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ON NEUMANN BOUNDARY VALUE PROBLEMS FOR SOME QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. We study the role played by the indefinite weight function $a(x)$ on the existence of positive solutions to the problem

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + b(x)|u|^{\gamma-2}u, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega\,, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, b changes sign, $1 < p < N$, $1 < \gamma < Np/(N-p)$ and $\gamma \neq p$. We prove that (i) if $\int_\Omega a(x)\,dx \neq 0$ and b satisfies another integral condition, then there exists some λ^* such that $\lambda^* \int_\Omega a(x)\,dx < 0$ and, for λ strictly between 0 and λ^*, the problem has a positive solution and (ii) if $\int_\Omega a(x)\,dx = 0$, then the problem has a positive solution for small λ provided that $\int_\Omega b(x)\,dx < 0$.

1. Introduction and results.

In this paper we study the existence of positive solutions of the Neumann boundary value problem

$$\begin{cases} -\Delta_p u + g(x,u) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$, where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $p > 1$, and $g(x,u)$ is a Caratheodory function.

A host of literature exists for this type of problem when $p = 2$; see, e.g., [AV], [GO1], [GO2], [G], [TA] and the references therein. Recently Li and Zhen [LZ] studied (1.1) with $p \geq 2$ and obtained some interesting results. In this paper

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we consider a special type of function $g(x,u)$ which was excluded in [LZ]. More precisely we investigate problems of the type

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u + b(x)|u|^{\gamma-2}u, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{$(1.2)_\lambda$}$$

where $a(x), b(x) \in L^\infty(\Omega)$, and $a(x)$ and $b(x)$ both may change sign. Also $1 < p$ and $1 < \gamma < p^*$, where $p^* = \infty$ if $p \geq N$ and $p^* = Np/(N-p)$ if $p < N$. Here we say a function $f(x)$ changes sign if the measures of the sets $\{x \in \Omega : f(x) > 0\}$ and $\{x \in \Omega : f(x) < 0\}$ are both positive.

We study the influence of the indefinite weight function $a(x)$ on the existence of positive solutions of $(1.2)_\lambda$. If $c_1 \geq a(x) \geq c_2 > 0$, then $\|u\|_{\lambda a} := (\int_\Omega (|\nabla u|^p - \lambda a|u|^p))^{1/p}$ defines an equivalent norm on $W^{1,p}(\Omega)$ for $\lambda < 0$. Then a standard variational method can be used to prove the existence of positive solutions to $(1.2)_\lambda$ (see the proof of Theorem 1 (ii) below). The case $-c_1 \leq a(x) \leq -c_2 < 0$ can be dealt with in the same way. The situation where $a(x)$ changes sign is more complicated because the related functional

$$I(u) = \frac{1}{p}\int (|\nabla u|^p - \lambda a|u|^p) - \frac{1}{\gamma}\int b|u|^\gamma$$

may not be coercive. Our method relies on the eigencurve theory developed in [BH1, BH2]. It turns out that the sign of the integral $\int_\Omega a$ plays an important role for the range of λ for which $(1.2)_\lambda$ has a positive solution.

To be more specific, we introduce some notations and recall some results. Consider the eigencurve problem

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u + \mu|u|^{p-2}u, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where we treat the eigenvalue μ associated with a positive eigenfunction as a function of λ. By taking

$$\mu(\lambda) := \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p - \lambda \int_\Omega a|u|^p}{\int_\Omega |u|^p}$$

we can establish the following (see, e.g., [BH1, BH2, H])

Proposition 1. *Assume that $a \in L^\infty(\Omega)$. Then $\mu(\lambda)$ is continuous and concave and $\mu(0) = 0$. If $a(x) > 0$, then $\mu(\lambda)$ is decreasing, and if $a(x) < 0$, then $\mu(\lambda)$ is increasing. Assume, now, that a changes sign in Ω. (i) If $\int_\Omega a < 0$, there exists a unique $\lambda_1^+ > 0$ such that $\mu(\lambda_1^+) = 0$ and $\mu(\lambda) > 0$ for $\lambda \in (0, \lambda_1^+)$. (ii) If $\int_\Omega a = 0$, then $\mu(0) = 0$ and $\mu(\lambda) < 0$ if $\lambda \neq 0$. (iii) If $\int_\Omega a > 0$, then there exists a unique $\lambda_1^- < 0$ such that $\mu(\lambda_1^-) = 0$ and $\mu(\lambda) > 0$ for $\lambda \in (\lambda_1^-, 0)$.*

Remark 1.1. It follows from this proposition that when a changes sign and $\int_\Omega a < 0$, the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega \end{cases}$$

has a positive eigenvalue λ_1^+ associated with a positive eigenfunction.

For a given weight function $a(x)$, we define

$$\lambda_1(a) = \begin{cases} +\infty, & \text{if } a(x) < 0, \\ \lambda_1^+, & \text{if } a \text{ changes sign and } \int_\Omega a < 0, \\ 0, & \text{if } \int_\Omega a = 0, \\ \lambda_1^-, & \text{if } a \text{ changes sign and } \int_\Omega a > 0, \\ -\infty, & \text{if } a(x) > 0, \end{cases}$$

where λ_1^+ and λ_1^- are given in Proposition 1. Let $\|\cdot\|$ denote the usual norm in $W^{1,p}(\Omega)$. When $\lambda_1(a)$ is a finite number, we choose a fixed eigenfunction $\varphi_1 > 0$ associated with $\lambda_1(a)$ and satisfying $\|\varphi_1\| = 1$. Note that if $\lambda_1(a) = 0$, then we can take $\varphi_1 \equiv 1$.

With these constructions, we have

Proposition 2. *Assume that a changes sign and $\int_\Omega a \neq 0$. Then for any λ strictly between 0 and $\lambda_1(a)$, the relation $\|u\|_{\lambda a} := (\int_\Omega(|\nabla u|^p - \lambda a|u|^p))^{1/p}$ defines an equivalent norm on $W^{1,p}(\Omega)$.*

Proof. Suppose the contrary. Then there exist $u_n \in W^{1,p}(\Omega)$ such that $\|u_n\| = 1$ and $\int_\Omega(|\nabla u_n|^p - \lambda a|u_n|^p) \to 0$. The variational characterization of $\mu(\lambda)$ then gives $\|u_n\|_{\lambda a}^p \geq \mu(\lambda)\int_\Omega |u_n|^p$. Since λ is between 0 and $\lambda_1(a)$, it follows that $\mu(\lambda) > 0$ so $u_n \to 0$ in $L^p(\Omega)$. This implies $\int_\Omega a|u_n|^p \to 0$ and hence $\int_\Omega |\nabla u_n|^p \to 0$. This contradicts the fact that $\|u_n\| = 1$. This proves the proposition. $\square$

Now we can state our main results. From now on we assume $1 < \gamma < p^*$, $\gamma \neq p$ and that a and b both change sign. We first consider the situation $\int_\Omega a \neq 0$.

Theorem 1. *Let $\int_\Omega a \neq 0$ and $\int b\varphi_1^\gamma < 0$. Then there exists a $\lambda^* \neq 0$ with $(\lambda_1(a) - \lambda^*) \cdot \int_\Omega a > 0$, such that for λ strictly between 0 and λ^*, $(1.2)_\lambda$ has a positive solution.*

The next result deals with the case where $\int a = 0$.

Theorem 2. *Assume $\int_\Omega a = 0$ and $\int b < 0$. Then for small enough $\lambda \neq 0$, $(1.2)_\lambda$ has a positive solution.*

When $\lambda = 0$, we have

Corollary 1. *Assume $\int_\Omega b < 0$. Then the problem*

$$\begin{cases} -\Delta_p u = b(x)|u|^{\gamma-2}u, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega \end{cases}$$

has a positive solution.

Throughout this paper we use c to denote various positive constants, and the integrals are always taken on Ω unless otherwise specified. We will use variational methods in a similar way to those in [DH]. The proof of Theorem 1 will be divided into three situations: (i) $\lambda = \lambda_1(a)$, (ii) $0 < |\lambda| < |\lambda_1(a)|$ and (iii) $|\lambda| > |\lambda_1(a)|$. The details are presented in Sections 2 and 3. We then study the case $\int a = 0$ in Section 4. We conclude with some remarks in Section 5.

2. The case $|\lambda| \leq |\lambda_1(a)|$.

We introduce the functional I on the space $W^{1,p}(\Omega)$ by

$$I(u) = \frac{1}{p}\int(|\nabla u|^p - \lambda a|u|^p) - \frac{1}{\gamma}\int b|u|^\gamma, \tag{2.1}$$

and we set

$$\begin{aligned}\Lambda &= \{u \in W^{1,p}(\Omega) : (I'(u), u) = 0\}\\ &= \{u \in W^{1,p}(\Omega) : \int(|\nabla u|^p - \lambda a|u|^p) = \int b|u|^\gamma\}.\end{aligned}$$

We can see that φ_1 does not belong to Λ since $\int b\varphi_1^\gamma < 0$. Note also that Λ is closed and for $u \in \Lambda$,

$$I(u) = (\frac{1}{p} - \frac{1}{\gamma})\int(|\nabla u|^p - \lambda a|u|^p) = (\frac{1}{p} - \frac{1}{\gamma})\int b|u|^\gamma. \tag{2.2}$$

The following lemma is needed.

Lemma 2.1. *Under the conditions of Theorem 1 or Theorem 2, any minimizer or maximizer z of the functional I on Λ with $I(z) \neq 0$ gives a solution of* $(1.2)_\lambda$.

Proof. If z is a nonzero maximizer or a minimizer of I on Λ, then there exists $\mu \in \mathbb{R}$ such that

$$\begin{aligned}&\int |\nabla z|^{p-2}\nabla z\nabla\varphi - \int \lambda a|z|^{p-2}z\varphi - \int b|z|^{\gamma-2}z\varphi\\ &= \mu\Big(p\int |\nabla z|^{p-2}\nabla z\nabla\varphi - p\int \lambda a|z|^{p-2}z\varphi - \gamma\int b|z|^{\gamma-2}z\varphi\Big),\end{aligned}$$

for any $\varphi \in W^{1,p}(\Omega)$. We claim that $\mu = 0$, which proves the lemma. If $\mu \neq 0$, then taking $\varphi = z$ and using the fact that $z \in \Lambda$ we get

$$(\gamma - p)\int(|\nabla z|^p - a|z|^p) = (\gamma - p)\int b|z|^\gamma = 0.$$

Since $I(z) \neq 0$, we obtain a contradiction. □

Proof of Theorem 1. (i) Here $\lambda = \lambda_1(a)$, and we start with the case $1 < \gamma < p$. We show that I satisfies the Palais-Smale condition on Λ, so we assume that $\{u_n\} \subset \Lambda$, $|I(u_n)| \leq c$ and $I'(u_n) \to 0$, and we show that $\{u_n\}$ contains a convergent subsequence.

We first prove that such $\{u_n\}$ is bounded. Suppose this is not true. Let $v_n = u_n/\|u_n\|$. Without loss of generality we may assume that $v_n \to v_0$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ and $L^\gamma(\Omega)$. We claim that $v_0 \neq 0$. Indeed, dividing $|I(u_n)| \leq c$ by $\|u_n\|^p$ yields

$$\int(|\nabla v_n|^p - \lambda a|v_n|^p) \to 0.$$

If $v_0 = 0$, similarly to the proof of Proposition 2, we have $v_n \to 0$ in $L^p(\Omega)$ and $\int|\nabla v_n|^p \to 0$. Since $\|v_n\| = 1$ and $\int|\nabla v_n|^p \to 0$, we must have $v_0 \neq 0$. This is a contradiction. The claim is proved. Thus by the variational characterization of $\lambda = \lambda_1(a)$ and the weak convergence of v_n to $v_0 \neq 0$, we have

$$0 = \mu(\lambda) \leq \int (|\nabla v_0|^p - \lambda a |v_0|^p) \leq \lim_{n\to\infty} \int (|\nabla v_n|^p - \lambda a|v_n|^p) = 0.$$

We conclude that

$$v_0 = k\varphi_1 \quad \text{for some nonzero constant } k. \tag{2.3}$$

On the other hand, dividing

$$\int (|\nabla u_n|^p - \lambda a|u_n|^p) = \int b|u_n|^\gamma \tag{2.4}$$

by $\|u_n\|^\gamma$ we obtain

$$0 \leq \|u_n\|^{p-\gamma} \int (|\nabla v_n|^p - \lambda a|v_n|^p) = \int b|v_n|^\gamma.$$

The fact that $v_n \to k\varphi_1$ strongly in $L^\gamma(\Omega)$ together with $\int b\varphi_1^\gamma < 0$ then implies that the right hand side of the above equality is negative for large n. This contradiction shows that $\{u_n\}$ is bounded.

We now can assume that $u_n \to u_0$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ and $L^\gamma(\Omega)$. Using $I'(u_n) \to 0$ we obtain

$$\begin{aligned}(I'(u_n) - I'(u_0), u_n - u_0) &= \int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)\nabla(u_n - u_0) \\ &\quad - \int \lambda a(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)(u_n - u_0) \\ &\quad - \int b(|u_n|^{\gamma-2}u_n - |u_0|^{\gamma-2}u_0)(u_n - u_0) \to 0.\end{aligned}$$

Due to the continuity of the Nemytskij operators $u \mapsto |u|^{p-2}u$ and $u \mapsto |u|^{\gamma-2}u$ from $L^p(\Omega)$ into $L^{p/(p-1)}(\Omega)$ and $L^p(\Omega)$ into $L^{\gamma/(\gamma-1)}(\Omega)$, respectively, the last two integrals approach zero. Hence, for $p' = p/(p-1)$, we have (cf. [DH])

$$\begin{aligned}&\Big\{\Big(\int |\nabla u_n|^p\Big)^{1/p'} - \Big(\int |\nabla u_0|^p\Big)^{1/p'}\Big\} \cdot \Big\{\Big(\int |\nabla u_n|^p\Big)^{1/p} - \Big(\int |\nabla u_0|^p\Big)^{1/p}\Big\} \\ &\leq \int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)\nabla(u_n - u_0) \to 0,\end{aligned}$$

i.e., $\int |\nabla u_n|^p$ converges to $\int |\nabla u_0|^p$. This together with the weak convergence of u_n to u_0 in $W^{1,p}(\Omega)$ implies that $u_n \to u_0$ strongly in $W^{1,p}(\Omega)$. Hence I satisfies the Palais-Smale condition on Λ.

We see that, for $1 < \gamma < p$, $I(u) < 0$ for $u \in \Lambda \setminus \{0\}$. We claim that I is bounded from below on Λ. If, on the contrary, there exists $u_n \in \Lambda$ such that $I(u_n) \to -\infty$, then clearly $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Dividing (2.4) by $\|u_n\|^p$ we obtain

$$\int (|\nabla v_n|^p - \lambda a|v_n|^p) = \int b|v_n|^\gamma \cdot \|u_n\|^{\gamma-p}. \tag{2.5}$$

It then follows from $\gamma < p$ and $\|v_n\| = 1$ that $\int(|\nabla v_n|^p - a|v_n|^p) \to 0$. As in the above proof of (2.3) we conclude that v_n converges weakly (and, without loss of generality, strongly in $L^\gamma(\Omega)$) to $k\varphi_1$ for some constant k. This implies that the right hand side of (2.5) is negative for large n since $\int b\varphi_1^\gamma < 0$, contradicting the variational characterization of $\lambda_1(a)$. Since any minimizer u of I on Λ now must satisfy $I(u) < 0$, we find a solution of $(1.2)_\lambda$ by Lemma 2.1. Observe that $|u|$ is also a minimizer of I on Λ. Then the Harnack inequality (cf. e.g. [TR]) implies that $u > 0$. Thus we obtain a positive solution.

For the case $\gamma > p$, let $\Lambda_0 := \Lambda \setminus \{0\}$. We note that in this case $I(u) > 0$ for $u \in \Lambda_0$. We first show that 0 is an isolated point of Λ. Suppose, for some $u_n \in \Lambda_0$, $u_n \to 0$. Let $v_n = u_n/\|u_n\|$. ¿From (2.5) and Sobolev's embedding theorem we obtain as for (2.3) that v_n converges in $L^\gamma(\Omega)$ to $k\varphi_1$ for some nonzero constant k. It then follows that the right hand side of (2.5) is negative when n is large, a contradiction. Now, since Λ is closed and 0 is isolated, Λ_0 is a closed set. Thus any minimizer of I on Λ_0 gives us a nontrivial solution. Its positivity is obtained exactly as for the case $1 < \gamma < p$.

(ii) Observe that, for the case $0 < |\lambda| < |\lambda_1(a)|$, λ is between 0 and $\lambda_1(a)$, so it follows from Proposition 2 that, for $u \in W^{1,p}(\Omega)$,

$$\int |u|^p \le c\int(|\nabla u|^p - \lambda a|u|^p).$$

Thus (2.2) shows that I is bounded from above on Λ if $\gamma < p$ and is bounded from below if $\gamma > p$. To prove that I satisfies the Palais-Smale condition, we note that if $|I(u_n)| < c$ then $\|u_n\|$ is bounded. Indeed, if $\|u_n\| \to \infty$, we divide $I(u_n)$ by $\|u_n\|^p$ and obtain $\int(|\nabla v_n|^p - \lambda a|v_n|^p) \to 0$, where $v_n = u_n/\|u_n\|$ and $\|v_n\| = 1$. But this is impossible since λ strictly between 0 and $\lambda_1(a)$ gives $\mu(\lambda) > 0$. The rest of the proof can be carried out in a similar manner to that of (i). □

3. The case $|\lambda| > |\lambda_1(a)|$.

We divide Λ into three subsets as follows:

$$\Lambda_\lambda^+(\text{resp. } \Lambda_\lambda^-, \Lambda_\lambda^0) = \{u \in \Lambda : \int(|\nabla u|^p - \lambda a|u|^p) > (\text{resp. } <, =)\frac{\gamma-1}{p-1}\int b|u|^\gamma\}.$$

We seek critical points of I on one of these sets. Observe that

$$\Lambda_\lambda^+(\text{resp. } \Lambda_\lambda^-, \Lambda_\lambda^0) = \{u \in \Lambda : (\gamma - p)\int b|u|^\gamma < (\text{resp. } >, =)0\}. \tag{3.1}$$

First we have

Lemma 3.1. *Let $\gamma > p$, $\int a \ne 0$ and $\int b\varphi_1^\gamma < 0$. Then there exists $|\lambda^*| > |\lambda_1(a)|$, such that for any λ strictly between $\lambda_1(a)$ and λ^*, Λ_λ^- is closed in $W^{1,p}(\Omega)$ and open in Λ_λ.*

Proof. The proof is similar to that of [DH, Lemma 3.3].

Assuming this is not true, there exist $\lambda_n \to \lambda_1(a)$ and $u_n \in \Lambda_{\lambda_n}^-$ such that $\int b|u_n|^\gamma \to 0$. Observe that, since $u_n \in \Lambda_{\lambda_n}^-$, we also have

$$0 < \int b|u_n|^\gamma = \int(|\nabla u_n|^p - \lambda_n a|u_n|^p) \to 0.$$

Let $v_n = u_n/\|u_n\|$. Then we have

$$0 < \int (|\nabla v_n|^p - \lambda_n a|v_n|^p) = \int b|v_n|^\gamma \cdot \|u_n\|^{\gamma-p} = \int b|u_n|^\gamma \cdot \|u_n\|^{-p},$$

which approaches zero regardless of whether $\|u_n\| \to \infty$ or not. We conclude, similarly to the proof of Theorem 1 (i), that $v_n \to k\varphi_1$ weakly in $W^{1,p}(\Omega)$ for some constant $k \neq 0$. In particular,

$$\int b|v_n|^{\gamma-2} v_n \varphi \to 0$$

for all $\varphi \in W^{1,p}(\Omega)$. Taking $\varphi = k\varphi_1$ in the above we obtain $\int b|k\varphi_1|^\gamma = 0$, a contradiction. Thus Λ_λ^- is closed. $\square$

Proof of Theorem 1. (iii) Assume first that $\gamma > p$. We observe that $0 \notin \Lambda_\lambda^-$, and for $u \in \Lambda_\lambda^-$,

$$I(u) = \frac{\gamma - p}{p\gamma} \int b|u|^\gamma = \frac{\gamma - p}{p\gamma} \int (|\nabla u|^p - \lambda a|u|^p) > 0.$$

Thus we look for a minimizer of I on the set Λ_λ^- when $\gamma > p$. We assume $\int a < 0$, i.e. $\lambda^* > \lambda_1(a)$. The other case can be treated similarly.

Next we verify that I satisfies the (P-S) condition on Λ_λ^- when λ is close enough to $\lambda_1(a)$. Let $\{u_n\}$ satisfy the hypotheses of the Palais-Smale condition, i.e., $\{u_n\} \subset \Lambda_\lambda^-$, $|I(u_n)| \le c$ and $I'(u_n) \to 0$.

We first show that there exist $\sigma > 0$ and $\lambda^* > \lambda_1(a)$ such that for $\lambda \in (\lambda_1(a), \lambda^*)$ and all $u \in \Lambda_\lambda^-$

$$\int |\nabla u|^p - \lambda \int a|u|^p \ge \sigma \|u\|^p. \tag{3.2}$$

Otherwise there are $\lambda_n > 0$ and $u_n \in \Lambda_{\lambda_n}^-$ such that

$$\int |\nabla v_n|^p - \lambda_n \int a|v_n|^p \to 0, \quad \text{and} \quad \lambda_n \to \lambda_1(a), \tag{3.3}$$

where $v_n = u_n/\|u_n\|$. Without loss of generality we can assume that $v_n \to v_0$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$, for some $v_0 \in W^{1,p}(\Omega)$. Thus $\int a|v_n|^p \to \int a|v_0|^p$ so (3.3) and the variational characterization of $\lambda_1(a)$ yield

$$0 \le \int (|\nabla v_0|^p - \lambda_1(a) a|v_0|^p) \le \liminf_{n\to\infty} \int (|\nabla v_n|^p - \lambda_n a|v_n|^p) = 0. \tag{3.4}$$

It follows from (3.4) that either $v_0 = 0$ or $\lambda_0 = \lambda_1(a)$ and $v_0 = \varphi_1$. The former case would imply that $v_n \to 0$ in $L^p(\Omega)$, a contradiction. In the latter case, $\|v_n\| = \|\varphi_1\| = 1$, so weak convergence of v_n to φ_1 implies that $v_n \to \varphi_1$ strongly in $W^{1,p}(\Omega)$, and hence strongly in $L^\gamma(\Omega)$. Since $u_n \in \Lambda_{\lambda_n}^-$, we get

$$0 < \int (|\nabla u_n|^p - \lambda_n a|u_n|^p) < \frac{\gamma - 1}{p - 1} \int b|u_n|^\gamma,$$

and consequently

$$0 < \int b|v_n|^\gamma \to \int b\varphi_1^\gamma < 0,$$

a contradiction.

Thus by (3.2) we have proved that $\{u_n\}$ is bounded. Now we can follow the proof of Theorem 1 (i, ii) to show that such $\{u_n\}$ contains a convergent subsequence. Thus we conclude that the Palais-Smale condition is satisfied.

The standard procedure then implies that the functional I has a minimizer, say z, on Λ_λ^-. Since $0 \notin \Lambda_\lambda^-$, $z \neq 0$. The fact that z is a positive solution of $(1.2)_\lambda$ then follows from Lemma 2.1 and the Harnack inequality as in the proof of Theorem 1 (i), so Theorem 1 (iii) is proved for the case $\gamma > p$.

For the case $\gamma < p$, we observe that the same procedure shows that Λ_λ^+ is a closed set. It is apparent that for $u \in \Lambda_\lambda^+$, $I(u) < 0$. Then we can find a nonzero maximizer z of I on Λ_λ^+ as above and it follows that this z is a positive solution of $(1.2)_\lambda$. This concludes the proof of Theorem 1 (iii). □

4. Proof of Theorem 2.

In this section we assume $\int a = 0$. Recall that

$$\Lambda_\lambda^+ (\text{resp. } \Lambda_\lambda^-, \Lambda_\lambda^0) = \{u \in \Lambda : (\gamma - p)\int b|u|^\gamma < (\text{resp. } >, =) 0\}.$$

Lemma 4.1. *If $\int b < 0$, then for sufficiently small λ, (i) Λ_λ^+ is closed in $W^{1,p}(\Omega)$ and open in Λ_λ when $\gamma > p$, and (ii) Λ_λ^- is closed in $W^{1,p}(\Omega)$ and open in Λ_λ when $\gamma < p$.*

Proof. (i) Suppose the contrary. Then there exist $\lambda_n \to 0$, $u_n \in \Lambda_{\lambda_n}^+$, such that

$$0 > \int (|\nabla u_n|^p - \lambda_n a|u_n|^p) = \int b|u_n|^\gamma \to 0. \tag{4.1}$$

Let $v_n = u_n/\|u_n\|$ and assume that $v_n \to v_0$ weakly in $W^{1,p}(\Omega)$ and strongly $L^p(\Omega)$ and $L^\gamma(\Omega)$ for some $v_0 \in W^{1,p}(\Omega)$. Dividing (4.1) by $\|u_n\|^p$ we obtain

$$0 \le \int |\nabla v_0|^p \le \liminf_{n\to\infty} \int (|\nabla v_n|^p - \lambda_n a|v_n|^p) = \liminf_{n\to\infty} \|u_n\|^{\gamma-p} \int b|v_n|^\gamma. \tag{4.2}$$

We claim that $\liminf \|u_n\|^{\gamma-p}\int b|v_n|^\gamma = 0$. Otherwise we obtain from (4.1) and (4.2) that

$$0 \le \int b|v_n|^\gamma < 0 \tag{4.3}$$

for certain n, which is a contradiction. Now, since the right hand side of (4.2) is zero, v_0 must be a constant. If $v_0 \neq 0$, then $\int b < 0$ gives $\int b|v_n|^\gamma \to \int b|v_0|^\gamma < 0$, so again we obtain the contradiction (4.3). If $v_0 = 0$, we have $\int |v_n|^p \to 0$ and $\int |\nabla v_n|^p \to 0$ (for a subsequence) from (4.2), contradicting $\|v_n\| = 1$. So, for λ sufficiently small, Λ_λ^+ is closed.

(ii) The case $\gamma < p$ is similar. The proof is complete. □

Lemma 4.2. *For sufficiently small λ, I satisfies the (P-S) condition on Λ_λ^+ if $\gamma > p$ and on Λ_λ^- if $\gamma < p$.*

Proof. We consider the case $\gamma > p$ only. The other case can be dealt with in a similar way. Assume that the contention is false. Then there exist $\lambda_n \to 0$ with an unbounded Palais-Smale sequence in each $\Lambda_{\lambda_n}^+$. Moreover (2.2) shows that we can scale the sequences so that $\|I(u)\|$ is bounded independently of n for each sequence. Thus, by a standard diagonal argument, we can find a sequence $u_n \in \Lambda_{\lambda_n}^+$ such that $I(u_n)$ is bounded, $I'(u_n) \to 0$ and $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Since $I(u_n)$ is bounded, it follows that $\int b|u_n|^\gamma$ is bounded and $\int b|v_n|^\gamma \to 0$. Thus (4.1) holds with u_n replaced by v_n, and we obtain a contradiction as in the proof of Lemma 4.1. We then conclude that the Palais-Smale sequences are bounded for sufficiently small λ. The rest of the proof is similar to that of Theorem 1 (i, ii). This concludes the proof. $\square$

Now we can find a nonzero maximizer of I on Λ_λ^+ if $\gamma > p$ and a nonzero minimizer of I on Λ_λ^- if $\gamma < p$, which gives a positive solution of $(1.2)_\lambda$. Theorem 2 is proved.

Proof of Corollary 1. Note that in this case we have, for $u \in \Lambda$,

$$I(u) = (\frac{1}{p} - \frac{1}{\gamma}) \int |\nabla u|^p = (\frac{1}{p} - \frac{1}{\gamma}) \int b|u|^\gamma.$$

We show that the functional satisfies the Palais-Smale condition on Λ. We first claim that any Palais-Smale sequence is bounded. Indeed, suppose for some $u_n \in \Lambda$, $|I(u_n)| \le c$, $I'(u_n) \to 0$, and $\|u_n\| \to \infty$. Then dividing $I(u_n)$ by $\|u_n\|^p$ we obtain $\int |\nabla v_n|^p \to 0$, where $v_n = u_n/\|u_n\|$. Let $\bar{v}_n = \int v_n/|\Omega|$. We have,

$$\int |v_n - \bar{v}_n|^p \le c \int |\nabla v_n|^p.$$

We then conclude that v_n converges strongly to some constant $v_0 \neq 0$ in $L^p(\Omega)$. Since Ω is bounded and has smooth boundary, it satisfies a uniform interior cone condition. The embedding theorem given in [GT, p. 158] then implies that $v_n \to v_0$ in $L^\gamma(\Omega)$ strongly. Now dividing $\int |\nabla u_n|^p = \int b|u_n|^\gamma$ by $\|u_n\|^\gamma$ we obtain

$$0 \le \|u_n\|^{p-\gamma} \int |\nabla v_n|^p = \int b|v_n|^\gamma.$$

It then follows from the strong convergence of $v_n \to v_0$ in $L^\gamma(\Omega)$ that $\int b \ge 0$, which contradicts the assumption that $\int b < 0$. We thus conclude that u_n must be bounded.

Now we can assume that u_n has a subsequence converging strongly to some u_0 in $L^p(\Omega)$ and $L^\gamma(\Omega)$ and weakly in $W^{1,p}(\Omega)$. The conclusion that u_n converges strongly to u_0 in $W^{1,p}(\Omega)$ then follows from similar arguments to those in the proof of Lemma 2.1. This shows that the functional I satisfies the Palais-Smale condition. The rest of the proof can be carried out as for that of Theorem 1 (iii). $\square$

5. Final Remarks.
(i) When $N = 1$, the existence of solutions for problem (1.1) with various boundary conditions can be found in [HM], where the Fučik spectrum was studied and employed.

(ii) Existence of positive solutions for Neumann problems when $p = 2$ has been studied in [BPT]. Part of our results in Theorem 1 (the case $\int_\Omega a(x)\,dx < 0$) is similar to those in [BPT]. Here we only deal with a power type "nonlinearity," but we allow a higher growth rate on the variable u. See [TA] for related results. A special form of Theorem 1 (for $p = 2$ with $\lambda = \lambda_1(a)$) has been given in [BCN]. We note that the proofs of our results originate with eigencurve theory and are different from those of [BPT] and [BCN]. Even for the case $p = 2$, our result for the case $\int_\Omega a(x)\,dx = 0$ is new.

(iii) An important sign condition on the nonlinear term $g(x, u)$, viz., a condition of the type either $g(x, u) \cdot u \geq 0$ for $|u| > c$ or $g(x, u) \cdot u \leq 0$ for $|u| > c$, has been employed extensively in the literature (see [G] and [ZL]). One easily sees that this does not hold in our case: when $a(x) \equiv 0$, which is the case studied in [LZ], $b(x)$ must change sign.

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