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Uniqueness of rapidly oscillating periodic solutions to a singularly perturbed differential-delay equation *

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Abstract

In this paper, we prove a uniqueness theorem for rapidly oscillating periodic solutions of the singularly perturbed differential-delay equation $\varepsilon\dot{x}(t) = -x(t) + f(x(t-1))$. In particular, we show that, for a given oscillation rate, there exists exactly one periodic solution to the above equation. Our proof relies upon a generalization of Lin's method, and is valid under generic conditions.

1 Introduction

The singularly perturbed differential-delay equation

$$\varepsilon\dot{x}(t) = -x(t) + f(x(t-1)) \tag{1.1}$$

has been studied in detail over the past twenty years. A great deal of research has concentrated upon the relationship between the map dynamics generated by (1.1) when $\varepsilon = 0$, and the dynamics of (1.1) when ε is small [5, 8]. If we formally set $\varepsilon = 0$, then (1.1) reduces to the discrete difference equation

$$x(t) = f(x(t-1)). \tag{1.2}$$

In general, (1.2) exhibits a rich dynamical structure; if the nonlinear feedback term f is chosen properly, then equation (1.2) will generate chaotic dynamics [4]. In [8], f is chosen so that (1.2) possesses a locally asymptotically stable, period 2 orbit when $\varepsilon = 0$. More specifically, it is assumed that (1.2) has an asymptotically stable solution of the form

$$x_0(t) = \begin{cases} a\,, & t \in (2n, 2n+1) \\ -b\,, & t \in (2n+1, 2n+2) \end{cases}$$

for n in $\mathbb{Z}$. It can be seen that $x_0(t)$ has jumps at the countable set of points $n \in \mathbb{Z}$ but is otherwise smooth. When $\varepsilon > 0$ but small, it is shown in [8]

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that (under general conditions) there exists a smooth periodic solution which is uniformly close to $x_0(t)$ away from the points of discontinuity. Such a solution is said to have a square wave profile. In [6], the result in [8] is proved, using local bifurcation analysis. We will describe the general approach, loosely called Lin's method, in the Sections 2 and 3. This method is developed for slowly oscillating periodic solutions in Sections 2 and 3. In this paper, we will make similar assumptions about the shape of the feedback function f as in [6], but will concentrate upon rapidly oscillating periodic solutions - solutions $x(t)$ which cross the $x = 0$ axis more than once per unit time interval. We prove that, for a fixed oscillation rate, there exists at most one square wave periodic solution to (1.2) in the limit as $\varepsilon \to 0$. Our uniqueness result will be stated and proven precisely in Section 3. Our result gives additional, detailed information about the global attractor associated with equation (1.1). In [7], it is shown that the oscillation rate (defined appropriately) of a solution to (1.1) decreases monotonically over time. In addition, numerical experiments [2] suggest that periodic solutions which oscillate about the t axis more than once per delay interval are locally unstable. For a given oscillation rate, we show in this paper (the original result appears as Theorem 5.1 in Section 5) that there exists at least one rapidly oscillating periodic solution to (1.1), and that this solution is unique in the limit as ε approaches 0. We will rely upon Melnikov-type methods to prove this statement.

In [8], a change of variables is made to eliminate the explicit dependence of equation (1.1) upon the parameter ε. In particular, we assume that the period of $x_\varepsilon(t)$ is an integer multiple of $p(\varepsilon) = 2 + O(\varepsilon) = 2 + 2\varepsilon r$, where $r = r(\varepsilon)$, $y(t) = x(-\varepsilon rt)$ and $z(t) = x(-\varepsilon rt - 1 - \varepsilon r)$. (We note that we can study slowly and rapidly oscillating periodic solutions using this rescaling.) Substituting y and z into (1.1), we obtain the system of transition-layer equations

$$\begin{aligned} \dot{y}(t) &= ry(t) - rf(z(t-1))\,, \\ \dot{z}(t) &= rz(t) - rf(y(t-1)). \end{aligned} \tag{1.3}$$

We observe that the period 2 orbit $\{-b, a\}$ of (1.2) corresponds to two equilibrium points for (1.3), namely $(-b, a)$ and $(a, -b) \in \mathbb{R}^2$. It is known that (see [1]), when f is monotone decreasing (and under other technical conditions), there exists a unique value $r(0) > 0$ under which (2.1) possesses a heteroclinic orbit $(p(t), q(t))$ connecting $(-b, a)$ to $(a, -b)$. From the symmetry of (1.3), we also have that the orbit $(q(t), p(t))$ connects $(a, -b)$ to $(-b, a)$. In addition, at the point r_0, the heteroclinic orbits $(p(t), q(t))$ and $(q(t), p(t))$ are unique. In our analysis (following [6]), we shall assume the conclusion of the above statement, without assuming that f is monotone.

We may formally regard (1.3) as an evolutionary system with respect to the phase space $C([-1, 0], \mathbb{R}^2)$, and shall look for a periodic solution $(y_t(\cdot), z_t(\cdot))$ of (2.1) which approaches a chain of heteroclinic solutions as $r \to r_0$. The heteroclinic chain consists of the equilibria $(-b, a)$ and $(a, -b)$ and the orbits $(p_t(\cdot), q_t(\cdot))$ and $(q_t(\cdot), p_t(\cdot))$. The periodic solutions we consider need not connect after only one loop, in contrast to [6].

Assuming that the period $4w$ of $(y_t(\cdot), z_t(\cdot))$ is large and positive, we know that ε, r and w are related by the equation $4\varepsilon rw = 2 + 2\varepsilon r$, or

$$\varepsilon = \frac{1}{(2w-1)r(w)}. \tag{1.4}$$

It is shown in [8] that, under appropriate conditions, there exists a periodic solution $(y_t(\cdot), z_t(\cdot)) \in C([-1,0], \mathbb{R}^2)$ which lies uniformly close to the heteroclinic chain given above, whenever $|r - r_0|$ is sufficiently small. In [6], Mel'nikov's method is generalized and applied, and we shall use some of the notation and technique of this paper.

2 Technical Assumptions

As in [6], we will need to make the following assumptions, labeled (A1)-(A5).

(A1) The equilibria $p_1 = (-b, a)$ and $p_2 = (a, -b)$ of (1.3) are hyperbolic for all $r \in [r_1, r_2]$, where $0 < r_1 < r_2$. In particular, we shall assume that, for $\Omega \subset \mathbb{C}$ the spectrum of the linear system

$$\begin{aligned} \dot{y}(t) &= r_0 y(t) - r_0 f'(p_i) z(t-1) \\ \dot{z}(t) &= r_0 z(t) - r_0 f'(p_i) y(t-1) \end{aligned} \tag{2.1}$$

with $i = 1, 2,\quad 0 < \rho = \min\{Re\lambda : \lambda \in \Omega\ ,\ \ Re\lambda > 0\}$, and $0 < \gamma = \min\{-Re\lambda : \lambda \in \Omega\ ,\ \ Re\lambda < 0\}$.

(A2) There exists a unique element $r_0 \in [r_1, r_2]$ such that (1.3), $r = r_0$, possesses a heteroclinic solution. The solutions $(p_t(\cdot), q_t(\cdot))$ and $(q_t(\cdot), p_t(\cdot))$ are the only orbits connecting $(-b, a), (a, -b)$ to $(a, -b), (-b, a)$, respectively.

(A3) The linear variational system

$$\begin{aligned} \dot{y}(t) &= r_0 y(t) - r_0 Df(q(t-1)) z(t-1) \\ \dot{z}(t) &= r_0 z(t) - r_0 Df(p(t-1)) y(t-1) \end{aligned} \tag{2.2}$$

possesses a 1-dimensional linear space of bounded solutions, spanned by $(\dot{p}(t), \dot{q}(t))$.

System (2.2) generates a non-autonomous linear semiflow operator, which we denote by

$$T(t,s) : C([-1,0], \mathbb{R}^2) \to C([-1,0], \mathbb{R}^2).$$

Also, from assumption (A1), we know that $T(t,s)$ possesses an exponential dichotomy on the intervals $(-\infty, -\tau]$ and $[\tau, \infty)$, where $\tau > 0$ is large. In particular, using the notation in [1], there exist projections

$$P_u(s), P_s(s) : C([-1,0], \mathbb{R}^2) \to C([-1,0], \mathbb{R}^2), s \in I_0$$

and also constants $K \geq 0$ and $\alpha > 0$ (dependent on τ), such that the following properties hold. Here I_0 is one of the intervals $(-\infty, -\tau]\ ,\ \ [\tau, \infty)$.

(i) $P_u(s) + P_s(s) = I$, the identity operator, for all $s \in I_0$.

(ii) P_u and P_s are strongly continuous in s.

(iii) $T(t,s)P_s(s) = P_s(t)T(t,s)$ for any $t \geq s$, $s, t \in I_0$.

(iv) $T(t,s) : \operatorname{Ran}(P_u(s)) \to \operatorname{Ran}(P_u(t))$ defines an isomorphism, with inverse $T(s,t) : \operatorname{Ran}(P_u(t)) \to \operatorname{Ran}(P_u(s))$.

(v) $|T(t,s)P_s(s)| \leq Ke^{-\alpha(t-s)}, t \geq s$, and $|T(s,t)P_u(t)(t)| \leq Ke^{-\alpha;(t-s)}, t \geq s$, where "$|\cdot|$" denotes the sup norm in $C([-1,0],\mathbb{R}^2)$.

We next define the backward evolutionary operator $T^*(s,t) : C^*([-1,0],\mathbb{R}^2) \to C^*([-1,0],\mathbb{R}^2)$ defined by

$$\langle \phi^*, T(t,s)\phi \rangle = \langle T^*(s,t)\phi^*, \phi \rangle \,,$$

where $\phi \in C([-1,0],\mathbb{R}^2)$ and $\phi^* \in C^*([-1,0],\mathbb{R}^2)$. For $\phi \in C([-1,0],\mathbb{R}^2)$ and $\psi \in C^*([-1,0],\mathbb{R}^2)$, we have used the convention

$$\langle \phi, \psi \rangle = \int_{-1}^{0} \phi(\theta)\psi(\theta)d\theta \,.$$

Since $T(t,s)$ admits an exponential dichotomy, it follows that $T^*(s,t)$ must also admit an exponential dichotomy on the intervals $(-\infty,-\tau]$ and $[\tau,\infty)$, with associated projections $P_u^*(s), P_s^*(s) : C^*([-1,0],\mathbb{R}^2) \to C^*([-1,0],\mathbb{R}^2)$. From assumption (A3) and property (iv), we deduce that $\dim\operatorname{Ran} P_u(-\tau) = \dim\operatorname{Ran} P_u(\tau) = 1$. We shall also assume that

(A4) $\operatorname{Ran} P_u(-\tau) \cap [T(\tau,-\tau)]^{-1}\operatorname{Ran} P_s(\tau)$ is a one dimensional subspace of $C([-1,0],\mathbb{R}^2)$ spanned by ψ_0.

We will choose $\alpha > 0$ to be as close to $\min\{\rho,\gamma\}$ as necessary in what follows. From (A4), we may define the operator $\mathcal{F} : \operatorname{Ran} P_u(-\tau) \times \operatorname{Ran} P_s(\tau) \to C([-1,0],\mathbb{R}^2)$ by $\phi = v - T(\tau,-\tau)u$. Thus, ϕ approximates (up to first order) the distance between the unstable manifold of each equilibrium, translated forward by an amount 2τ, and the stable manifold of the other equilibrium. We also define the adjoint operator $\mathcal{F}^*$ of $\mathcal{F}$ by $\mathcal{F}^* : C^*([-1,0],\mathbb{R}^2) \to \operatorname{Ran} P_u^*(-\tau) \times \operatorname{Ran} P_s^*(\tau), \mathcal{F}^* : \phi \longmapsto (u^*, v^*)$, and $\mathcal{F}^*\phi^* = (-T^*(-\tau,\tau)\phi^*, \phi^*)$. The following technical lemma, which appears in [6], is stated without proof.

Lemma 2.1 $\mathcal{F} : \operatorname{Ran} P_u(-\tau) \times \operatorname{Ran} P_s(\tau) \to C([-1,0],\mathbb{R}^2)$ *is a Fredholm operator, with* $\dim\ker\mathcal{F} = \operatorname{codim}\operatorname{Ran}\mathcal{F} = 1$. *Thus the index of* $\mathcal{F}$ *is zero,* $\operatorname{ind}(\mathcal{F}) = 0$. *In particular,*

(i) $\ker\mathcal{F} = \big\{(u,r) \in \operatorname{Ran} P_u(-\tau) \times \operatorname{Ran} P_s(\tau) : u = \xi u_0, r = T(\tau,-\tau)u,\ \xi \in \mathbb{R}\big\}$*;*

(ii) $\ker\mathcal{F}^* = \big\{\xi u_0^* \in \operatorname{Ran} P_u(-\tau) : T^*(-\tau,\tau)u_0^* \in \operatorname{Ran} P_s^*(-\tau), \xi \in \mathbb{R}\big\}$*;*

(iii) $\operatorname{Ran}\mathcal{F} = \{\phi \in C([-1,0],\mathbb{R}^2) :< u_0, \phi\rangle = 0\}$.

We will construct $\mathcal{F}^*$ from the semiflow $T^*(s,t)$, where T^* is the solution map associated with the formal adjoint system

$$\begin{aligned} \dot{y}(t) &= -r_0 y(t) + r_0 Df(p(t))z(t+1) \\ \dot{z}(t) &= -r_0 z(t) + r_0 Df(q(t))y(t+1). \end{aligned} \tag{2.3}$$

Given assumption (A3), it follows from the Fredholm alternative that System (2.3) also possesses a one-dimensional linear subspace of globally bounded solutions. Indeed, $\dim\ker\mathcal{F}^* = \dim\ker\mathcal{F} = 1$. We denote a basis for $\ker\mathcal{F}^*$ by $\psi_t(\cdot) \in C([-1,0],\mathbb{R}^2)$.
Lastly, we impose the Mel'nikov-type condition

(A5) $\int_{-\infty}^{\infty}\psi(t)\cdot(p(t),q(t))dt = C \neq 0$.

This condition allows us to perform the local bifurcation analysis that we need in this section and the next.

3 A Brief Review of Lin's results - The Slowly Oscillating Case

We start by reformulating equation (2.1) in a functional-analytic setting. If we set $\gamma(t) = (y(t), z(t))^T$ and $F(\gamma) = (f(z), f(y))^T$, we can rewrite (2.1) as

$$\dot{\gamma}(t) = r\gamma(t) - rF(\gamma(t-1)). \tag{3.1}$$

Suppose that $\gamma_1(t)$ satisfies (3.1) and that J is the permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It follows that $\gamma_2(t) = J\gamma_1(t)$ must also satisfy (3.1), since

$$\begin{aligned} \dot{\gamma}_2(t) &= J\dot{\gamma}_1(t) = J(r\gamma_1(t) - rF(\gamma_1(t-1))) \\ &= rJ\gamma_1(t) - rF(J\gamma_1(t-1)) = r\gamma_2(t) - rF(\gamma_2(t-1)). \end{aligned}$$

Next we set $r = r_0$ and denote the (unique) heteroclinic solutions $(p(t), q(t))^T$ of (3.1) by $W_1(t)$ and $W_2(t) = JW_1(t)$. If we define $\gamma_1(t) = W_1(t) + \eta_1(t)$ and $\gamma_2(t) = W_2(t) + \eta_2(t)$, where $\eta_1(t)$ and $\eta_2(t)$ are small for all t, we may rewrite (3.1) in variational form as

$$\dot{\eta}_i(t) = r_0\eta_i(t) - r_0 DF(W_i(t-1))\eta_i(t-1) + N(\eta_i(t-1), r, t-1) \tag{3.2}$$

with remainder term

$$\begin{aligned} N(\eta_i(t), r, t) &= -rF(W_i(t) + \eta_i(t)) + r_0F(W_i(t)) + r_0DF(W_i(t))\eta_i^2(t) \\ &\quad +(r - r_0)(W_i(t) + \eta_i(t)) = O(|\eta_i(t)| + |r - r_0|). \end{aligned} \tag{3.3}$$

Here $i \equiv i \bmod (4n+2)$, and (3.2) is valid for $\eta_1(t)$ close to $W_1(t)$, and $\eta_2(t)$ close to $W_2(t)$, respectively (we choose $i \equiv i \bmod (4n+2)$ to cover the general case where a solution undergoes 2n+1 oscillations before repeating). We next define

$$\eta_{it}(\cdot) = \eta_i(t+\cdot)\ ,\ W_{it}(\cdot) = W_i(t+\cdot) \in C([-1,0],\mathbb{R}^2);$$

hence, using the abstract variation-of-constants formula in $C([-1,0],\mathbb{R}^2)$, we may rewrite (3.2) in integral form as

$$\eta_{it} = T^i(t,\sigma)\eta_{i\sigma} + \int_\sigma^t T^i(t,s)X_0N(\eta_{is}(-1),r,s-1)\,ds, \tag{3.4}$$

with boundary conditions

$$\eta_{(i-1)w} - \eta_{i(-w)} = W_{i(-w)} - W_{(i-1)w} = b_i \in C([-1,0],\mathbb{R}^2), \tag{3.5}$$

where $T^i(t,s) : C([-1,0],\mathbb{R}^2) \to C([-1,0],\mathbb{R}^2)$ is the linear solution map associated with the equation

$$\dot\eta_i(t) = r_0\eta_i(t) - r_0DF(W_i(t-1))\eta_i(t-1) \tag{3.6}$$

and $X_0 : C([-1,0],\mathbb{R}^2) \to \mathbb{R}^2$ is the evaluation operator defined by $X_0\phi(\cdot) = \phi(0)$ for any $\phi \in C([-1,0],\mathbb{R}^2)$. The boundary conditions in (3.5) follow from the continuity condition

$$\gamma_{(i-1)w} = \eta_{(i-1)w} + W_{(i-1)w} = \eta_{i(-w)} + W_{i(-w)} = \gamma_{i(-w)}.$$

Definition 3.1 For $i \in \mathbb{Z}$, define $E([-w_i,w_i],\Delta)$ as the linear space of functions $\underline{\eta} = (\eta_{1\cdot},\eta_{2\cdot})$, where $\eta_{it}(\cdot) \in C([-1,0],\mathbb{R}^2)$ for each $t \in [-w_i,w_i]\backslash[\tau,\tau+1)$ and η_i has jumps at $t=\tau$ along the direction Δ_i. In this space we define the norm $\|\underline{\eta}\|_E = \max_i \sup_{t\in[-w_i,w_i]} |z_{it}(\cdot)|$, where $|\cdot|$ denotes the supremum norm in $C([-1,0],\mathbb{R}^2)$.

Definition 3.2 A neighborhood $U_{\varepsilon_1,\varepsilon_2}(0)$ in $E([-w_i,w_i],\Delta_i)\times\mathbb{R}$ is defined as

$$U_{\varepsilon_1,\varepsilon_2} = \left\{(\underline{\eta},r) : \underline{\eta} \in E([-w_i,w_i],\Delta), r\in\mathbb{R}, \|\underline{\eta}\|_E < \varepsilon_1, |r-r_0| < \varepsilon_2\right\}.$$

We are now ready to review the main results in [6], where existence and uniqueness properties of slowly oscillating periodic solutions to (1.1) (for $\varepsilon > 0$ small) are studied.

Lemma 3.1 ([6]) *Suppose that (A1)-(A5) are valid. Then there exist constants $\hat w$, $\varepsilon_0 > 0$ with the following property. If $w > \hat w$ and $|r-r_0| < \varepsilon_0$, then there exists a unique piecewise continuous solution $\underline{\eta} \in E([-w,w],\Delta)\times\mathbb{R}$ of (3.4), (3.5) with $\langle\psi_i,\eta_{i(-\tau)}\rangle = 0$, $i=1,2$.*

Let ξ be a real number such that $\eta_{i\tau-} - \eta_{i\tau+} = \xi_i\Delta_i$. Then

$$\xi_i = \int_{-w}^{w} \psi_{is}(-1)N(\eta_{is}(-1),r,s-1)\,ds + \langle\psi_{i(-w)}(\cdot),\eta_{i(-w)}(\cdot)\rangle - \langle\psi_{iw}(\cdot),\eta_{iw}(\cdot)\rangle.$$

The piecewise continuous solution of (3.4), (3.5) will be denoted by $(\underline{\eta}, r)$, with $\eta_{it} = x_i(t; \underline{b}, r, w)$.

Since ξ_i depends upon the parameters w, r, we set $\xi_i = G_i(w, r)$. $G_i : \mathbb{R}^2 \to \mathbb{R}$ is called a Mel'nikov function and measures the magnitude of the jump in $\eta_{it}(\cdot)$ at time $t = \tau$ along the direction Δ_i. We next review the two fundamental theorems which appear in §5 of [6].

Theorem 3.1 *Suppose that (A1)-(A5) are satisfied. Then there exist positive constants $\hat{w}$, ε, ε_0, and a continuous function $r^* : (\hat{w}, \infty) \to (r - \varepsilon_0, r + \varepsilon_0)$ with the following property. For each $w > \hat{w}$, system (1.3) possesses a $4w$-periodic solution $(y(t), z(t))$ with $y(t + 2w) = z(t)$ and $|y(t) - p(t)| + |z(t) - q(t)| < \varepsilon$, $t \in [-w, w]$, if and only if $r = r^*(w)$. In addition, for $r = r^*(w)$, the periodic solution is unique up to time translations.*

Proof: From Lemma 3.1, we need to solve the bifurcation equations

$$\begin{aligned} G_1(w,r) \quad &= \quad \int_{-w}^{w} \psi_{1s}(-1) N(\eta_{1s}(-1), r, s-1)\, ds \\ &\quad + \langle \psi_{1(-w)}(\cdot), \eta_{1(-w)}(\cdot)\rangle - \langle \psi_{1w}(\cdot), \eta_{1w}(\cdot)\rangle \end{aligned}$$

and

$$\begin{aligned} G_2(w,r) \quad &= \quad \int_{-w}^{w} \psi_{2s}(-1) N(\eta_{2s}(-1), r, s-1)\, ds \\ &\quad + \langle \psi_{2(-w)}(\cdot), \eta_{2(-w)}(\cdot)\rangle - \langle \psi_{2w}(\cdot), \eta_{2w}(\cdot)\rangle \end{aligned}$$

for w fixed. This follows from the fact that, for any pair (w, r), there exists a solution $\underline{\eta} \in E([-w, w], \Delta)$ to (3.4), (3.5) with $\langle \eta_{1\tau^-} - \eta_{1\tau^+}, \Delta_1\rangle = G_1(w, r)$ and $\langle \eta_{2\tau^-} - \eta_{2\tau^+}, \Delta_2\rangle = G_2(w, r)$. Now we know that $\psi_2 = J\psi_1$ and $\eta_2 = J\eta_1$. Since

$$\begin{aligned} &N(\eta_{2t}(-1), r, t-1) \\ &\quad = \quad N(J\eta_{1t}(-1), r, t-1) \\ &\quad = \quad -rF(JW_{1t}(-1) + J\eta_{1t}(-1)) + r_0 F(JW_{1t}(-1)) \\ &\qquad\quad + r_0 DF(JW_{1t}(-1)) J\eta_{1t}(-1) + (r - r_0)(JW_{1t}(-1) + J\eta_{1t}(-1)) \\ &\quad = \quad JN(\eta_{1t}(-1), r, t-1)\,, \end{aligned}$$

it follows that

$$\begin{aligned} G_2(w,r) \quad &= \quad \int_{-w}^{w} J\psi_{1s}(-1) JN(\eta_{1s}(-1), r, s-1)\, ds \\ &\quad + \langle J^2\psi_{1(-w)}(\cdot), \eta_{1(-w)}(\cdot)\rangle - \langle J^2\psi_{1w}(\cdot), \eta_{1w}(\cdot)\rangle \\ &= \quad G_1(w,r). \end{aligned}$$

Notice here that $DF(JW_{1t}(-1)) = DF(W_{1t}(-1))$ and $J^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Given the above calculations, it is sufficient to solve the Mel'nikov-type equation $G_1(w,r)$ alone, since (by symmetry) $G_1(w,r)=0$ implies that $G_2(w,r)=0$. Hence, we apply the implicit function theorem and condition (A5). Given that $G_1(\infty,r_0)=0$, we compute $\frac{\partial G_1(\infty,r_0)}{\partial r}$. We have

$$\begin{aligned}
&\frac{\partial G_1(w,r)}{\partial r}\\
&= \frac{\partial}{\partial r}\Big[\int_{-\infty}^{\infty}\psi_{1s}(-1)(-rF(W_{1s}(-1)+\eta_{1s}(-1))+r_0F(W_{1s}(-1))\\
&\quad +r_0DF(W_{1s}(-1))\eta_{1s}(-1))\eta_{1s}(-1)+(r-r_0)(W_{1s}(-1)+\eta_{1s}(-1))\,ds\\
&\quad +\langle\psi_{1(-w)}(\cdot),\eta_{1(-w)}(\cdot)\rangle-\langle\psi_{1w}(\cdot)\rangle\Big]
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial G_1(\infty,r_0)}{\partial r} &= \int_{-\infty}^{\infty}\psi_{1s}(-1)\frac{\partial}{\partial r}\Big[-rF(W_{1s}(-1)+\eta_{1s}(-1))+r_0F(W_{1s}(-1))\\
&\quad +r_0DF(W_{1s}(-1))\eta_{1s}(-1)+(r-r_0)(W_{1s}(-1)+\eta_{1s}(-1))\Big]\,ds
\end{aligned}$$

since $\psi_{1t}(\cdot)$ is independent of r and $\lim_{t\to\pm\infty}|\psi_{1t}(\cdot)|=0$. Computing further, we obtain

$$\begin{aligned}
&\frac{\partial G_1(\infty,r_0)}{\partial r}\\
&= \int_{-\infty}^{\infty}\psi_{1s}(-1)[-F(W_{1s}(-1)+\eta_{1s}(-1))+(W_{1s}(-1)+\eta_{1s}(-1))]\,ds\\
&= \int_{-\infty}^{\infty}\psi_{1s}(-1)[-F(W_{1s}(-1))+W_{1s}(-1)]\,ds\\
&\quad +\int_{-\infty}^{\infty}\psi_{1s}(-1)[-DF(W_{1s}(-1))\eta_{1s}(-1)+O(|\eta_{1s}(-1)|^2)+\eta_{1s}(-1)]\,ds.
\end{aligned}$$

Since $\|\underline{\eta}\|_E$ approaches 0 as w goes to ∞, we must have that

$$\begin{aligned}
\frac{\partial G_1(\infty,r_0)}{\partial r} &= \int_{-\infty}^{\infty}\psi_{1s}(-1)[-F(W_{1s}(-1))+W_{1s}(-1)]\,ds\\
&= -\int_{-\infty}^{\infty}\psi_{1s}(-1)\cdot(q(s-1),p(s-1))\,ds\\
&= \int_{-\infty}^{\infty}\psi_1(s-1)\cdot(p(s-1),q(s-1))\,ds=C\neq 0\,,
\end{aligned}$$

from assumption (A5). Thus there exists a unique function $r=r^*(w)$ which satisfies the bifurcation equation $G_1(w,r)=0$ for all $w>\hat{w}$, and the proof of Theorem 3.1 is complete. $\diamondsuit$

The next theorem allows us to establish a bijection between the solutions $(y(t),z(t))$ of (1.3) with long period $4w$ and square wave solutions $x_\varepsilon(t)$ of (1.1) with period $2+2r\varepsilon$. The crux of the proof is to show that ε is strictly decreasing in w. It will turn out that the technique of proof can be modified to show that the period $p(\varepsilon)$ of $x_\varepsilon(t)$ is monotone increasing in ε for $\varepsilon > 0$ small.

Theorem 3.2 ([6]) *Suppose that (A1)-(A5) are valid. Then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1)$, there exist unique $w \in (\hat{w}, \infty)$ and $r \in (r_0 - \varepsilon_0, r_0 + \varepsilon_0)$ with the following property. Equation (1.1) admits a unique periodic solution $x_\varepsilon(t)$ with period $p(\varepsilon) = 2 + 2\varepsilon r$ that satisfies the estimate $|x_\varepsilon(-\varepsilon rt) - p(t)| + |x_\varepsilon(-\varepsilon rt - 1 - \varepsilon r) - q(t)| < \varepsilon_2$ for all $t \in [-w, w]$.*

The proof of Theorem 3.2 relies upon the following lemma, which again appears in [6].

Lemma 3.2 *Suppose that $\eta^1_{it} = \eta_i(t; \underline{b}, r, w_1)$ and $\eta^2_{it} = \eta_i(t; \underline{b}, r, w_2)$ satisfy (3.4), (3.5), with $w_1, w_2 > \hat{w}$. Also suppose that, for any $\eta_1 \in E([-w_j, w_j], \Delta_j)$, $j = 1, 2$, we define the σ-weighted norm $\|\eta_i\|_\sigma = \|\eta_i\|_{E_j}(e^{-\sigma(w_i+\cdot)} + e^{-\sigma(w_i-\cdot)})^{-1}$, where $0 < \sigma < \alpha$. It follows that $\|\eta^2_i - \eta^1_i\|_\sigma = O(w_2 - w_1)$.*

Proof of Theorem 3.2 We start by writing an exponential estimate for the quantity $G_1(w_2, r) - G_1(w_1, r)$. In particular,

$$
\begin{aligned}
&|G_1(w_2,r) - G_1(w_1,r)| \\
&\quad\leq \Big| \int_{-w_2}^{w_2} \psi_{1s}(-1) \left[N(\eta^2_{1s}(-1), r, s-1) - N(\eta^1_{1s}(-1), r, s-1) \right] ds \Big| \\
&\qquad + \Big| \langle \psi_{1(-w_2)}(\cdot), \eta^2_{1(-w_2)}(\cdot) \rangle - \langle \psi_{1w_2}(\cdot), \eta^2_{1w_2}(\cdot) \rangle \\
&\qquad - \langle \phi_{1(-w_1)}(\cdot), \eta^1_{1(-w_1)}(\cdot) \rangle + \langle \psi_{1w_1}(\cdot), \eta^1_{1w_1}(\cdot) \rangle \Big|.
\end{aligned}
$$

We call the first term in absolute value I and the second II, and estimate these terms separately. Assuming without loss of generality that $\hat{w} < w_1 < w_2$, we have

$$
\begin{aligned}
I \quad \leq\quad & | \int_{-w_2}^{-w_1} \psi_{1s}(-1) N(\eta^2_{1s}(-1), r, s-1) ds | \\
& + | \int_{w_1}^{w_2} \psi_{1s}(-1) N(\eta^2_{1s}(-1), r, s-1)\, ds | \\
& + | \int_{-w_1}^{w_1} \psi_{1s}(-1) [N(\eta^2_{1s}(-1), r, s-1) - N(\eta^1_{1s}(-1), r, s-1)]\, ds | \\
\leq\quad & (w_2 - w_1) \big[\sup_{s \in [-w_1, w_2]} |\psi_{1s}(-1)| |N(\eta^2_{1s}(-1), r, s-1)| \\
& + \sup_{s \in [w_1, w_2]} |\psi_{1s}(-1)| |N(\eta^2_{1s}(-1), r, s-1)| \big] \\
& + | \int_{-w_1}^{w_1} \psi_{1s}(-1) [-r(F(W_{1s}(-1) + \eta^2_{1s}(-1)) - F(W_{1s}(-1) + \eta^1_{1s}(-1))) \\
& + r_0 DF(W_{1s}(-1))(\eta^2_{1s}(-1) - \eta^1_{1s}(-1)) + (r - r_0)(\eta^2_{1s}(-1) - \eta^1_{1s}(-1))]\, ds |
\end{aligned}
$$

$$
\begin{aligned}
&\le C(w_2-w_1)e^{-\alpha w_1}\\
&\quad +\Big|\int_{-w_1}^{w_1}\psi_{1s}(-1)[C|\eta^2_{1s}(-1)|+C|\eta^2_{1s}(-1)-\eta^1_{1s}(-1)|^2]\,ds\Big|\\
&\le C(w_2-w_1)e^{-\alpha w_1}\\
&\quad \times C\int_{-w_1}^{w_1}|\psi_{1s}(-1)|\|\eta^2_{1s}(-1)-\eta^1_{1s}(-1)\|_\sigma(e^{-\sigma(w_1-s)}+e^{-\sigma(w_1+s)})\,ds\\
&\le C(w_2-w_1)e^{-\alpha w_1}+C\int_{-w_1}^{w_1}e^{-\alpha|s|}(w_2-w_1)(e^{-\sigma(w_1-s)}+e^{-\sigma(w_1+s)})\,ds,
\end{aligned}
$$

using Lemma 3.2. Formally integrating the last term yields

$$
I\le C(w_2-w_1)e^{-\alpha w_1}+C(w_2-w_1)e^{-\sigma w_1}\le C(w_2-w_1)e^{-\sigma w_1}
$$

for any $\sigma\in(0,\alpha)$. Now we estimate II.

$$
\begin{aligned}
II &\le |\langle\psi_{1(-w_2)}(\cdot),\eta^2_{1(-w_2)}(\cdot)\rangle-\langle\psi_{1(-w_1)}(\cdot),\eta^1_{1(-w_1)}(\cdot),\eta^1_{1(-w_1)}(\cdot)\rangle|\\
&\quad +|\langle\psi_{1w_2}(\cdot),\eta^2_{1w_2}(\cdot)\rangle-\langle\psi_{1w_1}(\cdot),\eta^1_{1w_1}(\cdot)\rangle|\\
&\le |\langle\psi_{1(-w_2)}(\cdot),\eta^2_{1(-w_2)}(\cdot)\rangle-\langle\psi_{1(-w_1)}(\cdot),\eta^2_{1(-w_2)}(\cdot)\rangle|\\
&\quad +|\langle\psi_{1(-w_1)}(\cdot),\eta^2_{1(-w_2)}(\cdot)-\langle\psi_{1(-w_1)}(\cdot),\eta^1_{1(-w_1)}(\cdot)\rangle|\\
&\quad +|\langle\psi_{1w_2}(\cdot),\eta^2_{1w_2}(\cdot)\rangle-\langle\psi_{1w_1}(\cdot),\eta^2_{1w_2}(\cdot)\rangle|\\
&\quad +|\langle\psi_{1w_1}(\cdot),\eta^2_{1w_2}(\cdot)\rangle-\langle\psi_{1w_1}(\cdot),\eta^1_{1w_1}(\cdot)\rangle|\\
&\le C\big(\sup_{w\ge w_1}|\dot\psi_{1w}(-1)|(w_2-w_1)+\sup_{w\ge w_1}|\psi_{1w}(-1)|(w_2-w_1)\big)\\
&\le C(w_2-w_1)e^{-\alpha w}.
\end{aligned}
$$

Thus $|G_1(w_2,r)-G_1(w_1,r)|\le(w_2-w_1)e^{-\sigma w}$. Next we apply assumption (A5) to estimate the quality $|G_1(w_2,r_2)-G_1(w_1,r_1)|$, where $G_1(w_1,r_1)=G_1(w_2,r_2)=0$. Since $\frac{\partial G_1(r_0,\infty)}{\partial r}=C\ne0$, we know that, for $\hat w$ sufficiently large and $\varepsilon_0>0$ sufficiently small, $\inf_{w,r}\left|\frac{\partial G_1(r,w)}{\partial r}\right|\ge\frac{c}{2}$, where $w>\hat w$ and $|r-r_0|<\varepsilon_0$. Thus, setting $r_1=r^*(w_1)$ and $r_2=r^*(w_2)$, we have

$$
\begin{aligned}
\frac{c}{2}|r_2-r_1| &\le |G_1(w_1,r_2)-G_1(w_1,r_1)|=|G_1(w_1,r_2)|\\
&= |G_1(w_2,r_2)-G_1(w_1,r_2)|\le Ce^{-\sigma w_1}(w_2-w_1).
\end{aligned}
$$

It follows directly that $\frac{|r^*(w_2)-r^*(w_1)|}{|w_2-w_1|}\le Ce^{-\sigma r_1}$ for some constant $c>0$.

The above estimate enables us to show that, for $w_2>w_1>\hat w$, $\varepsilon(w_1)-\varepsilon(w_2)>0$ and thus ε is monotone decreasing in $w>\hat w$. In particular,

$$
\begin{aligned}
\varepsilon(w_1)-\varepsilon(w_2) &= \frac{1}{(2w_1-1)r^*(w_1)}-\frac{1}{(2w_2-1)r^*(w_2)}\\
&= \frac{(2w_2-1)r^*(w_2)-(2w_1-1)r^*(w_1)}{(2w_1-1)(2w_2-1)r^*(w_1)r^*(w_2)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(2w_2-1)r^*(w_2)-(2w_1-1)(r^*(w_2)+O((w_2-w_1)e^{-\sigma w_1}))}{(2w_1-1)(2w_2-1)r^*(w_1)r^*(w_2)} \\
&= \frac{2(w_2-w_1)r^*(w_2)+(w_2-w_1)O(w_1e^{-\sigma w_1})}{(2w_1-1)(2w_2-1)r^*(w_1)r^*(w_2)}.
\end{aligned}
$$

Since r is a continuous function of w, it follows that $r(w_2)$ is bounded away from 0 for w_2 large. Thus for $w_1 > \hat{w}$ and sufficiently large, $O(w_1e^{-\sigma w_1})$ is small and the quotient is strictly positive. This completes the proof.

4 Existence of Rapidly Oscillating Periodic Solutions

In Section 3, we summarized Lin's existence and uniqueness proof for slowly oscillating periodic solutions to (1.1) when ϵ is small and positive. Here we move our attention to existence and uniqueness properties of rapidly oscillating periodic solutions. The analysis in this section leads to a new uniqueness result in Section 5.

We start by giving a precise description of what it means for a solution to be *rapidly oscillating* [7]. We fix $t \in \mathbb{R}$, also set $x(t,\varphi) = x_t(\varphi, 0)$ and $\sigma = \sigma(t) = \inf\{s : s \geq t,\ x(s,\varphi) = 0\}$. We then define the integer-valued functional, $V : A \times \mathbb{R} \to \mathbb{N} \cup (+\infty)$, $V(x_t(\varphi,\cdot))$ as the number of elements in the set $s = \{s_0 \in (\sigma-1,\sigma] : x(s_0,\varphi) = 0\}$. V defines the oscillation rate of a solution to (1.1). A periodic solution $x_\varepsilon(t,\varphi)$ to (1.1) is said to be rapidly oscillating if $V(x_t(\varphi,\cdot)) > 1$ for all t (given the initial condition φ). We would like to establish conditions under which a solution $(y_t(\cdot), z_t(\cdot)) \in C([-1,0],\mathbb{R}^2)$ satisfies (1.3) and oscillates $(2n+1)$-times before repeating exists. Equivalently, we would like to find conditions under which the system of boundary-value problems

$$
\eta_{it} = T^i(t,\sigma)\eta_{i\sigma} + \int_\sigma^t T^i(t,s)X_0N(\eta_{is}(-1), r, s-1)\,ds \tag{4.1}
$$

$$
\eta_{(i-1)w_i} - \eta_{i(-w_i)} = W_{i(-w_i)} - W_{(i-1)w_i} = b_i \tag{4.2}
$$

with $t \in [-w_i, w_{i+1}]$, $b_i \in C([-1,0],\mathbb{R}^2)$, and $i \equiv i \bmod (4n+2)$ possesses a solution in

$$
E([-w_1,w_2],\Delta_1) \times \cdots \times E([-w_{4n+2},w_1],\Delta_{4n+2})
$$

without any jump discontinuities along the directions Δ_i (defined in Lemma 4.1). Lemma 4.1 is proved using the contraction mapping theorem as done in [6] for finite dimensions. We shall not repeat the proof here.

Lemma 4.1 *Suppose that (A1)-(A5) are valid. Then there exist positive constants $\hat{w}, \varepsilon_0$ with the following property. If $\{w_i\}_{i=1}^{4n+2}$ is a sequence of real numbers with each $w_i > \hat{w}$, and $|r - r_0| < \varepsilon_0$, then there exists a piecewise continuous solution $\underline{\eta} \in E([-w,w],\Delta) \times \mathbb{R}$ of (4.2), (4.3) with $\langle \psi_i, \eta_{i(-\tau)} \rangle = 0$, $i =$*

$1, 2, \cdots, 4n+2$. *Let* ξ *be such that* $\eta_{i\tau-} - \eta_{i\tau+} = \xi_i \Delta_i$. *Then*

$$\begin{aligned} \xi_i &= \int_{-w_1}^{w_{i+1}} \psi_s(-1) N(\eta_{is}(-1), r, s-1)\, ds \\ &\quad + \langle \psi_{i(-w_i)}(\cdot), \eta_{i(-w_2)}(\cdot) \rangle - \langle \psi_{iw_{i+1}}(\cdot), \eta_{iw_{i+1}} \rangle . \end{aligned} \tag{4.4}$$

When looking for rapidly oscillating periodic solutions $(y_t(\cdot), z_t(\cdot))$ of (1.3), which satisfy the symmetry condition $(y_{t+2w}(\cdot), z_{t+2w}(\cdot)) = (z_t(\cdot), y_t(\cdot))$ and whose period is $4w$, it is possible to make the simplifying assumption

$$w_i \in \mathbb{Z}_{2n+1}, \tag{4.5}$$

where $\mathbb{Z}_{2n+1}$ is the quotient group of integers modulo $2n+1$ under addition. This assumption will reduce the number of bifurcation equations to be solved by a factor of 2. We shall define the Mel'nikov function $G : \mathbb{R}^{4n+2} \times \mathbb{R} \to \mathbb{R}$ by $\xi_i = G(w_1, \cdots, w_{4n+2}, r) = G(\underline{w}, r)$ to emphasize the dependence of ξ_i on $\underline{w} \in \mathbb{R}^{4n+2}$ and r.

The following theorem indicates that there exists at least one rapidly oscillating periodic solution of (1.3) whose lap number is $2n+1, n \in \mathbb{Z}$. The solution we will construct has zero crossings which are equally spaced, and the proof reduces to the slowly oscillating periodic case, as in [6].

Theorem 4.1 *Suppose that (A1)-(A5) are valid. Then there exists* $\varepsilon_0 > 0$ *with the following property. For each* $\varepsilon \in (0, \varepsilon_0)$ *and each odd integer* $2n+1, n \in \mathbb{Z}$, *there exists a rapidly oscillating periodic solution* $x_\varepsilon(t)$ *of (1.3) whose period is* $2/(2n+1) + O(\varepsilon)$.

Proof: We shall fix $2n+1$ and choose $w_i = w/(2n+1) > \hat{W}$, where $\hat{w}$ is independent of i and depends continuously on w. In this case, we need to solve the bifurcation equations (here $\eta_i = \eta_{i+2}, \psi_i = \psi_{i+2}, J\eta_i = \eta_{i+1}$, $\xi_i = \xi_{i+2}$ and $J\psi_i = \psi_{i+1}$, where $i \equiv i \bmod (4n+2)$)

$$\begin{aligned} \xi_i &= \int_{-\frac{w}{2n+1}}^{\frac{w}{2n+1}} \psi_s(-1) N(\eta_{is}(-1), r, s-1)\, ds \\ &\quad + \langle \psi_{i(-\frac{w}{2n+1})}(\cdot), \eta_{i(-\frac{w}{2n+1})}(\cdot) \rangle - \langle \psi_{i(\frac{w}{2n+1})}(\cdot), \eta_{i(\frac{w}{2n+1})}(\cdot) \rangle \ , \ i = 1, 2. \end{aligned}$$

However, if n is fixed then for w large, the conditions in Theorem 3.1 are satisfied, since all of the bifurcation equations are equal to 0 if any one is equal to 0. More precisely, for

$$\tilde{r}(w) = r^* \left(\frac{w}{2n+1} \right) \ , \ \xi_1 = G_1(w, \tilde{r}(w)) = 0.$$

We now have that, for $r = \tilde{r}(w)$, there exists a $\frac{4w}{2n+1}$-periodic solution $(y(t), z(t))$ of equation (1.3). Now, from the proof of Theorem 3.2, there corresponds a unique, $\left(\frac{2}{2n+1} + O(\varepsilon)\right)$-periodic solution $x_\varepsilon(t)$ of (1.1) to each $\frac{4w}{2n+1}$-periodic solution $(y(t), z(t))$ of (1.3) (after a time rescaling). This completes the proof.

In the next section we will determine whether it is possible to find other types of rapidly oscillating periodic solutions to (1.3) when ε is small and positive.

5 A Uniqueness Theorem for Rapidly Oscillating Periodic Solutions

Suppose that $x_\varepsilon(t)$ is a rapidly oscillating periodic solution of (1.1) with oscillation rate 3 (i.e., $V(x_\varepsilon(t)) = 3$ for any time t). Also suppose that $x_\varepsilon(t) = 0$ at intervals given by $a(\varepsilon), b(\varepsilon)$ and $c(\varepsilon)$. That is, if $\{t_i\}$ is an ordered sequence of times for which $x_\varepsilon(t) = 0$, then $x_\varepsilon(t_{i+1}) - x_\varepsilon(t_i) = a(\varepsilon)$ if $i \equiv 0 \pmod 3$, $x_\varepsilon(t_{i+1}) - x_\varepsilon(t_i) = b(\varepsilon)$ if $i \equiv 1 \pmod 3$, and $x_\varepsilon(t_{i+1}) - x_\varepsilon(t_i) = c(\varepsilon)$ if $i \equiv 2 \pmod 3$.

In this section we will prove that $\lim_{\varepsilon\to 0^+} a(\varepsilon)/b(\varepsilon) = \lim_{\varepsilon\to 0^+} b(\varepsilon)/c(\varepsilon) = 1$, so that the distance between successive oscillations tends to a constant as ε tends to 0. The notation we use will be general enough to cover the case where the rapidly oscillating periodic solution $x_\varepsilon(t)$ has arbitrary lap number $2n+1$.

Suppose, then, that $\underline{\eta} \in E([-w,w],\Delta)$ satisfies the system of integral equations (4.1) with boundary conditions (4.2), $i \in \{1,\cdots,4n+2\}$, and $w_i = w_{i+(2n+1)}$. In order to emphasize that each w_i depends on w through the relation $\sum_{i=1}^{2n+1} w_i = 2w$, we shall set $w_i = a_i(w)$. We wish to show that $\lim_{w\to\infty} a_i(w)/a_j(w) = 1$, independent of $i,j \in \{1,\cdots,4n+2\}$. We shall need the technical Lemma 5.2 (proved by Lin in 1990) and Lemmas 5.1, 5.3 (proved here by the author) for our uniqueness result. We suppose that assumptions (A1)-(A5) are valid throughout.

Lemma 5.1 *Consider the bifurcation functions $G_i(\underline{w},r)$ defined by (4.4), where $w_i = a_i(w), w \in \{1,\cdots,2n+1\}$ and $\sum_{i=1}^{2n+1} a_i(w) = 2w$. Then there exists a unique, continuous function $r = r^*(w) : (\hat{w},+\infty) \to (r_0-\varepsilon_0, r_0+\varepsilon_0)$ such that $G(\underline{w},r) = \sum_{i=1}^{2n+1} G_i(\underline{w},r) = 0$.*

Proof: The proof uses assumption (A5) and is a direct application of the implicit function theorem. We know that $G(\infty,r_0) = 0$ (i.e. $w_i = \infty$ for all i) and compute $\frac{\partial G(\infty,r_0)}{\partial r}$. But now

$$\begin{aligned}
\frac{\partial G(\infty,r_0)}{\partial r} &= \frac{\partial}{\partial r}\Big[\sum_{i=1}^{2n+1}\int_{-\infty}^{\infty}\psi_s(-1)N(\eta_{is}(-1),r,s-1)\,ds\Big] \\
&= \sum_{i=1}^{2n+1}\int_{-\infty}^{\infty}\psi_s(-1)\frac{\partial}{\partial r}N(\eta_{is}(-1),r,s-1)\,ds \\
&= \sum_{i=1}^{2n+1}\int_{-\infty}^{\infty}\psi_s(-1)(-F(W_s(-1))+W_s(-1))\,ds \\
&= \frac{1}{r_0}\sum_{i=1}^{2n+1}\int_{-\infty}^{\infty}(\psi_2^1(-1)\dot{p}_s(-1)+\psi_s^2(-1)\dot{q}_s(-1))\,ds \\
&= \frac{(2n+1)}{r_0}C \neq 0\,,
\end{aligned}$$

and the proof is complete. ♢

The following lemma, which is identical to Lemma 3.3 in [6], gives a precise estimate for the bifurcation function $G_i(\underline{w}, r)$ when $r = r_0$ and will play a significant part in the proof of our main theorem. An important consequence of this lemma is that the boundary term $\langle\psi_{i(-w_i)}(\cdot), \eta_{i(-w_i)}(\cdot)\rangle - \langle\psi_{iw_{i+1}}(\cdot), \eta_{iw_{i+1}}(\cdot)\rangle$ is (generically) large relative to the integral term $\int_{-w_i}^{w_{i+1}} \psi_s(-1)N(\eta_{is}(-1), r, s-1)\,ds$ in (4.4) at the point $r = r_0$. In order to give a compact representation for $G_i(\underline{w}, r_0)$, we need the following notation. Suppose that X, Y are subspaces of $C([-1,0], \mathbb{R}^2)$ with $X \oplus Y = C([-1,0], \mathbb{R}^2)$ and $X \cap Y = \{0\}$; we then define $P(X,Y) : C([-1,0], \mathbb{R}^2) \to X$ to be the orthogonal projection of C onto X. Hence $\operatorname{Ran} P = X$ and $\ker P = Y$. Lemma 5.2 is presented without proof.

Lemma 5.2 ([6]) *Suppose that (A1)-(A5) hold. Then, for any $\sigma \in (0, \alpha)$ and any $i \in \mathbb{Z}_{2n+1}$, $i \equiv i \bmod (2n+1)$, the following equality is valid.*

$$\begin{aligned}
&G_i(\underline{w}, r_0)\\
&\quad= -\langle\psi_{i(-w_i)}(\cdot), P(\operatorname{Ran} P_s^i(-w_i), \operatorname{Ran} P_u^{i-1}(w_i))b_i(\cdot)\rangle\\
&\qquad -\langle\psi_{iw_{i+1}}(\cdot), P(\operatorname{Ran} P_u^i(w_{i+1}), \operatorname{Ran} P_s^{i+1}(-w_{i+1}))b_{i+1}(\cdot)\rangle\\
&\qquad +O(|\psi_{i(-w_i)}(\cdot)|(|b_i|^2 + |b|^2 + |\underline{b}|(e^{-2\sigma w_{i-1}} + e^{-2\sigma w_i} + e^{-2\sigma w_{i+1}})))\\
&\qquad +O(|\psi_{iw_{i+1}}(\cdot)|(|b_{i+1}|^2 + |\underline{b}|(e^{-2\sigma w_i} + e^{-2\sigma w_{i+1}} + e^{-2\sigma w_{i+2}}))) \qquad (5.1)\\
&\qquad +O(\{|b_i|^2 + |b_{i+1}|^2 + |\underline{b}|^2(e^{-4\sigma w_i} + e^{-4\sigma w_{i+1}} + e^{-4\sigma w_{i+2}})\}\\
&\qquad \times(e^{-\sigma w_i} + e^{-\sigma w_{i+1}}).
\end{aligned}$$

In (5.1), we have set $|\phi| = \langle\phi(\cdot), \phi(\cdot)\rangle^{1/2}$, where $\phi(\cdot) \in C([-1,0], \mathbb{R}^2)$. The following lemma will be vital in the proof of Theorem 5.1.

Lemma 5.3 *Suppose that (A1)-(A5) are valid and that $w_i = a_i(w) : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{Z}_{2n+1}$. Further suppose that there exists a function $r^* = r(w) : (\hat{w}, \infty) \to \mathbb{R}$ such that $G_i(\underline{w}, r^*) = 0$ for all $i \in \mathbb{Z}_{2n+1}$ and that, for every pair $j, k \in \mathbb{Z}$, $\lim_{w\to\infty} G_j(\underline{w}, r_0)/G_k(\underline{w}, r_0) \neq 0$. It follows that $\lim_{w\to\infty} G_j(\underline{w}, r_0)/G_k(\underline{w}, r_0) = 1$, independent of $j, k \in \mathbb{Z}_{2n+1}$.*

Proof: We shall use assumption (A5) and the fact that $G_i(\underline{w}, r)$ is jointly continuous in $\underline{w}, r$. First we set $\underline{w} = (a_1(w), \cdots, a_{2n+1}(w))$ and suppose that there exists a continuous function $r^* = r^*(w) : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $G_k(w, r^*) = 0$ for all $w > \hat{w}$ sufficiently large. Then there exist continuous functions $\delta_1, \delta_2 : (\hat{w}, \infty) \to \mathbb{R}$ with $\lim_{w\to\infty}\delta_1(w) = \lim_{w\to\infty}\delta_2(w) = 0$ such that

$$\begin{aligned}
G_k(w, r^*) &= G_k(w, r_0) + (1 + \delta_1(w))\frac{\partial G_k(\infty, r_0)}{\partial r}(r^* - r_0)\\
&= G_k(w, r_0) + C(1 + \delta_1(w))(r^* - r_0)\,,\\
G_j(w, r^*) &= G_j(w, r_0) + (1 + \delta_2(w))\frac{\partial G_j(\infty, r_0)}{\partial r}(r^* - r_0)\\
&= G_j(w, r_0) + c(1 + \delta_2(w))(r^* - r_0)\,.
\end{aligned}$$

Since $G_k(w,r^*)=0$, we gain

$$(r^*-r_0)=\frac{1}{C(1+\delta_1(w))}(G_k(w,r^*)-G_k(w,r_0))=-\frac{G_k(w,r_0)}{C(1+\delta_1(w))}.$$

Substituting for (r^*-r_0) into our expression for G_j yields

$$\begin{aligned}G_j(w,r^*) &= G_j(w,r_0)-\frac{(1+\delta_2(w))}{(1+\delta_1(w))}G_k(w,r_0)\\ &= G_j(w,r_0)=G_k(w,r_0)+\frac{\delta_2(w)-\delta_1(w))}{(1+\delta_1(w))}G_k(w,r_0)\\ &= (1-\frac{G_k(w,r_0)}{G_j(w,r_0)})G_j(w,r_0)+\frac{(\delta_2(w)-\delta_1(w))}{(1+\delta_1(w))}G_k(w,r_0).\end{aligned}$$

Now suppose, by way of contradiction, that $\lim_{w\to\infty}G_j(w,r_0)/G_k(\underline{w},r_0)=\tilde{c}\neq 1$. Then there exists a continuous function $\delta_3=\delta_3(w):(\hat{w},\infty)\to\mathbb{R}$ with $\lim_{w\to\infty}\delta_3(w)=0$ such that

$$G_j(w,r^*)=\left(1-\frac{(1+\delta_3(w))}{\tilde{c}}\right)G_j(w,r_0)+\frac{(\delta_2(w)-\delta_1(w))}{(1+\delta_1(w))}G_k(w,r_0).$$

From the hypotheses of our lemma we know that $G_k(w,r_0)=O(G_j(w,r_0))$. Hence, for $w>\hat{w}$ sufficiently large, we know that $G_j(w,r^*)\sim\left(1-\frac{1}{\tilde{c}}\right)G_j(w,r_0)\neq 0$. Thus there cannot exist a continuous function $r^*:(\hat{w},\infty)\to\mathbb{R}$ such that $G_i(r^*,w)=0$ for all i unless $\lim_{w\to\infty}G_j(w,r_0)/G_k(w,r_0)=1$ for all pairs j,k, and the proof is complete. $\diamondsuit$

We will require the following two assumptions in addition to (A1)-(A5).

(A6) There exist non-negative integers h_1,h_2,ℓ_1 and ℓ_2, and strictly positive real numbers c_1,c_2,d_1 and d_2 such that, for all $i\in\mathbb{Z}_{2n+1}$,

$$\lim_{t\to-\infty}\frac{|\psi_{it}(\cdot)|}{|t|^{h_1}e^{\gamma t}}=c_1\,,\quad \lim_{t\to+\infty}\frac{|\psi_{it}(\cdot)|}{t^{\ell_1}e^{-\rho t}}=d_1\,,$$
$$\lim_{t\to-\infty}\frac{|W_{it}(\cdot)|}{|t|^{h_2}e^{\rho t}}=c_2\,,\quad \lim_{t\to+\infty}\frac{|W_{it}(\cdot)|}{t^{\ell_2}e^{-\gamma t}}=d_2\,.$$

(A7) There exists a positive constant c_0 such that, for $w_i=w>\hat{w}$ and $i\in\mathbb{Z}_{2n+1}$,

$$\frac{|\langle\psi_{i(-w_i)}(\cdot),W_{(i-1)w_i}(\cdot)\rangle-\langle\psi_{iw_{i+1}}(\cdot)\,,\,W_{(i+1)(-w_{i+1})}(\cdot)\rangle|}{|\psi_{i(-w_i)}(\cdot)||W_{i-1)w_i}(\cdot)|+|\psi_{iw_{i+1}}(\cdot)||W_{(i+1)(-w_{i+1})}(\cdot)|}>c_0$$

whenever $\inf_{i\in\mathbb{Z}}w_i>\hat{w}$ is sufficiently large.

Conditions (A6)-(A7) are not new but are rather applications of Sil'nikov's conditions for the bifurcation of periodic orbits from homoclinic orbits [6] to bifurcations from heteroclinic chains. Condition (A6) guarantees that ψ and W decay at an appropriate exponential rate and condition (A7) stipulates that the local stable and unstable manifolds in (2.2.4) must intersect transversely. Both (A6) and (A7) are generic conditions.

Theorem 5.1 *Suppose that (A1)-(A7) are valid and that equation (1.1) possesses a rapidly oscillating periodic solution $x_\varepsilon(t)$ satisfying the following conditions:*

1. *$p(\varepsilon)$ is an integer multiple of the period of $x_\varepsilon(t)$; thus, for all $t \in \mathbb{R}, x_\varepsilon(t + p(\varepsilon)) = x_\varepsilon(t)$, where $p(\varepsilon) = 2 + 2r\varepsilon$*
2. *There exists an element $n \in \mathbb{Z}^+$ such that $V(x_{\varepsilon t}(\cdot)) = 2n + 1$ for all $t \in \mathbb{R}$*
3. *$x_\varepsilon(t) = 0$ if and only if $t \in \{a_i(\varepsilon)\}_{i\in\mathbb{Z}}$, where $a_i(\varepsilon) < a_j(\varepsilon)$ for any pair $i < j$*
4. *Uniform closeness condition: there exists a continuous function $\varepsilon_1(\varepsilon) : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\varepsilon\to 0^+} \varepsilon_1(\varepsilon) = 0$ and a sequence $\{\tilde{w}_i\}_{i=1}^{2n+1}$ defined by the recursive system of equations $\tilde{w}_1 = 0$, $\tilde{w}_i = \tilde{w}_{i-1} + w_{i-1} + w_i$, $i = 2, \cdots, 2n + 1$ such that*

$$|x_\varepsilon(-\varepsilon r(t - \tilde{w}_i) - p(t)| + |x_\varepsilon(-\varepsilon r(t - \tilde{w}_i) - 1 - \varepsilon r) - q(t)| < \varepsilon_1$$

   for i odd, and for i even:

$$|x_\varepsilon(-\varepsilon r(t - \tilde{w}_i) - q(t)| + |x_\varepsilon(-\varepsilon r(t - \tilde{w}_i) - 1 - \varepsilon r) - p(t)| < \varepsilon_1 \,.$$

Then there exists a positive constant c_1, independent of $i \in \mathbb{Z}_{2n+1}$, such that $\lim_{\varepsilon\to 0^+}(a_{i+1}(\varepsilon) - a_i(\varepsilon)) = c_1$.

Proof: From Lemma 3.1, for any sequence $(\underline{w}, r) = (w_1, \cdots, w_{2n+1}, r) \in \mathbb{R}^{2n+2}$, there exists a unique piecewise continuous function $\underline{\eta} \in E([-w, w], \Delta)$ which satisfies (3.4), (3.5). A necessary and sufficient condition for the existence of a globally continuous solution $\underline{\eta}$ of (2.2.1) is $G_i(\underline{w}, r) = 0$ for all $i \in \mathbb{Z}_{2n+1}$. We shall characterize those pairs $(\underline{w}, r)$ such that every G_i is identically 0. Using Theorem 1.3.2 and Lemma 2.3.1, we may uniquely define $w_i = b_i(w)$ for each $i \in \{1, \cdots, 2n+1\}$, where $b_i(w) : (\hat{w}, \infty) \to (\hat{w}, \infty)$. It is immediately seen that $\lim_{\varepsilon\to 0^+}(a_{i+1}(\varepsilon) - a_i(\varepsilon)) = c_1$ for all i if and only if $\lim_{w\to\infty} b_i(w)/b_{i+1}(w) = 1$, independent of $i \equiv i \bmod 2n + 1$. We shall prove the latter statement by contradiction.

Let us suppose, then, that there exists an element $i^* \in \{1, \cdots, 2n + 1\}$ such that $\lim_{w\to\infty} a_{i^*}(w)/a_i(w) \le 1$ for all $i \in \mathbb{Z}_{2n+1}$ and also $\lim_{w\to\infty} a_{i^*}(w)/a_{i^*+1}(w) = c_2 < 1$. Without loss of generality, it is possible to choose $i^* = 1$, for the following reason. Supposing that $X = (\hat{w}, \infty)$ and $\Sigma : X^{2n+1} \to X^{2n+1}$ is the left shift map defined by $\Sigma(a_1(w), \cdots, a_{2n+1}(w)) = (a_2(w), \cdots, a_{2n+1}(w), a_1(w))$, it follows directly from equation (2.2.3) that $G_i(\Sigma\underline{w}, r) = G_{i+1}(\underline{w}, r)$. Thus we need only apply $\Sigma(2n + 2 - i^*)$- times to $\underline{w}$ in order to ensure that $\lim_{w\to\infty} a_1(w)/a_i(w) \le 1$ and $\lim_{w\to\infty} a_1(w)/a_2(w) = c_2 < 1$. Again, without loss of generality , we assume that $0 < \gamma \le \rho$. There are now two subcases to consider, first the case where $\gamma < \rho$ and second the case where $\gamma = \rho$; we consider these separately.

Case I ($\gamma < \rho$)**:** We show that $G_2(\underline{w}, r_0) = o(G_1(\underline{w}, r_0))$. From Lemma 5.2 and assumptions (A6)-(A7), we have that

$$\begin{aligned} G_1(\underline{w}, r_0) &= -\langle \psi_{1(-a_1(w))}(\cdot)\,,\; P(\operatorname{Ran} P_s^1(-a_1(w)),\, \operatorname{Ran} P_u^{2n+1}(a_1(w)))b_1(\cdot)\rangle \\ &\quad -\langle \psi_{1a_2(w)}(\cdot)\,,\; P(\operatorname{Ran} P_u^1(a_2(w))\,, \operatorname{Ran} P_s^2(-a_2(w)))b_2(\cdot)\rangle \\ &\quad +O(G_1(\underline{w}, r)) \\ &= \langle \psi_{1(-a_1(w))}(\cdot),\, (W_{(2n+1)a_1(w)}(\cdot) + o(W_{(2n+1)a_1(w)}(\cdot))\rangle \\ &\quad -\langle \psi_{1a_2(w)}(\cdot),\, (W_{2(-a_2(w))}(\cdot) + o(W_{2(-a_2(w))}(\cdot))\rangle \\ &\sim c_1 d_2 |a_1(w)|^{h_1+\ell_2} e^{-2\gamma a_1(w)}. \end{aligned}$$

Although we did not, a priori, know that the remainder terms were smaller than the boundary terms in Lemma 5.2, it turns out that this is the case from assumption (A6). It is thus justifiable to treat the remainder terms as $o(G_1(\underline{w}, r))$ in the expression above. Proceeding in the same way, we can show that

$$\begin{aligned} G_2(\underline{w}, r_0) &\sim c_1 d_2 |a_2(w)|^{h_1+\ell_2} e^{-2\gamma a_2(w)} - c_2 d_1 |a_3(w)|^{h_2+\ell_1} e^{-2\rho a_3(w)} \\ &= o(G_1(\underline{w}, r_0)). \end{aligned}$$

Using the same argument as in the proof of Lemma 5.3, it follows that if $G_1(\underline{w}, r^*) = 0$ for some function $r^* = r^*(w) : X \to \mathbb{R}$, then $G_2(\underline{w}, r^*) \neq 0$, a contradiction. In particular, $G_1(\underline{w}, r^*) = 0$ if and only if

$$r^* - r_0 = -\frac{c_1 d_2 |a_1(w)|^{h_1+\ell_2} e^{-2\gamma a_1(w)}}{c(1+\delta_4(w))}$$

for some continuous function $\delta_4 : X \to \mathbb{R}$ with $\lim_{w\to\infty} \delta_4(w) = 0$ and $G_2(w, r^*) \sim c_1 d_2 |a_1(w)|^{h_1+\ell_2} e^{-2\gamma a_1(w)}$.

Case II ($\gamma = \rho$)**:** We have $\gamma = \rho$, $\lim_{w\to\infty} a_1(w)/a_i(w) \leq 1$ for $i \in \mathbb{Z}_{2n+1}$ and $\lim_{w\to\infty} a_1(w)/a_2(w) < 1$ and shall show that, if $G_1(\underline{w}, r^*) = G_2(\underline{w}, r^*) = \cdots = G_{2n}(\underline{w}, r^*) = 0$ for some function $r^* : X \to \mathbb{R}$, $r^* = r^*(w)$, then $G_{2n+1}(\underline{w}, r^*) = G_0(\underline{w}, r^*) \neq 0$. From Lemma 5.3, a necessary condition for the existence of a function r^* such that $G_1(\underline{w}, r^*) = G_2(\underline{w}, r^*) = \cdots = G_{2n}(\underline{w}, r^*) = 0$ is $\lim_{w\to\infty} \frac{G_j(\underline{w}, r_0)}{G_k(\underline{w}, r_0)} = 1$ for all pairs $j, k \in \mathbb{Z}$, $j, k \not\equiv 0 \bmod 2n+1$ (where $G_j(\underline{w}, r_0)$, $G_k(\underline{w}, r_0) \neq 0$). Since

$$G_1(\underline{w}, r_0) \sim c_1 d_2 |a_1(w)|^{h_1+\ell_2} e^{-2\gamma a_1(w)},$$
$$G_2(\underline{w}, r_0) \sim c_1 d_2 |a_2(w)|^{h_1+\ell_2} e^{-2\gamma a_2(w)} - c_2 d_1 |a_3(w)|^{h_2+\ell_1} e^{-2\gamma a_3(w)},$$

we must then have $\lim_{w\to\infty} \frac{a_1(w)}{a_3(w)} = 1$, $h_1 + \ell_2 = h_2 + \ell_1$ and $c_2 d_1 = -c_1 d_2$; by induction, we further have that, for all integers $(2i+1) \in \mathbb{Z}_{2n+1}$ with $i \in \mathbb{Z}$, $\lim_{w\to\infty} \frac{a_1(w)}{a_{(2i+1)}(w)} = 1$. It is now possible to establish a contradiction by showing that $\lim_{w\to\infty} \frac{G_1(\underline{w}, r_0)}{G_0(\underline{w}, r_0)} \neq 1$. In particular, by direct computation,

$$\begin{aligned} G_0(\underline{w}, r_0) &= G_{2n+1}(\underline{w}, r_0) \\ &\sim c_1 d_2 |a_{2n+1}(w)|^{h_1+\ell_2} e^{-2\gamma a_{2n+1}(w)} - c_2 d_1 |a_1(w)|^{h_2+\ell_1} e^{-2\gamma a_1(w)} \\ &\sim 2G_1(\underline{w}, r_0), \end{aligned}$$

and the proof is complete. ◇

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References

[1] J. K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, 1993.

[2] U. an der Heiden, *Delays in physiological systems*. J. Math. Biol. 8 (1979), 345–364.

[3] U. an der Heiden and M.C. Mackey, *The dynamics of production and destruction: analytic insight into complex behavior*. J. Math. Biol. 16 (1982), 75–101.

[4] U. an der Heiden and H.-O. Walther, *Existence of chaos in control systems with delayed feedback*. J. Diff. Eqns. 47 (1983), 273–295.

[5] A. F. Ivanov and A.N. Sharkovsky, *Oscillations in singularly perturbed delay equations*. Dynamics Reported (New Series) 1 (1991), 165–224.

[6] X.-B. Lin, *Using Mel'nikov's method to solve Sil'nikov's problems*. Proc. Roy. Soc. Edin 116A (1990), 295–325.

[7] J. Mallet-Paret, *Morse decompositions for delay-differential equations*. J. Diff. Eqns. 72 (1988), 270–315.

[8] J. Mallet-Paret and R.D. Nussbaum, *Global continuation and asymptotic behavior for periodic solutions of a differential-delay equation*. Ann. Mat. Pura. App. (IV) CXLV (1986), 33-128.

[9] J. T. Schwartz, *Nonlinear Functional Analysis*, Gordon and Breach, New York, 1969.

[10] H.-O. Walther, *An invariant manifold of slowly oscillating solutions for* $\dot{x}(t) = -\mu x(t) + f(x(t-1))$. J. Reine Angew. Math. 414 (1991), 67–112.

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