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VARYING DOMAINS IN A GENERAL CLASS OF SUBLINEAR ELLIPTIC PROBLEMS

SANTIAGO CANO-CASANOVA & JULIÁN LÓPEZ-GÓMEZ

Abstract. In this paper we use the linear theory developed in [8] and [9] to show the continuous dependence of the positive solutions of a general class of sublinear elliptic boundary value problems of mixed type with respect to the underlying domain. Our main theorem completes the results of Daners and Dancer [12] –and the references there in–, where the classical Robin problem was dealt with. Besides the fact that we are working with mixed non-classical boundary conditions, it must be mentioned that this paper is considering problems where bifurcation from infinity occurs; now a days, analyzing these general problems, where the coefficients are allowed to vary and eventually vanishing or changing sign, is focusing a great deal of attention –as they give rise to *metasolutions* (e.g.,[20])–.

1. Introduction

In this paper we analyze the continuous dependence with respect to the domain Ω of the positive solutions of the following sublinear weighted elliptic boundary value problem of mixed type

$$\begin{gathered}\mathcal{L}u = \lambda W(x)u - a(x)f(x,u)u \quad \text{in } \Omega\,,\\ \mathcal{B}(b)u = 0 \quad \text{on } \partial\Omega\,,\end{gathered}\tag{1.1}$$

where $a \in L_\infty(\Omega)$ belongs to a certain large class of nonnegative potentials, to be introduced later, and $W \in L_\infty(\Omega)$.

Throughout this paper we make the following assumptions:

(a) The domain Ω is bounded in $\mathbb{R}^N$, $N \geq 1$, and of class $\mathcal{C}^2$, i.e., $\bar\Omega$ is an N-dimensional compact connected submanifold of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $\mathcal{C}^2$.

(b) $\lambda \in \mathbb{R}$, $W \in L_\infty(\Omega)$ and the differential operator

$$\mathcal{L} := -\sum_{i,j=1}^N \alpha_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^N \alpha_i(x)\frac{\partial}{\partial x_i} + \alpha_0(x)\tag{1.2}$$

is uniformly strongly elliptic of second order in Ω with

$$\alpha_{ij} = \alpha_{ji} \in \mathcal{C}^1(\bar\Omega)\,, \quad \alpha_i \in \mathcal{C}(\bar\Omega)\,, \quad \alpha_0 \in L_\infty(\Omega)\,, \quad 1 \leq i\,,\, j \leq N\,.\tag{1.3}$$

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Subsequently, we denote by $\mu > 0$ the ellipticity constant of $\mathcal{L}$ in Ω. Then, for any $\xi \in \mathbb{R}^N \setminus \{0\}$ and $x \in \bar{\Omega}$ we have that

$$\sum_{i,j=1}^{N} \alpha_{ij}(x)\,\xi_i\,\xi_j \geq \mu\,|\xi|^2\,.$$

(c) The boundary operator is

$$\mathcal{B}(b)u := \begin{cases} u & \text{on } \Gamma_0\,, \\ \partial_\nu u + bu & \text{on } \Gamma_1\,, \end{cases} \tag{1.4}$$

where Γ_0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega$ with $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, $b \in \mathcal{C}(\Gamma_1)$, and

$$\nu = (\nu_1, \dots, \nu_N) \in \mathcal{C}^1(\Gamma_1; \mathbb{R}^N)$$

is an outward pointing nowhere tangent vector field. Necessarily, Γ_0 and Γ_1 possess finitely many components. Note that $\mathcal{B}(b)$ is the Dirichlet boundary operator on Γ_0, denoted in the sequel by $\mathcal{D}$, and the Neumann or a first order regular oblique derivative boundary operator on Γ_1. It should be pointed out that either Γ_0 or Γ_1 might be empty.

(d) The function $f : \bar{\Omega} \times [0,\infty) \to \mathbb{R}$ satisfies

$$f \in \mathcal{C}^1(\bar{\Omega} \times [0,\infty); \mathbb{R})\,, \quad \lim_{u \nearrow \infty} f(\cdot, u) = +\infty \quad \text{uniformly in } \bar{\Omega}\,, \tag{1.5}$$

$$\partial_u f(\cdot, u) > 0 \quad \text{for all} \quad u \geq 0\,. \tag{1.6}$$

Thanks to (1.5), for each $M > 0$ there exists $C_M > 0$ such that

$$f(x,\xi) > M \quad \text{for each} \quad (x,\xi) \in \bar{\Omega} \times [C_M, \infty)\,. \tag{1.7}$$

In the sequel, given $M > 0$, we denote by C_M any fixed positive constant satisfying (1.7). It should be noted that $f(\cdot, 0) \in \mathcal{C}^1(\bar{\Omega}; \mathbb{R})$ and that there is no sign restriction on $f(\cdot, 0)$ in Ω. Moreover, (1.6) implies

$$f(\cdot, 0) = \inf_{\xi > 0} f(\cdot, \xi)\,.$$

In the sequel, for each $\lambda \in \mathbb{R}$, we denote

$$\mathcal{L}(\lambda) := \mathcal{L} - \lambda W\,, \quad \mathcal{L}_f := \mathcal{L} + a f(\cdot, 0)\,, \quad \mathcal{L}_f(\lambda) := \mathcal{L}_f - \lambda W\,. \tag{1.8}$$

These operators are uniformly strongly elliptic in Ω with the same ellipticity constant $\mu > 0$ as $\mathcal{L}$.

As far as to the weight function $a \in L_\infty(\Omega)$ concerns, we assume that $a \in \mathfrak{A}(\Omega)$ where $\mathfrak{A}(\Omega)$ is the class of nonnegative bounded measurable real weight functions a in Ω for which there exist an open subset Ω_a^0 of Ω and a compact subset $K = K_a$ of $\bar{\Omega}$ with Lebesgue measure zero such that

$$K \cap (\bar{\Omega}_a^0 \cup \Gamma_1) = \emptyset\,, \tag{1.9}$$

$$\Omega_a^+ := \{\, x \in \Omega : a(x) > 0 \} = \Omega \setminus (\bar{\Omega}_a^0 \cup K)\,, \tag{1.10}$$

and each of the following four conditions is satisfied:

(A1) Ω_a^0 possesses a finite number of components of class $\mathcal{C}^2$, say $\Omega_a^{0,j}$, $1 \le j \le m$, such that $\bar\Omega_a^{0,i} \cap \bar\Omega_a^{0,j} = \emptyset$ if $i \neq j$, and

$$\operatorname{dist}(\Gamma_1, \partial\Omega_a^0 \cap \Omega) > 0\,. \tag{1.11}$$

Thus, if we denote by Γ_1^i, $1 \le i \le n_1$, the components of Γ_1, then for each $1 \le i \le n_1$ either $\Gamma_1^i \subset \partial\Omega_a^0$ or else $\Gamma_1^i \cap \partial\Omega_a^0 = \emptyset$. Moreover, if $\Gamma_1^i \subset \partial\Omega_a^0$, then Γ_1^i must be a component of $\partial\Omega_a^0$. Indeed, if $\Gamma_1^i \cap \partial\Omega_a^0 \neq \emptyset$ but Γ_1^i is not a component of $\partial\Omega_a^0$, then

$$\operatorname{dist}(\Gamma_1^i, \partial\Omega_a^0 \cap \Omega) = 0$$

and, hence, (1.11) fails.

(A2) Let $\{i_1, \dots, i_p\}$ denote the subset of $\{1, \dots, n_1\}$ for which

$$\Gamma_1^j \cap \partial\Omega_a^0 = \emptyset \quad \iff \quad j \in \{i_1, \dots, i_p\}\,.$$

Then, a is bounded away from zero on any compact subset of

$$\Omega_a^+ \cup \bigcup_{j=1}^p \Gamma_1^{i_j}\,.$$

Note that if $\Gamma_1 \subset \partial\Omega_a^0$, then we are only imposing that a is bounded away from zero on any compact subset of Ω_a^+.

(A3) Let Γ_0^i, $1 \le i \le n_0$, denote the components of Γ_0, and let $\{i_1, \dots, i_q\}$ be the subset of $\{1, \dots, n_0\}$ for which

$$(\partial\Omega_a^0 \cup K) \cap \Gamma_0^j \neq \emptyset \quad \iff \quad j \in \{i_1, \dots, i_q\}\,.$$

Then, a is bounded away from zero on any compact subset of

$$\Omega_a^+ \cup [\bigcup_{j=1}^q \Gamma_0^{i_j} \setminus (\partial\Omega_a^0 \cup K)]\,.$$

Note that if $(\partial\Omega_a^0 \cup K) \cap \Gamma_0 = \emptyset$, then we are only imposing that a is bounded away from zero on any compact subset of Ω_a^+.

(A4) For any $\eta > 0$ there exist a natural number $\ell(\eta) \ge 1$ and $\ell(\eta)$ open subsets of $\mathbb{R}^N$, G_j^η, $1 \le j \le \ell(\eta)$, with $|G_j^\eta| < \eta$, $1 \le j \le \ell(\eta)$, such that

$$\bar G_i^\eta \cap \bar G_j^\eta = \emptyset \quad \text{if} \quad i \neq j\,, \quad K \subset \bigcup_{j=1}^{\ell(\eta)} G_j^\eta\,,$$

and for each $1 \le j \le \ell(\eta)$ the open set $G_j^\eta \cap \Omega$ is connected and of class $\mathcal{C}^2$.

More precisely, under the previous assumptions it will be said that $a \in \mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$. In this case, the abstract theory developed by the authors in [8] and [7] can be applied to deal with (1.1).

Subsequently, we also consider the class of weight functions $\mathfrak{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$ consisting of the elements $a \in \mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$ for which $\Omega_a^0 = \emptyset$. Note that if $a \in \mathfrak{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$ then (1.9) and (1.10) become to

$$K \cap \Gamma_1 = \emptyset\,, \quad \Omega_a^+ := \{\, x \in \Omega : a(x) > 0 \,\} = \Omega \setminus K\,.$$

Moreover, if we denote by Γ_0^i, $1 \le i \le n_0$, the components of Γ_0 and by $\{i_1, \dots, i_q\}$ the subset of $\{1, \dots, n_0\}$ for which $K \cap \Gamma_0^j \neq \emptyset$ if and only if $j \in \{i_1, \dots, i_q\}$, then

a is bounded away from zero on any compact subset of

$$\Omega_a^+ \cup \Gamma_1 \cup \big(\bigcup_{j=1}^{q} \Gamma_0^{i_j} \setminus K\big) .$$

When, in addition, we assume that $K \cap \Gamma_0 = \emptyset$, then we are only imposing that a is bounded away from zero on any compact subset of $\Omega_a^+ \cup \Gamma_1$. Also, (A4) is satisfied if $a \in \mathfrak{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$.

In Figure 1 we have represented a typical configuration for which $a \in \mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$. In this case,

$$\Gamma_1 = \Gamma_1^1 \cup \Gamma_1^2 , \quad \Gamma_0 = \Gamma_0^1 \cup \Gamma_0^2 ,$$

and Ω_a^+ –dark area–, as well as Ω_a^0 –white area–, consists of two components; the compact set K consisting of a compact arc of curve.

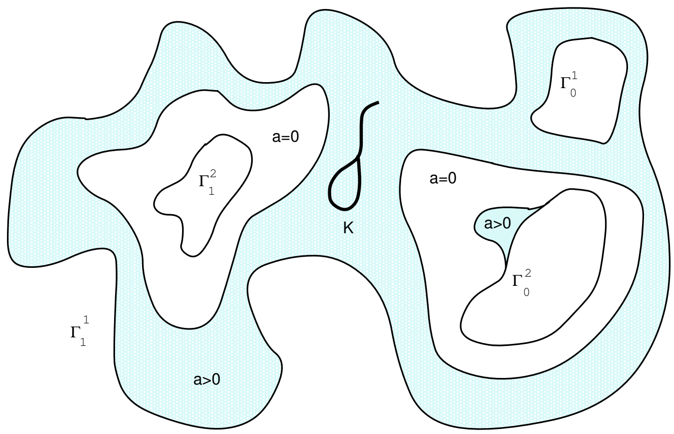

FIGURE 1. An admissible configuration

For the special configuration shown in Figure 1, conditions (A1) and (A4) are trivially satisfied. Moreover, condition (A2) is satisfied if, and only if, a is bounded away from zero in any compact subset of $\Omega_a^+ \cup \Gamma_1^1$, and condition (A3) holds if, and only if, a is bounded away from zero in any compact subset of $\Omega_a^+ \cup (\Gamma_0^2 \setminus \partial\Omega_a^0)$. We point out that a can vanish on the component Γ_0^1.

The main result of this paper shows that the positive solutions of (1.1) vary continuously with the domain Ω when Ω is perturbed through some of the components of Γ_0, keeping fixed, simultaneously, all the components of Γ_1. We point out that the coefficient $b(x)$ arising in the formulation of the boundary operator can vanish and change of sign. Thus, as a result of the theory developed by Hale and his associates (cf., e.g., Hale and Vegas [17] and Arrieta et al. [5]), the positive solutions of (1.1) do not vary continuously with Ω when $\Gamma_1 \neq \emptyset$ and $b = 0$ on some of the components of Γ_1 –in general–; from this point of view, the theory developed in this paper is optimal. It must be mentioned that Dancer and Daners [12] treated the same problem that we are dealing with here, but for a more restrictive family of nonlinear elliptic boundary value problems. Also, their theory requires the coefficient $b(x)$ to be positive and bounded away from zero and, hence, it cannot be applied straight away to treat (1.1).

The problem of the continuous variation of the positive solutions of a linear, or semilinear, boundary value problem with respect to the perturbations of the underlying domain has a very long and fruitful tradition since –at least– the memoir of J. Hadamard [16] sew the light and the results obtained by R. Courant were disseminated through his joint books with D. Hilbert [10], but this paper is far from being the best place for discussing the history of the theory. Otherwise, one should considerably enlarge the list of closely related references and discussing about the many ramifications of the abstract theory (cf., e.g., [3], [6], [11], [14], [18], [22], [24], [26], [28], and the references there in), so substantially enlarging this rather long and, necessarily, technical paper.

It must be mentioned that, however being truly classical the general problem tackled in it, this paper is certainly pioneer in two directions. Namely, because it treats a nonlinear problem subject to a very general class of boundary operators of mixed type where the coefficient $b(x)$ is allowed to vanish and change of sign –this allows applying our theory, e.g., to deal with problems subject to nonlinear boundary operators–, and because, for the class of potentials $a(x)$ considered in this paper, (1.1) exhibits bifurcation from infinity if $\Omega_a^0 \neq \emptyset$. Actually, if we regard fixed the domain Ω, the characterization of the existence and the uniqueness of the positive solutions for (1.1) is a very recent result by one of the authors, [7], who substantially extended the theory developed by Fraile et al. in [13]; the corresponding linear analysis will be published in [8] – it has been already summarized in [9]–.

Being so classical as the problem under study is, one of the reasons why it has not been solved yet is because of the lack of adequate comparison techniques to treat it adequately. Besides a very sharp analysis of the equation itself is imperative in order to get uniform L_p-estimates with respect to the underlying domain support and its admissible perturbations, the main ingredients in obtaining our result consist of the generalization of the strong maximum principle found by Amann in [2] and its characterization in terms of the existence of positive strict supersolutions coming from [21] and [4]. Such characterization has shown to be a very fruitful and powerful tool in dealing with these and other related problems.

To prove the continuous dependence of the positive solution of (1.1) with respect to the domain Ω, we first show the exterior continuous dependence. Then, we prove the interior continuous dependence and, finally, we conclude the *absolute continuous dependence* of the positive solutions with respect to any regular perturbation through the Dirichlet boundary of the domain. A crucial trouble to be overcome in this analysis comes from the problem of ascertaining whether or not being in the class of potentials $\mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$ is an hereditary property from Ω to some adequate class of subdomains of Ω. Section 3 carries out this analysis. Although the corresponding proofs are far from difficult they run rather lengthily and, actually, are quite tedious. Consequently, the reader may choose not to delve into all the technical details of Section 3, but merely give it at first glance a cursory reading to get its general flavor before reading the remaining sections of the paper.

The precise distribution of this paper is the following. Section 2 collects some known results, crucial to carry out our mathematical analysis. Section 3 shows that being in the class $\mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$ is an hereditary property. Section 4 shows the continuous dependence of the positive solutions of (1.1) with respect to any admissible exterior perturbation of the domain and Section 5 shows the continuous dependence from the interior. Finally, Section 6 shows the global continuous dependence.

2. Preliminaries, notation and previous results

In this section we fix some notation and collect some of the main results of [2], [8] and [7] that are going to be used throughout the rest of this paper. For each $p>1$ we consider

$$\begin{gathered} W^2_{p,\mathcal{B}(b)}(\Omega) := \{\, u \in W^2_p(\Omega) : \mathcal{B}(b)u = 0 \,\}\,, \\ W^2_{\mathcal{B}(b)}(\Omega) := \bigcap_{p>1} W^2_{p,\mathcal{B}(b)}(\Omega) \subset H^2(\Omega)\,, \end{gathered}$$

and use the natural product order in $L_p(\Omega)\times L_p(\partial\Omega)$,

$$(f_1,g_1) \geq (f_2,g_2) \quad \iff \quad f_1 \geq f_2 \ \wedge \ g_1 \geq g_2\,.$$

It will be said that $(f_1,g_1) > (f_2,g_2)$ if $(f_1,g_1) \geq (f_2,g_2)$ and $(f_1,g_1) \neq (f_2,g_2)$.

Since $b \in \mathcal{C}(\Gamma_1)$, it follows from the theory of [23] that, for each $p>1$,

$$\mathcal{B}(b) \in \mathcal{L}(W^2_p(\Omega); W_p^{2-\frac{1}{p}}(\Gamma_0) \times W_p^{1-\frac{1}{p}}(\Gamma_1))\,.$$

Moreover, for any $V \in L_\infty(\Omega)$ the linear eigenvalue problem

$$\begin{gathered} (\mathcal{L}+V)\varphi = \lambda\varphi \quad \text{in } \Omega\,, \\ \mathcal{B}(b)\varphi = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{2.1}$$

possesses a least real eigenvalue, denoted in the sequel by $\sigma[\mathcal{L}+V,\mathcal{B}(b),\Omega]$ and called the *principal eigenvalue of* $(\mathcal{L}+V,\mathcal{B}(b),\Omega)$. The principal eigenvalue is simple and associated with it there is a positive eigenfunction, unique up to multiplicative constants. This eigenfunction is called the *principal eigenfunction of* $(\mathcal{L}+V,\mathcal{B}(b),\Omega)$. Thanks to Theorem 12.1 of [2], the principal eigenfunction, subsequently denoted by φ, satisfies

$$\varphi \in W^2_{\mathcal{B}(b)}(\Omega) \subset H^2(\Omega)$$

and it is *strongly positive in* Ω in the sense that $\varphi(x)>0$ for each $x \in \Omega\cup\Gamma_1$ and $\partial_\nu\varphi(x) < 0$ if $x \in \Gamma_0$. Moreover, $\sigma[\mathcal{L}+V,\mathcal{B}(b),\Omega]$ is the unique eigenvalue of (2.1) with a positive eigenfunction and it is *dominant*, i.e.,

$$\operatorname{Re}\lambda > \sigma[\mathcal{L}+V,\mathcal{B}(b),\Omega]$$

for any other eigenvalue λ of (2.1). Furthermore, setting

$$(\mathcal{L}+V)_p := (\mathcal{L}+V)|_{W^2_{p,\mathcal{B}(b)}(\Omega)}\,,$$

we have that for each $\omega > -\sigma[\mathcal{L}+V,\mathcal{B}(b),\Omega]$ and $p>N$ the operator

$$[\omega + (\mathcal{L}+V)_p]^{-1} \in \mathcal{L}(L_p(\Omega))$$

is positive, compact and irreducible (cf. [25, V.7.7]).

Throughout this paper, given any proper subdomain Ω_0 of Ω of class $\mathcal{C}^2$ with

$$\operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0 \tag{2.2}$$

we shall denote by $\mathcal{B}(b,\Omega_0)$ the boundary operator defined from $\mathcal{B}(b)$ through

$$\mathcal{B}(b,\Omega_0)\varphi := \begin{cases} \varphi & \text{on } \partial\Omega_0 \cap \Omega\,, \\ \mathcal{B}(b)\varphi & \text{on } \partial\Omega_0 \cap \partial\Omega\,. \end{cases} \tag{2.3}$$

Also, we set $\mathcal{B}(b,\Omega) := \mathcal{B}(b)$. It should be noted that if $\bar\Omega_0 \subset \Omega$, then $\partial\Omega_0 \subset \Omega$ and, hence,

$$\mathcal{B}(b,\Omega_0)u = u\,,$$

by definition. Thus, in this case $\mathcal{B}(b, \Omega_0)$ becomes the Dirichlet boundary operator, subsequently denoted by $\mathcal{D}$. Also, $\sigma[\mathcal{L}+V, \mathcal{B}(b, \Omega_0), \Omega_0]$ will stand for the principal eigenvalue of the linear boundary value problem

$$\begin{aligned} (\mathcal{L}+V)\varphi &= \lambda\varphi \quad \text{in } \Omega_0\,, \\ \mathcal{B}(b, \Omega_0)\varphi &= 0 \quad \text{on } \partial\Omega_0\,. \end{aligned} \tag{2.4}$$

We now recall the concept of *principal eigenvalue* for a domain with several components.

Definition 2.1. Suppose Ω_0 is an open subset of Ω with a finite number of components of class $\mathcal{C}^2$, say Ω_0^j, $1 \le j \le m$, such that $\bar{\Omega}_0^i \cap \bar{\Omega}_0^j = \emptyset$ if $i \ne j$, and

$$\operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0\,. \tag{2.5}$$

Then, the principal eigenvalue of $(\mathcal{L}+V, \mathcal{B}(b, \Omega_0), \Omega_0)$ is defined through

$$\sigma[\mathcal{L}+V, \mathcal{B}(b, \Omega_0), \Omega_0] := \min_{1\le j\le m} \sigma[\mathcal{L}+V, \mathcal{B}(b, \Omega_0^j), \Omega_0^j]\,. \tag{2.6}$$

Remark 2.2. Since Ω_0 is of class $\mathcal{C}^2$, it follows from (2.5) that each of the principal eigenvalues $\sigma[\mathcal{L}+V, \mathcal{B}(b, \Omega_0^j), \Omega_0^j]$, $1 \le j \le m$, is well defined, which shows the consistency of Definition 2.1.

Suppose $p > N$ and $V \in L_\infty(\Omega)$. Then, a function $\bar{u} \in W_p^2(\Omega)$ is said to be a *positive strict supersolution* of $(\mathcal{L}+V, \mathcal{B}(b), \Omega)$ if $\bar{u} \ge 0$ and $((\mathcal{L}+V)\bar{u}, \mathcal{B}(b)\bar{u}) > 0$. A function $u \in W_p^2(\Omega)$ is said to be *strongly positive* if $u(x) > 0$ for each $x \in \Omega \cup \Gamma_1$ and $\partial_\beta u(x) < 0$ for each $x \in \Gamma_0$ where $u(x) = 0$ and any outward pointing nowhere tangent vector field $\beta \in \mathcal{C}^1(\Gamma_0; \mathbb{R}^N)$. Finally, $(\mathcal{L}+V, \mathcal{B}(b), \Omega)$ is said to satisfy the *strong maximum principle* if $p > N$, $u \in W_p^2(\Omega)$, and $((\mathcal{L}+V)u, \mathcal{B}(b)u) > 0$ imply that u is strongly positive. It should be recalled that for any $p > N$

$$W_p^2(\Omega) \hookrightarrow \mathcal{C}^{2-\frac{N}{p}}(\bar{\Omega}) \tag{2.7}$$

and that any function $u \in W_p^2(\Omega)$ is a.e. in Ω twice differentiable (cf. [27, Theorem VIII.1]).

The following characterization of the strong maximum principle provides us with one of the main technical tools to make most of the comparisons of this paper. It goes back to [21] and [19], thought the version given here comes from [4].

Theorem 2.3. *For any $V \in L_\infty(\Omega)$, the following assertions are equivalent:*

- $\sigma[\mathcal{L}+V, \mathcal{B}(b), \Omega] > 0$*;*
- $(\mathcal{L}+V, \mathcal{B}(b), \Omega)$ *possesses a positive strict supersolution;*
- $(\mathcal{L}+V, \mathcal{B}(b), \Omega)$ *satisfies the strong maximum principle.*

Now, we collect some of the main properties of $\sigma[\mathcal{L}+V, \mathcal{B}(b), \Omega]$; they are taken from [8] (cf. Propositions 3.2 and 3.3 therein).

Proposition 2.4. *Let Ω_0 be a proper subdomain of Ω of class $\mathcal{C}^2$ satisfying* (2.2)*. Then,*

$$\sigma[\mathcal{L}+V, \mathcal{B}(b), \Omega] < \sigma[\mathcal{L}+V, \mathcal{B}(b, \Omega_0), \Omega_0]\,,$$

where $\mathcal{B}(b, \Omega_0)$ is the boundary operator defined by (2.3)*.*

Proposition 2.5. *Let V_1, $V_2 \in L_\infty(\Omega)$ such that $V_1 \le V_2$ and $V_1 < V_2$ in a set of positive Lebesgue measure. Then,*

$$\sigma[\mathcal{L}+V_1, \mathcal{B}(b), \Omega] < \sigma[\mathcal{L}+V_2, \mathcal{B}(b), \Omega]\,.$$

A fundamental result which will be crucial for the mathematical analysis carried out in the next sections is the continuous dependence of the principal eigenvalue $\sigma[\mathcal{L}+V,\mathcal{B}(b),\Omega]$ with respect to the perturbations of the domain around its Dirichlet boundary. To state it we need introducing the following concept.

Definition 2.6. Let Ω_0 be a bounded domain of $\mathbb{R}^N$ with boundary $\partial\Omega_0=\Gamma_0^0\cup\Gamma_1$ such that $\Gamma_0^0\cap\Gamma_1=\emptyset$, where Γ_0^0 satisfies the same requirements as Γ_0, and consider a sequence Ω_n, $n\geq 1$, of bounded domains of $\mathbb{R}^N$ with boundaries $\partial\Omega_n=\Gamma_0^n\cup\Gamma_1$ of class $\mathcal{C}^2$ such that

$$\Gamma_0^n\cap\Gamma_1=\emptyset\,,\quad n\geq 1\,,$$

and Γ_0^n, $n\geq 1$, satisfies the same requirements as Γ_0. Then:

(*a*) It is said that Ω_n converges to Ω_0 from the exterior if for each $n\geq 1$

$$\Omega_0\subset\Omega_{n+1}\subset\Omega_n\quad\text{and}\quad\bigcap_{n=1}^{\infty}\bar\Omega_n=\bar\Omega_0\,.$$

(*b*) It is said that Ω_n converges to Ω_0 from the interior if for each $n\geq 1$

$$\Omega_n\subset\Omega_{n+1}\subset\Omega_0\quad\text{and}\quad\bigcup_{n=1}^{\infty}\Omega_n=\Omega_0\,.$$

(*c*) It is said that Ω_n converges to Ω_0 is there exist two sequences of domains, Ω_n^I and Ω_n^E, $n\geq 1$, whose boundaries satisfy the same requirements as those of Ω_n, and such that Ω_n^I converges to Ω_0 from the interior, Ω_n^E converges to Ω_0 from the exterior and

$$\Omega_n^I\subset\Omega_0\cap\Omega_n\subset\Omega_0\cup\Omega_n\subset\Omega_n^E\,,\quad n\geq 1\,.$$

Subsequently, we denote by $H^1_{\Gamma_0}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega\cup\Gamma_1)$ in $H^1(\Omega)$; $\mathcal{C}_c^\infty(\Omega\cup\Gamma_1)$ stands for the space of functions of class $\mathcal{C}^\infty$ with compact support in $\Omega\cup\Gamma_1$. The following result is a very sharp version of Theorem 3.7 in [29] going back to [8].

Theorem 2.7. *Let Ω be a bounded domain of $\mathbb{R}^N$ of class $\mathcal{C}^1$ with boundary*

$$\partial\Omega=\Gamma_0\cup\Gamma_1\,,\quad\Gamma_0\cap\Gamma_1=\emptyset\,,$$

and consider any proper subdomain $\Omega_0\subset\Omega$ of class $\mathcal{C}^1$ with boundary

$$\partial\Omega_0=\Gamma_0^0\cup\Gamma_1\,,\quad\Gamma_0^0\cap\Gamma_1=\emptyset\,,$$

where Γ_0^0 satisfies the same requirements as Γ_0. Then,

$$H^1_{\Gamma_0^0}(\Omega_0)=\{\,u\in H^1(\Omega):\operatorname{supp}u\subset\bar\Omega_0\,\}\,.$$

For the rest of this paper, $\nu=(\nu_1,\dots,\nu_N)$ is said to be the conormal vector field if

$$\nu_i:=\sum_{j=1}^N\alpha_{ij}n_j\,,\quad 1\leq i\leq N\,,\tag{2.8}$$

where $n=(n_1,\dots,n_N)$ is the outward unit normal to Ω on Γ_1. In this case ∂_ν will be called the *conormal derivative*. Let $\mu>0$ denote the ellipticity constant of $\mathcal{L}$ and assume (2.8). Then,

$$\langle\nu,n\rangle=\sum_{i,j=1}^N\alpha_{ij}n_jn_i\geq\mu\,|n|^2=\mu>0$$

and, therefore, ν is an outward pointing nowhere tangent vector field. Note that $\nu \in \mathcal{C}^1(\Gamma_1;\mathbb{R}^N)$, since $\alpha_{ij} \in \mathcal{C}^1(\bar{\Omega})$, $1 \le i,j \le N$, and Γ_1 is of class $\mathcal{C}^2$.

Now, we can state the continuous dependence of the principal eigenvalues with respect to the perturbations of the domains along their Dirichlet boundaries. The following results are Theorems 7.1, 7.3 of [8], respectively.

Theorem 2.8 (Exterior continuous dependence). *Suppose* (2.8) *and* $V \in L_\infty(\Omega)$. *Let* Ω_0 *be a proper subdomain of* Ω *with boundary of class* $\mathcal{C}^2$ *such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1\,, \quad \Gamma_0^0 \cap \Gamma_1 = \emptyset\,,$$

where Γ_0^0 *is assumed to satisfy the same requirements as* Γ_0, *and consider a sequence* Ω_n, $n \ge 1$, *of bounded domains of* $\mathbb{R}^N$ *of class* $\mathcal{C}^2$ *converging to* Ω_0 *from the exterior such that* $\Omega_n \subset \Omega$, $n \ge 1$. *For each* $n \ge 0$, *let* $\mathcal{B}_n(b)$ *denote the boundary operator defined through*

$$\mathcal{B}_n(b)u := \begin{cases} u & \text{on } \Gamma_0^n \\ \partial_\nu u + bu & \text{on } \Gamma_1 \end{cases} \tag{2.9}$$

where $\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1$, $n \ge 0$, *and denote by* $(\sigma[\mathcal{L}+V,\mathcal{B}_n(b),\Omega_n],\varphi_n)$ *the principal eigen-pair of* $(\mathcal{L}+V,\mathcal{B}_n(b),\Omega_n)$, *where* φ_n *is assumed to be normalized so that*

$$\|\varphi_n\|_{H^1(\Omega_n)} = 1\,, \quad n \ge 0\,.$$

Then, $\varphi_0 \in W^2_{\mathcal{B}_0(b)}(\Omega_0)$ *and*

$$\lim_{n\to\infty} \sigma[\mathcal{L}+V,\mathcal{B}_n(b),\Omega_n] = \sigma[\mathcal{L}+V,\mathcal{B}_0(b),\Omega_0]\,, \quad \lim_{n\to\infty} \|\varphi_n|_{\Omega_0} - \varphi_0\|_{H^1(\Omega_0)} = 0\,.$$

Theorem 2.9 (Interior continuous dependence). *Suppose* (2.8) *and* $V \in L_\infty(\Omega)$. *Let* Ω_0 *be a proper subdomain of* Ω *with boundary of class* $\mathcal{C}^2$ *such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1\,, \quad \Gamma_0^0 \cap \Gamma_1 = \emptyset\,,$$

where Γ_0^0 *is assumed to satisfy the same requirements as* Γ_0, *and let* Ω_n, $n \ge 1$, *be a sequence of bounded domains of* $\mathbb{R}^N$ *of class* $\mathcal{C}^2$ *converging to* Ω_0 *from the interior. For each* $n \ge 0$, *let* $\mathcal{B}_n(b)$ *denote the boundary operator defined by* (2.9) *where* $\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1$, $n \ge 0$, *and denote by* $(\sigma[\mathcal{L}+V,\mathcal{B}_n(b),\Omega_n],\varphi_n)$ *the principal eigen-pair of* $(\mathcal{L}+V,\mathcal{B}_n(b),\Omega_n)$, *where* φ_n *is assumed to be normalized so that*

$$\|\varphi_n\|_{H^1(\Omega_n)} = 1\,, \quad n \ge 0\,.$$

Then, $\varphi_0 \in W^2_{\mathcal{B}_0(b)}(\Omega_0)$ *and*

$$\lim_{n\to\infty} \sigma[\mathcal{L}+V,\mathcal{B}_n(b),\Omega_n] = \sigma[\mathcal{L}+V,\mathcal{B}_0(b),\Omega_0]\,, \quad \lim_{n\to\infty} \|\tilde{\varphi}_n - \varphi_0\|_{H^1(\Omega_0)} = 0\,,$$

where, for each $n \ge 0$,

$$\tilde{\varphi}_n := \begin{cases} \varphi_n & \text{in } \Omega_n \\ 0 & \text{in } \Omega_0 \setminus \Omega_n \end{cases}$$

Combining the previous results it readily follows the next theorem; it is Theorem 7.4 of [8].

Theorem 2.10 (Continuous dependence). *Suppose* (2.8) *and* $V \in L_\infty(\Omega)$. *Let* Ω_0 *be a proper subdomain of* Ω *with boundary of class* $\mathcal{C}^2$ *such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1\,, \quad \Gamma_0^0 \cap \Gamma_1 = \emptyset\,,$$

where Γ_0^0 *is assumed to satisfy the same requirements as* Γ_0, *and let* Ω_n, $n \ge 1$, *be a sequence of bounded domains of* $\mathbb{R}^N$ *of class* $\mathcal{C}^2$ *converging to* Ω_0. *For each* $n \ge 0$, *let $\mathcal{B}_n(b)$ denote the boundary operator defined by (2.9) where $\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1$, $n \geq 0$. Then,*

$$\lim_{n\to\infty} \sigma[\mathcal{L}+V,\mathcal{B}_n(b),\Omega_n] = \sigma[\mathcal{L}+V,\mathcal{B}_0(b),\Omega_0]\,.$$

The following result entails that $(\mathcal{L}+V,\Omega,\mathcal{B}(b))$ satisfies the strong maximum principle if b is sufficiently large and $|\Omega|$ is sufficiently small. It goes back to Theorems 9.1, 10.1 of [8]. Hereafter, $|\cdot|$ will stand for the Lebesgue measure of $\mathbb{R}^N$.

Theorem 2.11. *Suppose $\Gamma_1 \neq \emptyset$, $V \in L_\infty(\Omega)$, and consider a sequence $b_n \in \mathcal{C}(\Gamma_1)$, $n \geq 1$, such that*

$$\lim_{n\to\infty} \min_{\Gamma_1} b_n = \infty\,.$$

For each $n \geq 1$ let φ_n denote the principal eigenfunction associated with $\sigma[\mathcal{L}+V,\mathcal{B}(b_n),\Omega]$, normalized so that $\|\varphi_n\|_{H^1(\Omega)} = 1$. Then,

$$\lim_{n\to\infty} \sigma[\mathcal{L}+V,\mathcal{B}(b_n),\Omega] = \sigma[\mathcal{L}+V,\mathcal{D},\Omega]\,, \quad \lim_{n\to\infty} \|\varphi_n - \varphi\|_{H^1(\Omega)} = 0\,,$$

where $(\sigma[\mathcal{L}+V,\mathcal{D},\Omega],\varphi)$ is the principal eigen-pair associated with the Dirichlet problem in Ω. Moreover,

$$\liminf_{|\Omega|\searrow 0} \sigma[\mathcal{L}+V,\mathcal{D},\Omega]\,|\Omega|^{\frac{2}{N}} \geq \mu\Sigma_1|B_1|^{\frac{2}{N}}\,,$$

where $B_1 := \{x \in \mathbb{R}^N : |x| < 1\}$, $\Sigma_1 := \sigma[-\Delta,\mathcal{D},B_1]$, and $\mu > 0$ is the ellipticity constant of $\mathcal{L}$.

Now, we state the concept of solution for problem (1.1) and collect the results of [7] characterizing the existence of positive solutions for (1.1). For the remaining of this section, it suffices impossing

$$\alpha_{ij} = \alpha_{ji} \in \mathcal{C}(\bar\Omega) \cap W^1_\infty(\Omega)\,, \quad 1 \leq i,j \leq N\,,$$

instead of $\alpha_{ij} = \alpha_{ji} \in \mathcal{C}^1(\bar\Omega)$.

A function $u \in H^1_{\Gamma_0}(\Omega)$ is said to be a weak solution of (1.1) if, for each $\xi \in \mathcal{C}^\infty_c(\Omega\cup\Gamma_1)$,

$$\sum_{i,j=1}^N \int_\Omega \alpha_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \xi}{\partial x_j} + \sum_{i=1}^N \int_\Omega \tilde\alpha_i \xi \frac{\partial u}{\partial x_i} + \int_\Omega \alpha_0 \xi u = \int_\Omega (\lambda W - af(\cdot,u))\xi u - \int_{\Gamma_1} bu\xi$$

where have denoted

$$\tilde\alpha_i := \alpha_i + \sum_{j=1}^N \frac{\partial \alpha_{ij}}{\partial x_j} \in \mathcal{C}(\bar\Omega)\,, \quad 1 \leq i \leq N\,. \tag{2.10}$$

A function u is said to be a *strong solution* of (1.1) if $u \in W^2_p(\Omega)$ for some $p > N$ and it satisfies (1.1). A function u is said to be a positive solution of (1.1) if it is a strong solution and $u > 0$ in Ω. The solutions of (1.1) will be regarded as solution couples (λ,u). Thus, it will be said that a couple (λ_0,u_0) is a solution of (1.1) if u_0 is a solution of (1.1) for $\lambda = \lambda_0$.

Lemma 2.12. *Suppose (λ_0,u_0) is a strong positive solution of (1.1). Then, u_0 is strongly positive in Ω and $u_0 \in W^2_{\mathcal{B}(b)}(\Omega)$. Moreover,*

$$\sigma[\mathcal{L}-\lambda_0 W + af(\cdot,u_0),\mathcal{B}(b),\Omega] = 0\,. \tag{2.11}$$

In particular, $u_0 \in \mathcal{C}^{1,\gamma}(\bar\Omega)$ for each $\gamma \in (0,1)$ and u_0 is a.e. in Ω twice continuously differentiable.

Proof. By definition, $p > N$ exists such that $u_0 \in W_p^2(\Omega)$. Thus, thanks to Morrey's theorem, $u_0 \in L_\infty(\Omega)$ and, hence, $af(\cdot, u_0) \in L_\infty(\Omega)$. Moreover,

$$\begin{gathered}(\mathcal{L} - \lambda_0 W + af(\cdot, u_0))u_0 = 0 \quad \text{in } \Omega \\ \mathcal{B}(b)u_0 = 0 \quad \text{on } \partial\Omega\,.\end{gathered}$$

Thus, u_0 is the principal eigenfunction associated with

$$\sigma[\mathcal{L} - \lambda_0 W + af(\cdot, u_0), \mathcal{B}(b), \Omega] = 0\,.$$

Therefore, $u_0 \in W^2_{\mathcal{B}(b)}(\Omega)$ and it is strongly positive in Ω (cf. [2, Theorem 12.1]). The remaining assertions follow from (2.7) and [27, Th.VIII.1]. □

The following result characterizes the existence of positive solutions for (1.1).

Theorem 2.13. *The following assertions are true:*

a) *Suppose $a \in \mathfrak{A}(\Omega) \setminus \mathfrak{A}^+(\Omega)$, i.e., $a \in \mathfrak{A}(\Omega)$ and $\Omega_a^0 \neq \emptyset$, and in addition,* (2.8) *is satisfied on $\Gamma_1 \cap \partial\Omega_a^0$. Then,* (1.1) *possesses a positive solution if, and only if,*

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}(b), \Omega] < 0 < \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, \Omega_a^0), \Omega_a^0]$$

(cf. (1.8)*). Moreover, the positive solution is unique if it exists.*

b) *Suppose $a \in \mathfrak{A}^+(\Omega)$. Then,* (1.1) *possesses a positive solution if, and only if,*

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}(b), \Omega] < 0\,.$$

Moreover, it is unique if it exists.

Part (a) goes back to [7, Theorem 4.2]. Part (b) can be easily accomplished by adapting the arguments of the proof of Part (a), and so we will omit the details herein. Actually, the proof of Part (b) is simpler than the proof of Part (a).

Definition 2.14. Given $p > N$ it is said that $u \in W_p^2(\Omega)$ is a *positive supersolution* (resp. *positive subsolution*) of (1.1) if $u > 0$ and

$$(\mathcal{L}u - \lambda W u + af(\cdot, u)u, \mathcal{B}(b)u) \geq 0 \quad (\text{resp. } (\mathcal{L}u - \lambda W u + af(\cdot, u)u, \mathcal{B}(b)u) \leq 0)\,.$$

Theorem 2.15. *Suppose we are under the assumptions of Theorem 2.13,* (1.1) *possesses a positive solution, $p > N$, and $u \in W_p^2(\Omega)$ is a positive supersolution (resp. subsolution) of* (1.1)*. Then $u \geq \theta$ (resp. $u \leq \theta$), where θ stands for the unique positive solution of* (1.1).

Proof. Suppose u is a positive supersolution of (1.1). If it is a solution, then, by the uniqueness obtained as an application of Theorem 2.13, $u = \theta$ and the proof is completed. So, suppose u is a positive strict supersolution of (1.1). Then, $u \neq \theta$ and

$$\begin{gathered}(\mathcal{L} + ag - \lambda W)(u - \theta) \geq 0 \quad \text{in } \Omega\,, \\ \mathcal{B}(u - \theta) \geq 0 \quad \text{on } \partial\Omega\,,\end{gathered} \tag{2.12}$$

where

$$g(x) := \begin{cases} \dfrac{u(x)f(x, u(x)) - \theta(x)f(x, \theta(x))}{u(x) - \theta(x)} & \text{if } u(x) \neq \theta(x) \\ f(x, u(x)) & \text{if } u(x) = \theta(x) \end{cases} \qquad x \in \bar{\Omega}\,.$$

Moreover, some of the inequalities of (2.12) must be strict. By the monotonicity of f on its second argument, it follows that $g > f(\cdot,\theta)$ in Ω, since $u \neq \theta$. Thus, thanks to Proposition 2.5 and Lemma 2.12, we find that

$$\sigma[\mathcal{L} + ag - \lambda W, \mathcal{B}(b), \Omega] \geq \sigma[\mathcal{L} + af(\cdot,\theta) - \lambda W, \mathcal{B}(b), \Omega] = 0\,. \tag{2.13}$$

It should be noted that it might happen $g = f(\cdot,\theta)$ in Ω_a^+. Hence, $\geq$ cannot be substituted by $>$ in (2.13) without some additional work. Suppose

$$\sigma[\mathcal{L} + ag - \lambda W, \mathcal{B}(b), \Omega] = 0$$

and let $\varphi > 0$ denote the principal eigenfunction of $(\mathcal{L} + ag - \lambda W, \mathcal{B}(b), \Omega)$. Then, it follows from (2.12) that for each $\kappa > 0$ the function

$$\bar{u} := u - \theta + \kappa\varphi$$

provides us with a strict supersolution of $(\mathcal{L} + ag - \lambda W, \mathcal{B}(b), \Omega)$. Moreover, $\bar{u} > 0$ if κ is sufficiently large, since φ is strongly positive. Thus, it follows from Theorem 2.3 that

$$\sigma[\mathcal{L} + ag - \lambda W, \mathcal{B}(b), \Omega] > 0\,. \tag{2.14}$$

Therefore, thanks to the strong maximum principle, $u-\theta$ is strongly positive. This argument can be easily adapted to show that $\theta - u$ is strongly positive if u is a positive strict subsolution of (1.1). $\square$

3. Belonging to the class $\mathfrak{A}(\Omega)$ is hereditary

In this section we prove that the fact of being in $\mathfrak{A}(\Omega)$ and $\mathfrak{A}^+(\Omega)$ inherits to any open subdomain of Ω satisfying the adequate structural properties. Subsequently, for any $a \in \mathfrak{A}(\Omega)$ and any open subset $\tilde{\Omega}$ of Ω such that $a \in \mathfrak{A}(\tilde{\Omega})$, we denote by $[\tilde{\Omega}]_a^0$ the maximal open subset of $\tilde{\Omega}$ where the potential a vanishes (remember the definition of the class $\mathfrak{A}(\tilde{\Omega})$).

Theorem 3.1. *Suppose $a \in \mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$ and let $\tilde{\Omega}$ be an open subdomain of Ω of class $\mathcal{C}^2$ such that*

$$\operatorname{dist}(\partial\Omega, \partial\tilde{\Omega} \cap \Omega) > 0\,. \tag{3.1}$$

Then, each of the following sets

$$\tilde{\Gamma}_0 := \partial\tilde{\Omega} \cap (\Gamma_0 \cup \Omega)\,, \quad \tilde{\Gamma}_1 := \partial\tilde{\Omega} \setminus \tilde{\Gamma}_0 = \partial\tilde{\Omega} \cap \Gamma_1\,,$$

is closed and open in $\partial\tilde{\Omega}$. Moreover, the following assertions are true:

(a) *If $\Omega_a^0 \cap \tilde{\Omega} \neq \emptyset$ is of class $\mathcal{C}^2$ and*

$$\partial\tilde{\Omega} \cap \Omega \cap \partial(\Omega_a^0 \cap \tilde{\Omega}) = \partial\tilde{\Omega} \cap \Omega \cap \bar{\Omega}_a^0\,, \tag{3.2}$$

then $a \in \mathfrak{A}_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\tilde{\Omega})$ and $[\tilde{\Omega}]_a^0 = \Omega_a^0 \cap \tilde{\Omega}$.

(b) *Suppose $\Omega_a^0 \cap \tilde{\Omega} = \emptyset$ and $\Gamma \cap K \neq \emptyset \implies \Gamma \setminus K \subset \Omega_a^+$ for any component Γ of $\partial\tilde{\Omega} \cap \Omega$. Then, $a \in \mathfrak{A}^+_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\tilde{\Omega})$. In particular,*

$$a \in \mathfrak{A}^+_{\Gamma_0,\Gamma_1}(\Omega) \quad \Longrightarrow \quad a \in \mathfrak{A}^+_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\tilde{\Omega})\,.$$

Proof. Firstly it should be noted that, thanks to (3.1), each component $\hat{\Gamma}$ of $\partial\Omega$ either it satisfies $\hat{\Gamma} \subset \partial\tilde{\Omega}$ or

$$\hat{\Gamma} \cap \partial\tilde{\Omega} = \emptyset\,.$$

Moreover, $\hat{\Gamma}$ must be a component of $\partial\tilde{\Omega}$ if $\hat{\Gamma} \subset \partial\tilde{\Omega}$. In particular, if we denote by Γ_1^i, $1 \leq i \leq n_1$, the components of Γ_1, then, for each $1 \leq i \leq n_1$, either $\Gamma_1^i \subset \partial\tilde{\Omega}$ or

$\Gamma_1^i \cap \partial\tilde{\Omega} = \emptyset$. Moreover, Γ_1^i must be a component of $\partial\tilde{\Omega}$ if $\Gamma_1^i \subset \partial\tilde{\Omega}$. Subsequently, when

$$\Gamma_1 \cap \partial\tilde{\Omega} \neq \emptyset\,,$$

$\{i_1,\dots,i_{\tilde{n}_1}\}$ denotes the subset of $\{1,\dots,n_1\}$ for which $\Gamma_1^i \subset \partial\tilde{\Omega} \iff i \in \{i_1,\dots,i_{\tilde{n}_1}\}$. Then, it is easy to see that

$$\tilde{\Gamma}_1 = \bigcup_{j=1}^{\tilde{n}_1} \Gamma_1^{i_j} \quad \wedge \quad \tilde{\Gamma}_0 = \partial\tilde{\Omega} \setminus \tilde{\Gamma}_1\,. \tag{3.3}$$

When $\Gamma_1 \cap \partial\tilde{\Omega} = \emptyset$, we take $\tilde{\Gamma}_1 = \emptyset$. In any of these cases, as $\tilde{\Gamma}_1$ is closed and open in $\partial\tilde{\Omega}$, the proof of the first claim of the theorem is completed.

We now prove (a). Suppose $\Omega_a^0 \cap \tilde{\Omega}$ is non empty and of class $\mathcal{C}^2$. Since $a \in \mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$, there exist an open subset Ω_a^0 of Ω and a compact subset K of $\bar{\Omega}$ with Lebesgue measure zero such that

$$K \cap (\bar{\Omega}_a^0 \cup \Gamma_1) = \emptyset\,, \tag{3.4}$$

$$\Omega_a^+ := \{\, x \in \Omega : a(x) > 0\} = \Omega \setminus (\bar{\Omega}_a^0 \cup K)\,, \tag{3.5}$$

and each of the four conditions $(\mathfrak{A}_1) - (\mathfrak{A}_4)$ of the introduction is satisfied. In particular,

$$\operatorname{dist}(\Gamma_1, \partial\Omega_a^0 \cap \Omega) > 0\,. \tag{3.6}$$

Note that, thanks to (3.6), each of the components Γ_1^i, $1 \le i \le n_1$, of Γ_1 satisfies either

$$\Gamma_1^i \subset \partial\Omega_a^0 \quad \text{or} \quad \Gamma_1^i \cap \partial\Omega_a^0 = \emptyset\,.$$

Moreover, Γ_1^i must be a component of $\partial\Omega_a^0$ if $\Gamma_1^i \subset \partial\Omega_a^0$. Setting

$$\tilde{\Omega}_a^0 := \Omega_a^0 \cap \tilde{\Omega}\,, \quad \tilde{K} := K \cap \bar{\tilde{\Omega}}\,, \quad \tilde{\Omega}_a^+ := \tilde{\Omega} \setminus (\bar{\tilde{\Omega}}_a^0 \cup \tilde{K})\,,$$

we shall show that the open set

$$[\tilde{\Omega}]_a^0 := \tilde{\Omega}_a^0$$

and the compact set $\tilde{K}$ satisfy all the requirements of the definition of the class $\mathfrak{A}_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\tilde{\Omega})$.

Let $\Omega_a^{0,i}$, $1 \le i \le m$, be the components of Ω_a^0 (cf. the definition of the class $\mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$). Since $a \in \mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$,

$$\bar{\Omega}_a^{0,i} \cap \bar{\Omega}_a^{0,j} = \emptyset \quad \text{if} \quad i \neq j\,. \tag{3.7}$$

Moreover, since Ω_a^0 is the maximal open subset of Ω where a vanishes, $\tilde{\Omega}_a^0$ is the maximal open subset of $\tilde{\Omega}$ where a vanishes. Furthermore, since we are assuming $\tilde{\Omega}_a^0$ to be of class $\mathcal{C}^2$ and

$$\tilde{\Omega}_a^0 = \Omega_a^0 \cap \tilde{\Omega} = \bigcup_{i=1}^{m} \left(\Omega_a^{0,i} \cap \tilde{\Omega}\right)\,, \tag{3.8}$$

it follows from (3.7) that, for each $1 \le i \le m$, $\Omega_a^{0,i} \cap \tilde{\Omega}$ is of class $\mathcal{C}^2$. It should be noted that some of the sets

$$\Omega_a^{0,i} \cap \tilde{\Omega}\,, \quad 1 \le i \le m\,,$$

might be empty. Nevertheless, since each of them is of class $\mathcal{C}^2$, any of them possesses finitely many components of class $\mathcal{C}^2$. Necessarily, their respective closures are mutually disjoint. Thus, thanks to (3.7) and (3.8), $\tilde{\Omega}_a^0$ possesses a finite number of components of class $\mathcal{C}^2$ –whose respective closures must be mutually disjoint–. Also, since K is a compact subset of $\bar\Omega$ with Lebesgue measure zero, $\tilde K$ is a compact subset of $\bar{\tilde\Omega}$ with Lebesgue measure zero, and, since

$$\tilde K \subset K \quad \wedge \quad \tilde\Omega^0_a \subset \Omega^0_a\,,$$

we have that

$$\tilde K \cap (\bar{\tilde\Omega}^0_a \cup \Gamma_1) \subset K \cap (\bar\Omega^0_a \cup \Gamma_1)\,.$$

Hence, (3.4) implies $\tilde K \cap (\bar{\tilde\Omega}^0_a \cup \Gamma_1) = \emptyset$ and, therefore,

$$\tilde K \cap (\bar{\tilde\Omega}^0_a \cup \tilde\Gamma_1) = \emptyset\,,$$

since $\tilde\Gamma_1 \subset \Gamma_1$. Moreover, thanks to (3.5),

$$\{x \in \tilde\Omega : a(x) > 0\} = \tilde\Omega \cap [\Omega \setminus (\bar\Omega^0_a \cup K)] = \tilde\Omega \setminus (\bar{\tilde\Omega}^0_a \cup \tilde K)\,,$$

by the definition of $\tilde\Omega^0_a$ and $\tilde K$. Therefore,

$$\tilde\Omega^+_a = \{x \in \tilde\Omega : a(x) > 0\} = \tilde\Omega \setminus ([\bar{\tilde\Omega}]^0_a \cup \tilde K)\,.$$

To complete the proof of Part (a) it remains to show that each of the properties (A1)-(A4) is satisfied.

Since $\tilde\Omega^0_a$ possesses a finite number of components of class $\mathcal{C}^2$ whose respective closures are mutually disjoint and $\tilde\Gamma_1 \subset \Gamma_1$, in order to prove (A1) it suffices to show that

$$\operatorname{dist}(\Gamma_1, \partial\tilde\Omega^0_a \cap \tilde\Omega) > 0\,. \tag{3.9}$$

Indeed, the inclusion

$$\partial\tilde\Omega^0_a = \partial(\Omega^0_a \cap \tilde\Omega) \subset \partial\Omega^0_a \cup \partial\tilde\Omega$$

implies

$$\partial\tilde\Omega^0_a \cap \tilde\Omega \subset (\partial\Omega^0_a \cup \partial\tilde\Omega) \cap \tilde\Omega = \partial\Omega^0_a \cap \tilde\Omega \subset \partial\Omega^0_a \cap \Omega\,,$$

since $\partial\tilde\Omega \cap \tilde\Omega = \emptyset$. Thus, $a \in \mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$ implies

$$\operatorname{dist}(\Gamma_1, \partial\tilde\Omega^0_a \cap \tilde\Omega) \geq \operatorname{dist}(\Gamma_1, \partial\Omega^0_a \cap \Omega) > 0\,,$$

which completes the proof of (3.9). This shows property (A1) in $\tilde\Omega$.

Now, note that thanks to (3.9), for each $i \in \{i_1, \dots, i_{\tilde n_1}\}$, the component Γ^i_1 of $\tilde\Gamma_1 = \Gamma_1 \cap \partial\tilde\Omega$ (cf. the beginning of the proof) satisfies either $\Gamma^i_1 \subset \partial\tilde\Omega^0_a$ or else $\Gamma^i_1 \cap \partial\tilde\Omega^0_a = \emptyset$. Moreover, if $\Gamma^i_1 \subset \partial\tilde\Omega^0_a$, then Γ^i_1 must be a component of $\partial\tilde\Omega^0_a$. When

$$\tilde\Gamma_1 \subset \partial\tilde\Omega^0_a$$

property (A2) is satisfied, since we are assuming that a is bounded away from zero on any compact subset of Ω^+_a, because $a \in \mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$. Thus, in order to prove (A2) we can assume, without lost of generality, that there exists $j \in \{1, \dots, \tilde n_1\}$ for which

$$\Gamma^{i_j}_1 \cap \partial\tilde\Omega^0_a = \emptyset\,.$$

Then, without lost of generality, we can assume that there exists a natural number $1 \leq \tilde p \leq \tilde n_1$ such that

$$\Gamma^{i_j}_1 \cap \partial\tilde\Omega^0_a = \emptyset \quad \Longleftrightarrow \quad j \in \{1, \dots, \tilde p\}\,.$$

By construction, we have

$$\partial\tilde\Omega^0_a \cap \bigcup_{j=1}^{\tilde p} \Gamma^{i_j}_1 = \emptyset\,, \quad \tilde\Gamma_1 = \Gamma_1 \cap \partial\tilde\Omega = \bigcup_{j=1}^{\tilde n_1} \Gamma^{i_j}_1\,, \quad \tilde\Gamma_1 \cap \partial\tilde\Omega^0_a = \bigcup_{j=\tilde p+1}^{\tilde n_1} \Gamma^{i_j}_1\,,$$

if $\tilde{p} < \tilde{n}_1$. Using this notation, to prove (A2) we must demonstrate that a is bounded away from zero on any compact subset of

$$\tilde{\Omega}_a^+ \cup \bigcup_{j=1}^{\tilde{p}} \Gamma_1^{i_j} .$$

To prove this, we shall use the following identity

$$\partial\Omega_a^0 \cap \bigcup_{j=1}^{\tilde{p}} \Gamma_1^{i_j} = \emptyset , \tag{3.10}$$

whose proof follows by contradiction. Assume that there exists $1 \le k \le \tilde{p}$ for which

$$\Gamma_1^{i_k} \cap \partial\Omega_a^0 \neq \emptyset .$$

Then, thanks to (3.6), $\Gamma_1^{i_k} \subset \partial\Omega_a^0$ and, hence,

$$\Gamma_1^{i_k} \subset \partial\Omega_a^0 \cap \partial\tilde{\Omega} , \tag{3.11}$$

since, by construction, $\Gamma_1^{i_k} \subset \partial\tilde{\Omega}$ (cf. the beginning of the proof of the theorem). Thus, since

$$\partial\Omega_a^0 \cap \partial\tilde{\Omega} \subset \partial(\Omega_a^0 \cap \tilde{\Omega}) = \partial\tilde{\Omega}_a^0 , \tag{3.12}$$

it follows from (3.11) and (3.12) that

$$\Gamma_1^{i_k} = \Gamma_1^{i_k} \cap \partial\Omega_a^0 \cap \partial\tilde{\Omega} \subset \Gamma_1^{i_k} \cap \partial\tilde{\Omega}_a^0 ,$$

which is impossible, since, by construction,

$$\Gamma_1^{i_k} \cap \partial\tilde{\Omega}_a^0 = \emptyset .$$

This contradiction proves (3.10). On the other hand,

$$\tilde{\Omega}_a^+ = \{x \in \tilde{\Omega} : a(x) > 0\} \subset \{x \in \Omega : a(x) > 0\} = \Omega_a^+ \tag{3.13}$$

and, therefore, (3.10) and (3.13) imply

$$\tilde{\Omega}_a^+ \cup \bigcup_{j=1}^{\tilde{p}} \Gamma_1^{i_j} \subset \Omega_a^+ \cup \bigcup_{j=1}^{\tilde{p}} \Gamma_1^{i_j} \subset \Omega_a^+ \cup (\Gamma_1 \setminus \partial\Omega_a^0) . \tag{3.14}$$

Since $a \in \mathfrak{A}(\Omega)$, a is bounded away from zero in any compact subset of

$$\Omega_a^+ \cup (\Gamma_1 \setminus \partial\Omega_a^0) .$$

Thus, thanks to (3.14), a is bounded away from zero in any compact subset of

$$\tilde{\Omega}_a^+ \cup \bigcup_{j=1}^{\tilde{p}} \Gamma_1^{i_j}$$

and, therefore, (A2) is satisfied in $\tilde{\Omega}$.

In order to show (A3) recall that, thanks to (3.1), each component $\hat{\Gamma}$ of $\partial\Omega$ either it satisfies $\hat{\Gamma} \subset \partial\tilde{\Omega}$ or $\hat{\Gamma} \cap \partial\tilde{\Omega} = \emptyset$. Moreover, $\hat{\Gamma}$ must be a component of $\partial\tilde{\Omega}$ if $\hat{\Gamma} \subset \partial\tilde{\Omega}$. Therefore,

$$\partial\tilde{\Omega} = (\partial\tilde{\Omega} \cap \Gamma_0) \cup (\partial\tilde{\Omega} \cap \Gamma_1) \cup (\partial\tilde{\Omega} \cap \Omega) = \tilde{\Gamma}_0 \cup \tilde{\Gamma}_1$$

and

$$\operatorname{dist}(\partial\tilde{\Omega} \cap \Gamma_0, \partial\tilde{\Omega} \cap \Gamma_1) > 0 , \quad \operatorname{dist}(\partial\tilde{\Omega} \cap \Gamma_i, \partial\tilde{\Omega} \cap \Omega) > 0 , \quad i \in \{0, 1\} .$$

Let Γ_0^i, $1 \le i \le n_0$, and Γ_1^i, $1 \le i \le n_1$, denote the components of Γ_0 and Γ_1, respectively. Without lost of generality we can rearrange them, if necessary, so that

$$\partial\tilde\Omega \cap \Gamma_0 = \bigcup_{i=1}^{\tilde n_0} \Gamma_0^i\,, \quad \partial\tilde\Omega \cap \Gamma_1 = \bigcup_{i=1}^{\tilde n_1} \Gamma_1^i\,, \quad \partial\tilde\Omega \cap \Omega = \bigcup_{i=1}^{\tilde n_{0,I}} \Gamma_{0,I}^i\,,$$

for some $0 \le \tilde n_0 \le n_0$, $0 \le \tilde n_1 \le n_1$, and $\tilde n_{0,I} \ge 1$. It should be noted that $\tilde\Omega = \Omega$ if $\partial\tilde\Omega \cap \Omega = \emptyset$ and that

$$\tilde\Gamma_0 = \bigcup_{i=1}^{\tilde n_0} \Gamma_0^i \cup \bigcup_{i=1}^{\tilde n_{0,I}} \Gamma_{0,I}^i \quad \wedge \quad \tilde\Gamma_1 = \bigcup_{i=1}^{\tilde n_1} \Gamma_1^i\,.$$

The sub-index "I" makes reference to the fact that each of $\Gamma_{0,I}^i$, $1 \le i \le \tilde n_{0,I}$, provides us with an internal component –within Ω– of the *Dirichlet boundary* $\tilde\Gamma_0$ of $\tilde\Omega$.

Let $\{i_1, \dots, i_q\}$ be the subset of $\{1, \dots, n_0\}$ for which

$$\Gamma_0^j \cap (\partial\Omega_a^0 \cup K) \ne \emptyset \quad \iff \quad j \in \{i_1, \dots, i_q\}\,.$$

Similarly, let $\{j_1, \dots, j_{\tilde q}\}$ be the subset of $\{1, \dots, \tilde n_0\}$ for which

$$\Gamma_0^j \cap (\partial\tilde\Omega_a^0 \cup \tilde K) \ne \emptyset \quad \iff \quad j \in \{j_1, \dots, j_{\tilde q}\}\,,$$

and let $\{k_1, \dots, k_{\tilde q_{0,I}}\}$ be the subset of $\{1, \dots, \tilde n_{0,I}\}$ for which

$$\Gamma_{0,I}^j \cap (\partial\tilde\Omega_a^0 \cup \tilde K) \ne \emptyset \quad \iff \quad j \in \{k_1, \dots, k_{\tilde q_{0,I}}\}\,.$$

We claim that

$$\{j_1, \dots, j_{\tilde q}\} \subset \{i_1, \dots, i_q\}\,. \tag{3.15}$$

In other words, $\Gamma_0^j \cap (\partial\Omega_a^0 \cup K) \ne \emptyset$ if $j \in \{j_1, \dots, j_{\tilde q}\}$. Indeed, for any $j \in \{1, \dots, n_0\}$ we have that

$$\Gamma_0^j \cap (\partial\tilde\Omega_a^0 \cup \tilde K) \subset \Gamma_0^j \cap (\bar\Omega_a^0 \cup K) \subset \Gamma_0^j \cap (\partial\Omega_a^0 \cup K)\,.$$

Thus,

$$\Gamma_0^j \cap (\partial\tilde\Omega_a^0 \cup \tilde K) = \emptyset \quad \text{if} \quad \Gamma_0^j \cap (\partial\Omega_a^0 \cup K) = \emptyset$$

and, therefore,

$$j \in \{1, \dots, n_0\} \setminus \{i_1, \dots, i_q\} \quad \Longrightarrow \quad j \in \{1, \dots, n_0\} \setminus \{j_1, \dots, j_{\tilde q}\}\,.$$

This completes the proof of (3.15).

Using these notation, to prove that a satisfies (A3) in $\tilde\Omega$ we must show that a bounded away from zero on any compact subset of

$$\tilde\Omega_a^+ \cup \Big[\bigcup_{i=1}^{\tilde q} \Gamma_0^{j_i} \setminus (\partial\tilde\Omega_a^0 \cup \tilde K)\Big] \cup \Big[\bigcup_{i=1}^{\tilde q_{0,I}} \Gamma_{0,I}^{k_i} \setminus (\partial\tilde\Omega_a^0 \cup \tilde K)\Big]\,.$$

By definition,

$$\bigcup_{i=1}^{\tilde q_{0,I}} \Gamma_{0,I}^{k_i} \subset \partial\tilde\Omega \cap \Omega\,,$$

and, hence,

$$\bigcup_{i=1}^{\tilde q_{0,I}} \Gamma_{0,I}^{k_i} \setminus (\partial\tilde\Omega_a^0 \cup \tilde K) \subset (\partial\tilde\Omega \cap \Omega) \setminus (\partial\tilde\Omega_a^0 \cup \tilde K) = (\partial\tilde\Omega \cap \Omega) \setminus [\partial(\Omega_a^0 \cap \tilde\Omega) \cup \tilde K]\,. \tag{3.16}$$

Moreover, thanks to (3.2),

$$(\partial\tilde\Omega\cap\Omega)\setminus[\partial(\Omega_a^0\cap\tilde\Omega)\cup\tilde K]=(\partial\tilde\Omega\cap\Omega)\setminus[\bar\Omega_a^0\cup\tilde K]\,. \tag{3.17}$$

Thus, since

$$\partial\tilde\Omega\cap\Omega\cap\tilde K=\partial\tilde\Omega\cap\Omega\cap K\cap\bar{\tilde\Omega}=\partial\tilde\Omega\cap\Omega\cap K\,,$$

it follows from (3.16) and (3.17) that

$$\bigcup_{i=1}^{\tilde q_{0,I}}\Gamma_{0,I}^{k_i}\setminus(\partial\tilde\Omega_a^0\cup\tilde K)\subset(\partial\tilde\Omega\cap\Omega)\setminus(\bar\Omega_a^0\cup K)\subset\Omega\setminus(\bar\Omega_a^0\cup K)=\Omega_a^+\,. \tag{3.18}$$

Thanks to (3.18), to complete the proof of (A3) it suffices to show that a is bounded away from zero on any compact subset of

$$\Omega_a^+\cup\Big[\bigcup_{i=1}^{\tilde q}\Gamma_0^{j_i}\setminus(\partial\tilde\Omega_a^0\cup\tilde K)\Big],$$

since $\tilde\Omega_a^+\subset\Omega_a^+$. By construction, for each $1\le i\le\tilde q$, $\Gamma_0^{j_i}\subset\partial\tilde\Omega\cap\Gamma_0$ and, hence,

$$\begin{aligned}\Gamma_0^{j_i}\cap(\partial\tilde\Omega_a^0\cup\tilde K)&=(\Gamma_0^{j_i}\cap\partial\tilde\Omega_a^0)\cup(\Gamma_0^{j_i}\cap\tilde K)\\&=\Big[\Gamma_0^{j_i}\cap\partial(\Omega_a^0\cap\tilde\Omega)\Big]\cup(\Gamma_0^{j_i}\cap K\cap\bar{\tilde\Omega})\\&=(\Gamma_0^{j_i}\cap\partial\Omega_a^0)\cup(\Gamma_0^{j_i}\cap K)\\&=\Gamma_0^{j_i}\cap(\partial\Omega_a^0\cup K)\,.\end{aligned}$$

Thus,

$$\bigcup_{i=1}^{\tilde q}\Gamma_0^{j_i}\setminus(\partial\tilde\Omega_a^0\cup\tilde K)=\bigcup_{i=1}^{\tilde q}\Gamma_0^{j_i}\setminus(\partial\Omega_a^0\cup K)$$

and, therefore, since a is bounded away from zero on any compact subset of

$$\Omega_a^+\cup\Big[\bigcup_{i=1}^{\tilde q}\Gamma_0^{j_i}\setminus(\partial\Omega_a^0\cup K)\Big],$$

because of (3.13), (3.15) and the fact that a satisfies (A3) in Ω. The proof of (A3) in $\tilde\Omega$ is completed.

To complete the proof of Part (a) it remains to show that (A4) is satisfied in $\tilde\Omega$. Fix $\eta>0$. Then, since $a\in\mathfrak{A}(\Omega)$, there exist a natural number $\ell(\eta)\ge1$ and $\ell(\eta)$ open subsets of $\mathbb{R}^N$, G_j^η, $1\le j\le\ell(\eta)$, with $|G_j^\eta|<\eta$, $1\le j\le\ell(\eta)$, such that

$$K\subset\bigcup_{j=1}^{\ell(\eta)}G_j^\eta\quad\text{and}\quad\bar G_i^\eta\cap\bar G_j^\eta=\emptyset\quad\text{if}\quad i\ne j\,. \tag{3.19}$$

Moreover, for each $1\le j\le\ell(\eta)$ the open set $G_j^\eta\cap\Omega$ is connected and of class $\mathcal{C}^2$.

Without lost of generality, we can assume that $G_j^\eta\cap\tilde\Omega\ne\emptyset$, $1\le j\le\ell(\eta)$. Since $\tilde\Omega$ and $G_j^\eta\cap\Omega$ are of class $\mathcal{C}^2$, for each $1\le j\le\ell(\eta)$ the set $G_j^\eta\cap\tilde\Omega$ possesses a finite number of components with mutually disjoint closures, although it might not be of class $\mathcal{C}^2$. For each $\eta>0$ and $1\le j\le\ell(\eta)$, let $N(\eta,j)$ denote the number of components of $G_j^\eta\cap\tilde\Omega$ and let

$$\{G_{j,k}^\eta:1\le k\le N(\eta,j)\}$$

be the set of such components. Now, for each $\varepsilon > 0$ let B_ε denote the ball of radius ε centered at the origin and consider the open neighborhoods

$$\tilde{G}^\eta_{j,k} := G^\eta_{j,k} + B_\varepsilon\,, \quad 1 \le k \le N(\eta, j)\,, \quad 1 \le j \le \ell(\eta)\,.$$

By construction, $\varepsilon_0 = \varepsilon_0(\eta) > 0$ exists such that for each $\varepsilon \in (0, \varepsilon_0)$

$$\bar{\tilde{G}}^\eta_{j,k} \cap \bar{\tilde{G}}^\eta_{i,h} = \emptyset \quad \text{if} \quad (j,k) \ne (i,h)\,. \tag{3.20}$$

Moreover, since

$$|G^\eta_{j,k}| \le |G^\eta_j| < \eta\,, \quad 1 \le k \le N(\eta, j)\,, \quad 1 \le j \le \ell(\eta)\,,$$

$\varepsilon_1 \in (0, \varepsilon_0)$ exists such that for each $\varepsilon \in (0, \varepsilon_1)$

$$|\tilde{G}^\eta_{j,k}| < \eta\,, \quad 1 \le k \le N(\eta, j)\,, \quad 1 \le j \le \ell(\eta)\,. \tag{3.21}$$

Furthermore, we find from (3.19) that

$$\tilde{K} = K \cap \bar{\tilde{\Omega}} \subset \bigcup_{j=1}^{\ell(\eta)} \Big(G^\eta_j \cap \bar{\tilde{\Omega}} \Big) \subset \bigcup_{j=1}^{\ell(\eta)} \bigcup_{k=1}^{N(\eta,j)} \tilde{G}^\eta_{j,k}\,. \tag{3.22}$$

Also, since $G^\eta_{j,k}$ is connected, $\tilde{G}^\eta_{j,k}$ is connected for each $1 \le k \le N(\eta, j)$ and $1 \le j \le \ell(\eta)$. Hence, there exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that $\tilde{G}^\eta_{j,k} \cap \tilde{\Omega}$ is connected for each $1 \le k \le N(\eta, j)$, $1 \le j \le \ell(\eta)$. Subsequently, $\varepsilon \in (0, \varepsilon_2)$ is fixed.

Suppose that, for each $1 \le k \le N(\eta, j)$ and $1 \le j \le \ell(\eta)$, $\tilde{G}^\eta_{j,k} \cap \tilde{\Omega}$ is of class $\mathcal{C}^2$. Then, thanks to (3.20), (3.21) and (3.22), there exists

$$1 \le \tilde{\ell}(\eta) \le \ell(\eta) N(\eta, j)$$

and $\tilde{\ell}(\eta)$ elements of

$$\{\tilde{G}^\eta_{j,k} \cap \tilde{\Omega} : 1 \le k \le N(\eta, j)\,, \ 1 \le j \le \ell(\eta)\}\,,$$

satisfying all the requirements of (A4) in $\tilde{\Omega}$.

Now, suppose that $\tilde{G}^\eta_{j,k} \cap \tilde{\Omega} \notin \mathcal{C}^2$ for some $1 \le k \le N(\eta, j)$ and $1 \le j \le \ell(\eta)$. Then, thanks to (3.22),

$$\operatorname{dist}(\tilde{K}, \partial \bigcup_{\substack{1 \le k \le N(\eta,j) \\ 1 \le j \le \ell(\eta)}} \tilde{G}^\eta_{j,k}) > 0$$

and, hence, there exists an open subset $\hat{G}^\eta_{j,k}$ of $\mathbb{R}^N$ such that $\hat{G}^\eta_{j,k} \cap \tilde{\Omega}$ is connected and

$$\tilde{K} \cap \bar{\tilde{G}}^\eta_{j,k} \subset \hat{G}^\eta_{j,k} \subset \bar{\hat{G}}^\eta_{j,k} \subset \tilde{G}^\eta_{j,k}\,, \quad \hat{G}^\eta_{j,k} \cap \tilde{\Omega} \in \mathcal{C}^2\,.$$

Substituting each of those $\tilde{G}^\eta_{j,k}$'s by the corresponding $\hat{G}^\eta_{j,k}$'s and arguing as in the previous case the proof of Part (a) is easily completed.

The details of the proof of Part (a) can be easily adapted to prove the first claim of Part (b). Finally, suppose $a \in \mathfrak{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$. Then, $\Omega^0_a = \emptyset$ and, in particular,

$$\Omega^0_a \cap \tilde{\Omega} = \emptyset\,.$$

Moreover, $\Omega^+_a = \Omega \setminus K$ and, hence,

$$\Big(\partial\tilde{\Omega} \cap \Omega \Big) \setminus K \subset \Omega \setminus K = \Omega^+_a\,.$$

Therefore, thanks to the first claim, $a \in \mathfrak{A}^+_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\tilde{\Omega})$. This completes the proof. $\square$

As an immediate consequence, from Theorem 3.1 we find the next corollary:

Corollary 3.2. *Suppose* a, $V \in \mathfrak{A}_{\Gamma_0,\Gamma_1}(\Omega)$ *with* Ω_V^0 *connected and*

$$\operatorname{dist}(\Gamma_0, \partial\Omega_V^0 \cap \Omega) > 0\,.$$

Then,

$$\tilde\Gamma_0 := \partial\Omega_V^0 \cap (\Gamma_0 \cup \Omega) \quad \text{and} \quad \tilde\Gamma_1 := \partial\Omega_V^0 \setminus \tilde\Gamma_0 = \partial\Omega_V^0 \cap \Gamma_1$$

are closed and open sets of class $\mathcal{C}^2$*, and each of the following assertions is true:*

(a) *If* $\Omega_a^0 \cap \Omega_V^0 \neq \emptyset$ *is of class* $\mathcal{C}^2$ *and*

$$\partial\Omega_V^0 \cap \Omega \cap \partial(\Omega_a^0 \cap \Omega_V^0) = \partial\Omega_V^0 \cap \Omega \cap \bar\Omega_a^0\,, \tag{3.23}$$

then $a \in \mathfrak{A}_{\tilde\Gamma_0,\tilde\Gamma_1}(\Omega_V^0)$ *and* $[\Omega_V^0]_a^0 = \Omega_a^0 \cap \Omega_V^0$.

(b) *Suppose* $\Omega_a^0 \cap \Omega_V^0 = \emptyset$ *and*

$$\Gamma \cap K_a \neq \emptyset \quad \Longrightarrow \quad \Gamma \setminus K_a \subset \Omega_a^+$$

for any component Γ *of* $\partial\Omega_V^0 \cap \Omega$*. Then,* $a \in \mathfrak{A}^+_{\tilde\Gamma_0,\tilde\Gamma_1}(\Omega_V^0)$*. In particular,*

$$a \in \mathfrak{A}^+_{\Gamma_0,\Gamma_1}(\Omega) \quad \Longrightarrow \quad a \in \mathfrak{A}^+_{\tilde\Gamma_0,\tilde\Gamma_1}(\Omega_V^0)\,.$$

4. Exterior continuous dependence

In this section we analyze the continuous dependence of the positive solutions of (1.1) with respect to exterior perturbations of the domain Ω around its Dirichlet boundary Γ_0 in the special case when ∂_ν is the conormal derivative with respect to $\mathcal{L}$. So, this section assumes (2.8).

Subsequently, we will refer to problem (1.1) as problem $P[\lambda, \Omega, \mathcal{B}(b)]$. Also, we will denote by $\Lambda[\Omega, \mathcal{B}(b)]$ the set of values of $\lambda \in \mathbb{R}$ for which $P[\lambda, \Omega, \mathcal{B}(b)]$ possesses a positive solution.

The following result will provide us with the *exterior continuous dependence* of the positive solutions of $P[\lambda, \Omega, \mathcal{B}(b)]$.

Theorem 4.1. *Suppose* (2.8)*. Let* Ω_0 *be a proper subdomain of* Ω *with boundary of class* $\mathcal{C}^2$ *such that* $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$, $\Gamma_0^0 \cap \Gamma_1 = \emptyset$*, where* Γ_0^0 *satisfies the same requirements as* Γ_0*, and let* $\Omega_n \subset \Omega$, $n \geq 1$*, be a sequence of bounded domains of* $\mathbb{R}^N$ *of class* $\mathcal{C}^2$ *converging to* Ω_0 *from the exterior. For each* $n \in \mathbb{N} \cup \{0\}$ *let* $\mathcal{B}_n(b)$ *denote the boundary operator defined by*

$$\mathcal{B}_n(b)u := \begin{cases} u & \text{on } \Gamma_0^n\,, \\ \partial_\nu u + bu & \text{on } \Gamma_1\,, \end{cases} \tag{4.1}$$

where $\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1$, $n \in \mathbb{N} \cup \{0\}$*. Suppose, in addition, that* $a \in \mathfrak{A}(\Omega_0)$, $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$ *and that* $n_0 \in \mathbb{N}$ *exists such that*

$$a \in \bigcap_{n=n_0}^{\infty} \mathfrak{A}(\Omega_n) \quad \text{and} \quad \lambda \in \bigcap_{n=n_0}^{\infty} \Lambda[\Omega_n, \mathcal{B}_n(b)]\,. \tag{4.2}$$

For each $n \geq 0$*, let* u_n *denote the unique positive solution of* $P[\lambda, \Omega_n, \mathcal{B}_n(b)]$*; it should be noted that the uniqueness is guaranteed by Theorem 2.13. Then,*

$$\lim_{n\to\infty} \|u_n|_{\Omega_0} - u_0\|_{H^1(\Omega_0)} = 0\,. \tag{4.3}$$

Proof. Suppose (4.2). Without lost of generality we can assume that $n_0 = 1$. Then, thanks to Theorem 2.13, the problem $P[\lambda, \Omega_n, \mathcal{B}_n(b)]$, $n \in \mathbb{N} \cup \{0\}$, has a unique positive solution, denoted in the sequel by u_n. Moreover, thanks to Lemma 2.12,

$$u_n \in W^2_{\mathcal{B}_n(b)}(\Omega_n) \subset H^2(\Omega_n)\,, \quad n \in \mathbb{N} \cup \{0\}\,,$$

and u_n is strongly positive in Ω_n. In the sequel for each $n \in \mathbb{N} \cup \{0\}$ we set

$$\tilde{u}_n := \begin{cases} u_n & \text{in } \Omega_n\,, \\ 0 & \text{in } \Omega \setminus \Omega_n\,. \end{cases}$$

Since $u_n \in H^1(\Omega_n)$ and $u_n = 0$ on Γ_0^n, we have that $\tilde{u}_n \in H^1(\Omega)$ and

$$\|\tilde{u}_n\|_{H^1(\Omega)} = \|u_n\|_{H^1(\Omega_n)}\,, \quad n \in \mathbb{N} \cup \{0\}\,. \tag{4.4}$$

Moreover, since u_n is strongly positive in Ω_n, $\Gamma_1 = \partial\Omega_n \setminus \Gamma_0^n$ for each $n \in \mathbb{N} \cup \{0\}$ and

$$\Omega_0 \subset \Omega_{n+1} \subset \Omega_n\,, \quad n \in \mathbb{N}\,,$$

it is easily seen that

$$\begin{gathered} \mathcal{L}u_n = \lambda W u_n - af(\cdot, u_n)u_n \quad \text{in } \Omega_{n+1} \quad n \in \mathbb{N}\,, \\ \mathcal{B}_{n+1}(b)u_n \geq 0 \quad \text{on } \partial\Omega_{n+1} \quad n \in \mathbb{N}\,, \end{gathered}$$

and

$$\begin{gathered} \mathcal{L}u_n = \lambda W u_n - af(\cdot, u_n)u_n \quad \text{in } \Omega_0 \quad n \in \mathbb{N} \\ \mathcal{B}_0(b)u_n \geq 0 \quad \text{on } \partial\Omega_0 \quad n \in \mathbb{N}\,. \end{gathered}$$

Thus, for each $n \in \mathbb{N}$ the function u_n is a positive supersolution of the problems $P[\lambda, \Omega_{n+1}, \mathcal{B}_{n+1}(b)]$ and $P[\lambda, \Omega_0, \mathcal{B}_0(b)]$. Hence, thanks to Theorem 2.15, we find that

$$u_n|_{\Omega_{n+1}} \geq u_{n+1} > 0\,, \quad u_n|_{\Omega_0} \geq u_0 > 0\,, \quad n \geq 1\,.$$

Therefore, in Ω we have that

$$0 < \tilde{u}_0 \leq \tilde{u}_{n+1} \leq \tilde{u}_n \leq \tilde{u}_1\,, \quad n \in \mathbb{N}\,. \tag{4.5}$$

Now, setting $M := \|\tilde{u}_1\|_{L_\infty(\Omega)}$, it follows from (4.5) that

$$\|\tilde{u}_n\|_{L_\infty(\Omega)} \leq M\,, \quad n \in \mathbb{N} \cup \{0\}\,, \tag{4.6}$$

and, hence,

$$\|\tilde{u}_n\|_{L_2(\Omega)} \leq M\,|\Omega|^{1/2}\,, \quad n \in \mathbb{N} \cup \{0\}\,. \tag{4.7}$$

Now, we will prove that $\hat{M} > 0$ exists such that

$$\|\tilde{u}_n\|_{H^1(\Omega)} \leq \hat{M}\,, \quad n \in \mathbb{N} \cup \{0\}\,. \tag{4.8}$$

Indeed, since $\Omega_n \subset \Omega$ for each $n \geq 0$ and $\mathcal{L}$ is strongly uniformly elliptic in $\bar{\Omega}$, integrating by parts and using $u_n = 0$ on Γ_0^n, $\tilde{u}_n = 0$ on $\Omega \setminus \Omega_n$, $\tilde{u}_n|_{\Omega_n} = u_n$ and $u_n \in H^2(\Omega_n)$, $n \geq 0$, gives

$$\begin{aligned} &\mu\|\nabla\tilde{u}_n\|^2_{L_2(\Omega)} \\ &\leq \sum_{i,j=1}^N \int_\Omega \alpha_{ij} \frac{\partial\tilde{u}_n}{\partial x_i} \frac{\partial\tilde{u}_n}{\partial x_j} = \sum_{i,j=1}^N \int_{\Omega_n} \alpha_{ij} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} \\ &= -\sum_{i,j=1}^N \int_{\Omega_n} \frac{\partial}{\partial x_j}(\alpha_{ij} \frac{\partial u_n}{\partial x_i}) u_n + \sum_{i,j=1}^N \int_{\Gamma_1} \alpha_{ij} \frac{\partial u_n}{\partial x_i} u_n n_j \end{aligned}$$

$$= -\sum_{i,j=1}^{N}\int_{\Omega_n}\alpha_{ij}\frac{\partial^2 u_n}{\partial x_i\partial x_j}u_n - \sum_{i,j=1}^{N}\int_{\Omega_n}\frac{\partial \alpha_{ij}}{\partial x_j}\frac{\partial u_n}{\partial x_i}u_n + \sum_{i,j=1}^{N}\int_{\Gamma_1}\alpha_{ij}\frac{\partial u_n}{\partial x_i}u_n n_j\,.$$

From this relation, taking into account that u_n is a solution of $P[\lambda,\Omega_n,\mathcal{B}_n(b)]$ we find that

$$\begin{aligned}
&\mu\|\nabla\tilde{u}_n\|^2_{L_2(\Omega)}\\
&\le \int_{\Omega_n}[(\lambda W - af(\cdot,u_n)-\alpha_0)u_n - \sum_{i=1}^{N}\tilde{\alpha}_i\frac{\partial u_n}{\partial x_i}]u_n + \sum_{i,j=1}^{N}\int_{\Gamma_1}\alpha_{ij}\frac{\partial u_n}{\partial x_i}u_n n_j\,,
\end{aligned}\tag{4.9}$$

where the function coefficients $\tilde{\alpha}_i \in \mathcal{C}(\bar{\Omega})$, $1\le i\le N$, are those given by (2.10). Thus, since $\tilde{u}_n = 0$ in $\Omega\setminus\Omega_n$ and $\tilde{u}_n\in H^1(\Omega)$ for each $n\in\mathbb{N}$, it follows from (4.9) that

$$\begin{aligned}
&\mu\|\nabla\tilde{u}_n\|^2_{L_2(\Omega)}\\
&\le \int_{\Omega}[\lambda W - af(\cdot,\tilde{u}_n)-\alpha_0]\tilde{u}_n^2 - \int_{\Omega}\sum_{i=1}^{N}\tilde{\alpha}_i\frac{\partial \tilde{u}_n}{\partial x_i}\tilde{u}_n + \sum_{i,j=1,}^{N}\int_{\Gamma_1}\alpha_{ij}\frac{\partial u_n}{\partial x_i}u_n n_j\,.
\end{aligned}\tag{4.10}$$

On the other hand, by construction we have that $\partial_\nu u_n + bu_n = 0$ on Γ_1, $n\in\mathbb{N}$, where $\nu=(\nu_1,\dots,\nu_N)$ satisfies

$$\nu_i := \sum_{j=1}^{N}\alpha_{ij}n_j\,,\quad 1\le i\le N\,,$$

since we are assuming (2.8). Thus, for any natural number $n\ge 1$ we have that

$$\sum_{i,j=1}^{N}\alpha_{ij}\frac{\partial u_n}{\partial x_i}n_j = \sum_{i=1}^{N}\nu_i\frac{\partial u_n}{\partial x_i} = \langle\nabla u_n,\nu\rangle = \partial_\nu u_n = -bu_n$$

and, hence,

$$\sum_{i,j=1}^{N}\alpha_{ij}\frac{\partial u_n}{\partial x_i}u_n n_j = -bu_n^2\,.\tag{4.11}$$

Now, substituting (4.11) into (4.10) and using $\tilde{u}_n|_{\Gamma_1} = u_n|_{\Gamma_1}$ gives

$$\mu\|\nabla\tilde{u}_n\|^2_{L_2(\Omega)} \le \int_{\Omega}[\lambda W - af(\cdot,\tilde{u}_n)-\alpha_0]\tilde{u}_n^2 - \int_{\Omega}\sum_{i=1}^{N}\tilde{\alpha}_i\frac{\partial \tilde{u}_n}{\partial x_i}\tilde{u}_n - \int_{\Gamma_1}b\tilde{u}_n^2\,.\tag{4.12}$$

We now proceed to estimate each of the terms of the right hand side of (4.12). Thanks to (4.6),

$$\Big|\int_{\Omega}[\lambda W - af(\cdot,\tilde{u}_n)-\alpha_0]\tilde{u}_n^2\Big| \le M_1 M^2|\Omega|\,,\tag{4.13}$$

where

$$M_1 := |\lambda|\|W\|_{L_\infty(\Omega)} + \|a\|_{L_\infty(\Omega)}\|f\|_{L_\infty(\bar{\Omega}\times[0,M])} + \|\alpha_0\|_{L_\infty(\Omega)}\,.$$

Moreover,

$$|\int_{\Gamma_1}b\tilde{u}_n^2| \le M^2\|b\|_{L_\infty(\Gamma_1)}|\Gamma_1|\,,\tag{4.14}$$

where $|\Gamma_1|$ stands for the $(N-1)$-dimensional Lebesgue measure of Γ_1. Now, setting

$$M_2 := \sum_{i=1}^N \|\tilde{\alpha}_i\|_{L_\infty(\Omega)}\,, \quad \varepsilon := \big(\frac{\mu}{M_2}\big)^{1/2}\,, \tag{4.15}$$

where $\mu > 0$ is the ellipticity constant of $\mathcal{L}$, and using Hölder inequality yields

$$\begin{aligned}\Big|\int_\Omega \sum_{i=1}^N \tilde{\alpha}_i \frac{\partial \tilde{u}_n}{\partial x_i}\tilde{u}_n\Big| &\le \sum_{i=1}^N \|\tilde{\alpha}_i\|_{L_\infty(\Omega)} \int_\Omega |\varepsilon \frac{\partial \tilde{u}_n}{\partial x_i}|\,|\varepsilon^{-1}\tilde{u}_n| \\ &\le M_2 \frac{\varepsilon^2}{2}\|\nabla \tilde{u}_n\|^2_{L_2(\Omega)} + \frac{M_2}{2\varepsilon^2}\|\tilde{u}_n\|^2_{L_2(\Omega)}\,.\end{aligned}$$

Thus, (4.15) implies

$$\Big|\int_\Omega \sum_{i=1}^N \tilde{\alpha}_i \frac{\partial \tilde{u}_n}{\partial x_i}\tilde{u}_n\Big| \le \frac{\mu}{2}\|\nabla \tilde{u}_n\|^2_{L_2(\Omega)} + \frac{M_2^2}{2\mu}\|\tilde{u}_n\|^2_{L_2(\Omega)}\,. \tag{4.16}$$

Hence, thanks to (4.7), (4.13), (4.14) and (4.16), we find from (4.12) that

$$\mu\|\nabla \tilde{u}_n\|^2_{L_2(\Omega)} \le M_3 + \frac{\mu}{2}\|\nabla \tilde{u}_n\|^2_{L_2(\Omega)}\,,$$

where

$$M_3 := M^2(M_1|\Omega| + \|b\|_{L_\infty(\Gamma_1)}|\Gamma_1|) + \frac{1}{2\mu}M_2^2 M^2|\Omega|\,.$$

Thus,

$$\|\nabla \tilde{u}_n\|^2_{L_2(\Omega)} \le \frac{2M_3}{\mu} \tag{4.17}$$

and, therefore, thanks to (4.7) and (4.17), we find that $\|\tilde{u}_n\|_{H^1(\Omega)} \le \hat{M}$, $n \in \mathbb{N}$, where

$$\hat{M} := \Big(M^2|\Omega| + \frac{2M_3}{\mu}\Big)^{1/2}\,.$$

This completes the proof of (4.8).

Now, thanks to and (4.5) and (4.8), along some subsequence, again labeled by n, we have that

$$0 < L := \lim_{n\to\infty} \|\tilde{u}_n\|_{H^1(\Omega)}\,. \tag{4.18}$$

In the sequel we restrict ourselves to deal with functions of that subsequence. Since $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$, it follows from (4.8) that $\tilde{u} \in L_2(\Omega)$ and a subsequence of $\tilde{u}_n$, $n \ge 1$, relabeled by n, exist such that

$$\lim_{n\to\infty} \|\tilde{u}_n - \tilde{u}\|_{L_2(\Omega)} = 0\,. \tag{4.19}$$

To complete the proof of the theorem it suffices to show that (4.18) and (4.19) imply

$$\lim_{n\to\infty} \|\tilde{u}_n - \tilde{u}\|_{H^1(\Omega)} = 0\,, \quad \text{supp}\ \tilde{u} \subset \bar{\Omega}_0\,, \quad \tilde{u}|_{\Omega_0} = u_0\,.$$

since this argument can be repeated along any subsequence. In fact, it suffices proving the validity of the first relation along some subsequence, since u_0 is the unique weak positive solution of problem $P[\lambda, \Omega_0, \mathcal{B}_0(b)]$. Set

$$\tilde{v}_n := \frac{\tilde{u}_n}{\|\tilde{u}_n\|_{H^1(\Omega)}}\,, \quad v_n := \tilde{v}_n|_{\Omega_n} = \frac{u_n}{\|u_n\|_{H^1(\Omega_n)}}\,, \quad n \in \mathbb{N}\cup\{0\}\,.$$

By construction, $\tilde{v}_n \in H^1(\Omega)$, $v_n \in H^2(\Omega_n)$,

$$\tilde{v}_n|_{\Omega\setminus\Omega_n} = 0\,, \quad \|\tilde{v}_n\|_{H^1(\Omega)} = \|v_n\|_{H^1(\Omega_n)} = 1\,, \quad n \in \mathbb{N}\cup\{0\}\,, \tag{4.20}$$

and v_n is a positive solution of

$$\begin{gathered} \mathcal{L}v_n = \lambda W v_n - af(\cdot,u_n)v_n \quad \text{in } \Omega_n \\ \mathcal{B}_n(b)v_n = 0 \quad \text{on } \partial\Omega_n\,, \end{gathered} \tag{4.21}$$

since u_n is a positive solution of $P[\lambda,\Omega_n,\mathcal{B}_n(b)]$. Moreover, (4.5) and (4.6) imply

$$\|\tilde{v}_n\|_{L_\infty(\Omega)} = \frac{\|\tilde{u}_n\|_{L_\infty(\Omega)}}{\|\tilde{u}_n\|_{H^1(\Omega)}} \le \frac{M}{\|\tilde{u}_n\|_{L_2(\Omega)}} \le \frac{M}{\|\tilde{u}_0\|_{L_2(\Omega)}}\,. \tag{4.22}$$

Now, since $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$, we find from (4.20) that there exist $\tilde{v} \in L_2(\Omega)$ and a subsequence of $\tilde{v}_n$, $n \ge 1$, labeled by n, such that

$$\lim_{n\to\infty} \|\tilde{v}_n - \tilde{v}\|_{L_2(\Omega)} = 0\,. \tag{4.23}$$

In particular,

$$\lim_{n\to\infty} \tilde{v}_n = \tilde{v} \quad \text{almost everywhere in} \quad \Omega\,. \tag{4.24}$$

In the sequel we restrict ourselves to consider that subsequence. We claim that

$$\operatorname{supp}\, \tilde{v} \subset \bar{\Omega}_0\,. \tag{4.25}$$

Indeed, pick

$$x \notin \bar{\Omega}_0 = \bigcap_{n=1}^{\infty} \bar{\Omega}_n\,.$$

Then, since $\bar{\Omega}_n$, $n \ge 1$, is a non-increasing sequence of compact sets, a natural number $n_0 \ge 1$ exists such that $x \notin \bar{\Omega}_n$ for each $n \ge n_0$. Thus, $\tilde{v}_n(x) = 0$ for each $n \ge n_0$, and, hence,

$$\lim_{n\to\infty} \tilde{v}_n(x) = 0 \quad \text{if} \quad x \notin \bar{\Omega}_0\,.$$

Therefore, the uniqueness of the limit in (4.24) gives $\tilde{v} = 0$ in $\Omega \setminus \bar{\Omega}_0$. This shows (4.25).

Note that $\tilde{v}_n(x) > 0$ for each $x \in \Omega_n \cup \Gamma_1$ and $n \in \mathbb{N}\cup\{0\}$, since $\tilde{v}_n$ is strongly positive in Ω_n. Hence, $\tilde{v}_n(x) > 0$ for each $x \in \Omega_0 \cup \Gamma_1$ and $n \in \mathbb{N}\cup\{0\}$, since $\Omega_0 \subset \Omega_n$. Thus, (4.24) implies

$$\tilde{v} \ge 0 \quad \text{in} \quad \Omega_0\,. \tag{4.26}$$

Now, we will analyze the limiting behavior of the traces of $\tilde{v}_n$, $n \ge 1$, on Γ_1. By our regularity requirements on $\partial\Omega_0$, it follows from the trace theorem (e.g. Theorem 8.7 of [29]) that the trace operator on Γ_1

$$\begin{array}{rcl} \gamma_1 : H^1(\Omega_0) & \longrightarrow & W_2^{1/2}(\Gamma_1) \\ u & \mapsto & \gamma_1 u := u|_{\Gamma_1} \end{array} \tag{4.27}$$

is well defined and it is a linear continuous operator. Now, for each $n \in \mathbb{N}$ let i_n denote the canonical injection

$$i_n : H^1(\Omega_n) \to H^1(\Omega_0)\,,$$

i.e., the restriction to Ω_0 of the functions of $H^1(\Omega_n)$. Note that for each $n \ge 1$

$$\|i_n\|_{\mathcal{L}(H^1(\Omega_n),H^1(\Omega_0))} \le 1\,. \tag{4.28}$$

Then, setting $T_n := \gamma_1 \circ i_n$, $n \geq 1$, we find from (4.28) that

$$\|T_n\|_{\mathcal{L}(H^1(\Omega_n),W_2^{1/2}(\Gamma_1))} \leq \|\gamma_1\|_{\mathcal{L}(H^1(\Omega_0),W_2^{1/2}(\Gamma_1))}, \quad n \geq 1\,.$$

Thus, the trace operators T_n, $n \geq 1$, are uniformly bounded. Moreover, for each $n \geq 1$ we have that

$$v_n|_{\Gamma_1} = T_n v_n \in W_2^{1/2}(\Gamma_1)\,.$$

Hence, (4.20) implies

$$\|\tilde{v}_n|_{\Gamma_1}\|_{W_2^{1/2}(\Gamma_1)} = \|v_n|_{\Gamma_1}\|_{W_2^{1/2}(\Gamma_1)} = \|T_n v_n\|_{W_2^{1/2}(\Gamma_1)} \leq \|\gamma_1\|_{\mathcal{L}(H^1(\Omega_0),W_2^{1/2}(\Gamma_1))},$$

for $n \geq 1$. Since the embedding

$$W_2^{1/2}(\Gamma_1) \hookrightarrow L_2(\Gamma_1)$$

is compact, because Γ_1 is compact (e.g. Theorem 7.10 of [29]), $v^* \in L_2(\Gamma_1)$ and a subsequence of $\tilde{v}_n$, $n \geq 1$, –again labeled by n– exist such that

$$\lim_{n\to\infty} \|\tilde{v}_n|_{\Gamma_1} - v^*\|_{L_2(\Gamma_1)} = 0\,. \tag{4.29}$$

In the sequel we restrict ourselves to consider that subsequence.

Now, we will show that $\tilde{v}_n$, $n \geq 1$, is a Cauchy sequence in $H^1(\Omega)$. Note that, thanks to (4.23), this implies

$$\lim_{n\to\infty} \|\tilde{v}_n - \tilde{v}\|_{H^1(\Omega)} = 0\,. \tag{4.30}$$

Indeed, let k and m be two natural numbers such that $1 \leq k \leq m$. Then, $\Omega_m \subset \Omega_k$ and, since $\mathcal{L}$ is strongly uniformly elliptic in Ω, integrating by parts and using $v_n = 0$ on Γ_0^n, $\tilde{v}_n = 0$ in $(\Omega \setminus \Omega_n) \cup \Gamma_0^n$ and $\tilde{v}_n|_{\Omega_n} = v_n$, $n \geq 1$, gives

$$\begin{aligned}
&\mu\|\nabla(\tilde{v}_k - \tilde{v}_m)\|^2_{L_2(\Omega)} \\
&\leq \sum_{i,j=1}^N \int_\Omega \alpha_{ij} \frac{\partial}{\partial x_i}(\tilde{v}_k - \tilde{v}_m) \frac{\partial}{\partial x_j}(\tilde{v}_k - \tilde{v}_m) \\
&= \sum_{i,j=1}^N \Big[\int_{\Omega_k} \alpha_{ij} \frac{\partial v_k}{\partial x_i} \frac{\partial v_k}{\partial x_j} + \int_{\Omega_m} \alpha_{ij} \frac{\partial v_m}{\partial x_i} \frac{\partial v_m}{\partial x_j} - 2 \int_{\Omega_m} \alpha_{ij} \frac{\partial v_k}{\partial x_i} \frac{\partial v_m}{\partial x_j} \Big] \\
&= -\sum_{i,j=1}^N \Big[\int_{\Omega_k} v_k \frac{\partial}{\partial x_j}(\alpha_{ij} \frac{\partial v_k}{\partial x_i}) + \int_{\Omega_m} v_m \frac{\partial}{\partial x_j}(\alpha_{ij} \frac{\partial v_m}{\partial x_i}) - 2 \int_{\Omega_m} v_m \frac{\partial}{\partial x_j}(\alpha_{ij} \frac{\partial v_k}{\partial x_i}) \Big] \\
&\quad + \sum_{i,j=1}^N \int_{\Gamma_1} \alpha_{ij} \left(v_k \frac{\partial v_k}{\partial x_i} + v_m \frac{\partial v_m}{\partial x_i} - 2 v_m \frac{\partial v_k}{\partial x_i} \right) n_j\,.
\end{aligned}$$

Thus, since v_n, $n \geq 1$, is a positive solution of (4.21), we find from the previous inequality that

$$\begin{aligned}\mu\,\|\nabla(\tilde{v}_k-\tilde{v}_m)\|^2_{L_2(\Omega)} \leq & \int_{\Omega_k}\Big[\lambda W v_k - af(\cdot,u_k)v_k - \sum_{i=1}^N \tilde{\alpha}_i \frac{\partial v_k}{\partial x_i} - \alpha_0 v_k\Big] v_k \\ & + \int_{\Omega_m}\Big[\lambda W v_m - af(\cdot,u_m)v_m - \sum_{i=1}^N \tilde{\alpha}_i \frac{\partial v_m}{\partial x_i} - \alpha_0 v_m\Big] v_m \\ & - 2\int_{\Omega_m}\Big[\lambda W v_k - af(\cdot,u_k)v_k - \sum_{i=1}^N \tilde{\alpha}_i \frac{\partial v_k}{\partial x_i} - \alpha_0 v_k\Big] v_m \\ & + \sum_{i,j=1}^N \int_{\Gamma_1} \alpha_{ij}\left(v_k\frac{\partial v_k}{\partial x_i} + v_m\frac{\partial v_m}{\partial x_i} - 2v_m\frac{\partial v_k}{\partial x_i}\right) n_j\,,\end{aligned} \tag{4.31}$$

where the functions $\tilde{\alpha}_i \in \mathcal{C}(\bar{\Omega})$, $1 \leq i \leq N$, are those given by (2.10). Rearranging terms in (4.31) gives

$$\begin{aligned}\mu\,\|\nabla(\tilde{v}_k-\tilde{v}_m)\|^2_{L_2(\Omega)} \leq & \int_{\Omega_k}(\lambda W-\alpha_0)(v_k-\tilde{v}_m)v_k + \int_{\Omega_m}(\lambda W-\alpha_0)(v_m-v_k)v_m \\ & + \int_{\Omega_k} af(\cdot,u_k)(\tilde{v}_m - v_k)v_k + \int_{\Omega_m} af(\cdot,u_k)(v_k-v_m)v_m \\ & + \int_{\Omega_m} a v_m^2[f(\cdot,u_k)-f(\cdot,u_m)] + \sum_{i=1}^N\int_{\Omega_k}\tilde{\alpha}_i(\tilde{v}_m-v_k)\frac{\partial v_k}{\partial x_i} \\ & + \sum_{i=1}^N\int_{\Omega_m}\tilde{\alpha}_i v_m \frac{\partial}{\partial x_i}(v_k-v_m) \\ & + \sum_{i,j=1}^N\int_{\Gamma_1}\alpha_{ij}[(v_k-v_m)\frac{\partial v_k}{\partial x_i} + v_m\frac{\partial}{\partial x_i}(v_m-v_k)]n_j\,.\end{aligned} \tag{4.32}$$

Now, we shall estimate each of the terms in the right hand side of (4.32). Note that (4.20) implies

$$\|\tilde{v}_n\|_{L_2(\Omega)} \leq 1\,, \quad \|\nabla\tilde{v}_n\|_{L_2(\Omega)} \leq 1\,, \quad n \in \mathbb{N}\cup\{0\}\,. \tag{4.33}$$

Thus, thanks to Hölder's inequality, we find from (4.6) and (4.33) that

$$\Big|\int_{\Omega_k}(\lambda W-\alpha_0)(v_k-\tilde{v}_m)v_k\Big| \leq \|\lambda W-\alpha_0\|_{L_\infty(\Omega)}\,\|\tilde{v}_k-\tilde{v}_m\|_{L_2(\Omega)}\,, \tag{4.34}$$

$$\Big|\int_{\Omega_m}(\lambda W-\alpha_0)(v_m-v_k)v_m\Big| \leq \|\lambda W-\alpha_0\|_{L_\infty(\Omega)}\,\|\tilde{v}_k-\tilde{v}_m\|_{L_2(\Omega)}\,, \tag{4.35}$$

$$\Big|\int_{\Omega_k} af(\cdot,u_k)(\tilde{v}_m-v_k)v_k\Big| \leq \|a\|_{L_\infty(\Omega)}\|f\|_{L_\infty(\Omega\times[0,M])}\|\tilde{v}_k-\tilde{v}_m\|_{L_2(\Omega)}\,, \tag{4.36}$$

$$\Big|\int_{\Omega_m} af(\cdot,u_k)(v_k-v_m)v_m\Big| \leq \|a\|_{L_\infty(\Omega)}\|f\|_{L_\infty(\Omega\times[0,M])}\|\|\tilde{v}_k-\tilde{v}_m\|_{L_2(\Omega)}\,, \tag{4.37}$$

$$\Big|\sum_{i=1}^N \int_{\Omega_k} \tilde\alpha_i(\tilde v_m - v_k)\frac{\partial v_k}{\partial x_i}\Big| \le \|\tilde v_m - \tilde v_k\|_{L_2(\Omega)} \sum_{i=1}^N \|\tilde\alpha_i\|_{L_\infty(\Omega)}\,. \tag{4.38}$$

Moreover, thanks to (4.6) and (1.5), it is easily seen that

$$|f(\cdot,u_k)-f(\cdot,u_m)| \le \|\partial_u f(\cdot,\cdot)\|_{L_\infty(\Omega\times[0,M])}|u_k-u_m|$$

and, hence,

$$\Big|\int_{\Omega_m} av_m^2[f(\cdot,u_k)-f(\cdot,u_m)]\Big| \le C\,\|\tilde v_m\|_{L_\infty(\Omega)}\int_{\Omega_m}|v_m(u_k-u_m)|\,,$$

where

$$C := \|a\|_{L_\infty(\Omega)}\|\partial_u f\|_{L_\infty(\Omega\times[0,M])}\,.$$

Thus, using Hölder's inequality we find from (4.22) and (4.33) that

$$\Big|\int_{\Omega_m} av_m^2[f(\cdot,u_k)-f(\cdot,u_m)]\Big| \le \frac{CM}{\|\tilde u_0\|_{L_2(\Omega)}}\|\tilde u_k - \tilde u_m\|_{L_2(\Omega)}\,. \tag{4.39}$$

To estimate the integrals over Γ_1 one should remember that $\partial_\nu v_n + b\,v_n = 0$ on Γ_1, $n\in\mathbb{N}$, since v_n is a positive solution of (4.21). Then, it follows from assumption (2.8) that for any $n\in\mathbb{N}$

$$\sum_{i,j=1}^N \alpha_{ij}\frac{\partial v_n}{\partial x_i}n_j = \sum_{i=1}^N \nu_i\frac{\partial v_n}{\partial x_i} = \langle\nabla v_n,\nu\rangle = \partial_\nu v_n = -b\,v_n$$

and, hence,

$$\sum_{i,j=1}^N \alpha_{ij}\frac{\partial}{\partial x_i}(v_m-v_k)n_j = -b\,(v_m-v_k)\,.$$

Therefore,

$$\begin{aligned}\Big|\sum_{i,j=1}^N \int_{\Gamma_1}\alpha_{ij}(v_k-v_m)\frac{\partial v_k}{\partial x_i}n_j\Big| &= \Big|\int_{\Gamma_1} b\,v_k\,(v_m-v_k)\Big|\\ &\le \|b\|_{L_\infty(\Gamma_1)}\|v_k|_{\Gamma_1}\|_{L_2(\Gamma_1)}\|(v_k-v_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)}\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}\Big|\sum_{i,j=1}^N \int_{\Gamma_1}\alpha_{ij}v_m\frac{\partial}{\partial x_i}(v_m-v_k)n_j\Big| &= \Big|\int_{\Gamma_1} b\,v_m\,(v_k-v_m)\Big|\\ &\le \|b\|_{L_\infty(\Gamma_1)}\|v_m|_{\Gamma_1}\|_{L_2(\Gamma_1)}\|(v_k-v_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)}\,.\end{aligned} \tag{4.41}$$

To complete the proof of the claim above, it only remains estimating the term

$$I_{mk} := \sum_{i=1}^N \int_{\Omega_m}\tilde\alpha_i v_m\frac{\partial}{\partial x_i}(v_k-v_m)\,. \tag{4.42}$$

Since $\tilde\alpha_i \in \mathcal{C}(\bar\Omega)$, $1\le i\le N$, in order to perform an integration by parts in (4.42) we must approach each of these coefficients by a sequence of smooth functions, say α_i^n, $n\ge 1$, $1\le i\le N$. Fix $\delta>0$ and consider the δ-neighborhood of Ω

$$\Omega_\delta := \bar\Omega + B_\delta(0)\,.$$

For each $1 \le i \le N$, let $\hat\alpha_i$ be a continuous extension of $\tilde\alpha_i$ to $\mathbb{R}^N$ such that

$$\hat\alpha_i \in \mathcal{C}_c(\Omega_\delta)\,, \quad \|\hat\alpha_i\|_{L_\infty(\mathbb{R}^N)} = \|\tilde\alpha_i\|_{L_\infty(\Omega)}\,. \tag{4.43}$$

Now, consider the function

$$\rho(x) := \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1\,, \\ 0 & \text{if } |x| \ge 1\,, \end{cases}$$

and the associated *approximation of the identity*

$$\rho_n := \Big(\int_{\mathbb{R}^N} \rho\Big)^{-1} n^N \rho(n\,\cdot)\,, \quad n \in \mathbb{N}\,.$$

Note that for each $n \ge 1$ the function ρ_n satisfies

$$\rho_n \in \mathcal{C}_c^\infty(\mathbb{R}^N)\,, \quad \operatorname{supp}\,\rho_n \subset B_{\frac{1}{n}}(0)\,, \quad \rho_n \ge 0\,, \quad \|\rho_n\|_{L_1(\mathbb{R}^N)} = 1\,.$$

Then, for each $1 \le i \le N$ the new sequence $\alpha_i^n := \rho_n * \hat\alpha_i$, $n \ge 1$, is of class $\mathcal{C}_c^\infty(\mathbb{R}^N)$ and it converges to $\hat\alpha_i$ uniformly on any compact subset of $\mathbb{R}^N$ (e.g. Theorem 8.1.3 of [15]). In particular,

$$\lim_{n\to\infty} \|\alpha_i^n|_\Omega - \tilde\alpha_i\|_{L_\infty(\Omega)} = 0\,, \quad 1 \le i \le N\,, \tag{4.44}$$

since $\hat\alpha_i|_\Omega = \tilde\alpha_i$. Moreover, thanks to (4.43), it follows from Young's inequality that for each $n \ge 1$

$$\|\alpha_i^n\|_{L_\infty(\mathbb{R}^N)} \le \|\rho_n\|_{L_1(\mathbb{R}^N)} \|\hat\alpha_i\|_{L_\infty(\mathbb{R}^N)} = \|\tilde\alpha_i\|_{L_\infty(\Omega)}\,, \quad 1 \le i \le N\,, \tag{4.45}$$

and

$$\|\frac{\partial \alpha_i^n}{\partial x_i}\|_{L_\infty(\mathbb{R}^N)} \le \|\frac{\partial \rho_n}{\partial x_i}\|_{L_1(\mathbb{R}^N)} \|\tilde\alpha_i\|_{L_\infty(\Omega)}\,, \quad 1 \le i \le N\,, \tag{4.46}$$

since

$$\frac{\partial \alpha_i^n}{\partial x_i} = \frac{\partial \rho_n}{\partial x_i} * \hat\alpha_i\,, \quad 1 \le i \le N\,, \quad n \ge 1\,.$$

Furthermore, since for each $1 \le i \le N$ and $n \ge 1$

$$\|\frac{\partial \rho_n}{\partial x_i}\|_{L_1(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} \rho\Big)^{-1} n\, \|\frac{\partial \rho}{\partial x_i}\|_{L_1(\mathbb{R}^N)}\,,$$

(4.46) implies

$$\|\frac{\partial \alpha_i^n}{\partial x_i}\|_{L_\infty(\mathbb{R}^N)} \le \Big(\int_{\mathbb{R}^N} \rho\Big)^{-1} n\, \|\frac{\partial \rho}{\partial x_i}\|_{L_1(\mathbb{R}^N)} \|\tilde\alpha_i\|_{L_\infty(\Omega)}\,, \quad 1 \le i \le N\,, \tag{4.47}$$

for each $n \ge 1$. Now, going back to (4.42) we find that for each $n \ge 1$

$$I_{mk} := \sum_{i=1}^N \int_{\Omega_m} (\tilde\alpha_i - \alpha_i^n) v_m \frac{\partial}{\partial x_i}(v_k - v_m) + \sum_{i=1}^N \int_{\Omega_m} \alpha_i^n v_m \frac{\partial}{\partial x_i}(v_k - v_m)\,. \tag{4.48}$$

We now estimate each of the terms in the right hand side of (4.48). Applying Hölder inequality and using (4.33) it is easily seen that

$$\begin{aligned} &\Big|\sum_{i=1}^N \int_{\Omega_m} (\tilde\alpha_i - \alpha_i^n) v_m \frac{\partial (v_k - v_m)}{\partial x_i}\Big| \\ &\le \Big(\sum_{i=1}^N \|\tilde\alpha_i - \alpha_i^n\|_{L_\infty(\Omega)}\Big) \|\tilde v_m\|_{L_2(\Omega)} \|\nabla(\tilde v_k - \tilde v_m)\|_{L_2(\Omega)} \end{aligned}$$

$$\leq 2\sum_{i=1}^{N}\|\tilde{\alpha}_i-\alpha_i^n\|_{L_\infty(\Omega)}\,.$$

Moreover, integrating by parts gives

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega_m}\alpha_i^n v_m\frac{\partial}{\partial x_i}(v_k-v_m)\\
&=-\sum_{i=1}^{N}\int_{\Omega_m}(v_k-v_m)\frac{\partial}{\partial x_i}(\alpha_i^n v_m)+\sum_{i=1}^{N}\int_{\Gamma_1}\alpha_i^n v_m(v_k-v_m)n_i
\end{aligned}$$

and, hence,

$$\begin{aligned}
&\Big|\sum_{i=1}^{N}\int_{\Omega_m}\alpha_i^n v_m\frac{\partial}{\partial x_i}(v_k-v_m)\Big|\\
&\leq\Big(\sum_{i=1}^{N}\|\alpha_i^n\|_{L_\infty(\mathbb{R}^N)}\Big)\|\nabla\tilde{v}_m\|_{L_2(\Omega)}\|\tilde{v}_k-\tilde{v}_m\|_{L_2(\Omega)}\\
&\quad+\Big(\sum_{i=1}^{N}\|\frac{\partial\alpha_i^n}{\partial x_i}\|_{L_\infty(\mathbb{R}^N)}\Big)\|\tilde{v}_m\|_{L_2(\Omega)}\|\tilde{v}_k-\tilde{v}_m\|_{L_2(\Omega)}\\
&\quad+\Big(\sum_{i=1}^{N}\|\alpha_i^n\|_{L_\infty(\mathbb{R}^N)}\Big)\|v_m|_{\Gamma_1}\|_{L_2(\Gamma_1)}\|(v_k-v_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)}\,.
\end{aligned}$$

Thus, substituting these estimates into (4.48) and using (4.33), (4.45) and (4.47) we find that

$$\begin{aligned}
|I_{mk}|\leq{}&\sum_{i=1}^{N}\big(2\|\tilde{\alpha}_i-\alpha_i^n\|_{L_\infty(\Omega)}+\|\tilde{\alpha}_i\|_{L_\infty(\Omega)}\|v_m|_{\Gamma_1}\|_{L_2(\Gamma_1)}\|(v_k-v_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)}\big)\\
&+\sum_{i=1}^{N}\Big(1+(\int_{\mathbb{R}^N}\rho)^{-1}n\,\|\frac{\partial\rho}{\partial x_i}\|_{L_1(\mathbb{R}^N)}\Big)\|\tilde{\alpha}_i\|_{L_\infty(\Omega)}\|\tilde{v}_k-\tilde{v}_m\|_{L_2(\Omega)}
\end{aligned}$$

for any $n\geq 1$. Now, fix $\epsilon>0$. Thanks to (4.44), there exists $n\geq 1$ such that

$$2\sum_{i=1}^{N}\|\tilde{\alpha}_i-\alpha_i^n\|_{L_\infty(\Omega)}\leq\frac{\epsilon}{4}\,.$$

Hence, thanks to (4.23) and (4.29), there exists $n_0\geq 1$ such that for any $n_0\leq k\leq m$

$$|I_{mk}|\leq\frac{\epsilon}{2}\,. \tag{4.49}$$

Therefore, substituting (4.34)-4.41 and (4.49) into (4.32) and using (4.19), (4.23) and (4.29), it is easily seen that there exists $k_0\geq n_0$ such that for any $k_0\leq k\leq m$

$$\mu\,\|\nabla(\tilde{v}_k-\tilde{v}_m)\|_{L_2(\Omega)}^2\leq\epsilon\,.$$

This shows that $\tilde{v}\in H^1(\Omega)$ and completes the proof of (4.30). Note that, thanks to (4.20),

$$\|\tilde{v}\|_{H^1(\Omega)}=\lim_{n\to\infty}\|\tilde{v}_n\|_{H^1(\Omega)}=1\,. \tag{4.50}$$

Moreover, if γ^1 stands for the trace operator of $H^1(\Omega)$ on Γ_1, then

$$\|\tilde{v}_n|_{\Gamma_1}-\tilde{v}|_{\Gamma_1}\|_{L_2(\Gamma_1)}=\|\gamma^1(\tilde{v}_n-\tilde{v})\|_{L_2(\Gamma_1)}\leq\|\gamma^1\|_{\mathcal{L}(H^1(\Omega),L_2(\Gamma_1))}\|\tilde{v}_n-\tilde{v}\|_{H^1(\Omega)}$$

and hence, (4.30) implies

$$\lim_{n\to\infty} \|\tilde{v}_n|_{\Gamma_1} - \tilde{v}|_{\Gamma_1}\|_{L_2(\Gamma_1)} = 0 .$$

Thus, thanks to (4.29), we find that

$$\tilde{v}|_{\Gamma_1} = v^* . \tag{4.51}$$

Now, set

$$v := \tilde{v}|_{\Omega_0} . \tag{4.52}$$

Since by construction $v_n|_{\Omega_0} = \tilde{v}_n|_{\Omega_0}$, it follows from (4.30) that $v \in H^1(\Omega_0)$ and

$$\lim_{n\to\infty} \|v_n|_{\Omega_0} - v\|_{H^1(\Omega_0)} = 0 . \tag{4.53}$$

Moreover, thanks to (4.25) and (4.50),

$$\|v\|_{H^1(\Omega_0)} = \|\tilde{v}\|_{H^1(\Omega)} = 1 . \tag{4.54}$$

On the other hand,

$$\begin{aligned} \Big\|\tilde{v}_n - \frac{\tilde{u}}{L}\Big\|_{L^2(\Omega)} &= \Big\|\frac{\tilde{u}_n}{\|\tilde{u}_n\|_{H^1(\Omega)}} - \frac{\tilde{u}}{L}\Big\|_{L^2(\Omega)} \\ &\le \frac{\|\tilde{u}_n - \tilde{u}\|_{L^2(\Omega)}}{\|\tilde{u}_n\|_{H^1(\Omega)}} + \Big|\frac{1}{\|\tilde{u}_n\|_{H^1(\Omega)}} - \frac{1}{L}\Big|\|\tilde{u}\|_{L^2(\Omega)} , \end{aligned}$$

where L is the constant defined through (4.18). Thus, it follows from (4.18) and (4.19) that

$$\lim_{n\to\infty} \Big\|\tilde{v}_n - \frac{\tilde{u}}{L}\Big\|_{L_2(\Omega)} = 0 .$$

Consequently, thanks to (4.19) and (4.30), we find that

$$\tilde{u} = L\tilde{v} \quad \text{in} \quad L_2(\Omega) . \tag{4.55}$$

Moreover, thanks to (4.25), (4.26), (4.53) and (4.55) we have that

$$\tilde{u} \in H^1(\Omega_0) , \quad \operatorname{supp} \tilde{u} \subset \bar{\Omega}_0 , \quad \tilde{u} > 0 . \tag{4.56}$$

Now, set $u := \tilde{u}|_{\Omega_0}$. Thanks to (4.53) and (4.55), we have that

$$u = Lv \quad \text{and} \quad \lim_{n\to\infty} \Big\|v_n|_{\Omega_0} - \frac{u}{L}\Big\|_{H^1(\Omega_0)} = 0 . \tag{4.57}$$

In the sequel we will show that u is a weak solution of $P[\lambda, \Omega_0, \mathcal{B}_0(b)]$. Indeed, since $\tilde{v} \in H^1(\Omega)$ and supp $\tilde{v} \subset \bar{\Omega}_0$, it follows from Theorem 2.7 that $\tilde{v} \in H^1_{\Gamma_0^0}(\Omega_0)$. Thus, $v \in H^1_{\Gamma_0^0}(\Omega_0)$ and hence $u = Lv \in H^1_{\Gamma_0^0}(\Omega_0)$. Now, pick

$$\xi \in \mathcal{C}_c^\infty(\Omega_0 \cup \Gamma_1) .$$

Then, multiplying the differential equations

$$\mathcal{L} v_n = \lambda W v_n - a f(\cdot, u_n) v_n , \quad n \ge 1 ,$$

by ξ, integrating in Ω_n, applying the formula of integration by parts and taking into account that $\operatorname{supp} \xi \subset \Omega_0 \cup \Gamma_1$ gives

$$\begin{aligned} &\sum_{i,j=1}^N \int_{\Omega_0} \alpha_{ij} \frac{\partial v_n}{\partial x_i} \frac{\partial \xi}{\partial x_j} + \sum_{i=1}^N \int_{\Omega_0} \tilde{\alpha}_i \frac{\partial v_n}{\partial x_i} \xi + \int_{\Omega_0} \alpha_0 v_n \xi \\ &= \int_{\Omega_0} (\lambda W - a f(\cdot, u_n)) v_n \xi + \sum_{i,j=1}^N \int_{\Gamma_1} \alpha_{ij} \frac{\partial v_n}{\partial x_i} \, \xi \, n_j , \end{aligned}$$

for each $n \geq 1$, where the coefficients $\tilde{\alpha}_i$, $1 \leq i \leq N$, are given by (2.10). Moreover, using $\partial_\nu v_n + b\, v_n = 0$ on Γ_1, $n \geq 1$, yields

$$\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial v_n}{\partial x_i}\, \xi\, n_j = \sum_{i=1}^{N} \nu_i \frac{\partial v_n}{\partial x_i} \xi = \langle \nabla v_n, \nu \rangle \xi = \partial_\nu v_n \xi = -b\, v_n\, \xi\,,$$

and, hence, for each $n \geq 1$ we find that

$$\begin{aligned} &\sum_{i,j=1}^{N} \int_{\Omega_0} \alpha_{ij} \frac{\partial v_n}{\partial x_i} \frac{\partial \xi}{\partial x_j} + \sum_{i=1}^{N} \int_{\Omega_0} \tilde{\alpha}_i \frac{\partial v_n}{\partial x_i} \xi + \int_{\Omega_0} \alpha_0 v_n \xi \\ &= \int_{\Omega_0} (\lambda W - a f(\cdot, u_n)) v_n \xi - \int_{\Gamma_1} b\, v_n\, \xi\,. \end{aligned} \tag{4.58}$$

Thus, using (4.6), $\tilde{v}|_{\Gamma_1} = v|_{\Gamma_1}$ and

$$\lim_{n\to\infty} \|u_n - u\|_{L^2(\Omega)} = 0\,, \quad \lim_{n\to\infty} \|v_n|_{\Omega_0} - v\|_{H^1(\Omega_0)} = 0\,,$$
$$\lim_{n\to\infty} \|v_n|_{\Gamma_1} - v|_{\Gamma_1}\|_{L_2(\Gamma_1)} = 0\,,$$

and passing to the limit as $n \to \infty$ in (4.58), the theorem of dominated convergence implies

$$\begin{aligned} &\sum_{i,j=1}^{N} \int_{\Omega_0} \alpha_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \xi}{\partial x_j} + \sum_{i=1}^{N} \int_{\Omega_0} \tilde{\alpha}_i \frac{\partial v}{\partial x_i} \xi + \int_{\Omega_0} \alpha_0 v \xi \\ &= \int_{\Omega_0} (\lambda W - a f(\cdot, u)) v \xi - \int_{\Gamma_1} b\, v\, \xi\,. \end{aligned} \tag{4.59}$$

Finally, multiplying (4.59) by L and taking into account that $u = Lv$ gives

$$\begin{aligned} &\sum_{i,j=1}^{N} \int_{\Omega_0} \alpha_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \xi}{\partial x_j} + \sum_{i=1}^{N} \int_{\Omega_0} \tilde{\alpha}_i \frac{\partial u}{\partial x_i} \xi + \int_{\Omega_0} \alpha_0 u \xi \\ &= \int_{\Omega_0} (\lambda W - a f(\cdot, u)) u \xi \\ &- \int_{\Gamma_1} b\, u\, \xi \end{aligned}$$

for each $\xi \in \mathcal{C}_c^\infty(\Omega_0 \cup \Gamma_1)$. Therefore, $u \in H^1_{\Gamma_0^0}(\Omega_0)$, $u > 0$, is a weak positive solution of $P[\lambda, \Omega_0, \mathcal{B}_0(b)]$. Since u_0 is the unique positive solution of $P[\lambda, \Omega_0, \mathcal{B}_0(b)]$, necessarily

$$u_0 = u = Lv\,. \tag{4.60}$$

Now, thanks to (4.53), it follows from (4.60) that

$$\lim_{n\to\infty} \|v_n|_{\Omega_0} - \frac{u_0}{L}\|_{H^1(\Omega_0)} = 0\,. \tag{4.61}$$

Moreover, since

$$u_n|_{\Omega_0} - u_0 = \|\tilde{u}_n\|_{H^1(\Omega)} \Big[\big(v_n|_{\Omega_0} - \frac{u_0}{L}\big) + \big(\frac{1}{L} - \frac{1}{\|\tilde{u}_n\|_{H^1(\Omega)}}\big) u_0 \Big]\,,$$

it follows from (4.8) that

$$\|u_n|_{\Omega_0} - u_0\|_{H^1(\Omega_0)} \leq \hat{M} \Big[\|v_n|_{\Omega_0} - \frac{u_0}{L}\|_{H^1(\Omega_0)} + \big|\frac{1}{L} - \frac{1}{\|\tilde{u}_n\|_{H^1(\Omega)}}\big| \|u_0\|_{H^1(\Omega_0)} \Big]\,.$$

Therefore, thanks to (4.18) and (4.61), we conclude that

$$\lim_{n\to\infty} \|u_n|_{\Omega_0} - u_0\|_{H^1(\Omega_0)} = 0\,.$$

This shows the validity of (4.3) along the subsequence we have been dealing with. As the previous argument works out along any subsequence, the proof is completed. $\square$

The following result provides us with some sufficient conditions ensuring that condition (4.2) is satisfied. Therefore, under these conditions the conclusion of Theorem 4.1 is satisfied.

Theorem 4.2. *Let Ω_0 be a proper subdomain of Ω with boundary of class $\mathcal{C}^2$ such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1\,, \quad \Gamma_0^0 \cap \Gamma_1 = \emptyset\,,$$

where Γ_0^0 satisfies the same requirements as Γ_0, and let $\Omega_n \subset \Omega$, $n \ge 1$, be a sequence of bounded domains of $\mathbb{R}^N$ of class $\mathcal{C}^2$ converging to Ω_0 from the exterior such that

$$\operatorname{dist}(\partial\Omega, \partial\Omega_n \cap \Omega) > 0\,, \quad n \ge 0\,. \tag{4.62}$$

For each natural number $n \ge 0$ let $\mathcal{B}_n(b)$ be the boundary operator defined by (4.1). Then, the following assertions are true:

(*a*) *Suppose (2.8) on $\Gamma_1 \cap \partial\Omega_a^0$ and $\emptyset \ne \Omega_a^0 \subset \Omega_0$. Then, for each $n \ge 0$,*

$$a \in \bigcap_{n=0}^{\infty} \mathfrak{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n) \quad \textit{and} \quad [\Omega_n]_a^0 = \Omega_a^0\,, \tag{4.63}$$

where $\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1$ and $[\Omega_n]_a^0$ is the corresponding open set of the definition of the class $\mathfrak{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n)$, $n \ge 0$. Suppose, in addition, that $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$. Then,

$$\lambda \in \bigcap_{n=0}^{\infty} \Lambda[\Omega_n, \mathcal{B}_n(b)]\,. \tag{4.64}$$

(*b*) *Suppose $\bar\Omega_0 \cap \bar\Omega_a^0 = \emptyset$. Then, $a \in \mathfrak{A}^+_{\Gamma_0^0,\Gamma_1}(\Omega_0)$. Moreover, $n_0 \in \mathbb{N}$ exists for which*

$$a \in \bigcap_{n=n_0}^{\infty} \mathfrak{A}^+_{\Gamma_0^n,\Gamma_1}(\Omega_n)\,. \tag{4.65}$$

Furthermore,

$$\lambda \in \bigcap_{n=n_0}^{\infty} \Lambda[\Omega_n, \mathcal{B}_n(b)] \tag{4.66}$$

if $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$.

(*c*) *Suppose $\bar\Omega_a^0 \cap \bar\Omega_0 \ne \emptyset$, $\Omega_0 \cap \Omega_a^0 = \emptyset$, and $n_0 \in \mathbb{N}$ exists for which $\Omega_n \cap \Omega_a^0$ is of class $\mathcal{C}^2$ and*

$$\partial\Omega_n \cap \Omega \cap \partial(\Omega_a^0 \cap \Omega_n) = \partial\Omega_n \cap \Omega \cap \bar\Omega_a^0\,, \quad n \ge n_0\,. \tag{4.67}$$

Suppose, in addition, that $\Gamma \cap K_a \ne \emptyset$ implies $\Gamma \setminus K_a \subset \Omega_a^+$ for any component Γ of Γ_0^0. Then, $a \in \mathfrak{A}^+_{\Gamma_0^0,\Gamma_1}(\Omega_0)$ and

$$a \in \bigcap_{n=n_0}^{\infty} \mathfrak{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n)\,, \quad [\Omega_n]_a^0 = \Omega_a^0 \cap \Omega_n\,, \quad n \ge n_0\,. \tag{4.68}$$

Suppose, in addition, that $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$. Then, $m_0 \in \mathbb{N}$, $m_0 \geq n_0$, exists for which

$$\lambda \in \bigcap_{n=m_0}^{\infty} \Lambda[\Omega_n, \mathcal{B}_n(b)] \,. \tag{4.69}$$

(*d*) *Suppose* (2.8) *on $\Gamma_1 \cap \partial[\Omega_0]_a^0$ and*
1. $\Omega_a^0 \cap \Omega_0 \neq \emptyset$ *is of class $\mathcal{C}^2$,*
2. $\Omega_a^0 \cap (\Omega \setminus \Omega_0) \neq \emptyset$,
3. $n_0 \in \mathbb{N}$ *exists such that $\Omega_a^0 \cap \Omega_n$ is a proper subdomain of Ω of class $\mathcal{C}^2$ if $n \geq n_0$,*
4. (3.2) *is satisfied for any $\tilde{\Omega} \in \{\Omega_0, \Omega_{n_0+j} : j \geq 0\}$.*

Then, $a \in \mathfrak{A}_{\Gamma_0^0,\Gamma_1}(\Omega_0)$ and $m_0 \geq n_0$ exists for which

$$a \in \bigcap_{n=m_0}^{\infty} \mathfrak{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n) \quad \wedge \quad [\Omega_n]_a^0 = \Omega_n \cap \Omega_a^0 \quad \textit{if} \quad n \in \{0, m_{0+j} : j \geq 0\} \,. \tag{4.70}$$

Moreover, if, in addition, $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$, then, for some $\ell_0 \geq m_0$,

$$\lambda \in \bigcap_{n=\ell_0}^{\infty} \Lambda[\Omega_n, \mathcal{B}_n(b)] \,. \tag{4.71}$$

(*e*) *Suppose $a \in \mathfrak{A}^+(\Omega)$, i.e. $\Omega_a^0 = \emptyset$. Then,*

$$a \in \bigcap_{n=0}^{\infty} \mathfrak{A}^+_{\Gamma_0^n,\Gamma_1}(\Omega_n) \,, \tag{4.72}$$

i.e., $a \in \mathfrak{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n)$ and $[\Omega_n]_a^0 = \emptyset$ for each $n \geq 0$. Moreover,

$$\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)] \quad \Longrightarrow \quad \lambda \in \bigcap_{n=0}^{\infty} \Lambda[\Omega_n, \mathcal{B}_n(b)] \,. \tag{4.73}$$

Furthermore, in any of the five previous cases, if (2.8) *is satisfied on Γ_1, $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$ and u_n stands for the unique positive solution of $P[\lambda, \Omega_n, \mathcal{B}_n(b)]$ –whose existence is guaranteed for n sufficiently large–, then, thanks to Theorem 4.1,*

$$\lim_{n\to\infty} \|u_n|_{\Omega_0} - u_0\|_{H^1(\Omega_0)} = 0 \,, \tag{4.74}$$

where u_0 is the unique positive solution of $P[\lambda, \Omega_0, \mathcal{B}_0(b)]$.

In most of the applications, in order to have the results of the theorem it suffices assuming that

$$\operatorname{dist}(\partial\Omega, \partial\Omega_0 \cap \Omega) > 0 \,,$$

instead of (4.62), since this condition implies (4.62) to hold for $n = 0$ and n sufficiently large.

Proof. Without lost of generality we can assume that Ω_{n+1} is a proper subset of Ω_n for each $n \geq 1$. Then, Ω_0 is a proper subset of Ω_n for any $n \geq 1$ and, for each $n \geq 1$,

$$\operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega_n) > 0 \,,$$

since $\partial\Omega_0 \cap \Omega_n \subset \Gamma_0^0$ and Ω_n converges from the exterior to Ω_0 as $n \to \infty$. Thus, thanks to Proposition 2.4,

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}_n(b), \Omega_n] < \sigma[\mathcal{L}_f(\lambda), \mathcal{B}_0(b), \Omega_0] \,, \quad n \geq 1 \,. \tag{4.75}$$

Now, we will proceed to prove each of the assertions of the theorem separately:
(a) Suppose $\emptyset \neq \Omega_a^0 \subset \Omega_0$. Then, for each $n \geq 0$, $\Omega_a^0 \cap \Omega_n = \Omega_a^0 \neq \emptyset$ is of class $\mathcal{C}^2$, since $\Omega_a^0 \subset \Omega_0 \subset \Omega_n$. Moreover, for each $n \geq 0$,

$$\partial\Omega_n \cap \Omega \cap \partial(\Omega_a^0 \cap \Omega_n) = \partial\Omega_n \cap \Omega \cap \bar{\Omega}_a^0 , \quad n \geq 0 ,$$

since $\Omega_a^0 \cap \Omega_n = \Omega_a^0$ and $\partial\Omega_n \cap \Omega \subset \Gamma_0^n$. Therefore, thanks to (4.62), it readily follows from Theorem 3.1(a) that (4.63) is satisfied. Note that

$$\sigma[\mathcal{L}(\lambda), \mathcal{B}(b, [\Omega_n]_a^0), [\Omega_n]_a^0] = \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, \Omega_a^0), \Omega_a^0] , \quad n \geq 0 . \tag{4.76}$$

Now, suppose $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$. Then, thanks to Theorem 2.13(a),

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}_0(b), \Omega_0] < 0 < \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, \Omega_a^0), \Omega_a^0] . \tag{4.77}$$

Therefore, thanks to (4.75) and (4.77), we find that, for each $n \geq 0$,

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}_n(b), \Omega_n] < 0 < \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, \Omega_a^0), \Omega_a^0] .$$

Consequently, (4.64) follows from Theorem 2.13(a).
(b) Let Γ be any component of Γ_0^0 satisfying $\Gamma \cap K \neq \emptyset$. Then, it follows from

$$\Gamma \setminus K \subset \Gamma_0^0 \subset \partial\Omega_0 \quad \wedge \quad \bar{\Omega}_0 \cap \bar{\Omega}_a^0 = \emptyset ,$$

that

$$(\Gamma \setminus K) \cap \bar{\Omega}_a^0 = \emptyset \quad \wedge \quad (\Gamma \setminus K) \cap K = \emptyset .$$

Thus, $(\Gamma \setminus K) \cap (\bar{\Omega}_a^0 \cup K) = \emptyset$ and, hence, (1.10) implies $\Gamma \setminus K \subset \Omega_a^+$. Therefore, thanks to (4.62), we find from Theorem 3.1(b) that

$$a \in \mathfrak{A}^+_{\Gamma_0^0, \Gamma_1}(\Omega_0) .$$

On the other hand, since

$$\bar{\Omega}_a^0 \cap \bar{\Omega}_0 = \emptyset \quad \wedge \quad \bigcap_{n=1}^{\infty} \bar{\Omega}_n = \bar{\Omega}_0 ,$$

$n_0 \in \mathbb{N}$ exists for which

$$\bar{\Omega}_a^0 \cap \bar{\Omega}_n = \emptyset , \quad n \geq n_0 . \tag{4.78}$$

Pick $n \geq n_0$ and let Γ be any component of Γ_0^n satisfying $\Gamma \cap K \neq \emptyset$. Then, it follows from

$$\Gamma \setminus K \subset \Gamma_0^n \subset \partial\Omega_n \quad \wedge \quad \bar{\Omega}_n \cap \bar{\Omega}_a^0 = \emptyset ,$$

that

$$(\Gamma \setminus K) \cap \bar{\Omega}_a^0 = \emptyset \quad \wedge \quad (\Gamma \setminus K) \cap K = \emptyset .$$

Thus, $(\Gamma \setminus K) \cap (\bar{\Omega}_a^0 \cup K) = \emptyset$ and, hence, (1.10) implies $\Gamma \setminus K \subset \Omega_a^+$. Therefore, thanks to (4.62), Theorem 3.1(b) implies (4.65).

Now, suppose $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$. Then, thanks to Theorem 2.13(b),

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}_0(b), \Omega_0] < 0 .$$

Thus, thanks to (4.75), for each $n \geq n_0$, we have that

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}_n(b), \Omega_n] < \sigma[\mathcal{L}_f(\lambda), \mathcal{B}_0(b), \Omega_0] < 0 , \quad n \geq n_0 ,$$

and, therefore, thanks again to Theorem 2.13(b), condition (4.66) holds.
(c) Since any component of $\partial\Omega_0 \cap \Omega$ must be a component of Γ_0^0, thanks to (4.62), it follows from Theorem 3.1(b) that $a \in \mathfrak{A}^+_{\Gamma_0^0, \Gamma_1}(\Omega_0)$. Similarly, thanks to Theorem 3.1(a),

$$a \in \mathfrak{A}_{\Gamma_0^n, \Gamma_1}(\Omega_n) , \quad [\Omega_n]_a^0 = \Omega_n \cap \Omega_a^0 , \quad n \geq n_0 . \tag{4.79}$$

In particular,

$$\lim_{n\to\infty}|[\Omega_n]_a^0| = \lim_{n\to\infty}|\Omega_a^0\cap\Omega_n| = 0\,,$$

since $\Omega_n \to \Omega_0$ from the exterior, as $n\to\infty$, and $\Omega_a^0\cap\Omega_0=\emptyset$. Here $|\cdot|$ stands for the N-dimensional Lebesgue measure. Therefore, thanks to Theorem 2.11, there exists $m_0\geq n_0$ such that

$$\sigma[\mathcal{L}(\lambda),\mathcal{D},[\Omega_n]_a^0] > 0\,,\quad n\geq m_0\,. \tag{4.80}$$

Now, we shall show that, for each $n\geq m_0$,

$$\Gamma_1\cap\partial[\Omega_n]_a^0=\emptyset \tag{4.81}$$

and that, consequently,

$$\mathcal{B}(b,[\Omega_n]_a^0)=\mathcal{D}$$

is the Dirichlet boundary operator. On the contrary assume that $\Gamma_1\cap\partial[\Omega_n]_a^0\neq\emptyset$ for some $n\geq m_0$ and let Γ_1^* be a component of Γ_1 such that

$$\Gamma_1^*\cap\partial[\Omega_n]_a^0\neq\emptyset\,.$$

Then, $\Gamma_1^*\subset\partial[\Omega_n]_a^0$, since $a\in\mathfrak{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n)$, and, hence, $a=0$ in a neighborhood of Γ_1^* in Ω_n. Thus, $\Gamma_1^*\subset\partial\Omega_a^0$ and, therefore, Γ_1^* cannot be a component of $\partial\Omega_0$, because $\Omega_a^0\cap\Omega_0=\emptyset$. This contradiction shows (4.81). Consequently, (4.80) can be written in the form

$$0<\sigma[\mathcal{L}(\lambda),\mathcal{B}(b,[\Omega_n]_a^0),[\Omega_n]_a^0]\,,\quad n\geq m_0\,. \tag{4.82}$$

Now, suppose $\lambda\in\Lambda[\Omega_0,\mathcal{B}_0(b)]$. Then, thanks to Theorem 2.13(b) and (4.75), we find that

$$\sigma[\mathcal{L}_f(\lambda),\mathcal{B}_n(b),\Omega_n]<\sigma[\mathcal{L}_f(\lambda),\mathcal{B}_0(b),\Omega_0]<0\,,\quad n\geq 1\,,$$

and, hence, (4.82) gives

$$\sigma[\mathcal{L}_f(\lambda),\mathcal{B}_n(b),\Omega_n]<0<\sigma[\mathcal{L}(\lambda),\mathcal{B}(b,[\Omega_n]_a^0),[\Omega_n]_a^0]\,,\quad n\geq m_0\,.$$

Therefore, thanks to Theorem 2.13(b), (4.69) is satisfied. This completes the proof of Part (c).

(d) Since $\Omega_a^0\cap\Omega_0\neq\emptyset$ is of class $\mathcal{C}^2$ and

$$\partial\Omega_0\cap\Omega\cap\partial(\Omega_a^0\cap\Omega_0)=\partial\Omega_0\cap\Omega\cap\bar{\Omega}_a^0\,,$$

it follows from Theorem 3.1(a) that

$$a\in\mathfrak{A}_{\Gamma_0^0,\Gamma_1}(\Omega_0)\,,\quad [\Omega_0]_a^0=\Omega_a^0\cap\Omega_0\,.$$

Moreover, since $\Omega_a^0\cap\Omega_n$ is a proper subdomain of Ω of class $\mathcal{C}^2$ and $\Omega_n\to\Omega_0$ from the exterior, as $n\to\infty$, it is easy to see that

$$\lim_{n\to\infty}\Omega_a^0\cap\Omega_n=\Omega_a^0\cap\Omega_0 \tag{4.83}$$

from the exterior. Furthermore, since $\Omega_a^0\cap\Omega_0\neq\emptyset$, there exists $m_0\geq n_0$ such that $\Omega_a^0\cap\Omega_n\neq\emptyset$ for each $n\geq m_0$. Therefore, thanks again to Theorem 3.1(a),

$$a\in\mathfrak{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n)\,,\quad [\Omega_n]_a^0=\Omega_a^0\cap\Omega_n\,,\quad n\geq m_0\,.$$

This completes the proof of (4.70).

Now, suppose $\lambda\in\Lambda[\Omega_0,\mathcal{B}_0(b)]$. Then, thanks to Theorem 2.13(a),

$$\sigma[\mathcal{L}_f(\lambda),\mathcal{B}_0(b),\Omega_0]<0<\sigma[\mathcal{L}(\lambda),\mathcal{B}(b,[\Omega_0]_a^0),[\Omega_0]_a^0]\,. \tag{4.84}$$

Thus, thanks to (4.75), (4.70) and (4.84), for each $n \geq m_0$ we have that

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}_n(b), \Omega_n] < 0 < \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, [\Omega_0]_a^0), [\Omega_0]_a^0]\,.$$

Moreover, thanks to (4.83), it follows from Theorem 2.10 that

$$\lim_{n\to\infty} \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, [\Omega_n]_a^0), [\Omega_n]_a^0] = \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, [\Omega_0]_a^0), [\Omega_0]_a^0]\,.$$

Therefore, $\ell_0 \geq m_0$ exists for which

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}_n(b), \Omega_n] < 0 < \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, [\Omega_0]_a^0), [\Omega_0]_a^0]\,, \quad n \geq \ell_0\,,$$

and, hence, thanks to Theorem 2.13(a), (4.71) holds.
(e) Suppose $a \in \mathfrak{A}^+(\Omega)$. Then, $\Omega_a^0 = \emptyset$ and, thanks to (4.62), condition (4.72) can be easily obtained from the definition of $\mathfrak{A}^+(\Omega_n)$, $n \geq 0$. Suppose, in addition, that $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$. Then, thanks to (4.75), Theorem 2.13 implies

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}_n(b), \Omega_n] < \sigma[\mathcal{L}_f(\lambda), \mathcal{B}_0(b), \Omega_0] < 0\,, \quad n \geq 0\,.$$

Therefore, thanks again to Theorem 2.13, (4.73) is satisfied. This completes the proof of the theorem. □

5. Interior continuous dependence

In this section we analyze the continuous dependence of the positive solutions of (1.1) respect to interior perturbations of the domain Ω around its Dirichlet boundary Γ_0 in the special case when ∂_ν is the conormal derivative with respect to $\mathcal{L}$. So, for the remaining of this section we assume (2.8). As in Section 4, we will refer to (1.1) as problem $P[\lambda, \Omega, \mathfrak{B}(b)]$. Also, we will denote by $\Lambda[\Omega, \mathcal{B}(b)]$ the set of values of $\lambda \in \mathbb{R}$ for which $P[\lambda, \Omega, \mathcal{B}(b)]$ possesses a positive solution.

The following result will provide us with the *interior continuous dependence* of the positive solutions of $P[\lambda, \Omega, \mathcal{B}(b)]$.

Theorem 5.1. *Suppose* (2.8)*. Let Ω_0 be a proper subdomain of Ω with boundary of class $\mathcal{C}^2$ such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1\,, \quad \Gamma_0^0 \cap \Gamma_1 = \emptyset\,,$$

where Γ_0^0 satisfies the same requirements as Γ_0, and let $\Omega_n \subset \Omega$, $n \geq 1$, be a sequence of bounded domains of $\mathbb{R}^N$ of class $\mathcal{C}^2$ converging to Ω_0 from its interior. For each $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{B}_n(b)$ denote the boundary operator defined by

$$\mathcal{B}_n(b)u := \begin{cases} u & \text{on } \Gamma_0^n \\ \partial_\nu u + bu & \text{on } \Gamma_1 \end{cases} \tag{5.1}$$

where $\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1$, $n \in \mathbb{N} \cup \{0\}$. Suppose in addition that

$$a \in \mathfrak{A}(\Omega_0)\,, \quad \lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$$

and that $n_0 \in \mathbb{N}$ exists for which

$$a \in \bigcap_{n=n_0}^{\infty} \mathfrak{A}(\Omega_n)\,, \quad \lambda \in \bigcap_{n=n_0}^{\infty} \Lambda[\Omega_n, \mathcal{B}_n(b)]\,. \tag{5.2}$$

For each $n \geq 0$, let u_n denote the unique positive solution of $P[\lambda, \Omega_n, \mathcal{B}_n(b)]$; it should be noted that the uniqueness is guaranteed by Theorem 2.13. Then,

$$\lim_{n\to\infty} \|\tilde{u}_n - u_0\|_{H^1(\Omega_0)} = 0 \tag{5.3}$$

where

$$\tilde{u}_n := \begin{cases} u_n & \textit{in } \Omega_n \\ 0 & \textit{in } \Omega_0 \setminus \Omega_n \end{cases} \qquad n \geq 1\,. \tag{5.4}$$

Proof. Suppose (5.2). Then, thanks to Theorem 2.13, the problem $P[\lambda, \Omega_n, \mathcal{B}_n(b)]$, $n \geq n_0$, has a unique positive solution, denoted in the sequel by u_n. Moreover, thanks to Lemma 2.12,

$$u_n \in W^2_{\mathcal{B}_n(b)}(\Omega_n) \subset H^2(\Omega_n)\,, \quad n \geq n_0\,,$$

and u_n is strongly positive in Ω_n. Since $u_n \in H^1(\Omega_n)$ and $u_n = 0$ on Γ_0^n, we have that $\tilde{u}_n \in H^1(\Omega_0)$ and

$$\|\tilde{u}_n\|_{H^1(\Omega_0)} = \|u_n\|_{H^1(\Omega_n)}\,, \quad n \geq n_0\,. \tag{5.5}$$

Moreover, since u_n is strongly positive in Ω_n, $\Gamma_1 = \partial\Omega_n \setminus \Gamma_0^n$ for each $n \geq 0$ and $\Omega_n \subset \Omega_{n+1} \subset \Omega_0$, $n \in \mathbb{N}$, it is easily seen that for $n \geq n_0$,

$$\begin{gathered} \mathcal{L}u_{n+1} = \lambda W u_{n+1} - af(\cdot, u_{n+1})u_{n+1} \quad \text{in } \Omega_n \\ \mathcal{B}_n(b)u_{n+1} \geq 0 \quad \text{on } \partial\Omega_n \end{gathered}$$

and

$$\begin{gathered} \mathcal{L}u_0 = \lambda W u_0 - af(\cdot, u_0)u_0 \quad \text{in } \Omega_n \\ \mathcal{B}_n(b)u_0 \geq 0 \quad \text{on } \partial\Omega_n\,. \end{gathered}$$

Thus, for each $n \geq n_0$ the function u_{n+1} is a positive supersolution of the problems $P[\lambda, \Omega_n, \mathcal{B}_n(b)]$ and u_0 is a positive supersolution of $P[\lambda, \Omega_n, \mathcal{B}_n(b)]$. Hence, thanks to Theorem 2.15, we find that

$$u_{n+1}|_{\Omega_n} \geq u_n > 0\,, \quad u_0|_{\Omega_n} \geq u_n > 0\,, \quad n \geq n_0\,.$$

Therefore, in Ω_0 we have that

$$0 < \tilde{u}_{n_0} \leq \tilde{u}_n \leq \tilde{u}_{n+1} \leq u_0\,, \quad n \geq n_0\,. \tag{5.6}$$

Now, setting $M := \|u_0\|_{L_\infty(\Omega_0)}$, it follows from (5.6) that

$$\|\tilde{u}_n\|_{L_\infty(\Omega_0)} \leq M\,, \quad n \geq n_0\,. \tag{5.7}$$

Now, changing Ω by Ω_0, the proof of Theorem 4.1 can be easily adapted to show that there exist $u \in H^1(\Omega_0)$ and a subsequence of $\tilde{u}_n$, $n \geq n_0$, labeled again by n, such that

$$\lim_{n\to\infty} \|\tilde{u}_n - u\|_{H^1(\Omega_0)} = 0\,.$$

Since $\tilde{u}_n \in H^1_{\Gamma_0^0}(\Omega_0)$, $n \geq n_0$, Theorem 2.7 implies $u \in H^1_{\Gamma_0^0}(\Omega_0)$. Moreover, it is easily seen that u provides us with a weak positive solution of $P[\lambda, \Omega_0, \mathcal{B}_0(b)]$. Since u can be regarded as a principal eigenfunction for a second order elliptic operator, u provides us with a positive solution of $P[\lambda, \Omega_0, \mathcal{B}_0(b)]$. Thus, thanks to the uniqueness of u_0, $u = u_0$. As the previous argument works out along any subsequence of $\tilde{u}_n$, $n \geq n_0$, the proof of the theorem is completed. □

The following result provides us with some sufficient conditions ensuring that condition (5.2) is satisfied. Therefore, under these conditions the conclusion of Theorem 5.1 is satisfied.

Theorem 5.2. *Suppose* (2.8)*. Let* Ω_0 *a proper subdomain of* Ω *with boundary of class* $\mathcal{C}^2$ *such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1 , \quad \Gamma_0^0 \cap \Gamma_1 = \emptyset ,$$

where Γ_0^0 *satisfies the same requirements as* Γ_0*, and let* Ω_n*,* $n \geq 1$ *be a sequence of bounded domains of* $\mathbb{R}^N$ *of class* $\mathcal{C}^2$ *converging to* Ω_0 *from its interior and satisfying* (4.62)*. For each* $n \geq 0$ *let* $\mathcal{B}_n(b)$ *denote the boundary operator defined by* (5.1)*. Then, the following assertions are true:*

(*a*) *Suppose* $\Omega_a^0 \cap \Omega_0 = \emptyset$ *and* $\Gamma \cap K \neq \emptyset$ *implies* $\Gamma \setminus K \subset \Omega_a^+$ *for any component* Γ *of* Γ_0^0*. Then,*

$$a \in \bigcap_{n=0}^{\infty} \mathfrak{A}^+_{\Gamma_0^n,\Gamma_1}(\Omega_n) . \tag{5.8}$$

Moreover, if $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$*, then there exists* $n_0 \geq 1$ *such that*

$$\lambda \in \bigcap_{n=n_0}^{\infty} \Lambda[\Omega_n, \mathcal{B}_n(b)] . \tag{5.9}$$

(*b*) *Suppose* $\Omega_0 \cap \Omega_a^0 \neq \emptyset$ *is of class* $\mathcal{C}^2$*,* $n_0 \in \mathbb{N}$ *exists such that* $\Omega_a^0 \cap \Omega_n$ *is of class* $\mathcal{C}^2$ *if* $n \geq n_0$*, and* (3.2) *is satisfied for any* $\tilde{\Omega} \in \{\Omega_0, \Omega_{n_0+j} : j \geq 0\}$*. Then,*

$$a \in \mathfrak{A}_{\Gamma_0^0,\Gamma_1}(\Omega_0) , \quad [\Omega_0]_a^0 = \Omega_a^0 \cap \Omega_0 , \tag{5.10}$$

and $m_0 \geq n_0$ *exists for which*

$$a \in \bigcap_{n=m_0}^{\infty} \mathfrak{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n) , \quad [\Omega_n]_a^0 = \Omega_a^0 \cap \Omega_n , \quad n \geq m_0 . \tag{5.11}$$

Moreover, if, in addition, $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$*, then*

$$\lambda \in \bigcap_{n=\ell_0}^{\infty} \Lambda[\Omega_n, \mathcal{B}_n(b)] \tag{5.12}$$

for some $\ell \geq m_0$*.*

Thanks to Theorem 5.1, in any of these cases we have that

$$\lim_{n\to\infty} \|\tilde{u}_n - u_0\|_{H^1(\Omega_0)} = 0 , \tag{5.13}$$

where $\tilde{u}_n$ *is the extension to* Ω_0 *defined by* (5.4) *and* u_0 *is the unique positive solution of the problem* $P[\lambda, \Omega_0, \mathcal{B}_0(b)]$*.*

Proof. Once proven parts (a) and (b), the relation (5.13) follows as a straightforward consequence from Theorem 5.1. Without lost of generality we can assume that Ω_n is a proper subset of Ω_{n+1} for each $n \geq 1$. Then, Ω_n is a proper subset of Ω_0 for any $n \geq 1$. Now, we proceed to prove each part of the theorem separately.
(a)]; Thanks to Theorem 3.1(b), $a \in \mathfrak{A}^+_{\Gamma_0^0,\Gamma_1}(\Omega_0)$. Moreover, since $\lim_{n\to\infty} \Omega_n = \Omega_0$ from its interior,

$$\Omega_a^0 \cap \Omega_n = \emptyset$$

for each $n \geq 1$. Furthermore, if Γ is a component of $\partial\Omega_n \cap \Omega$ for which $\Gamma \cap K \neq \emptyset$, then it follows from (1.10) that

$$\Gamma \setminus K \subset \Omega_0 \setminus K \subset \Omega_a^+ .$$

Therefore, Theorem 3.1(b) implies $a \in \mathfrak{A}^+_{\Gamma_0^n,\Gamma_1}(\Omega_n)$, $n \geq 1$. This completes the proof of (5.8).

Now, suppose $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$. Then, thanks to Theorem 2.13(b),

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}(b, \Omega_0), \Omega_0] < 0\,. \tag{5.14}$$

Moreover, thanks to Theorem 2.9,

$$\lim_{n\to\infty} \sigma[\mathcal{L}_f(\lambda), \mathcal{B}(b, \Omega_n), \Omega_n] = \sigma[\mathcal{L}_f(\lambda), \mathcal{B}(b, \Omega_0), \Omega_0]\,. \tag{5.15}$$

Thus, thanks to (5.14) and (5.15), $n_0 \geq 1$ exists for which

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}(b, \Omega_n), \Omega_n] < 0 \quad \text{if} \quad n \geq n_0\,.$$

This completes the proof of (5.9).

(b) Condition (5.10) follows from Theorem 3.1(b). Moreover, since $\lim_{n\to\infty} \Omega_n = \Omega_0$ from its interior, $\Omega_n \cap \Omega_a^0 \neq \emptyset$ for large enough $n \geq 1$. Thus, $m_0 \geq n_0$ exists for which $\Omega_n \cap \Omega_a^0 \neq \emptyset$ is of class $\mathcal{C}^2$ for each $n \geq m_0$. Therefore, thanks to Theorem 3.1(b), (5.11) is satisfied. In particular, each of the principal eigenvalues

$$\sigma[\mathcal{L}(\lambda), \mathcal{B}(b, [\Omega_n]_a^0), [\Omega_n]_a^0]\,, \quad n \geq m_0\,,$$

is well defined. Now, suppose $\lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]$. Then, thanks to Theorem 2.13(a),

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}(b, \Omega_0), \Omega_0] < 0 < \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, [\Omega_0]_a^0), [\Omega_0]_a^0]\,. \tag{5.16}$$

Moreover, since $\lim_{n\to\infty} \Omega_n = \Omega_0$ from its interior,

$$\lim_{n\to\infty} [\Omega_n]_a^0 = [\Omega_0]_a^0$$

from its interior. Hence, Theorem 2.9 implies

$$\lim_{n\to\infty} \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, [\Omega_n]_a^0), [\Omega_n]_a^0] = \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, [\Omega_0]_a^0), [\Omega_0]_a^0]\,. \tag{5.17}$$

Thus, thanks to (5.16) and (5.17), $\ell_0 \geq m_0$ exists for which

$$\sigma[\mathcal{L}_f(\lambda), \mathcal{B}(b, \Omega_n), \Omega_n] < 0 < \sigma[\mathcal{L}(\lambda), \mathcal{B}(b, [\Omega_n]_a^0), [\Omega_n]_a^0]$$

if $n \geq \ell_0$. Therefore, thanks to Theorem 2.13(a), the proof of (5.12) is completed.

As already mentioned above, (5.13) follows from Theorem 5.1. This completes the proof. □

6. Continuous dependence

As an easy consequence from Theorems 4.1 and 5.1 the next result follows.

Theorem 6.1. *Suppose* (2.8). *Let Ω_0 be a proper subdomain of Ω with boundary of class $\mathcal{C}^2$ such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1\,, \quad \Gamma_0^0 \cap \Gamma_1 = \emptyset\,,$$

where Γ_0^0 satisfies the same requirements as Γ_0, and let $\Omega_n \subset \Omega$, $n \geq 1$, be a sequence of bounded domains of $\mathbb{R}^N$ of class $\mathcal{C}^2$ converging to Ω_0.

Let Ω_n^I and Ω_n^E, $n \geq 1$, two sequences of bounded domains in Ω such that Ω_n^I, $n \geq 1$, converges to Ω_0 from the interior, Ω_n^E, $n \geq 1$, converges to Ω_0 from the exterior and

$$\Omega_n^I \subset \Omega_0 \cap \Omega_n\,, \quad \Omega_0 \cup \Omega_n \subset \Omega_n^E\,, \quad n \geq 1\,.$$

For each $\tilde\Omega \in \{\,\Omega_0\,, \Omega_n\,, \Omega_n^I\,, \Omega_n^E : n \geq 1\,\}$ let $\mathcal{B}(b, \tilde\Omega)$ denote the boundary operator defined by

$$\mathcal{B}(b, \tilde\Omega)u := \begin{cases} u & \text{on } \partial\tilde\Omega \setminus \Gamma_1\,, \\ \partial_\nu u + bu & \text{on } \Gamma_1\,. \end{cases} \tag{6.1}$$

Suppose, in addition, that

$$a \in \mathfrak{A}_{\Gamma_0^0,\Gamma_1}(\Omega_0)\,, \quad \lambda \in \Lambda[\Omega_0, \mathcal{B}_0(b)]\,, \tag{6.2}$$

and that there exists $n_0 \geq 1$ *such that*

$$a \in \bigcap_{n=n_0}^{\infty} \left[\mathfrak{A}_{\partial\Omega_n\setminus\Gamma_1,\Gamma_1}(\Omega_n) \cap \mathfrak{A}_{\partial\Omega_n^I\setminus\Gamma_1,\Gamma_1}(\Omega_n^I) \cap \mathfrak{A}_{\partial\Omega_n^E\setminus\Gamma_1,\Gamma_1}(\Omega_n^E)\right] \tag{6.3}$$

and

$$\lambda \in \bigcap_{n=n_0}^{\infty} \left(\Lambda[\Omega_n, \mathcal{B}(b,\Omega_n)] \cap \Lambda[\Omega_n^I, \mathcal{B}(b,\Omega_n^I)] \cap \Lambda[\Omega_n^E, \mathcal{B}(b,\Omega_n^E)]\right)\,. \tag{6.4}$$

Let u_0 *denote the unique positive solution of* $P[\lambda, \Omega_0, \mathcal{B}(b,\Omega_0)]$ *and for each* $n \geq n_0$ *let* u_n, u_n^I, u_n^E *denote the unique positive solutions of*

$$P[\lambda, \Omega_n, \mathcal{B}(b,\Omega_n)]\,, \quad P[\lambda, \Omega_n^I, \mathcal{B}(b,\Omega_n^I)]\,, \quad P[\lambda, \Omega_n^E, \mathcal{B}(b,\Omega_n^E)]\,,$$

respectively. Now, for $n \geq 1$*, set*

$$\tilde{u}_n^I := \begin{cases} u_n & \text{in } \Omega_n^I \\ 0 & \text{in } \Omega_0 \setminus \Omega_n^I, \end{cases} \tag{6.5}$$

$$\tilde{u}_n := \begin{cases} u_n & \text{in } \Omega_n \\ 0 & \text{in } \Omega \setminus \Omega_n \end{cases} \qquad n \geq 1\,. \tag{6.6}$$

Then,

$$\lim_{n\to\infty} \|u_n^E|_{\Omega_0} - u_0\|_{H^1(\Omega_0)} = 0\,, \quad \lim_{n\to\infty} \|\tilde{u}_n^I - u_0\|_{H^1(\Omega_0)} = 0\,, \tag{6.7}$$

$$\tilde{u}_n^I \leq u_0 \leq u_n^E|_{\Omega_0}\,, \quad \tilde{u}_n^I \leq \tilde{u}_n|_{\Omega_0} \leq u_n^E|_{\Omega_0}\,, \quad \text{in } \Omega_0\,, \quad n \geq n_0\,. \tag{6.8}$$

Therefore, for each $p \in [1,\infty)$*,*

$$\lim_{n\to\infty} \|\tilde{u}_n|_{\Omega_0} - u_0\|_{L_p(\Omega_0)} = 0\,. \tag{6.9}$$

Proof. Relations (6.7) follow straight away from Theorem 4.1 and Theorem 5.1. Relations (6.8) follow very easily combining the uniqueness of the positive solutions with Theorem 2.15. Now, thanks to (6.7) and (6.8),

$$\lim_{n\to\infty} \|\tilde{u}_n|_{\Omega_0} - u_0\|_{L_2(\Omega_0)} = 0\,, \tag{6.10}$$

and, hence,

$$\lim_{n\to\infty} \|\tilde{u}_n|_{\Omega_0} - u_0\|_{L_p(\Omega_0)} = 0\,, \quad 1 \leq p \leq 2\,. \tag{6.11}$$

On the other hand, arguing as in beginning of the proof of Theorem 4.1, we find from (6.8) and Theorem 2.15 that

$$\tilde{u}_n|_{\Omega_0} \leq u_n^E|_{\Omega_0} \leq u_{n_0}^E|_{\Omega_0}\,, \quad n \geq n_0\,. \tag{6.12}$$

Thus, setting $M := \|u_{n_0}^E\|_{L_\infty(\Omega_0)}$, (6.12) implies that

$$\|\tilde{u}_n\|_{L_\infty(\Omega_0)} \leq M\,, \quad n \geq n_0\,.$$

Finally, combining this uniform estimate with (6.10) gives

$$\lim_{n\to\infty} \|\tilde{u}_n|_{\Omega_0} - u_0\|_{L_p(\Omega_0)} = 0\,, \quad 2 \leq p < \infty\,.$$

This completes the proof of the theorem. □

Adapting the proofs of Theorem 4.2 and Theorem 5.2 one can easily obtain rather simple conditions on a and the Ω_n's so that (6.2) imply (6.3) and (6.4).

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Santiago Cano-Casanova
Departamento de Matemática Aplicada y Computación, Universidad Pontificia Comillas de Madrid, 28015-Madrid, Spain
E-mail address: scano@dmc.icai.upco.es

Julián López-Gómez
Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040-Madrid, Spain
E-mail address: Lopez_Gomez@mat.ucm.es