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SOLUTION TO A SEMILINEAR PSEUDOPARABOLIC PROBLEM WITH INTEGRAL CONDITIONS

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ABSTRACT. In this article, we use the Rothe time-discretization method to prove the well-posedness of a mixed problem with integral conditions for a third order semilinear pseudoparabolic equation. Also we establish the convergence of the method and an error estimate for a semi-discrete approximation.

1. STATEMENT OF THE PROBLEM

This paper concerns the problem of finding a function $v = v(x,t)$ satisfying, in a weak sense, the semilinear pseudoparabolic equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \eta \frac{\partial^3 v}{\partial x^2 \partial t} = F(x,t,v)\,, \quad (x,t) \in (0,1) \times [0,T], \tag{1.1}$$

subject to the initial condition

$$v(x,0) = V_0(x), \quad 0 \le x \le 1, \tag{1.2}$$

and to the integral conditions

$$\int_0^1 v(x,t)dx = E(t), \quad 0 \le t \le T, \tag{1.3}$$

$$\int_0^1 xv(x,t)dx = G(t), \quad 0 \le t \le T, \tag{1.4}$$

where F, V_0, E and G are given functions which are sufficiently regular, and T and η are positive constants.

This problem has a practical relevance, for instance in the context of soil thermophysics, (1.1) describes the dynamics of moisture transfer in a subsoil layer $0 < x < 1$ for $t \in [0,T]$, while (1.3)-(1.4) represent the moisture moments (see [5] and references therein). Equations of type (1.1) (with eventually variable coefficients and additional nonlinear terms) have also many other applications in various physical situations, notably in the non-steady flows of second order fluids [23, 8]; in the infiltration of homogeneous fluids through fissured rocks [1]; in the diffusion of imprisoned resonant radiation through a gas [15, 16, 22] (which has applications in the analysis of certain laser systems [18]); in the theory of the two temperatures

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in heat conduction [7]; in the monodirectional propagation of nonlinear dispersive long waves [2, 10], and so forth. This is the main reason for which the investigation of (classical) mixed problems for such equations have been the subject of many works for a long time, (see, e.g. [3, 11, 13, 17, 20, 21, 24, 25]).

Recently, mixed problems with integral condition(s) for some generalizations of equation (1.1) have been treated by the second author in [5, 6] using the energy-integral method. Differently to these works, in the present paper we use a constructive method (Rothe time-discretization method) to build the solution, which is more suitable for numerical computations. It is interesting to note that the application of Rothe method to this nonlocal problem is made possible thanks, essentially, to the use of a nonclassical function space (see also [14]).

By the the transformation

$$u(x,t) := v(x,t) - r(x,t), \quad (x,t) \in (0,1) \times [0,T],$$

where

$$r(x,t) = 6(2G(t) - E(t))x - 2(3G(t) - 2E(t)),$$

problem (1.1)-(1.4) with inhomogeneous integral conditions (1.3) and (1.4) is converted to the following equivalent problem with homogeneous conditions for the new unknown function u:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \eta \frac{\partial^3 u}{\partial x^2 \partial t} = f(x,t,u)\,, \quad (x,t) \in (0,1) \times I, \tag{1.5}$$

$$u(x,0) = U_0(x), \quad 0 \le x \le 1, \tag{1.6}$$

$$\int_0^1 u(x,t)dx = 0, \quad t \in I, \tag{1.7}$$

$$\int_0^1 xu(x,t)dx = 0, \quad t \in I, \tag{1.8}$$

where the notation $I := [0,T]$ is used and

$$\begin{gathered} f(x,t,u) := F(x,t,u+r) - \frac{\partial r}{\partial t}(x,t), \\ U_0(x) := V_0(x) - r(x,0). \end{gathered}$$

Hence, instead of looking for the function v, we seek the function u. The solution of problem (1.1)-(1.4) will be simply given by the formula $v = u + r$.

This paper is organized as follows: In Section 2, we introduce function spaces needed in our investigation and recall an auxiliary result. We also state the assumptions on data and make precise concept of the solution. In Section 3, approximate solutions of problem (1.5)-(1.8) are constructed by solving the corresponding linearized time-discretized problems. Then, some a priori estimates for the approximations are derived in Section 4, while the convergence of the method and the well-posedness of the problem under study are established in Section 5.

2. Preliminaries and main result

Let $H^2(0,1)$ be the (real) second order Sobolev space on $(0,1)$ with norm $\|\cdot\|_{H^2(0,1)}$ and let $(\cdot,\cdot)$ and $\|\cdot\|$ be the usual inner product and the corresponding norm respectively in $L^2(0,1)$. The nature of the boundary conditions (1.7)-(1.8) suggests to introduce the following space

$$V := \{\phi \in L^2(0,1) : \int_0^1 \phi(x)dx = \int_0^1 x\phi(x)dx = 0\} \tag{2.1}$$

which is clearly a Hilbert space for $(\cdot,\cdot)$.

Our analysis requires the use of the nonclassical function space $B_2^1(0,1)$ defined for example in [4] as the completion of the space $C_0(0,1)$ of real continuous functions with compact support in $(0,1)$, for the inner product

$$(u,v)_{B_2^1} = \int_0^1 \Im_x u \cdot \Im_x v \, dx, \tag{2.2}$$

and the associated norm

$$\|v\|_{B_2^1} = \sqrt{(v,v)_{B_2^1}},$$

where $\Im_x v := \int_0^x v(\xi)d\xi$ for $x \in (0,1)$. We recall that, for $v \in L^2(0,1)$, the inequality

$$\|v\|_{B_2^1}^2 \leq \frac{1}{2}\|v\|^2 \tag{2.3}$$

holds, implying the continuity of the embedding $L^2(0,1) \to B_2^1(0,1)$. Moreover, we will work in the standard functional spaces $C(I,X)$, $C^{0,1}(I,X)$ and $L^2(I,X)$ where X is a Banach space, the main properties of which can be found in [12].

The notation $\theta(t)$ is automatically used for the same function $\theta(x,t)$ considered as an abstract function of the variable $t \in I$ into some functional space on $(0,1)$. Strong or weak convergence are denoted by $\to$ or $\rightharpoonup$ respectively.
The Gronwall Lemma in the following continuous and discrete forms will be very useful to us thereafter.

Lemma 2.1. (i) *Let $x(t) \geq 0$, $h(t), y(t)$ be real integrable functions on the interval $[a,b]$. If*

$$y(t) \leq h(t) + \int_a^t x(\tau)\, y(\tau) d\tau, \quad \forall t \in [a,b],$$

then

$$y(t) \leq h(t) + \int_a^t h(\tau)\, x(\tau)\, \exp\Big(\int_\tau^t x(s)ds\Big) d\tau, \quad \forall t \in [a,b].$$

In particular, if $x(\tau) \equiv C$ is a constant and $h(\tau)$ is nondecreasing, then

$$y(t) \leq h(t) e^{C(t-a)}, \quad \forall t \in [a,b].$$

(ii) *Let $\{a_i\}$ be a sequence of real nonnegative numbers satisfying*

$$a_1 \leq a,$$
$$a_i \leq a + bh\sum_{k=1}^{i-1} a_k, \quad \forall i = 2, \dots,$$

where a, b and h are positive constants. Then

$$a_i \leq a e^{b(i-1)h}, \quad \forall i = 1, 2, \dots .$$

Proof. The proof of assertion (i) is the same as in [9, Lemma 1.3.19]. To establish assertion (ii), we use induction on i giving

$$a_i \le a(1+bh)^{i-1}, \quad \forall i = 1, 2, \dots .$$

from where, the desired inequality follows thanks to the elementary inequality $1 + t \le e^t$, for all $t \in \mathbb{R}_+$. $\square$

We shall work under the following hypotheses:

(H1) $f(t,w) \in L^2(0,1)$ for each pair $(t,w) \in I \times L^2(0,1)$ and the Lipschitz condition

$$\|f(t,w) - f(t',w')\|_{B_2^1} \le l\big[|t-t'|(1 + \|w\|_{B_2^1} + \|w'\|_{B_2^1}) + \|w-w'\|_{B_2^1}\big],$$

is satisfied for all $t, t' \in I$ and $w, w' \in V$, where l is some positive constant.

(H2) $U_0 \in H^2(0,1)$

(H3) the compatibility condition: $U_0 \in V$, i.e. $\int_0^1 U_0(x)dx = \int_0^1 xU_0(x)dx = 0$.

We look for a weak solution in the following sense:

Definition 2.2. By a weak solution of Problem (1.5)-(1.8), we mean a function $u : I \to L^2(0,1)$ such that

(i) $u \in C^{0,1}(I,V)$;
(ii) u has (a.e. in I) a strong derivative $\frac{du}{dt} \in L^\infty(I, L^2(0,1))$;
(iii) $u(0) = U_0$ in V;
(iv) the equality

$$\big(\frac{du}{dt}(t), \phi\big)_{B_2^1} + \big(u(t), \phi\big) + \eta\big(\frac{du}{dt}(t), \phi\big) = \big(f(t, u(t)), \phi\big)_{B_2^1}, \tag{2.4}$$

holds for all $\phi \in V$ and all $t \in I$.

We remark that since $u \in C^{0,1}(I,V) \subset C(I,V)$ the condition (iii) makes sense, and in view of (i), (ii) and Assumption (H1) each term in (2.4) is well defined. On the other hand, the fulfillment of the integral conditions (1.7) and (1.8) is included in the fact that $u(t) \in V$, for all $t \in I$.

In this paper, we will demonstrated the following main result.

Theorem 2.3. *Assuming (H1)–(H3), problem* (1.5)-(1.8) *admits a unique weak solution u in the sense of Definition 2.2, that depends continuously upon the data f and U_0. Moreover, u is the limit as $n \to \infty$ of the sequence of Rothe functions* (3.13) *in the following sense:*

$$u^{(n)} \to u \quad \text{in } C(I,V), \quad (\text{with convergence order } O(n^{-1/2})),$$
$$\frac{du^{(n)}}{dt} \rightharpoonup \frac{du}{dt} \quad \text{in } L^2(I, L^2(0,1)).$$

3. Rothe approximations

To solve problem (1.5)-(1.8) by the Rothe method, we divide the time interval I into n subintervals $[t_{j-1}, t_j]$, $j = 1, \dots, n$, where $t_j = jh$ and $h := T/n$ is the time-step. Then, replacing $\frac{\partial u}{\partial t}$, at each point $t = t_j$, $j = 1, \dots, n$, by the difference quotient $\delta u_j := \frac{u_j - u_{j-1}}{h}$, where u_j is destined to be an approximation of $u(\cdot, t_j)$, we are conducted to *solve successively for* $j = 1, \dots, n$ the linearized problem

$$\delta u_j - \frac{d^2 u_j}{dx^2} - \eta \frac{d^2 \delta u_j}{dx^2} = f_j\,, \quad x \in (0,1), \tag{3.1}$$

$$\int_0^1 u_j(x)dx = 0, \tag{3.2}$$

$$\int_0^1 xu_j(x)dx = 0, \tag{3.3}$$

where $f_j := f(t_j, u_{j-1})$, starting from

$$u_0 = U_0. \tag{3.4}$$

To this purpose, it is astute to introduce the following auxiliary functions

$$w_j = u_j + \eta\delta u_j, \quad j = 1, \dots, n. \tag{3.5}$$

In this case, we have

$$u_j = \frac{h}{\eta + h} w_j + \frac{\eta}{\eta + h} u_{j-1}, \quad j = 1, \dots, n,$$

from which, it follows

$$\delta u_j = \frac{1}{\eta + h}(w_j - u_{j-1}), \quad j = 1, \dots, n, \tag{3.6}$$

so that, problem (3.1)-(3.3) is seen to be equivalent to the problem of *finding the function* $w_j : (0,1) \to \mathbb{R}$ satisfying:

$$-\frac{d^2 w_j}{dx^2} + \frac{1}{\eta + h} w_j = f_j + \frac{1}{\eta + h} u_{j-1}\,, \quad x \in (0,1), \tag{3.7}$$

$$\int_0^1 w_j(x)dx = \int_0^1 xw_j(x)dx = 0, \tag{3.8}$$

with the update

$$u_j = \frac{h}{\eta + h} w_j + \frac{\eta}{\eta + h} u_{j-1}, \quad j = 1, \dots, n. \tag{3.9}$$

Of course, this coupled problem has to be solved successively for $j = 1, \dots, n$ starting from $u_0 = U_0$.

Developing an idea of [19], we, first, look for a function $w'_j(x) = w'_j(x; \lambda, \mu)$ which solves equation (3.7) supplemented by the classical Dirichlet boundary conditions

$$w'_j(0) = \lambda \quad \text{and} \quad w'_j(1) = \mu, \tag{3.10}$$

instead of nonlocal conditions (3.8), where (λ, μ) is for the moment an arbitrary fixed ordered pair of real numbers.

Since

$$f_1 + \frac{1}{\eta + h} u_0 := f(t_1, U_0) + \frac{1}{\eta + h} U_0 \in L^2(0,1),$$

the Lax-Milgram Lemma guarantees the existence and uniqueness of a strong solution $w'_1 \in H^2(0,1)$ to the elliptic problem (3.7) and (3.10) with $j = 1$. Let us show that the parameters λ and μ can be chosen in a suitable way such that the corresponding function $w'_1(\cdot; \lambda, \mu)$ is also a solution of problem (3.7)-(3.8) with $j = 1$ provided that n is large enough.

In fact, the function $w_1'(\cdot;\lambda,\mu)$ shall be a solution to problem (3.7)-(3.8), with $j=1$, if and only if the pair (λ,μ) satisfies

$$\begin{aligned}\int_0^1 w_1'(x;\lambda,\mu)dx = 0,\\ \int_0^1 x w_1'(x;\lambda,\mu)dx = 0,\end{aligned} \tag{3.11}$$

thus, the above equation will provide all the solutions to problem (3.7)-(3.8) with $j=1$. But,

$$w_1'(x;\lambda,\mu) = w_1'(x;0,0) + \widetilde{w}_1(x;\lambda,\mu), \quad x \in (0,1),$$

where $\widetilde{w}_1$ is the solution to the problem:

$$-\frac{d^2\widetilde{w}_1}{dx^2} + \frac{1}{\eta+h}\widetilde{w}_1 = 0, \quad x \in (0,1),$$
$$\widetilde{w}_1(0) = \lambda,\ \widetilde{w}_1(1) = \mu.$$

One can readily find that

$$\widetilde{w}_1(x) = k_1 e^{x/\sqrt{\eta+h}} + k_2 e^{-x/\sqrt{\eta+h}},$$

where

$$k_1 = \frac{\mu - \lambda e^{-1/\sqrt{\eta+h}}}{e^{1/\sqrt{\eta+h}} - e^{-1/\sqrt{\eta+h}}}, \quad k_2 = \frac{\lambda e^{1/\sqrt{\eta+h}} - \mu}{e^{1/\sqrt{\eta+h}} - e^{-1/\sqrt{\eta+h}}}.$$

Then, replacing in (3.11) and performing some computations and elementary simplifications, we finally obtain the following equivalent linear algebraic system

$$\begin{aligned}\lambda+\mu &= \frac{\sinh(1/\sqrt{\eta+h})}{\sqrt{\eta+h}(1-\cosh(1/\sqrt{\eta+h}))}\int_0^1 w_1'(x;0,0)dx,\\ (1-\sqrt{\eta+h}\sinh\frac{1}{\sqrt{\eta+h}})\lambda &+ (\sqrt{\eta+h}\sinh\frac{1}{\sqrt{\eta+h}} - \cosh\frac{1}{\sqrt{\eta+h}})\mu\\ &= \frac{\sinh(1/\sqrt{\eta+h})}{\sqrt{\eta+h}}\int_0^1 x w_1'(x;0,0)dx\end{aligned} \tag{3.12}$$

whose determinant is

$$D(h) = 2\sqrt{\eta+h}\,\sinh\frac{1}{\sqrt{\eta+h}} - \cosh\frac{1}{\sqrt{\eta+h}} - 1.$$

It can be shown that the real function $\Phi(s) := 2\sqrt{s}\sinh\frac{1}{\sqrt{s}} - \cosh\frac{1}{\sqrt{s}} - 1$ possesses a unique real root $\overline{s} \simeq 3.448\times 10^{15}$. Therefore, if $\eta \geq \overline{s}$ then $D(h) \neq 0$ for all $h>0$ and the system (3.12) which is equivalent to (3.11) admits a unique solution $(\lambda_1,\mu_1)\in\mathbb{R}^2$, hence problem (3.7)-(3.8), with $j=1$, is uniquely solvable. In the case where $\eta < \overline{s}$, then $D(h)$ vanishes only for $h = \overline{s}-\eta$, consequently the system (3.12) which is equivalent to (3.11) has a unique solution for every $h < \overline{s}-\eta$ and so is problem (3.7)-(3.8) with $j=1$. To summarize, let n_0 be the smallest positive integer satisfying $T/n_0 \leq h_0$ where

$$h_0 := \begin{cases} T, & \text{if } \eta \geq \overline{s},\\ \min\{\overline{s}-\eta, T\}, & \text{if } \eta < \overline{s}.\end{cases}$$

Then we have shown that problem (3.7)-(3.8), with $j=1$, admits a unique solution $w_1 = w_1'(\cdot;\lambda_1,\mu_1) \in H^2(0,1)$ and consequently the solution $u_1 \in H^2(0,1)$ of problem (3.1)-(3.3), with $j=1$, is uniquely determined via the formula (3.9) provided that $n > n_0$ holds. Replacing in (3.7) with $j = 2$, this latter is seen to admit a unique strong solution $w_2' \in H^2(0,1)$ satisfying (3.10) wiht $j = 2$. Thanks to Lax-Milgram Lemma since $f_2 + \frac{1}{\eta+h}u_1 \in L^2(0,1)$. Similarly as above, the function $w_2'(\cdot;\lambda,\mu)$ is seen to be the unique solution of problem (3.7)-(3.8) with $j = 2$ for a suitable choice of (λ,μ) if $n > n_0$ is true. Accordingly, the solution $u_2 \in H^2(0,1)$ of problem (3.1)-(3.3) with $j = 2$ is uniquely determined due to relation (3.9). Proceeding in this way step by step, we will be able to state the following result:

Theorem 3.1. *There exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ and for all $j = 1,\dots,n$, the problems* (3.7)-(3.8) *and* (3.1)-(3.3) *admit a unique solution $w_j \in H^2(0,1)$ and $u_j \in H^2(0,1)$ respectively.*

So, for all $n > n_0$, we can define the Rothe approximation $u^{(n)} : I \to H^2(0,1) \cap V$ as a piecewise linear interpolation of the u_j $(j = 1,\dots,n)$ with respect to time, i.e.

$$u^{(n)}(t) = u_{j-1} + \delta u_j(t - t_{j-1}), \quad t \in [t_{j-1}, t_j], \quad j = 1,\dots,n, \tag{3.13}$$

as well as the corresponding step function $\overline{u}^{(n)} : I \to H^2(0,1) \cap V$:

$$\overline{u}^{(n)}(t) = \begin{cases} u_j & \text{for } t \in (t_{j-1}, t_j],\ j = 1,\dots,n. \\ U_0 & \text{for } t \in [\frac{-T}{n}, 0] \end{cases} \tag{3.14}$$

4. A priori estimates for the approximations

In this section, we will derive some a priori estimates which are the key points to establish Theorem 2.3. Note that, in the rest of the paper, positive constants are denoted by C, C_i $(i = 1,2,\dots)$.

Lemma 4.1. *There exist $C > 0$ such that, for all $n > n_0$, the solutions u_j of the time-discretized problems* (3.1)-(3.3)*, $j = 1,\dots,n$, satisfy the estimates*
- (i) $\|u_j\| \le C$
- (ii) $\|\delta u_j\| \le C$.

Proof. First of all, we write problem (3.7)-(3.8) in a variational form. Suppose that $n > n_0$ and let ϕ be any function from the space V defined in (2.1). A standard integration by parts yields

$$\int_0^x (x-\xi)\phi(\xi)d\xi = \Im_x^2\phi, \quad \text{for all } x \in (0,1), \tag{4.1}$$

where

$$\Im_x^2\phi := \Im_x(\Im_\xi\phi) = \int_0^x d\xi \int_0^\xi \phi(\eta)d\eta.$$

Hence, taking $x = 1$ in (4.1), we get

$$\Im_1^2\phi = \int_0^1 (1-\xi)\phi(\xi)d\xi = \int_0^1 \phi(\xi)d\xi - \int_0^1 \xi\phi(\xi)d\xi = 0. \tag{4.2}$$

Next, taking for all $j = 1,\dots,n$, the inner product in $L^2(0,1)$ of equation (3.7) and $\Im_x^2\phi$, we get

$$-\int_0^1 \frac{d^2w_j}{dx^2}(x)\Im_x^2\phi dx + \frac{1}{\eta+h}\int_0^1 w_j(x)\Im_x^2\phi dx = \int_0^1 (f_j(x) + \frac{1}{\eta+h}u_{j-1}(x))\Im_x^2\phi dx. \tag{4.3}$$

Carrying out some integrations by parts and invoking (4.2), we obtain for each term in (4.3):

$$\begin{aligned}\int_0^1 \frac{d^2w_j}{dx^2}(x)\Im_x^2\phi dx &= \frac{dw_j}{dx}(x)\Im_x^2\phi\big|_{x=0}^{x=1} - \int_0^1 \frac{dw_j}{dx}(x)\Im_x\phi dx\\ &= -w_j(x)\Im_x\phi\big|_{x=0}^{x=1} + \int_0^1 w_j(x)\phi(x)dx\\ &= (w_j,\phi),\end{aligned}$$

$$\begin{aligned}\int_0^1 w_j(x)\Im_x^2\phi dx &= \int_0^1 \frac{d}{dx}(\Im_x w_j)\Im_x^2\phi dx\\ &= \Im_x w_j \Im_x^2\phi\big|_{x=0}^{x=1} - \int_0^1 \Im_x w_j \Im_x\phi dx\\ &= -(w_j,\phi)_{B_2^1},\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 (f_j(x) + \frac{1}{\eta+h}u_{j-1}(x))\Im_x^2\phi dx\\ &= \int_0^1 \frac{d}{dx}\big[\Im_x\big(f_j + \frac{1}{\eta+h}u_{j-1}\big)\big]\Im_x^2\phi dx\\ &= \Im_x\big(f_j + \frac{1}{\eta+h}u_{j-1}\big)\Im_x^2\phi\big|_{x=0}^{x=1} - \int_0^1 \Im_x\big(f_j + \frac{1}{\eta+h}u_{j-1}\big)\Im_x\phi dx\\ &= -\big(f_j + \frac{1}{\eta+h}u_{j-1},\phi\big)_{B_2^1}.\end{aligned}$$

So that (4.3) becomes

$$(w_j,\phi) + \frac{1}{\eta+h}(w_j,\phi)_{B_2^1} = \big(f_j + \frac{1}{\eta+h}u_{j-1},\phi\big)_{B_2^1}, \quad \forall j = 1,\dots,n. \tag{4.4}$$

Now, testing this identity with $\phi = w_j$ which is in V thanks to (3.8), with the help of the Cauchy-Schwarz inequality we obtain

$$\|w_j\|^2 + \frac{1}{\eta+h}\|w_j\|_{B_2^1}^2 \le \big[\|f_j\|_{B_2^1} + \frac{1}{\eta+h}\|u_{j-1}\|_{B_2^1}\big]\|w_j\|_{B_2^1},$$

from where we deduce

$$\|w_j\| \le \|f_j\|_{B_2^1} + \frac{1}{\eta+h}\|u_{j-1}\|_{B_2^1}, \tag{4.5}$$

as well as

$$\|w_j\|_{B_2^1} \le (\eta+h)\|f_j\|_{B_2^1} + \|u_{j-1}\|_{B_2^1}. \tag{4.6}$$

Hence, (3.9) gives for all $j = 1,\dots,n$,

$$\begin{aligned}\|u_j\|_{B_2^1} &\le \frac{h}{\eta+h}\|w_j\|_{B_2^1} + \frac{\eta}{\eta+h}\|u_{j-1}\|_{B_2^1}\\ &\le \frac{h}{\eta+h}((\eta+h)\|f_j\|_{B_2^1} + \|u_{j-1}\|_{B_2^1}) + \frac{\eta}{\eta+h}\|u_{j-1}\|_{B_2^1},\end{aligned}$$

i.e.,

$$\|u_j\|_{B_2^1} \le h\|f_j\|_{B_2^1} + \|u_{j-1}\|_{B_2^1},$$

then, iterating this last inequality, we may arrive at

$$\|u_j\|_{B_2^1} \le h \sum_{i=1}^{i=j} \|f_i\|_{B_2^1} + \|U_0\|_{B_2^1}, \quad \forall j = 1, \dots, n. \tag{4.7}$$

But, for all $i \ge 1$ we have in view of Assumption (H1):

$$\|f_i\|_{B_2^1} \le \|f(t_i, u_{i-1}) - f(t_i, 0)\|_{B_2^1} + \|f(t_i, 0)\|_{B_2^1} \le l\|u_{i-1}\|_{B_2^1} + M, \tag{4.8}$$

where $M := \max_{t\in I} \|f(t, 0)\|_{B_2^1}$. So that substituting in (4.7),

$$\begin{aligned}\|u_j\|_{B_2^1} &\le h \sum_{i=1}^{i=j} (l\|u_{i-1}\|_{B_2^1} + M) + \|U_0\|_{B_2^1}\\ &= jhM + (1 + lh)\|U_0\|_{B_2^1} + lh \sum_{i=2}^{i=j} \|u_{i-1}\|_{B_2^1}\\ &\le TM + (1 + lh_0)\|U_0\|_{B_2^1} + lh \sum_{i=1}^{i=j-1} \|u_i\|_{B_2^1},\end{aligned}$$

from where it comes due to the discrete Gronwall's Lemma

$$\|u_j\|_{B_2^1} \le (TM + (1 + lh_0)\|U_0\|_{B_2^1})e^{l(j-1)h}.$$

Then

$$\|u_j\|_{B_2^1} \le C_1, \quad j = 1, \dots, n, \tag{4.9}$$

with $C_1 := (TM + (1+lh_0)\|U_0\|_{B_2^1})e^{lT}$. Then, From (3.6), (4.6) and (4.8), we have the estimate

$$\begin{aligned}\|\delta u_j\|_{B_2^1} &= \frac{1}{\eta + h}\|w_j - u_{j-1}\|_{B_2^1}\\ &\le \frac{1}{\eta}(\|w_j\|_{B_2^1} + \|u_{j-1}\|_{B_2^1})\\ &\le \frac{1}{\eta}\Big((\eta + h)\|f_j\|_{B_2^1} + 2\|u_{j-1}\|_{B_2^1}\Big)\\ &\le \frac{1}{\eta}\Big(((\eta + h)l + 2)\|u_{j-1}\|_{B_2^1} + (\eta + h)M\Big),\end{aligned}$$

finally, due to (4.9),

$$\|\delta u_j\|_{B_2^1} \le C_2, \quad j = 1, \dots, n, \tag{4.10}$$

where $C_2 := \frac{1}{\eta}([(\eta + h_0)l + 2]\, C_1 + (\eta + h_0)M)$. Now, combining (4.5) and (4.8),

$$\|w_j\| \le \big(l + \frac{1}{\eta + h}\big)\|u_{j-1}\|_{B_2^1} + M.$$

Consequently by (4.9), we get

$$\|w_j\| \le C_3, \quad j = 1, \dots, n, \tag{4.11}$$

with $C_3 := (l + \frac{1}{\eta})C_1 + M$. On the other hand, iterating (3.9) we may obtain for $j = 1, \dots, n$

$$\begin{aligned}u_j &= \frac{h}{\eta + h}w_j + \frac{\eta}{\eta + h}u_{j-1}\\ &= \frac{h}{\eta + h}w_j + \frac{\eta}{\eta + h}\Big(\frac{h}{\eta + h}w_{j-1} + \frac{\eta}{\eta + h}u_{j-2}\Big)\end{aligned}$$

$$\begin{aligned}
&= \frac{h}{\eta+h}\big(w_j + \frac{\eta}{\eta+h} w_{j-1}\big) + \big(\frac{\eta}{\eta+h}\big)^2 u_{j-2} \\
&= \dots \\
&= \frac{h}{\eta+h}\Big[w_j + \frac{\eta}{\eta+h} w_{j-1} + (\frac{\eta}{\eta+h})^2 w_{j-2} + \dots + (\frac{\eta}{\eta+h})^{j-1} w_1\Big] + (\frac{\eta}{\eta+h})^j U_0.
\end{aligned}$$

So that by (4.11), we have

$$\begin{aligned}
\|u_j\| &\le \frac{h}{\eta+h}\Big[\|w_j\| + \frac{\eta}{\eta+h}\|w_{j-1}\| + (\frac{\eta}{\eta+h})^2\|w_{j-2}\| + \dots + (\frac{\eta}{\eta+h})^{j-1}\|w_1\|\Big] \\
&\quad + (\frac{\eta}{\eta+h})^j \|U_0\| \\
&\le \frac{C_3 h}{\eta+h}\Big[1 + \frac{\eta}{\eta+h} + (\frac{\eta}{\eta+h})^2 + \dots + (\frac{\eta}{\eta+h})^{j-1}\Big] + \|U_0\|,
\end{aligned}$$

since $\frac{\eta}{\eta+h} < 1$. But

$$\begin{aligned}
1 + \frac{\eta}{\eta+h} + (\frac{\eta}{\eta+h})^2 + \dots + (\frac{\eta}{\eta+h})^{j-1} &= \frac{1-(\frac{\eta}{\eta+h})^j}{1-\frac{\eta}{\eta+h}} \\
&\le \frac{1}{1-\frac{\eta}{\eta+h}} = \frac{\eta+h}{h},
\end{aligned}$$

then

$$\|u_j\| \le \frac{C_3 h}{\eta+h}\frac{\eta+h}{h} + \|U_0\| = C_3 + \|U_0\|, \quad \text{for } j = 1,\dots,n, \tag{4.12}$$

which proves estimate (i) with $C := C_3 + \|U_0\|$. Finally, using (3.5), (4.11) and (4.12), we derive

$$\|\delta u_j\| \le \frac{1}{\eta}(\|w_j\| + \|u_j\|) \le \frac{1}{\eta}(2C_3 + \|U_0\|), \quad \text{for } j = 1,\dots,n.$$

Thus, estimate (ii) is proved with $C := \frac{1}{\eta}(2C_3 + \|U_0\|)$, and so the proof is complete. □

We deduce the following estimates that we shall use later.

Corollary 4.2. *For all $n > n_0$, the functions $u^{(n)}$ and $\overline{u}^{(n)}$ satisfies the inequalities*

(i) $\|u^{(n)}(t)\| \le C$, $\|\overline{u}^{(n)}(t)\| \le C$, *for all $t \in I$,*
(ii) $\|\frac{du^{(n)}}{dt}(t)\| \le C$, *a.e. in I,*
(iii) $\|\overline{u}^{(n)}(t) - u^{(n)}(t)\| \le \frac{C}{n}$, *for all $t \in I$*
(iv) $\|\overline{u}^{(n)}(t) - \overline{u}^{(n)}(t - \frac{T}{n})\| \le \frac{C}{n}$, *for all $t \in I$.*

Proof. Inequalities (i) follow immediately from Lemma 4.1 (i) with the same constant, whereas inequality (ii) is an easy consequence of Lemma 4.1 (ii), also with the same constant, noting that we have

$$\frac{du^{(n)}}{dt}(t) = \delta u_j, \quad \forall t \in (t_{j-1}, t_j],\ 1 \le j \le n.$$

As for inequalities (iii) and (iv), since we have

$$\overline{u}^{(n)}(t) - u^{(n)}(t) = (t_j - t)\delta u_j, \quad \forall t \in (t_{j-1}, t_j],\ 1 \le j \le n,$$

and

$$\overline{u}^{(n)}(t) - \overline{u}^{(n)}(t - \frac{T}{n}) = u_j - u_{j-1}, \quad \forall t \in (t_{j-1}, t_j],\ 1 \le j \le n,$$

it follows that

$$\|\overline{u}^{(n)}(t) - u^{(n)}(t)\| \le h \max_{1\le j\le n} \|\delta u_j\|, \quad \forall\, t \in I,$$

and

$$\|\overline{u}^{(n)}(t) - \overline{u}^{(n)}(t - \frac{T}{n})\| \le h \max_{1\le j\le n} \|\delta u_j\|, \quad \forall\, t \in I.$$

Hence, applying Lemma 4.1 (ii), we get the desired inequalities (iii) and (iv) with $C := \frac{T}{\eta}(2C_3 + \|U_0\|)$. □

5. Convergence and Existence result

Using relations (3.5) and (3.6), we can rearrange the variational equations (4.4) as follows

$$(\delta u_j, \phi)_{B_2^1} + (u_j, \phi) + \eta(\delta u_j, \phi) = (f_j, \phi)_{B_2^1}, \quad \forall \phi \in V,\ j = 1, \dots, n.$$

If we define, for all $n > n_0$, the abstract step function $\overline{f}^{(n)} : I \times V \to L^2(0,1)$ by

$$\overline{f}^{(n)}(t, v) = f(t_j, v), \quad t \in (t_{j-1}, t_j],\ j = 1, \dots, n,$$

the previous equations may be rewritten as

$$\Big(\frac{du^{(n)}}{dt}(t), \phi\Big)_{B_2^1} + \big(\overline{u}^{(n)}(t), \phi\big) + \eta\Big(\frac{du^{(n)}}{dt}(t), \phi\Big) = \big(\overline{f}^{(n)}(t, \overline{u}^{(n)}(t - \frac{T}{n})), \phi\big)_{B_2^1}, \tag{5.1}$$

for all $\phi \in V$, $t \in (0, T]$. Before performing a limiting process in the approximation scheme (5.1), we have to prove some convergence assertions.

Theorem 5.1. *The sequence $\{u^{(n)}\}_{n>n_0}$ converges under the the norm of $C(I, V)$ to some function $u \in C(I, V)$ and the error estimate*

$$\|u^{(n)} - u\|_{C(I,V)} \le \frac{C}{n^{1/2}}, \tag{5.2}$$

takes place for all $n > n_0$.

Proof. The main idea of the proof is to show that $\{u^{(n)}\}_{n>n_0}$ is a Cauchy sequence in the Banach space $C(I, V)$. Let $u^{(n)}$ and $u^{(m)}$ be the Rothe functions (3.13) corresponding to the step lengths $h_n = \frac{T}{n}$ and $h_m = \frac{T}{m}$ respectively with $m > n > n_0$. Testing the difference of (5.1) for n and m, with $\phi = u^{(n)}(t) - u^{(m)}(t)\ (\in V)$, we get for all $t \in (0, T]$:

$$\begin{aligned}
&\Big(\frac{d}{dt}\big(u^{(n)}(t) - u^{(m)}(t)\big), u^{(n)}(t) - u^{(m)}(t)\Big)_{B_2^1} \\
&+ \big(\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t), u^{(n)}(t) - u^{(m)}(t)\big) \\
&+ \eta\Big(\frac{d}{dt}\big(u^{(n)}(t) - u^{(m)}(t)\big), u^{(n)}(t) - u^{(m)}(t)\Big) \\
&= \Big(\overline{f}^{(n)}(t, \overline{u}^{(n)}(t - \frac{T}{n})) - \overline{f}^{(m)}(t, \overline{u}^{(m)}(t - \frac{T}{m})), u^{(n)}(t) - u^{(m)}(t)\Big)_{B_2^1},
\end{aligned}$$

or after some rearrangement

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^{(n)}(t) - u^{(m)}(t)\|_{B_2^1}^2 + \frac{\eta}{2}\frac{d}{dt}\|u^{(n)}(t) - u^{(m)}(t)\|^2 + \|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\|^2 \\
&= \big(\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t), (\overline{u}^{(n)}(t) - u^{(n)}(t)) + (u^{(m)}(t) - \overline{u}^{(m)}(t))\big) \\
&\quad + \Big(\overline{f}^{(n)}(t, \overline{u}^{(n)}(t - \frac{T}{n})) - \overline{f}^{(m)}(t, \overline{u}^{(m)}(t - \frac{T}{m})), u^{(n)}(t) - u^{(m)}(t)\Big)_{B_2^1}.
\end{aligned} \tag{5.3}$$

But, from the fact that

$$\overline{f}^{(n)}(t,\overline{u}^{(n)}(t-\frac{T}{n}))=f(t_j,u_{j-1}):=f_j,\quad \forall t\in(t_{j-1},t_j],\ j=1,\dots,n,$$

we deduce, in view of (4.8), that

$$\begin{aligned}\|\overline{f}^{(n)}\big(t,\overline{u}^{(n)}(t-\frac{T}{n})\big)\|_{B_2^1}&\le \max_{1\le j\le n}\|f_j\|_{B_2^1}\\ &\le l\max_{1\le j\le n}\|u_{j-1}\|_{B_2^1}+M,\quad \forall t\in I.\end{aligned}$$

Therefore, due to (4.9),

$$\|\overline{f}^{(n)}\big(t,\overline{u}^{(n)}(t-\frac{T}{n})\big)\|_{B_2^1}\le lC_1+M,\quad \forall t\in I. \tag{5.4}$$

Thus, from the identity

$$(\overline{u}^{(n)}(t),\phi)=\Big(\overline{f}^{(n)}\big(t,\overline{u}^{(n)}(t-\frac{T}{n})\big)-\frac{du^{(n)}}{dt}(t),\phi\Big)_{B_2^1}-\eta\big(\frac{du^{(n)}}{dt}(t),\phi\big),$$

for all $t\in I$, $\phi\in V$, which follows from (5.1), due to (4.10), (5.4) and Corollary 4.2(ii), we obtain

$$\begin{aligned}|(\overline{u}^{(n)}(t),\phi)|&\le\Big[\|\overline{f}^{(n)}\big(t,\overline{u}^{(n)}(t-\frac{T}{n})\big)\|_{B_2^1}+\|\frac{du^{(n)}}{dt}(t)\|_{B_2^1}+\eta\|\frac{du^{(n)}}{dt}(t)\|\Big]\|\phi\|\\ &\le C_4\|\phi\|,\quad \forall t\in I,\ \forall\phi\in V,\end{aligned} \tag{5.5}$$

with $C_4:=lC_1+M+C_2+2C_3+\|U_0\|$. Applying (5.5) for

$$\phi=(\overline{u}^{(n)}(t)-u^{(n)}(t))+(u^{(m)}(t)-\overline{u}^{(m)}(t)),$$

together with Corollary 4.2 (iii), we can dominate the first term in the right-hand side of (5.3) as follows

$$\begin{aligned}&\big(\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t),(\overline{u}^{(n)}(t)-u^{(n)}(t))+(u^{(m)}(t)-\overline{u}^{(m)}(t))\big)\\ &\le 2C_4\big(\|\overline{u}^{(n)}(t)-u^{(n)}(t)\|+\|\overline{u}^{(m)}(t)-u^{(m)}(t)\|\big)\\ &\le C_5(\frac{1}{n}+\frac{1}{m}),\quad \forall t\in I,\end{aligned} \tag{5.6}$$

with $C_5:=\frac{2C_4T}{\eta}(2C_3+\|U_0\|)$. It remains to majorize the second term in the right hand side in (5.3). To this end, we use the Cauchy inequality

$$\alpha\beta\le\frac{\varepsilon}{2}\alpha^2+\frac{1}{2\varepsilon}\beta^2,\quad \forall\alpha,\beta\in\mathbb{R},\quad \forall\varepsilon\in\mathbb{R}_+^*,$$

for every $\varepsilon>0$:

$$\begin{aligned}&\Big(\overline{f}^{(n)}(t,\overline{u}^{(n)}(t-\frac{T}{n}))-\overline{f}^{(m)}(t,\overline{u}^{(m)}(t-\frac{T}{m})),u^{(n)}(t)-u^{(m)}(t)\Big)_{B_2^1}\\ &\le\|\overline{f}^{(n)}(t,\overline{u}^{(n)}(t-\frac{T}{n}))-\overline{f}^{(m)}(t,\overline{u}^{(m)}(t-\frac{T}{m}))\|_{B_2^1}\|u^{(n)}(t)-u^{(m)}(t)\|_{B_2^1}\\ &\le\frac{\varepsilon}{2}\|\overline{f}^{(n)}(t,\overline{u}^{(n)}(t-\frac{T}{n}))-\overline{f}^{(m)}(t,\overline{u}^{(m)}(t-\frac{T}{m}))\|_{B_2^1}^2\\ &\quad+\frac{1}{2\varepsilon}\|u^{(n)}(t)-u^{(m)}(t)\|_{B_2^1}^2,\quad \forall t\in I.\end{aligned} \tag{5.7}$$

Now, let t be arbitrary but fixed in $(0,T]$. Then there exist two integers k and i corresponding to the subdivision of I into n and m subintervals respectively, such that $t \in (t_{k-1}, t_k] \cap (t_{i-1}, t_i]$. Hence thanks to the assumed Lipschitz continuity of f,

$$
\begin{aligned}
&\|\overline{f}^{(n)}\big(t, \overline{u}^{(n)}(t - \frac{T}{n})\big) - \overline{f}^{(m)}\big(t, \overline{u}^{(m)}(t - \frac{T}{m})\big)\|^2_{B^1_2} \\
&= \|f\big(t_k, \overline{u}^{(n)}(t - \frac{T}{n})\big) - f\big(t_i, \overline{u}^{(m)}(t - \frac{T}{m})\big)\|^2_{B^1_2} \\
&\le l^2 \Big[|t_k - t_i| \big\{ 1 + \|\overline{u}^{(n)}(t - \frac{T}{n})\|_{B^1_2} + \|\overline{u}^{(m)}(t - \frac{T}{m})\|_{B^1_2} \big\} \\
&\quad + \|\overline{u}^{(n)}(t - \frac{T}{n}) - \overline{u}^{(m)}(t - \frac{T}{m})\|_{B^1_2} \Big]^2 \\
&\le l^2 \Big[(h_n + h_m)(1 + \|u_{k-1}\|_{B^1_2} + \|u_{i-1}\|_{B^1_2}) + \|\overline{u}^{(n)}(t - \frac{T}{n}) - \overline{u}^{(n)}(t)\|_{B^1_2} \\
&\quad + \|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\|_{B^1_2} + \|\overline{u}^{(m)}(t) - \overline{u}^{(m)}(t - \frac{T}{m})\|_{B^1_2} \Big]^2 .
\end{aligned}
$$

Then follows with consideration to (4.9) and Corollary 4.2 (iv) that

$$
\begin{aligned}
&\|\overline{f}^{(n)}\big(t, \overline{u}^{(n)}(t - \frac{T}{n})\big) - \overline{f}^{(m)}\big(t, \overline{u}^{(m)}(t - \frac{T}{m})\big)\|^2_{B^1_2} \\
&\le l^2 \big[T(\frac{1}{n} + \frac{1}{m})(1 + 2C_1) + \frac{T}{\eta}(2C_3 + \|U_0\|)(\frac{1}{n} + \frac{1}{m}) + \|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\|_{B^1_2} \big]^2 \\
&= l^2 \big[T(1 + 2C_1 + \frac{1}{\eta}(2C_3 + \|U_0\|))(\frac{1}{n} + \frac{1}{m}) + \|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\|_{B^1_2} \big]^2 \\
&\le l^2 \Big[C_6^2(\frac{1}{n} + \frac{1}{m})^2 + 2C_6(\frac{1}{n} + \frac{1}{m})(\|\overline{u}^{(n)}(t)\|_{B^1_2} + \|\overline{u}^{(m)}(t)\|_{B^1_2}) \\
&\quad + \|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\|^2_{B^1_2} \Big] \\
&\le (lC_6)^2(\frac{1}{n} + \frac{1}{m})^2 + 4l^2 C_6 C_1(\frac{1}{n} + \frac{1}{m}) + l^2 \|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\|^2_{B^1_2}, \quad \forall t \in I,
\end{aligned}
$$

with $C_6 := T\big(1 + 2C_1 + \frac{1}{\eta}(2C_3 + \|U_0\|)\big)$. Thus, setting $C_7 := (lC_6)^2$ and $C_8 := 4l^2 C_6 C_1$, we write

$$
\begin{aligned}
&\|\overline{f}^{(n)}(t, \overline{u}^{(n)}(t - \frac{T}{n})) - \overline{f}^{(m)}(t, \overline{u}^{(m)}(t - \frac{T}{m}))\|^2_{B^1_2} \\
&\le C_7(\frac{1}{n} + \frac{1}{m})^2 + C_8(\frac{1}{n} + \frac{1}{m}) + l^2 \|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\|^2_{B^1_2}, \quad \forall t \in I;
\end{aligned}
\tag{5.8}
$$

therefore, going back to (5.7), we have

$$
\begin{aligned}
&\Big(\overline{f}^{(n)}(t, \overline{u}^{(n)}(t - \frac{T}{n})) - \overline{f}^{(m)}(t, \overline{u}^{(m)}(t - \frac{T}{m})), u^{(n)}(t) - u^{(m)}(t)\Big)_{B^1_2} \\
&\le \frac{\varepsilon}{2} C_7(\frac{1}{n} + \frac{1}{m})^2 + \frac{\varepsilon}{2} C_8(\frac{1}{n} + \frac{1}{m}) + \frac{\varepsilon}{2} l^2 \|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\|^2_{B^1_2} \\
&\quad + \frac{1}{2\varepsilon} \|u^{(n)}(t) - u^{(m)}(t)\|^2_{B^1_2}, \quad \forall t \in I.
\end{aligned}
\tag{5.9}
$$

Now, combining (5.3), (5.6), (5.9) and (2.3), we get

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|u^{(n)}(t) - u^{(m)}(t)\|^2_{B^1_2} + \eta \|u^{(n)}(t) - u^{(m)}(t)\|^2 \Big) + 2\|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\|^2 \\
&\le \varepsilon C_7(\frac{1}{n} + \frac{1}{m})^2 + (\varepsilon C_8 + 2C_5)(\frac{1}{n} + \frac{1}{m}) + \frac{\varepsilon l^2}{2} \|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\|^2
\end{aligned}
$$

$$+\frac{1}{2\varepsilon}\|u^{(n)}(t)-u^{(m)}(t)\|^2, \quad \forall t \in I.$$

Hence

$$\begin{aligned}&\eta\frac{d}{dt}\|u^{(n)}(t)-u^{(m)}(t)\|^2+(2-\frac{\varepsilon l^2}{2})\|\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t)\|^2\\&\le \varepsilon C_7(\frac{1}{n}+\frac{1}{m})^2+(\varepsilon C_8+2C_5)(\frac{1}{n}+\frac{1}{m})+\frac{1}{2\varepsilon}\|u^{(n)}(t)-u^{(m)}(t)\|^2.\end{aligned}$$

Choosing $\varepsilon > 0$ so that $2-\frac{\varepsilon l^2}{2}=0$, i.e. $\varepsilon=\frac{4}{l^2}$ and integrating the just obtained inequality between 0 and t taking into account the fact that $u^{(n)}(0)=u^{(m)}(0)=U_0$, we get for all $t\in I$:

$$\begin{aligned}&\|u^{(n)}(t)-u^{(m)}(t)\|^2\\&\le \frac{4C_7T}{\eta l^2}(\frac{1}{n}+\frac{1}{m})^2+\frac{2T}{\eta}(\frac{2C_8}{l^2}+C_5)(\frac{1}{n}+\frac{1}{m})+\frac{l^2}{8\eta}\int_0^t\|u^{(n)}(\tau)-u^{(m)}(\tau)\|^2d\tau.\end{aligned}$$

Then, by Gronwall's Lemma,

$$\|u^{(n)}(t)-u^{(m)}(t)\|^2\le\big[C_9(\frac{1}{n}+\frac{1}{m})^2+C_{10}(\frac{1}{n}+\frac{1}{m})\big]e^{\frac{l^2}{8\eta}t} \quad \forall t\in I,$$

with $C_9:=\frac{4C_7T}{\eta l^2}$ and $C_{10}:=\frac{2T}{\eta}(\frac{2C_8}{l^2}+C_5)$. Accordingly

$$\|u^{(n)}(t)-u^{(m)}(t)\|\le\big[C_9(\frac{1}{n}+\frac{1}{m})^2+C_{10}(\frac{1}{n}+\frac{1}{m})\big]^{1/2}e^{\frac{l^2T}{16\eta}}, \quad \forall t\in I.$$

Then, taking the upper bound with respect to t in the left-hand side of this inequality,

$$\|u^{(n)}-u^{(m)}\|_{C(I,V)}\le\big[C_9(\frac{1}{n}+\frac{1}{m})^2+C_{10}(\frac{1}{n}+\frac{1}{m})\big]^{1/2}e^{\frac{l^2T}{16\eta}}, \tag{5.10}$$

which shows that $\{u^{(n)}\}_{n>n_0}$ is a Cauchy sequence in $C(I,V)$. This implies the existence of a function $u\in C(I,V)$ such that $u^{(n)}\to u$ in this space. Moreover, letting $m\to\infty$ in (5.10) we obtain the error estimate (5.2) with $C=\sqrt{C_9+C_{10}}e^{\frac{l^2T}{16\eta}}$, what completes the proof. □

We write down some results for the limit-function u.

Corollary 5.2. *The function u possesses the following properties:*

(i) $u\in C^{0,1}(I,V)$;
(ii) *u is strongly differentiable a.e. in I and* $\frac{du}{dt}\in L^\infty(I,L^2(0,1))$;
(iii) $\overline{u}^{(n)}(t)\to u(t)$ *in V for all* $t\in I$;
(iv) $\frac{du^{(n)}}{dt}\rightharpoonup\frac{du}{dt}$ *in* $L^2(I,L^2(0,1))$.

Proof. On the basis of Corollary 4.2 (i) and (ii), uniform convergence statement from Theorem 5.1 and the continuous embedding $V\hookrightarrow Y:=L^2(0,1)$, [9, Lemma 1.3.15] is valid for our special situation yielding assertions (i), (ii) and (iv) of the present Corollary. The remaining assertion (iii) is an immediate consequence of the combination of Corollary 4.2 (iii) with the convergence result stated in Theorem 5.1. □

Collecting all the obtained results, we can state our existence theorem.

Theorem 5.3. *The limit function u from Theorem 5.1 is the unique weak solution to problem* (1.5)-(1.8) *in the sense of Definition 2.2. Moreover, u depends continuously upon data f and U_0, namely the inequality*

$$\max_{0\le s\le t}\|u^*(s)-u^{**}(s)\|\le C\Big(\|U_0^*-U_0^{**}\|+\int_0^t\|f^*(s,u^*(s))-f^{**}(s,u^{**}(s))\|_{B_2^1}ds\Big), \tag{5.11}$$

holds for all $t\in I$, with some positive constant C depending only on η.

Proof. **Existence.** It suffices to check all the points (i)-(iv) of Definition 2.2. Obviously, in light of Corollary 5.2, the first two points of Definition 2.2 are already fulfilled. Moreover, since $u^{(n)}\to u$ in $C(I,V)$ as $n\to\infty$ and, by definition, $u^{(n)}(0)=U_0$, it follows that $u(0)=U_0$ holds in V so the initial condition (1.6) is also fulfilled, that is point (iii) of Definition 2.2 takes place. To show that u obeys the integral equation (2.4), we investigate the behaviour as $n\to\infty$ of the integral relation

$$\begin{aligned}&\big(u^{(n)}(t)-U_0,\phi\big)_{B_2^1}+\int_0^t\big(\overline{u}^{(n)}(\tau),\phi\big)d\tau+\eta\big(u^{(n)}(t)-U_0,\phi\big)\\&=\int_0^t\Big(\overline{f}^{(n)}\big(\tau,\overline{u}^{(n)}(\tau-\frac{T}{n})\big),\phi\Big)_{B_2^1}d\tau,\quad\forall\phi\in V,\ \forall t\in I,\end{aligned}\tag{5.12}$$

which results from (5.1) by integration over $(0,t)\subset I$ noting that $u^{(n)}(0)=U_0$. This requires some additional convergence statements.

Firstly, since $u^{(n)}\to u$ in $C(I,V)$ and since for all fixed $\phi\in V$, the linear functional $v\mapsto(v,\phi)_{B_2^1}$ is continuous on V, we deduce that

$$\big(u^{(n)}(t),\phi\big)\underset{n\to\infty}{\longrightarrow}\big(u(t),\phi\big),\quad\forall\phi\in V,\ \forall t\in I, \tag{5.13}$$

$$\big(u^{(n)}(t),\phi\big)_{B_2^1}\underset{n\to\infty}{\longrightarrow}\big(u(t),\phi\big)_{B_2^1},\quad\forall\phi\in V,\ \forall t\in I. \tag{5.14}$$

Secondly, by virtue of (5.5) the Lebesgue Theorem of dominated convergence may be applied to the convergence statement (iii) from Corollary 5.2 giving

$$\int_0^t\big(\overline{u}^{(n)}(\tau),\phi\big)d\tau\underset{n\to\infty}{\longrightarrow}\int_0^t\big(u(\tau),\phi\big)d\tau,\quad\forall\phi\in V,\ \forall t\in I. \tag{5.15}$$

Thirdly, in view of Assumption (H1), we have

$$\begin{aligned}&\|\overline{f}^{(n)}\big(\tau,\overline{u}^{(n)}(\tau-\frac{T}{n})\big)-f(\tau,u(\tau))\|_{B_2^1}\\&=\|f\big(t_j,\overline{u}^{(n)}(\tau-\frac{T}{n})\big)-f(\tau,u(\tau))\|_{B_2^1}\\&\le l\big[|t_j-\tau|(1+\|u_{j-1}\|_{B_2^1}+\|u(\tau)\|_{B_2^1})+\|\overline{u}^{(n)}(\tau-\frac{T}{n})-u(\tau)\|_{B_2^1}\big],\end{aligned}$$

for all $\tau\in(t_{j-1},t_j]$, $1\le j\le n$; therefore

$$\|\overline{f}^{(n)}\big(\tau,\overline{u}^{(n)}(\tau-\frac{T}{n})\big)-f(\tau,u(\tau))\|_{B_2^1}\le\frac{C}{n}+l\|\overline{u}^{(n)}(\tau-\frac{T}{n})-u(\tau)\|_{B_2^1},$$

for all $\tau\in I$, where $C:=lT(1+C_1+\|u\|_{C(I,V)})$. However, with consideration to estimates (iii)-(iv) from Corollary 4.2 and inequality (5.2), we can write

$$\|\overline{u}^{(n)}(\tau-\frac{T}{n})-u(\tau)\|_{B_2^1}\le\|\overline{u}^{(n)}(\tau-\frac{T}{n})-\overline{u}^{(n)}(\tau)\|$$

$$+\|\overline{u}^{(n)}(\tau)-u^{(n)}(\tau)\|+\|u^{(n)}(\tau)-u(\tau)\|$$
$$\le C(\frac{1}{n}+\frac{1}{n^{1/2}}),\quad \forall\tau\in I,$$

whence

$$\|\overline{f}^{(n)}\big(\tau,\overline{u}^{(n)}(\tau-\frac{T}{n})\big)-f(\tau,u(\tau))\|_{B_2^1}\le\frac{C}{n^{1/2}},\quad\forall\tau\in I,$$

and then

$$\overline{f}^{(n)}\big(\tau,\overline{u}^{(n)}(\tau-\frac{T}{n})\big)\underset{n\to\infty}{\longrightarrow}f(\tau,u(\tau))\quad\text{in } B_2^1(0,1),\quad\forall\tau\in I. \tag{5.16}$$

Now, due to (5.4) the function $|(\overline{f}^{(n)}\big(\tau,\overline{u}^{(n)}(\tau-\frac{T}{n})\big),\phi)_{B_2^1}|$ is uniformly bounded with respect to both τ and n. So the Lebesgue Theorem of dominated convergence may be applied again to (5.16) yielding

$$\int_0^t(\Big(\overline{f}^{(n)}\big(\tau,\overline{u}^{(n)}(\tau-\frac{T}{n})\big),\phi\Big)_{B_2^1}d\tau\underset{n\to\infty}{\longrightarrow}\int_0^t(f(\tau,u(\tau)),\phi)_{B_2^1}d\tau, \tag{5.17}$$

for all $\phi\in V$ and all $t\in I$. Then, by (5.13), (5.14) , (5.15) and (5.17), a limiting process $n\to\infty$ in (5.12) leads to

$$\big(u(t)-U_0,\phi\big)_{B_2^1}+\int_0^t\big(u(\tau),\phi\big)d\tau+\eta\big(u(t)-U_0,\phi\big)=\int_0^t(f(\tau,u(\tau)),\phi)_{B_2^1}d\tau,$$

for all $\phi\in V$ and $t\in I$. Finally, the differentiation of this last identity with respect to t recalling that $u:I\to L^2(0,1)$ is strongly differentiable for *a.e.* $t\in I$, leads to the required identity (2.4) by the aid of the equalities $\frac{d}{dt}(u(t),\phi)_{B_2^1}=(\frac{du}{dt}(t),\phi)_{B_2^1}$ and $\frac{d}{dt}(u(t),\phi)=(\frac{du}{dt}(t),\phi)$ which hold for all $t\in I$ and all $\phi\in V$. Thus, u weakly solves problem (1.5)-(1.8).

Uniqueness and continuous dependence upon data. Let u^* and u^{**} be two weak solutions of problem (1.5)-(1.8) corresponding respectively to (U_0^*,f^*) and (U_0^{**},f^{**}) instead of (U_0,f). Subtracting the identity (2.4) written for u^{**} from the same identity written for u^* and inserting $\phi=u^*(t)-u^{**}(t)$ in the resulting relation, we get by integration over $(0,\tau)$, with $\tau\in I$:

$$\begin{aligned}
&\frac12\|u^*(\tau)-u^{**}(\tau)\|_{B_2^1}^2-\frac12\|u^*(0)-u^{**}(0)\|_{B_2^1}^2+\int_0^\tau\|u^*(t)-u^{**}(t)\|^2dt\\
&+\frac{\eta}{2}\|u^*(\tau)-u^{**}(\tau)\|^2-\frac{\eta}{2}\|u^*(0)-u^{**}(0)\|^2\\
&=\int_0^\tau\Big(f^*(t,u^*(t))-f^{**}(t,u^{**}(t)),u^*(t)-u^{**}(t)\Big)_{B_2^1}dt,
\end{aligned}$$

hence, ignoring the first and the third terms in the left hand side, we obtain

$$\begin{aligned}
&\|u^*(\tau)-u^{**}(\tau)\|^2\\
&\le\frac{1}{\eta}\|u^*(0)-u^{**}(0)\|_{B_2^1}^2+\|u^*(0)-u^{**}(0)\|^2\\
&\quad+\frac{2}{\eta}\int_0^\tau\|f^*(t,u^*(t))-f^{**}(t,u^{**}(t))\|_{B_2^1}\|u^*(t)-u^{**}(t)\|_{B_2^1}dt\\
&\le(\frac{1}{2\eta}+1)\|u^*(0)-u^{**}(0)\|^2+\frac{\sqrt2}{\eta}\max_{0\le t\le\tau}\|u^*(t)-u^{**}(t)\|\\
&\times\int_0^\tau\|f^*(t,u^*(t))-f^{**}(t,u^{**}(t))\|_{B_2^1}dt
\end{aligned}$$

$$
\begin{aligned}
&\leq (\frac{1}{2\eta}+1)\|U_0^* - U_0^{**}\| \max_{0\leq t\leq\tau}\|u^*(t)-u^{**}(t)\| + \frac{\sqrt{2}}{\eta}\max_{0\leq t\leq\tau}\|u^*(t)-u^{**}(t)\| \\
&\quad\times \int_0^\tau \|f^*(t,u^*(t)) - f^{**}(t,u^{**}(t))\|_{B_2^1}dt \\
&\leq \Big[(\frac{1}{2\eta}+1)\|U_0^* - U_0^{**}\| + \frac{\sqrt{2}}{\eta}\int_0^\tau \|f^*(t,u^*(t)) - f^{**}(t,u^{**}(t))\|_{B_2^1}dt\Big] \\
&\quad\times \max_{0\leq t\leq\tau}\|u^*(t)-u^{**}(t)\|,
\end{aligned}
$$

where (2.3) has been used. Consequently for all $s \in [0,\tau]$, we have

$$
\begin{aligned}
&\|u^*(s)-u^{**}(s)\|^2 \\
&\leq \Big[(\frac{1}{2\eta}+1)\|U_0^* - U_0^{**}\| + \frac{\sqrt{2}}{\eta}\int_0^\tau \|f^*(t,u^*(t)) - f^{**}(t,u^{**}(t))\|_{B_2^1}dt\Big] \\
&\quad\times \max 0\leq t\leq\tau\|u^*(t)-u^{**}(t)\|,
\end{aligned}
$$

whence

$$
\begin{aligned}
&\max_{0\leq s\leq\tau}\|u^*(s)-u^{**}(s)\|^2 \\
&\leq \Big[(\frac{1}{2\eta}+1)\|U_0^* - U_0^{**}\| + \frac{\sqrt{2}}{\eta}\int_0^\tau \|f^*(t,u^*(t)) - f^{**}(t,u^{**}(t))\|_{B_2^1}dt\Big] \\
&\quad\times \max_{0\leq t\leq\tau}\|u^*(t)-u^{**}(t)\|,
\end{aligned}
$$

from which the estimate (5.11) follows with $C := \max(\frac{1}{2\eta}+1, \frac{\sqrt{2}}{\eta})$. This implies the uniqueness as well as the continuous dependence of the solution of (1.5)-(1.8) upon data. So the proof is complete. □

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