{4}\|\overline{\chi}\|^2 + (\frac{1}{4} - \frac{1}{2^{**}})\int_{\mathbb{R}^N} |\overline{\chi}|^{2^{**}}dx + \frac{\lambda}{4}\int_{\mathbb{R}^N} \left(C_1|\overline{\chi}|^2 + C_2|\overline{\chi}|^p\right)dx \\
&\le \frac{1}{4}\|\lambda_1\chi^+\|^2 + (\frac{1}{4} - \frac{1}{2^{**}})\int_{\mathbb{R}^N} |\lambda_1\chi^+|^{2^{**}}dx \\
&\quad + \frac{\lambda}{4}\int_{\mathbb{R}^N} \left(C_1|\lambda_1\chi^+|^2 + C_2|\lambda_1\chi^+|^p\right)dx \\
&\quad + \frac{1}{4}\|\lambda_2\chi^-\|^2 + (\frac{1}{4} - \frac{1}{2^{**}})\int_{\mathbb{R}^N} |\lambda_2\chi^-|^{2^{**}}dx \\
&\quad + \frac{\lambda}{4}\int_{\mathbb{R}^N} \left(C_1|\lambda_2\chi^-|^2 + C_2|\lambda_2\chi^-|^p\right)dx \\
&:= C^*,
\end{aligned}$$

where C^* is a positive constant independent of b. Thus, as $n \to \infty$, it follows that

$$C^* + 1 \ge I_{b_n}^\lambda(u_{b_n}) = I_{b_n}^\lambda(u_{b_n}) - \frac{1}{4}\langle (I_{b_n}^\lambda)'(u_{b_n}), u_{b_n}\rangle \ge \frac{1}{4}\|u_{b_n}\|^2,$$

that is, $\{u_{b_n}\}$ is bounded in E.

Step 2. In this step, we prove that problem (1.10) possesses one nodal solution u_0. Since $\{u_{b_n}\}$ is bounded in E, thanks to Step 1, up to a subsequence, there exists $u_0 \in E$ such that

$$\begin{gathered}
u_{b_n} \rightharpoonup u_0 \quad \text{in} E, \\
u_{b_n} \to u_0 \quad \text{in } L^p(\mathbb{R}^N) \text{ for } p \in [2, 2^{**}), \\
u_{b_n} \to u_0 \quad \text{a.e. in} \mathbb{R}^N.
\end{gathered} \tag{3.4}$$

Given that $\{u_{b_n}\}$ is a weak solution to (1.1) with $b = b_n$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (\Delta u\Delta\phi + \nabla u\nabla\phi + V(x)u\phi)\,dx + b_n\int_{\mathbb{R}^N}|\nabla u|^2dx\int_{\mathbb{R}^N}\nabla u\nabla\phi\,dx \\ &= \lambda\int_{\mathbb{R}^N} f(u)\phi\,dx + \int_{\mathbb{R}^N}|u|^{2^{**}-2}u\phi\,dx \end{aligned} \tag{3.5}$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$.

Combing (3.4), (3.5) and Step 1, we find that

$$\begin{aligned} &\int_{\mathbb{R}^N} (\Delta u_0\Delta\phi + \nabla u_0\nabla\phi + V(x)u_0\phi)\,dx + b_n\int_{\mathbb{R}^N}|\nabla u_0|^2dx\int_{\mathbb{R}^N}\nabla u_0\nabla\phi\,dx \\ &= \lambda\int_{\mathbb{R}^N} f(u_0)\phi\,dx + \int_{\mathbb{R}^N}|u_0|^{2^{**}-2}u_0\phi\,dx \end{aligned} \tag{3.6}$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$, which in turn implies that u_0 is a weak solution to (1.10). Analogous to the process of Lemma 2.3, we obtain that $u_0^\pm \neq 0$. Thus, we have completed the proof of this step.

Step 3. In this step, we prove that problem (1.10) possesses a least energy nodal solution v_0, and that there exists a unique pair $(\alpha_{b_n}, \beta_{b_n}) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfying $\alpha_{b_n}v_0^+ + \beta_{b_n}v_0^- \in \mathcal{N}_{b_n}^\lambda$. Also we prove that $(\alpha_{b_n}, \beta_{b_n}) \to (1,1)$ as $n \to \infty$.

Similar to the proof of Theorem 1.2, we can reach the conclusion that problem (1.10) possesses a least energy nodal solution v_0, where $I_0^\lambda(v_0) = c_0$ and $(I_0^\lambda)'(v_0) = 0$. Then, in view of Lemma 2.1, we can obtain with ease the existence and uniqueness of the pair $(\alpha_{b_n}, \beta_{b_n})$ such that $\alpha_{b_n}v_0^+ + \beta_{b_n}v_0^- \in \mathcal{N}_{b_n}^\lambda$. Besides, we know $\alpha_{b_n} > 0$ and $\beta_{b_n} > 0$. To complete the proof, we just establish that $(\alpha_{b_n}, \beta_{b_n}) \to (1,1)$ as $n \to \infty$. Actually, given that $\alpha_{b_n}v_0^+ + \beta_{b_n}v_0^- \in \mathcal{N}_{b_n}^\lambda$, we have

$$\begin{aligned} &\alpha_{b_n}^2\|v_0^+\|^2 + \alpha_{b_n}\beta_{b_n}\int_{\mathbb{R}^N}\Delta v_0^+\Delta v_0^-\,dx \\ &+ \alpha_{b_n}^2 b_n\int_{\mathbb{R}^N}|\nabla v_0^+|^2dx\Big(\alpha_{b_n}^2\int_{\mathbb{R}^N}|\nabla v_0^+|^2dx + \beta_{b_n}^2\int_{\mathbb{R}^N}|\nabla v_0^-|^2dx\Big) \\ &= \lambda\int_{\mathbb{R}^N} f(\alpha_{b_n}v_0^+)\alpha_{b_n}v_0^+\,dx + \int_{\mathbb{R}^N}|\alpha_{b_n}v_0^+|^{2^{**}}dx \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &\beta_{b_n}^2\|v_0^-\|^2 + \alpha_{b_n}\beta_{b_n}\int_{\mathbb{R}^N}\Delta v_0^+\Delta v_0^-\,dx \\ &+ \beta_{b_n}^2 b_n\int_{\mathbb{R}^N}|\nabla v_0^-|^2dx\Big(\beta_{b_n}^2\int_{\mathbb{R}^N}|\nabla v_0^-|^2dx + \alpha_{b_n}^2\int_{\mathbb{R}^N}|\nabla v_0^+|^2dx\Big) \\ &= \lambda\int_{\mathbb{R}^N} f(\beta_{b_n}v_0^-)\beta_{b_n}v_0^-\,dx + \int_{\mathbb{R}^N}|\beta_{b_n}v_0^-|^{2^{**}}dx. \end{aligned} \tag{3.8}$$

a Since $b_n \searrow 0$, we conclude that the sequences $\{\alpha_{b_n}\}$ and $\{\beta_{b_n}\}$ are bounded. Assume, up to a subsequence, $\alpha_{b_n} \to \alpha_0$ and $\beta_{b_n} \to \beta_0$. Then by (3.7) and (3.8), we have

$$\alpha_0^2\|v_0^+\|^2 + \alpha_0\beta_0\int_{\mathbb{R}^N}\Delta v_0^+\Delta v_0^-\,dx = \lambda\int_{\mathbb{R}^N} f(\alpha_0 v_0^+)\alpha_0 v_0^+\,dx + \int_{\mathbb{R}^N}|\alpha_0 v_0^+|^{2^{**}}dx \tag{3.9}$$

and

$$\beta_0^2\|v_0^+\|^2 + \alpha_0\beta_0\int_{\mathbb{R}^N}\Delta v_0^+\Delta v_0^-\,dx = \lambda\int_{\mathbb{R}^N} f(\beta_0 v_0^-)\beta_0 v_0^-\,dx + \int_{\mathbb{R}^N}|\beta_0 v_0^-|^{2^{**}}dx. \tag{3.10}$$

Noticing that v_0 is a nodal solution to problem (1.10), we obtain

$$\|v_0^+\|^2 + \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- \,dx = \lambda \int_{\mathbb{R}^N} f(v_0^+) v_0^+ \,dx + \int_{\mathbb{R}^N} |v_0^+|^{2^{**}} dx, \tag{3.11}$$

$$\|v_0^+\|^2 \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- \,dx = \lambda \int_{\mathbb{R}^N} f(v_0^-) v_0^- \,dx + \int_{\mathbb{R}^N} |v_0^-|^{2^{**}} dx. \tag{3.12}$$

Therefore, from (3.9)-(3.12), we can easily obtain that $(\alpha_0, \beta_0) = (1,1)$, and thus Step 3 follows.

We can now complete the proof of Theorem 1.4. We claim that u_0 obtained in Step 2 is a least energy solution to problem (1.10). In fact, according to Step 3 and Lemma 2.1, we see that

$$\begin{aligned}
I_0^\lambda(v_0) \le I_0^\lambda(u_0) &= \lim_{n\to\infty} I_{b_n}^\lambda(u_{b_n}) \\
&\le \lim_{n\to\infty} I_{b_n}^\lambda(\alpha_{b_n} v_0^+ + \beta_{b_n} v_0^-) \\
&= \lim_{n\to\infty} I_0^\lambda(v_0^+ + v_0^-) \\
&= I_0^\lambda(v_0),
\end{aligned}$$

which yields completest the proof of Theorem 1.4. □

Acknowledgments. H. Pu was supported by the Graduate Scientific Research Project of Changchun Normal University (SGSRPCNU No. 2020-51). S. Liang was supported by the Foundation for China Postdoctoral Science Foundation (Grant no. 2019M662220), Scientific research projects for Department of Education of Jilin Province, China (JJKH20210874KJ), Natural Science Foundation of Changchun Normal University (No. 2017-09). D. D. Repovš was supported by the Slovenian Research Agency (No. P1-0292, N1-0114, N1-0083, N1-0064, and J1-8131). We want to thank the anonymous referees for their comments and suggestions.

References

[1] C. O. Alves, A. B. Nóbrega; *Nodal ground state solution to a biharmonic equation via dual method*, J. Differential Equations, **260** (2016), 5174–5201.

[2] C. O. Alves, M. A. S. Souto; *Existence of least energy nodal solution for a Schrödinger-Poisson system in bounded domains*, Z. Angew. Math. Phys., **65** (2014), 1153–1166.

[3] T. Bartsch, Z. Liu, T. Weth; *Sign changing solutions of superlinear Schrödinger equations*, Comm. Partial Differential Equations, **29** (2004), 25–42.

[4] T. Bartsch, Z. Q. Wang; *Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$*, Comm. Partial Differential Equations, **20** (1995), 1725–1741.

[5] T. Bartsch, T. Weth; *Three nodal solutions of singularly perturbed elliptic equations on domains without topology*, Ann Inst H Poincaré Anal Non Linéaire, **22** (2005), 259–281.

[6] G. F. Carrier; *On the nonlinear vibration problem of the elastic string*, Quart. Appl. Math., **3** (1945), 157–165.

[7] S. Chen, B. Zhang, X. Tang; *Existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity*, Adv. Nonlinear Anal., **9** (2020), 148–167.

[8] Y. B. Deng, W. Shuai; *Sign-changing multi-bump solutions for Kirchhoff-type equations in* $\mathbb{R}^3$, Discrete Contin. Dyn. Syst. Ser. A, **38** (2018), 3139–3168.

[9] A. Fiscella, P. Pucci, B.L. Zhang; *p-fractional Hardy-Schrödinger-Kirchhoff systems with critical nonlinearities*, Adv. Nonlinear Anal., **8** (2019), 1111–1131.

[10] X. M. He, W. M. Zou; *Ground states for nonlinear Kirchhoff equations with critical growth*, Ann. Mat. Pura Appl., **193** (2014), 473–500.

[11] G. Kirchhoff; *Mechanik*, Teubner, Leipzig, 1883.

[12] F. Y. Li, C. Gao, X. Zhu; *Existence and concentration of sign-changing solutions to Kirchhoff-type system with Hartree-type nonlinearity*, J. Math. Anal. Appl., **448** (2017), 60–80.

[13] S. Liang, P. Pucci, B. Zhang; *Multiple solutions for critical Choquard-Kirchhoff type equations*, Adv. Nonlinear Anal., **10** (2021), 400–419.
[14] S. Liang, V. D. Rădulescu; *Least-energy nodal solutions of critical Kirchhoff problems with logarithmic nonlinearity*, Anal. Math. Phys., **10:45** (2020), 1–31.
[15] S. Liang, S. Shi; *Soliton solutions to Kirchhoff type problems involving the critical growth in* $\mathbb{R}^N$, Nonlinear Anal., **81** (2013), 31–41.
[16] S. Liang, J. Zhang; *Existence of solutions for Kirchhoff type problems with critical nonlinearity in* $\mathbb{R}^3$, Nonlinear Anal. Real World Applications, **17** (2014), 126–136.
[17] S. Liang, J. Zhang; *Existence and multiplicity of solutions for fourth-order elliptic equations of Kirchhoff type with critical growth in* $\mathbb{R}^N$, J. Math. Phys., **57** (2016), 111505.
[18] S. Liang, J. Zhang; *Multiplicity of solutions for the noncooperative Schrödinger-Kirchhoff system involving the fractional* $p-$*laplacian in* $\mathbb{R}^N$, Z. Angew. Math. Phys., **68:63** (2017), 1–18.
[19] J. L. Lions; *On some questions in boundary value problems of mathematical physics ScienceDirect*, North-Holland Mathematics Studies, **30** (1978), 284–346.
[20] S. Lu; *Signed and sign-changing solutions for a Kirchhoff-type equation in bounded domains*, J. Math. Anal. Appl., **432** (2015), 965–982.
[21] T. F. Ma, J. E. Munoz Rivera; *Positive solutions for a nonlinear nonlocal elliptic transmission problem*, Appl. Math. Lett., **16** (2003), 243–248.
[22] C. Miranda; *Un'osservazione su un teorema di Brouwer*, Boll Un Mat Ital, **3** (1940), 5–7.
[23] G. Molica Bisci, V.D. Rădulescu, R. Servadei; *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics and its Applications, vol. 162, Cambridge University Press, Cambridge, 2016.
[24] D. Oplinger; *Frequency response of a nonlinear stretched string*, J. Acoust. Soc. Am., **32** (1960), 1529–1538.
[25] P. Pucci, M. Xiang, B. Zhang; *Existence and multiplicity of entire solutions for fractional p–Kirchhoff equations*, Adv. Nonlinear Anal., **5** (2016), 27–55.
[26] W. Shuai; *Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains*, J. Differ. Equations, **259** (2015), 1256–1274.
[27] J. Sun, L. Li, M. Cencelj, B. Gabrovšek; *Infinitely many sign-changing solutions for Kirchhoff type problems in* $\mathbb{R}^3$, Nonlinear Anal., **186** (2018), 33–54.
[28] X. H. Tang and B. Cheng; *Ground state sign-changing solutions for Kirchhoff type problems in bounded domains*, J. Differ. Equations, **261** (2016), 2384–2402.
[29] D.B. Wang; *Least energy sign-changing solutions of Kirchhoff-type equation with critical growth*, J. Math. Phys., **61** (2020), 011501.
[30] F. Wang, M. Avci, Y. An; *Existence of solutions for fourth order elliptic equations of Kirchhoff type*, J. Math. Anal. Appl., **409** (2014), 140–146.
[31] T. Weth; *Energy bounds for entire nodal solutions of autonomous superlinear equations*, Calc. Var. Partial Differential Equations, **27** (2006), 421-437.
[32] M. Willem; *Minimax Theorems*, Birkhäuser, Boston, 1996.
[33] M. Q. Xiang, B.L. Zhang, V. D. Rădulescu; *Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian*, Nonlinearity, **29** (2016), 3186–3205.
[34] M. Q. Xiang, M. Q. Zhang, V. D. Rădulescu; *Superlinear Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian and critical exponent*, Adv. Nonlinear Anal., **9** (2020), 690–709.
[35] M. Q. Zhang, V. D. Rădulescu, L. Wang; *Existence results for Kirchhoff-type superlinear problems involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A., **149** (2019), 1061–1081.
[36] W. Zhang, X. Tang, B. Cheng, J. Zhang; *Sign-changing solutions for fourth order elliptic equations with Kirchhoff-type*, Comm. Pure Appl. Anal., **15** (6) (2016), 2161–2177.
[37] Z. T. Zhang, K. Perera; *Sign changing solutions of Kirchhoff type problems via invariant sets of descentow*, J. Math. Anal. Appl., **317** (2006), 456–463.

Hongling Pu
College of Mathematics, Changchun Normal University, Changchun 130032, China
Email address: pauline_phl@163.com

Shiqi Li
College of Mathematics, Changchun Normal University, Changchun 130032, China
Email address: lishiqi59@126.com

Sihua Liang
College of Mathematics, Changchun Normal University, Changchun 130032, China
Email address: liangsihua@163.com

Dušan D. Repovš
Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana & Institute of Mathematics, Physics and Mechanics, Ljubljana, 1000, Slovenia
Email address: dusan.repovs@guest.arnes.si