

SEMILINEAR HYPERBOLIC SYSTEMS IN ONE SPACE DIMENSION WITH STRONGLY SINGULAR INITIAL DATA

KIRSTEN E. TRAVERS

ABSTRACT. In this article interactions of singularities in semilinear hyperbolic partial differential equations are studied. Consider a simple non-linear system of three equations in \mathbb{R}^2 with derivatives of Dirac delta functions as initial data. As the micro-local linear theory prescribes, the initial singularities propagate along forward bicharacteristics. However, there are also anomalous singularities created when these characteristics intersect. Their regularity satisfies the following “sum law”: the “strength” of the anomalous singularity equals the sum of the “strengths” of the incoming singularities. Hence the solution to the system becomes more singular as time progresses.

1. INTRODUCTION

This paper is devoted to the study of a typical example of a semilinear hyperbolic system of partial differential equations:

$$(\partial_t + \partial_x)u = 0 \tag{1}$$

$$(\partial_t - \partial_x)v = 0 \tag{2}$$

$$\partial_t w = uv. \tag{3}$$

The Cauchy problem has been studied by Rauch and Reed [4], [5] when the initial data are either classical or have jump discontinuities. It was proven that, as in a linear systems of partial differential equations, the singularities in the initial data propagate along characteristics. However, the nonlinearity in (3) causes anomalous singularities to be created when singular characteristics intersect. This may be contrasted with a linear system, where the principle of superposition and disjoint null bicharacteristic strips ensure that no interaction of singularities occurs. Similar problems were also studied by Rauch and Reed in [3] when the initial data were distributions, but when the non-linearity was sublinear. Their method was to solve the problem for approximating smooth initial data, then pass to a limit, obtaining traveling “delta waves”. A similar problem was addressed by Oberguggenberger and Wang [2], but where the nonlinearity was again sublinear or bounded.

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In this paper, the initial data will be extended to distributions with support at finitely many points. The Structure Theorem for Distributions of point support states that any such distribution may be written as the finite sum of Dirac delta functions and derivatives thereof. Initial data will be given by:

$$u(x, 0) = \sum_{i=1}^k \delta^{(m_i)}(x - x_i) \text{ for some } m_i \in \mathbb{N}, x_i \in \mathbb{R} \quad (1')$$

$$v(x, 0) = \sum_{j=1}^l \delta^{(n_j)}(x - x_j) \text{ for some } n_j \in \mathbb{N}, x_j \in \mathbb{R} \quad (2')$$

$$w(x, 0) = 0, \quad (3')$$

where $\delta^{(k)}(x)$ is the distribution given by $\int_{-\infty}^{\infty} \delta^{(k)}(x)\phi(x) dx = (-1)^k \phi^{(k)}(0)$ for all $\phi \in \mathbf{C}_0^\infty(\mathbb{R})$.

Then the problem (1),(2),(3) with(1'), (2') and (3') is solved in the sense of distributions. It will be shown that the new "anomalous" singularities that are created are "stronger" (in a sense to be defined) than the initial singularities. In other words, the solution gets more and more singular as time progresses. This is different from previous work (eg. [4]), where the anomalous singularities produced were weaker.

The motivation for studying this problem is clear. Many physical phenomena are modeled by non-linear systems of partial differential equations. For example, the equations governing fluid and gas dynamics and quantum field theory are quasilinear. They develop shocks, and are very difficult to analyze. However, many of the properties of non-linear equations are also present in semilinear systems, such as this, where singularities also interact, but shocks do not occur, simplifying the analysis. It is the hope that analysis of problems such as this will help to understand more difficult non-linear phenomena. The strongly singular initial data model the very high peaks or oscillations demonstrated by point charges, dipoles etc.

First, for simplicity of exposition, the initial data are simplified to having support at just two points. The key elements of the argument are still present in this case. Hence, consider:

$$u(x, 0) = \delta^{(m)}(x + 1) \text{ for some } m \in \mathbb{N} \quad (1')$$

$$v(x, 0) = \delta^{(n)}(x - 1) \text{ for some } n \in \mathbb{N} \quad (2')$$

$$w(x, 0) = 0. \quad (3')$$

In Part 2, the first two equations can be solved separately, since they are each decoupled from the system. By changing coordinates, it is seen that u and v are solved by translation of the initial distributions in \mathbb{R} :

$$u(x, t) = \delta^{(m)}(x + 1 - t) \text{ and } v(x, t) = \delta^{(n)}(x + t - 1),$$

where the derivative is in the x -direction.

Then in Part 3, the solutions to u and v may be substituted into (3), giving:

$$\partial_t w = \delta^{(m)}(x + 1 - t) \cdot \delta^{(n)}(x - 1 + t), \quad (3'')$$

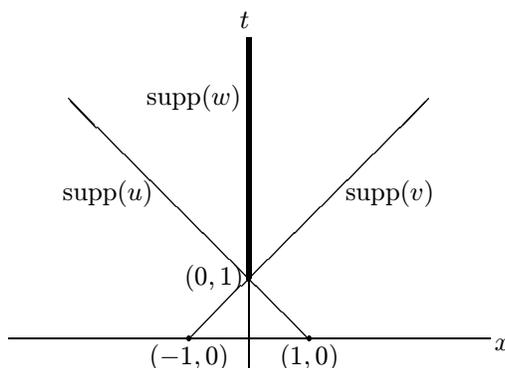
where all derivatives are in the x -direction. It must be checked that it does indeed make sense to multiply u and v . This involves conditions on their respective wave front sets. Once this product well-defined as a distribution in the plane, it will be rewritten in a form which will make (3'') simpler to solve. Finally, this p.d.e. is

solved, giving a sum of a propagating term, with singularity stronger than those in the initial data, and a localized part.

Theorem 1.1. *The solution to the system is given by*

$$\begin{aligned} u(x, t) &= \delta^{(m)}(x + 1 - t) \\ v(x, t) &= \delta^{(n)}(x - 1 + t) \\ w(x, t) &= \frac{1}{2^{m+n+1}} \delta^{(m+n)}(x) \otimes H(t - 1) + \text{terms with support at } (0, 1). \end{aligned}$$

So the supports of the solutions $\{u, v, w\}$ are as follows:



Then in part 4, the question of the “strengths” of the singularities of the solution (u, v, w) is addressed.

Definition 1.2. *Let the strength $-s_T \leq 0$ of the distribution T be defined such that s_T is the least natural number so that T is the $s_T + 2$ nd distributional derivative in the x -direction of a function continuous in x .*

Theorem 1.3. *The strength of u is $-m$; the strength of v is $-n$; the strength of w is $-(m + n)$.*

This generalizes the result in Rauch and Reed’s paper [4] which proved this formula when the initial data had singularities of at worst jump-discontinuities. However, notice in this case that the new singularities are stronger than those in the initial data. This is different from the results in the aforementioned papers, where the strengths were positive, so the new singularities were weaker than in the initial data. The threshold case is when initial data are Dirac delta functions (with strength zero), in which case the solution also has strength zero, and so the solution at a later time t has the same regularity as the initial data (like the linear case).

Finally, in Part 5, the initial data will be generalized to be sums of many delta functions again. Using superposition properties of the u and v linear equations, it will be shown that solutions of the above form may be added to give a solution to the original more general question. The sum law for the relative strengths of solutions u, v and w still holds.

Given these results, it is natural to ask whether similar results hold when initial data comprises of countably many Dirac delta functions. This question is answered in subsequent work, where the Cauchy data is that of measures of compact support and, more generally, arbitrary distributions. The same sum law holds.

2. THE u AND v FUNCTIONS

This section solves the two decoupled distributional partial differential equations

$$(\partial_t + \partial_x)u = 0 \quad u(x, 0) = \delta^{(m)}(x + 1) \text{ for some } m \in \mathbb{N} \quad (1)$$

$$(\partial_t - \partial_x)v = 0 \quad v(x, 0) = \delta^{(n)}(x - 1) \text{ for some } n \in \mathbb{N}, \quad (2)$$

where $\delta^{(k)}(x)$ is the distribution given by $\int_{-\infty}^{\infty} \delta^{(k)}(x)\phi(x) dx = (-1)^k \phi^{(k)}(0)$ for all $\phi \in C_0^\infty(\mathbb{R})$.

In order to solve these equations, some theory is recalled about defining distributions on submanifolds of \mathbb{R}^2 . In particular, we should like to extend the Dirac delta function and its derivatives to the lines $t = x + 1$ and $t = -x + 1$. If our initial data were functions, then the solutions to (1) and (2) would be given by the d'Alembertian. Indeed, $\delta^{(m)}(x - t + 1)$ and $\delta^{(n)}(x + t - 1)$ solve (1) and (2) above, when interpreted in the correct sense.

Definition 2.1. Let K be the distribution in \mathbb{R}^2 given by

$$\iint K(x, t)\phi(x, t) dx dt = \int \phi(0, t) dt, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2).$$

This may be informally thought of as the Dirac delta function $\delta(x)$, supported along the t -axis.

Let G be the distribution in \mathbb{R}^2 given by

$$\iint G(x, t)\phi(x, t) dx dt = \int \phi(x, 0) dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2).$$

This may be informally thought of as the Dirac delta function $\delta(t)$, supported along the x -axis.

Definition 2.2. Let $X_1, X_2 \subset \mathbb{R}^2$ be open sets, $u \in \mathcal{D}'(X_2)$, and let $f : X_1 \rightarrow X_2$ be a smooth invertible map such that its derivative is surjective. Then the **pullback of u by f** , f^*u , is the unique continuous linear map: $\mathcal{D}'(X_2) \rightarrow \mathcal{D}'(X_1)$ such that

$$(f^*u)(\phi) = u(|J(f^{-1})|(\phi \circ f^{-1})),$$

where $J(f^{-1})$ is the Jacobian matrix of f^{-1} .

Remark 1. It is an extension of the map $f^*u = u \circ f$ when $u \in C^0(\mathbb{R}^2)$. When u is continuous, this means that $f^*(u) = u \circ f$, and the usual calculus for change of variables is used:

$$\iint (f^*u)(x, t)\phi(x, t) dx dt := \iint u(y, z)\phi(f^{-1}(y, z))|\det J^{-1}| dy dz$$

where J is the Jacobian (change of basis) matrix taking $(x, t) \rightarrow (y, z)$.

Example 2.3. Using the definition of G and K in 2.1, new distributions in the plane $\delta(x - t + 1)$ and $\delta(x - 1 + t)$ may be defined as follows: Let $f : \mathbb{R}^2 \hookrightarrow \mathbb{R}^2$ be the transformation in the plane such that f^{-1} maps the x -axis to the line $t = x + 1$ and the t -axis to $t = -x + 1$. i.e.

$$f : (x, t) \rightarrow (x + t - 1, x - t + 1); \quad f^{-1} : (x, t) \rightarrow \left(\frac{x+t}{2}, \frac{x-t}{2} + 1\right).$$

So f' may be written as the matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and f has Jacobian $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2$. Then define $\delta(x - t + 1) := f^*G$, and $\delta(x + t - 1) := f^*K$.

Evaluating the distribution $\delta(x - t + 1)$ on a test function,

$$\langle \delta(x - t + 1), \phi(x, t) \rangle := \langle f^*G, \phi \rangle = \frac{1}{2} \langle G, \phi \circ f^{-1} \rangle := \frac{1}{2} \int (\phi \circ f^{-1})(\alpha, 0) d\alpha$$

by definition of G . But $f^{-1}(\alpha, 0) = (\frac{\alpha}{2}, \frac{\alpha}{2} + 1)$, so

$$\langle \delta(x - t + 1), \phi \rangle = \frac{1}{2} \int \phi(\frac{\alpha}{2}, \frac{\alpha}{2} + 1) d\alpha = \int \phi(\lambda, \lambda + 1) d\lambda.$$

Similarly, $\delta(x + t - 1)$ acts as follows:

$$\langle \delta(x + t - 1), \phi(x, t) \rangle = \int \phi(\lambda, -\lambda + 1) d\lambda.$$

To restrict a distribution to the initial line $\{t = 0\}$, the following definitions and theorem are needed.

Definition 2.4. A distribution $T \in \mathcal{D}'(\mathbb{R}^2)$ is **microlocally smooth** at (x, t, ξ, η) ($(\xi, \eta) \neq 0$) if when T is localized about (x, t) by $\phi \in C_0^\infty(\mathbb{R}^2)$ with $\phi(x, t) \equiv 1$, the Fourier Transform of ϕT is rapidly decreasing in an open cone about (ξ, η) . The **wave front set of T** , $WF(T)$, is the complement in \mathbb{R}^4 of the set of microlocally smooth points.

Example 2.5. To evaluate $WF(K)$, first notice that K is certainly microlocally smooth in all directions at (x_0, t_0) for $x_0 \neq 0$, since $K \equiv 0$ in all sufficiently small neighbourhoods there. Let $\phi \in C_0^\infty(\mathbb{R}^2)$ be such that $\phi(0, t_0) \equiv 1$. Then

$$\widehat{\phi K}(\xi, \eta) = (\phi K)(e^{-ix\xi - it\eta}) = K(\phi e^{-ix\xi - it\eta}) = \int \phi(0, t)e^{-it\eta} dt$$

If $\eta \neq 0$, integration by parts gives:

$$\eta^N \int \phi(0, t)e^{-it\eta} dt = (i)^N \int (\partial_t^N \phi)(0, t)e^{-it\eta} < \infty$$

So in any direction with a non-zero η -component the integral is rapidly decreasing. However, if $\eta = 0$, then the integral cannot be rapidly decreasing, since

$$\xi^N \int \phi(0, t)e^{-it\eta} dt \rightarrow \infty, \text{ as } \xi \rightarrow \infty.$$

Hence $WF(K) = \{(0, t, \xi, 0) : \xi \neq 0\}$.

Analogously, $WF(G) = \{(x, 0, 0, \eta) : \eta \neq 0\}$

Theorem 2.6. [1, 8.2.7] Let X be a manifold and Y a submanifold with normal bundle denoted by $N(Y)$. For every distribution u in X with $WF(u)$ disjoint from $N(Y)$, the restriction $u|_Y$ to Y is a well-defined distribution on Y , the pullback by the inclusion $Y \hookrightarrow X$.

Lemma 2.7. $u(x, t) = \delta^{(m)}(x + 1 - t)$ is the unique weak solution to (1), and $v(x, t) = \delta^{(n)}(x + t - 1)$ is the unique weak solution to (2).

Proof. Consider the first statement. It must be shown that

$$\langle (\partial_t + \partial_x)\delta_x^{(m)}(x - t + 1), \phi \rangle = 0, \forall \phi \in C_0^\infty(\mathbb{R}^2)$$

Well, the left hand side of the above is equal to

$$-\langle \delta_x^{(m)} f^*G, \phi_t + \phi_x \rangle = (-1)^{m+1} \langle f^*G, (\partial_x)^m \partial_t \phi \rangle + (-1)^{m+1} \langle (\partial_x)^{m+1} f^*G, \phi \rangle.$$

Now let $\psi = \partial_x^m \phi$. Then ψ is also in C_0^∞ . So the left hand side becomes

$$(-1)^{m+1} \langle f^* G, \psi_t + \psi_x \rangle = 0,$$

as required, since $\delta(x+1-t)$ is a distribution in $(x-t)$, and is therefore a weak solution to $(\partial_t + \partial_x)z = 0$.

$\delta^{(m)}(x-t+1)$ may be restricted to the submanifold $t=0$, because if $u \in \mathcal{D}'$ and P is a partial differential operator, then $WF(Pu) \subseteq WF(u)$. So

$$WF((f^*G)^{(m)}) \subseteq WF(f^*G) \subseteq \{(\lambda, \lambda+1, \xi, -\xi) : \xi \neq 0\},$$

which does not intersect the normal bundle of $\{t=0\}$. (For details, see [1, 8.2.4].) When $t=0$, $u(x,t) = \delta^{(m)}(x+1)$, satisfying the initial condition (1'). To show uniqueness, suppose we have two such solutions u_1 and u_2 . Then let $z = u_1 - u_2$. Then z satisfies

$$(\partial_t + \partial_x)z = 0, \quad z(x,0) = 0.$$

So $z = 0$ by [1, 3.1.4]. Hence the solution u to (1) and (1') is unique. This completes the proof. Similarly, $\delta^{(n)}(x+t-1)$ satisfies (2) and (2'). \square

3. THE w FUNCTION:

This section solves the third distributional partial differential equation

$$\partial_t w = uv \tag{3}$$

$$w(x,0) = 0 \tag{3'}$$

By Part 2, $u(x,t) = \delta^{(m)}(x+1-t)$ and $v(x,t) = \delta^{(n)}(x-1+t)$, so this problem turns into:

$$i\partial_t w = \delta^{(m)}(x+1-t) \cdot \delta^{(n)}(x-1+t). \tag{3''}$$

This part consists of four sections. First, it must be checked that u and v may be multiplied. Then understanding how $\delta^{(m)}(x+1-t) \cdot \delta^{(n)}(x-1+t)$ acts on test functions as a distribution in the plane defines this product. To simplify calculations with $\delta^{(m)}(x+1-t) \cdot \delta^{(n)}(x-1+t)$, it is rewritten as a sum of tensor products of distributions of one variable. This then enables (3'') to be solved. It will be shown that the solution is a distribution in the plane, that it is unique, and satisfies the initial condition (3').

Theorem 3.1. [1, 5.1.1] *If $w_1 \in \mathcal{D}'(X_1)$ and $w_2 \in \mathcal{D}'(X_2)$ then there is a unique distribution $w \in \mathcal{D}'(X_1 \times X_2)$ such that*

$$w(\phi_1 \otimes \phi_2) = w_1(\phi_1)w_2(\phi_2)$$

for $\phi_i \in C_0^\infty(X_i)$. Then

$$w(\phi) = w_1[w_2(\phi(x_1, x_2))] = w_2[w_1(\phi(x_1, x_2))]$$

for $\phi \in C_0^\infty(X_1 \times X_2)$, where w_i acts on the following function of x_i only. Then w is called the **tensor product**, $w = w_1 \otimes w_2$.

Definition 3.2. Let $\mathbf{WF}(u) \oplus \mathbf{WF}(v)$ be defined to be

$$\{(x, t, \xi_1 + \xi_2, \eta_1 + \eta_2) \text{ such that } (x, t, \xi_1, \eta_1) \in \mathbf{WF}(u); (x, t, \xi_2, \eta_2) \in \mathbf{WF}(v)\}$$

Theorem 3.3. [1, 8.2.10] *If $u, v \in \mathcal{D}'(X)$ then the product uv can be defined as the pullback of the tensor product $u \otimes v$ by the diagonal map $\delta : X \rightarrow X \times X$ unless there exists an element $(x, t, 0, 0) \in \mathbf{WF}(u) \oplus \mathbf{WF}(v)$.*

Corollary 3.4. $\delta^{(m)}(x+1-t) \cdot \delta^{(n)}(x+t-1)$ exists.

Proof. $WF(u) \oplus WF(v) = \{(0, 1, \xi_1 + \xi_2, -\xi_1 + \xi_2) : \xi_1, \xi_2 \neq 0\}$. Hence there is no element in $WF(u) \oplus WF(v)$ of the form $(x, t, 0, 0)$. \square

Lemma 3.5.

$$\delta^{(m)}(x+1-t)\delta^{(n)}(x-1+t) = \frac{1}{2^{m+n+1}} \sum_{i=0}^{m+n} c_i \delta^{(i)}(x) \otimes \delta^{(m+n-i)}(t-1)$$

where c_i are constants with $c_{m+n} = 1$.

Proof. Using the definitions 2.1 and 2.3, consider $\delta^{(m)}(x+1-t) \cdot \delta^{(n)}(x+t-1)$. Denote this by L . Then

$$L = \partial_x^{(m)}(f^*G) \cdot \partial_x^{(n)}(f^*K).$$

Let $f^{-1}(x, t) = (\alpha, \beta)$. So by the Chain Rule for Distributions [1, 6.1.2],

$$(\partial_x)^m(f^*G) = \frac{1}{2^m} f^*(\partial_\beta^m G), \quad (\partial_x)^n(f^*K) = \frac{(-1)^n}{2^n} f^*(\partial_\alpha^n K).$$

However, $(f^*A)(f^*B) = f^*(AB)$, so

$$L = \frac{(-1)^n}{2^{m+n}} f^*(\partial_\alpha^m G)(\partial_\beta^n K).$$

But $\partial_\beta K = 0$ and $\partial_\alpha G = 0$. Therefore,

$$\begin{aligned} \langle L, \phi \rangle &= \frac{(-1)^n}{2^{m+n}} \langle f^*(\partial_\alpha^m \partial_\beta^n GK), \phi \rangle \\ &= \frac{(-1)^n}{2^{m+n+1}} \langle \partial_\alpha^m \partial_\beta^n GK, \phi \circ f^{-1} \rangle \\ &= \frac{(-1)^m}{2^{m+n+1}} \langle GK, \partial_\alpha^m \partial_\beta^n (\phi \circ f^{-1}) \rangle \\ &= \frac{(-1)^m}{2^{m+n+1}} (\partial_\alpha^m \partial_\beta^n \phi) \circ f^{-1}(0, 0) \\ &= \frac{(-1)^m}{2^{m+n+1}} (\partial_x + \partial_t)^m (\partial_t - \partial_x)^n \phi(0, 1) \end{aligned}$$

Hence by expanding the binomial terms,

$$\langle L, \phi \rangle = \frac{(-1)^{m+n}}{2^{m+n+1}} \left(\sum_{k=0}^m \binom{m}{k} \partial_x^k \partial_t^{m-k} \right) \cdot \left(\sum_{l=0}^n (-1)^{n-l} \binom{n}{l} \partial_x^l \partial_t^{n-l} \right) \phi(0, 1).$$

Therefore,

$$\begin{aligned} L &= \frac{1}{2^{m+n+1}} \left(\sum_{k=0}^m \binom{m}{k} \partial_x^k \partial_t^{m-k} \right) \cdot \left(\sum_{l=0}^n (-1)^{n-l} \binom{n}{l} \partial_x^l \partial_t^{n-l} \right) \delta(x) \otimes \delta(t-1) \\ &= \frac{1}{2^{m+n+1}} \sum_{i=0}^{m+n} c_i (\partial_x^i \partial_t^{m+n-i}) \delta(x) \delta(t-1) \\ &= \frac{1}{2^{m+n+1}} \sum_{i=0}^{m+n} c_i \delta^{(i)}(x) \otimes \delta^{(m+n-i)}(t-1), \end{aligned}$$

where c_i are constants obtained from the multinomial expansion of the summations above, with $c_{m+n} = 1$, as required. \square

So now (3) becomes

$$\partial_t w = \frac{1}{2^{m+n+1}} \sum_{i=0}^{m+n} c_i \delta^{(i)}(x) \otimes \delta^{(m+n-i)}(t-1)$$

Theorem 3.6. [1, 8.2.9] For distributions S and T ,

$$WF(S \otimes T) \subseteq (WF(S) \times WF(T)) \cup ((\text{supp}(S) \times \{0\}) \times WF(T)) \cup (WF(S) \times (\text{supp}(T) \times \{0\})).$$

Theorem 3.7.

$$w(x, t) = \frac{1}{2^{m+n+1}} (\delta^{(m+n)}(x) \otimes H(t-1) + \sum_{i=0}^{m+n-1} c_i \delta^i(x) \otimes \delta^{m+n-i-1}(t-1))$$

uniquely satisfies (3) and (3'), where c_i are as in 3.5.

Proof. w given above must be checked to satisfy the differential equation and initial condition, and be unique.

$$\begin{aligned} \langle w_t, \phi \rangle &= - \langle w, \phi_t \rangle \\ &= - \frac{1}{2^{m+n+1}} \langle (\delta^{(m+n)}(x) \otimes H(t-1) + \sum_{i=0}^{m+n-1} c_i \delta^i(x) \otimes \delta^{m+n-i-1}(t-1)), \phi_t \rangle \\ &= - \frac{1}{2^{m+n+1}} \langle \delta^{(m+n)}(x), (\int_1^\infty \phi_t(x, t) dt) \rangle \\ &\quad + \frac{1}{2^{m+n+1}} \sum_{i=0}^{m+n} c_i \langle \delta^i(x) \otimes \delta^{m+n-i-1}(t-1), \phi_t \rangle \\ &= - \frac{1}{2^{m+n+1}} \langle \delta^{(m+n)}(x), (\phi(x, \infty) - \phi(x, 1)) \rangle \\ &\quad + \frac{1}{2^{m+n+1}} \sum_{i=0}^{m+n} c_i \langle \delta^i(x) \otimes \delta^{m+n-i}(t-1), \phi \rangle \\ &= \frac{1}{2^{m+n+1}} \langle \delta^{(m+n)}(x), \phi(x, 1) \rangle + \frac{1}{2^{m+n+1}} \sum_{i=0}^{m+n} c_i \langle \delta^i(x) \otimes \delta^{m+n-i}(t-1), \phi \rangle \\ &= \frac{1}{2^{m+n+1}} \langle \delta^{(m+n)}(x) \otimes \delta(t-1), \phi(x, t) \rangle \\ &\quad + \frac{1}{2^{m+n+1}} \sum_{i=0}^{m+n} c_i \langle \delta^i(x) \otimes \delta^{m+n-i}(t-1), \phi \rangle \\ &= \frac{1}{2^{m+n+1}} \langle \sum_{i=0}^{m+n} c_i \delta^i(x) \otimes \delta^{m+n-i}(t-1), \phi \rangle \end{aligned}$$

To check the initial condition (3') on w , the distribution needs to be restricted to the submanifold $t = 0$. It must be checked that this can be done, using theorems 2.6 and 3.6.

Let $R = \delta^{(i)}(x)$, $S = \delta^k(t-1)$ and $T = H(t-1)$.

If u is a distribution and P is a partial differential operator, then

$$WF(Pu) \subset WF(u).$$

So $WF(R) \subseteq WF(\delta(x)) = \{(0, \lambda) : \lambda \neq 0\}$, $WF(S) \subseteq \{(1, \mu) : \mu \neq 0\}$ and $WF(T) = \{(t, \mu) : t \geq 1; \mu \neq 0\}$. So by 3.6,

$$WF(R \otimes S) \subseteq \{(0, 1, \lambda, \mu)\} \cup \{(0, 1, 0, \mu)\} \cup \{(0, 1, \lambda, 0)\}$$

$$WF(R \otimes T) \subseteq \{(0, t, \lambda, \mu)\} \cup \{(0, t, 0, \mu)\} \cup \{(0, t, \lambda, 0)\},$$

where $t \geq 1, \lambda \neq 0, \mu \neq 0$. i.e. $WF(R \otimes S)$ and $WF(R \otimes T)$ are supported on $\{(0, t) : t \geq 1\}$, not intersecting the normal bundle to $\{t = 0\}$. So by 2.6, w may be restricted to this submanifold, where it is the distribution $\delta^{(m+n)}(x)H(-1) \equiv 0$, satisfying (3').

All that remains is to prove uniqueness of the solution. Suppose that there are two solutions w_1 and w_2 . Then let $z = w_1 - w_2$. Then z satisfies: $z_t = 0, z(x, 0) = 0$. So $z \equiv 0$, again by [1, 3.1.4]. Hence, $w_1 = w_2$, and the solution to (3) and (3') is unique. □

However, all but one of the terms in the solution $w(x, t)$ in 3.7 have support at $(0, 1)$, and hence do not propagate as time evolves. $\frac{1}{2^{m+n+1}}\delta^{(m+n)}(x) \otimes H(t - 1)$ is the only propagating term in the solution to (3). Hence, from now on, the solution to (3) and (3') will be taken to be

$$w(x, t) = \frac{1}{2^{m+n+1}}\delta^{(m+n)}(x) \otimes H(t - 1), \text{ with } (x, t) \neq (0, 1).$$

4. STRENGTHS:

This section considers the strengths of the solutions (u, v, w) . Let **strength** be defined as in 1.2. For example, $\delta(x)$ has strength zero, being the second distributional derivative of $(0 \vee |x|)$.

Lemma 4.1. *The strength of u is $-m$; the strength of v is $-n$.*

Proof. Recall that $u = \delta^{(m)}(x+1-t)$, where derivatives are taken in the x -direction. It is sufficient to prove that u is the $(m+2)nd$ distributional derivative of a function in the plane continuous in x . In order to do this, it must be checked that u can be restricted to any line parallel to the x -axis. This can be done, using 2.6.

Clearly, u is the m th derivative of $\delta(x + 1 - t)$. It is sufficient to prove that this distribution has strength zero in the x -direction, i.e. that it is the second derivative of a function in the plane continuous (but not differentiable) in x .

Now $\partial_x H(x + 1 - t) = \delta(x - 1 + t)$ and $\partial_x(0 \vee |x + 1 - t|) = H(x - 1 + t)$, in the sense of distributions, where $(0 \vee |x + 1 - t|)$ is continuous but non-differentiable in x . Similarly, it is easily seen that v has strength $-n$. □

Lemma 4.2. *w has strength $-(m + n)$ in the x -direction.*

Proof. Recall that $w = \delta^{(m+n)}(x) \otimes H(t - 1)$. Hence, $w \equiv 0$ when $t \leq 1$. w is the $(m + n)th$ derivative of $\delta(x) \otimes H(t - 1)$. Again, by 2.6, this may be restricted to any horizontal line $\{t = \text{constant}\}$, where it is a distribution of point support in \mathbb{R} , $\delta(x)$. By elementary distribution theory, this is the second derivative of $0 \vee |x|$, which is a continuous (but non-differentiable) function in x , as required. □

Corollary 4.3. *The strength of w is the sum of the strengths of u and v .*

Note that w is more singular than u or v , since its strength is smaller.

5. GENERALISATION OF INITIAL DATA:

The problem will now be generalised to the following:

$$(\partial_t + \partial_x)u = 0 \quad (4)$$

$$(\partial_t - \partial_x)v = 0 \quad (5)$$

$$\partial_t w = uv \quad (6)$$

with initial conditions

$$u(x, 0) = \sum_{i=1}^k \delta^{(m_i)}(x - x_i) \text{ for some } m, k, m_i \in \mathbb{N} \quad (4')$$

$$v(x, 0) = \sum_{j=1}^l \delta^{(n_j)}(x - x_j) \text{ for some } n, l, n_j \in \mathbb{N} \quad (5')$$

$$w(x, 0) = 0. \quad (6')$$

In the same way as in Part 2, the first two equations may be solved using techniques of linear partial differential equations, giving solutions:

$$u(x, t) = \sum_{i=1}^k \delta^{(m_i)}(x - x_i - t), \quad v(x, t) = \sum_{j=1}^l \delta^{(n_j)}(x - x_j + t).$$

Then to solve (6), an analogous technique to Part 3 is used to rewrite $u \cdot v$,

$$\begin{aligned} & \sum_i \sum_j \delta^{(m_i)}(x - x_i - t) \cdot \delta^{(n_j)}(x - x_j + t) \\ &= \sum_i \sum_j c_{ij} \frac{1}{2^{m_i+n_j+1}} \delta^{(m_i+n_j)}(x) \otimes \delta(t-1) \\ & \quad + \text{terms of the form } c_l \delta^l(x) \otimes \delta^{m_i+n_j-l}(t-1) \end{aligned}$$

where $0 < l \leq m_i + n_j$.

Then the property of superposition of linear waves is used to consider each of these terms separately in solving (6). Part 3 can be used to solve each individual term indexed by the pair (i, j) , then summing these solutions gives the solution to (6) above.

Hence, analogously to Part 3, (6) has solution:

$$w(x, t) = \sum \sum \frac{1}{2^{m_i+n_j+1}} \delta^{(m_i+n_j)}(x) \otimes H(t-1) \quad \forall (x, t) \neq 0.$$

Now that the system is solved, the question of strengths remains.

Lemma 5.1. *If f has strength a and g has strength b then $f + g$ has strength $\min\{a, b\}$.*

Proof. The strength is defined to be 2 plus the minimum number of derivatives that need to be taken in the x -direction in order for the resulting function to be continuous. The sum of two continuous functions is continuous. \square

Corollary 5.2. *The strength of w is the sum of the strengths of u and v .*

Proof. The strength of $u(x, t) = \sum_{i=1}^k \delta^{(m_i)}(x - x_i - t)$ is $\min_i(m_i)$, where $m_i = \text{strength}(\delta^{m_i}(x - x_i - t))$.

The strength of $v(x, t) = \sum_{j=1}^l \delta^{(n_j)}(x - x_j + t)$ is $\min_j(m_j)$, where $m_j = \text{strength}(\delta^{(n_j)}(x - x_j + t))$.

Since

$$w(x, t) = \sum_i \sum_j \frac{1}{2^{m_i+n_j+1}} \delta^{(m_i+n_j)}(x) \otimes H(t-1),$$

the strength of w equals

$$\begin{aligned} \min_{i,j}(\text{strength}(\delta^{(m_i+n_j)}(x) \otimes H(t-1))) &= \min_{i,j}(m_i + n_j) \\ &= \min_i(m_i) + \min_j(n_j) \\ &= \text{strength}(u) + \text{strength}(v) \end{aligned}$$

as required. \square

Thus, the results of Part 4 still hold after generalizing the initial data to sums of delta functions.

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DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY, DURHAM, NC 27708. USA
E-mail address: kirsten@math.duke.edu