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IMPROVED LIFESPAN OF SOLUTIONS TO AN INVISCID SURFACE QUASI-GEOSTROPHIC MODEL

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Abstract. This article consider the two-dimensional (2D) inviscid surface quasi-geostrophic (SQG) model. By studying the decay estimate of the operator $e^{\mathcal{R}_1^2 t}$, we obtain an improved lifespan of the solutions to the corresponding model. More precisely, if the initial data is of size ϵ, then the lifespan satisfies $T_\epsilon \simeq \epsilon^{-4/3}$, which improves the result obtained via hyperbolic methods.

1. Introduction

The classical 2D inviscid surface quasi-geostrophic (SQG) equation has the form

$$\begin{gathered} \partial_t\theta + (u\cdot\nabla)\theta = 0, \\ u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \\ \theta(x,0) = \theta_0(x), \end{gathered} \tag{1.1}$$

where the real scalar function $\theta = \theta(x,t)$ represents the potential temperatures of the fluid, and

$$\mathcal{R}_1 := \partial_{x_1}\Lambda^{-1}, \mathcal{R}_2 := \partial_{x_2}\Lambda^{-1} \quad (\Lambda := (-\Delta)^{1/2})$$

are the standard 2D Riesz transforms. The SQG equation is an important model in geophysical fluid dynamics. In particular, it is the special case of the general quasi-geostrophic approximations for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers, see [7, 16] and the references cited there. Mathematically, as pointed out by Constantin, Majda and Tabak [7], the inviscid SQG equation shares many parallel properties with those of the 3D Euler equations such as the vortex-stretching mechanism and thus serves as a lower-dimensional model of the 3D Euler equations.

The mathematical study on the SQG equation is divided into two major cases. The first case is the dissipative SQG equation with fractional Laplacian, namely (1.1) adding Λ^α, which has recently attracted enormous attention and significant progress has been made on the global well-posedness issue. The global regularity problem for the SQG equation with subcritical ($\alpha > 1$) can be found in [9, 17] Constantin, Córdoba and Wu in [6] first addresses the global regularity issue for the critical case ($\alpha = 1$) and obtained a small data global existence result. Since then, small data global existence results have been obtained in various functional

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settings (see, e.g., [10, 5, 4]). Recently, the global regularity without small condition for the SQG equation with critical dissipation has been successfully resolved (see, e.g., [1, 8, 14, 15]). The global regularity issue for the supercritical case ($0 < \alpha < 1$) remains outstandingly open (see [11, 18] for the eventual regularity).

The second case is the inviscid case (1.1) which is probably the simplest dynamical scalar equation, however, the global regularity problem still remains open. The local well-posedness and blow-up criterion of (1.1) were first established in the Sobolev spaces by Constantin, Majda and Tabak [7]. Subsequently, there are various results available in different function spaces (see for instance [19, 3, 17, 21]). We remark that aside from local well-posedness and breakdown criteria not much is known about the well-posedness of the inviscid SQG equation. As a matter of fact, the global small data well-posednes result of the inviscid case is also an unsolved problem. Recently, Wu-Xu-Ye [20] established the global smooth solutions to the damped SQG equation with small initial data.

Cannone, Miao, and Xue [2] proved the existence of global strong solutions to the following dispersive SQG equation with supercritical dissipation

$$\partial_t\theta + (u\cdot\nabla)\theta + \Lambda^\alpha\theta + K\mathcal{R}_1\theta = 0, \quad 0 < \alpha < 1.$$

More precisely, they show that for given initial data θ_0 there exists K large enough such that the solution is global. Very recently, Elgindi and Widmayer [13] considered the following dispersive SQG equation (without any dissipation)

$$\begin{gathered}
\partial_t\theta + (u\cdot\nabla)\theta = \mathcal{R}_1\theta, \\
u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \\
\theta(x,0) = \theta_0(x).
\end{gathered} \tag{1.2}$$

By studying the anisotropic linear semigroup $e^{\mathcal{R}_1 t}$ and using the stationary phase lemma, the lifespan of the above system (1.2) was given in [13] as follows

$$T_\epsilon \simeq \epsilon^{-4/3}. \tag{1.3}$$

Let us also point out that Elgindi [12] considered the following inviscid porous medium equation

$$\begin{gathered}
\partial_t\theta + (u\cdot\nabla)\theta = 0, \\
u = -\nabla p - \theta e_2, \\
\nabla\cdot u = 0, \\
\theta(x,0) = \theta_0(x).
\end{gathered} \tag{1.4}$$

By classifying all stationary solutions of the inviscid porous medium equation under mild conditions, he proved that sufficiently regular perturbations which are also small must be globally regular and strongly converge to a steady state.

In this article, we consider system (1.2) when replacing $\mathcal{R}_1\theta$ by $\mathcal{R}_1^2\theta$. More precisely, we consider

$$\begin{gathered}
\partial_t\theta + (u\cdot\nabla)\theta = \mathcal{R}_1^2\theta, \\
u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \\
\theta(x,0) = \theta_0(x).
\end{gathered} \tag{1.5}$$

By establishing the decay estimate of the operator $e^{\mathcal{R}_1^2 t}$ (see Lemma 2.1), we also obtain the lifespan (1.3) for the above system (1.5). Precisely, the main result can be stated as follows.

Theorem 1.1. *Let $s_1 > 4$, $s_2 > 3$ and ϵ be a sufficiently small positive constant. If $\|\theta_0\|_{H^{s_1}} + \|\theta_0\|_{W^{s_2,1}} \le \epsilon$, then there exists a unique solution $\theta \in C([0, T_\epsilon]; H^{s_1}(\mathbb{R}^2))$ of the system* (1.5)*, where T_ϵ satisfies*

$$T_\epsilon \simeq \epsilon^{-4/3}.$$

Moreover,

$$\|\theta(t)\|_{H^{s_1}} \lesssim \epsilon, \quad \forall t \in [0, T].$$

It is well-known that by using hyperbolic methods, it is easy to get the maximal existence time T_ϵ with $T_\epsilon \ge \frac{C}{\epsilon}$. Thus, it is clear that Theorem 1.1 improves this result.

Remark 1.2. Theorem 1.1 still holds for the system

$$\begin{gathered} \partial_t\theta + (u \cdot \nabla)\theta = \mathcal{R}_2^2\theta, \\ u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \\ \theta(x, 0) = \theta_0(x). \end{gathered} \tag{1.6}$$

The proof is the same as that of Theorem 1.1.

2. Proof of Theorem 1.1

We first apply $(I + \Lambda)^s$ $(s > 0)$ to the equation (1.5) and multiply the resultant by $(I + \Lambda)^s\theta$, add them up to to conclude that

$$\frac{d}{dt}\|\theta(t)\|_{H^s}^2 + \|\mathcal{R}_1\theta\|_{H^s}^2 = -\int (I + \Lambda)^s(u \cdot \nabla\theta)(I + \Lambda)^s\theta\, dx, \tag{2.1}$$

where we have used

$$\int (I + \Lambda)^s\mathcal{R}_1^2\theta(I + \Lambda)^s\theta\, dx = -\int (I + \Lambda)^s\mathcal{R}_1\theta\,(I + \Lambda)^s\mathcal{R}_1\theta\, dx = \|\mathcal{R}_1\theta\|_{H^s}^2.$$

Thanks to the Kato-Ponce inequality and the divergence condition, we have

$$\begin{aligned} \int (I + \Lambda)^s(u \cdot \nabla\theta)(I + \Lambda)^s\theta\, dx &= \int [(I + \Lambda)^s,\, u \cdot \nabla]\theta(I + \Lambda)^s\theta\, dx \\ &\le C(\|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty})\|\theta\|_{H^s}^2. \end{aligned} \tag{2.2}$$

Putting (2.2) into (2.1) yields

$$\frac{d}{dt}\|\theta(t)\|_{H^s}^2 + \|\mathcal{R}_1\theta\|_{H^s}^2 \le C(\|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty})\|\theta\|_{H^s}^2.$$

By integrating the above inequality in time, we obtain

$$\|\theta(t)\|_{H^s}^2 + \int_0^t \|\mathcal{R}_1\theta(\tau)\|_{H^s}^2\, d\tau \le \|\theta_0\|_{H^s}^2 e^{C\int_0^t(\|\nabla u(\tau)\|_{L^\infty} + \|\nabla\theta(\tau)\|_{L^\infty})\, d\tau}. \tag{2.3}$$

Next, our goal is to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla\theta\|_{L^\infty}$ at the right-hand side of (2.3). To this end, we apply the Duhamel principle to the first equation of the system (1.5) to show

$$\theta(x, t) = e^{\mathcal{R}_1^2 t}\theta_0 - \int_0^t e^{\mathcal{R}_1^2(t-\tau)}(u \cdot \nabla\theta)(\tau)\, d\tau. \tag{2.4}$$

The following lemma concerns the decay estimate of the operator $e^{\mathcal{R}_1^2 t}$, which is a key component in proving our main result.

Lemma 2.1. *For any $\rho > 2$, it holds*

$$\|e^{\mathcal{R}_1^2 t} f\|_{L^\infty} \le C(1+t)^{-1/2}\|f\|_{W^{\rho,1}}, \tag{2.5}$$

$$\|e^{\mathcal{R}_1^2 t}\mathcal{R}_1 f\|_{L^\infty} \le C(1+t)^{-1/2}\|f\|_{W^{\rho,1}}, \tag{2.6}$$

$$\|e^{\mathcal{R}_1^2 t}\mathcal{R}_2 f\|_{L^\infty} \le C(1+t)^{-1/2}\|f\|_{W^{\rho,1}}. \tag{2.7}$$

Proof of Lemma 2.1. We first prove (2.5). Using the polar coordinates $\xi_1 = r\cos\alpha$, $\xi_2 = r\sin\alpha$, we get the estimate

$$\begin{aligned}\|e^{\mathcal{R}_1^2 t} f\|_{L^\infty} &\le \|e^{-\frac{\xi_1^2}{|\xi|^2}t}\widehat{f}(\xi)\|_{L^1}\\ &\le C\int_0^{2\pi} e^{-t\cos^2\alpha}\underbrace{\int_0^\infty |\widehat{f}(\xi)| r\,dr}_{N}\,d\alpha.\end{aligned}$$

Now we show that N can be bounded by

$$N \le C\|f\|_{W^{\rho,1}}.$$

As a matter of fact, it is not hard to see that

$$\begin{aligned}N &= \int_0^\infty \frac{|(1+|\xi|)^\rho \widehat{f}(\xi)| r}{(1+|\xi|)^\rho}\,dr\\ &= \int_0^\infty \frac{|\widehat{(1+\Lambda)^\rho f}(\xi)| r}{(1+r)^\rho}\,dr\\ &\le C\int_0^\infty \frac{\|(1+\Lambda)^\rho f\|_{L^1}\, r}{(1+r)^\rho}\,dr\\ &\le C\|f\|_{W^{\rho,1}}\int_0^\infty \frac{r}{(1+r)^\rho}\,dr \le C\|f\|_{W^{\rho,1}}.\end{aligned}$$

It directly gives

$$\|e^{\mathcal{R}_1^2 t} f\|_{L^\infty} \le C\|f\|_{W^{\rho,1}}\int_0^{2\pi} e^{-t\cos^2\alpha} d\alpha. \tag{2.8}$$

By the simple calculation, we get

$$\begin{aligned}\int_0^{2\pi} e^{-t\cos^2\alpha} d\alpha &= 4\int_0^{\pi/2} e^{-t\cos^2\alpha} d\alpha\\ &= 4\int_0^{\pi/4} e^{-t\cos^2\alpha} d\alpha + 4\int_{\pi/4}^{\pi/2} e^{-t\cos^2\alpha} d\alpha\\ &\le 4\int_0^{\pi/4} e^{-t(\frac{\sqrt{2}}{2})^2} d\alpha + 4\sqrt{2}\int_{\pi/4}^{\pi/2} e^{-t\cos^2\alpha}\sin\alpha d\alpha\\ &\le \pi e^{-t/2} - 4\sqrt{2}t^{-1/2}\int_{\pi/4}^{\pi/2} e^{-t\cos^2\alpha} d(t^{1/2}\cos\alpha)\\ &\le C(1+t)^{-1/2}.\end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9) implies the desired estimate (2.5). Noting the simple fact that $|\sin\alpha|$, $|\cos\alpha| \le 1$, the estimates (2.6) and (2.7) follow directly from the proof of (2.6). Thus, the proof is complete. $\square$

The estimate (2.6) can be improved as

$$\|e^{\mathcal{R}_1^2 t}\mathcal{R}_1 f\|_{L^\infty} \le C(1+t)^{-1}\|f\|_{W^{\rho,1}}.$$

The estimate (2.6) itself would suffice our purpose.

If we denote $L_1 = \nabla\nabla^\perp\Lambda^{-1}$ and $L_2 = \nabla$, then $L_1\theta = \nabla u$ and $L_2\theta = \nabla\theta$. Applying L_i $(i = 1,2)$ to the first equation of the system (1.5) and using the Duhamel principle yield

$$L_i\theta(x,t) = e^{\mathcal{R}_1^2 t}L_i\theta_0 - \int_0^t e^{\mathcal{R}_1^2(t-\tau)}L_i(u\cdot\nabla\theta)(\tau)\,d\tau. \tag{2.10}$$

Recalling the above mentioned estimates of Lemma 2.1 and invoking some simple embedding allows us to deduce

$$\begin{aligned}
&\|L_i\theta\|_{L^\infty}\\
&\le \|e^{\mathcal{R}_1^2 t}L_i\theta_0\|_{L^\infty} + \int_0^t \|e^{\mathcal{R}_1^2(t-\tau)}L_i(u\cdot\nabla\theta)(\tau)\|_{L^\infty}\,d\tau\\
&\le C(1+t)^{-1/2}\|\nabla\theta_0\|_{W^{s_2-1,1}} + C\int_0^t (1+t)^{-1/2}\|\nabla(u\cdot\nabla\theta)(\tau)\|_{W^{s_1-2,1}}\,d\tau\\
&\le C(1+t)^{-1/2}\|\theta_0\|_{W^{s_2,1}} + C\int_0^t (1+t)^{-1/2}\|(u\cdot\nabla\theta)(\tau)\|_{W^{s_1-1,1}}\,d\tau\\
&\le C(1+t)^{-1/2}\|\theta_0\|_{W^{s_2,1}} + C\int_0^t (1+t)^{-1/2}\|\theta(\tau)\|_{H^{s_1}}^2\,d\tau.
\end{aligned} \tag{2.11}$$

If we assume for any $t\in[0,T]$ that $\|\theta(t)\|_{H^{s_1}} \le 2\epsilon$, then from (2.11) we deduce that

$$\|L_i\theta\|_{L^\infty} \le \widetilde{C}\epsilon t^{-1/2} + \widetilde{C}\epsilon^2 t^{1/2}, \tag{2.12}$$

where $\widetilde{C} > 0$ is an absolute constant. The above estimate together with the estimate (2.3) yields for any $t\in[0,T]$ that

$$\begin{aligned}
\|\theta(t)\|_{H^{s_1}} &\le \|\theta_0\|_{H^{s_1}}\,e^{\frac{C}{2}\int_0^t(\|\nabla u(\tau)\|_{L^\infty}+\|\nabla\theta(\tau)\|_{L^\infty})\,d\tau}\\
&\le \|\theta_0\|_{H^{s_1}}\,e^{\frac{C}{2}\int_0^t(\widetilde{C}\epsilon\tau^{-1/2}+\widetilde{C}\epsilon^2\tau^{1/2})\,d\tau}\\
&\le \|\theta_0\|_{H^{s_1}}\,e^{\frac{C}{2}(2\widetilde{C}\epsilon t^{1/2}+\frac{2\widetilde{C}}{3}\epsilon^2 t^{3/2})}.
\end{aligned} \tag{2.13}$$

Thus, if

$$\|\theta(t)\|_{H^{s_1}} \le 2\epsilon, \quad \forall\, t\in[0,T],$$

then it suffices that

$$C\widetilde{C}\epsilon T^{1/2} + \frac{C\widetilde{C}}{3}\epsilon^2 T^{3/2} = \ln 2,$$

which further implies $T \simeq \epsilon^{-4/3}$. This completes the proof of Theorem 1.1

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