Electronic Journal of Differential Equations, Vol. 2006(2006), No. 31, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

MULTIPLICITY OF SOLUTIONS FOR A QUASILINEAR PROBLEM WITH SUPERCRITICAL GROWTH

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Abstract. The multiplicity and concentration of positive solutions are established for the equation

$$-\epsilon^p \Delta_p u + V(z)|u|^{p-2}u = |u|^{q-2}u + \lambda|u|^{s-2}u \quad \text{in } \mathbb{R}^N,$$

where $1 < p < N$, $\epsilon > 0$, $p < q < p^* \leq s$, $p^* = \frac{Np}{N-p}$, $\lambda \geq 0$ and V is a positive continuous function.

1. Introduction

This article concerns the multiplicity and concentration of positive solutions for the problem

$$\begin{gathered} -\epsilon^p \Delta_p u + V(z)|u|^{p-2}u = |u|^{q-2}u + \lambda|u|^{s-2}u \quad \text{in } \mathbb{R}^N \\ u \in W^{1,p}(\mathbb{R}^N) \quad \text{with } 1 < p < N \\ u(z) > 0, \quad \text{for } z \in \mathbb{R}^N, \end{gathered} \tag{1.1}$$

$\epsilon > 0$, $p < q < p^* \leq s$, $p^* = \frac{Np}{N-p}$, $\lambda \geq 0$ and $\Delta_p u$ is the p-Laplacian operator; that is,

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\Big).$$

We assume that V is a continuous function satisfying

$$V(x) \geq V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0 \quad \text{for } x \in \mathbb{R}^N; \tag{1.2}$$

Also assume that there exists an open and bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$V_0 < \min_{\partial\Omega} V. \tag{1.3}$$

In recent years, much attention has been paid to the existence and multiplicity of solutions for both subcritical and critical cases and to the concentration behavior of solutions for problem

$$-\epsilon^2 \Delta u + V(z)u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

2000 *Mathematics Subject Classification.* 35A15, 35H30, 35B40.
Key words and phrases. Variational method; penalization method; Moser iteration method.

Submitted October 26, 2005. Published March 16, 2006.

when ϵ is small. Interesting results may be found, for example, in [3, 5, 6, 8, 10, 14, 17] and their references.

Cingolani & Lazzo [9], using Lusternik-Schnirelman category and involving the sets

$$M = \{x \in \Omega : V(x) = V_0\},$$
$$M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \le \delta\}, \quad \delta > 0,$$

showed a result of multiplicity of positive solutions for (1.4), where $\Omega = \mathbb{R}^N$, $f(u) = |u|^{q-2}u$ with $q \in (2, 2^*)$, and

$$V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{\mathbb{R}^N} V(x) > 0. \tag{1.5}$$

Recall that for a closed subset Y of a topological space X, the Lusternik-Schnirelman category, denoted by $\operatorname{cat}_X Y$, is the least number of closed and contractible sets in X which cover Y.

Alves & Souto [4] showed an existence and concentration result for (1.4) with $f(u) = u^{q-1} + u^{2^*-1}$ assuming that condition (1.5) holds.

Alves & Figueiredo [1] (see also [12]) proved a multiplicity result for

$$-\epsilon^p \Delta_p u + V(z)|u|^{p-2}u = f(u) \quad in \quad \mathbb{R}^N \tag{1.6}$$

using again Lusternik-Schinirelman category and assuming that condition (1.5) holds, $2 \le p < N$ and f belongs to a large class which includes the model $f(u) = |u|^{q-2}u$ with $q \in (p, p^*)$. Moreover, the authors showed that each solution of (P_{**}) has a phenomenon of concentration near a point of minimum of the potential V. The case with critical growth was proved in [13].

del Pino & Felmer [11] proved that if the conditions (1.2) and (1.3) hold, problem (1.4) has a positive solution for small ϵ, which has a phenomenon of concentration near of one minimum point of potential V.

Alves & Figueiredo [2], using the penalization method and Lusternik-Schnirelman category theory, showed again a multiplicity and concentration result for (1.6), using now the conditions (1.2) and (1.3) with $1 < p < N$.

In this work, motivated by [2] and by some ideas developed [16], [15] and [7], we prove the multiplicity and concentration of positive solutions to (1.1) using Lusternik-Schnirelman category. For $\lambda = 0$ and $p = 2$, we have the result obtained in [9]. Hence the results of this paper complete those [9] in three senses: because we deal with $1 < p < N$ instead of $p = 2$, because we do restrict the behavior of V at infinity, and because we have $f(u) = |u|^{q-2}u + \lambda|u|^{s-2}u$ with $s \ge p^*$. Moreover, in the present paper, we continue the study of [2] and [13], because we consider supercritical nonlinearities. To our knowledge there is no results on existence of solutions to problem (P_λ) via the penalization method, and multiplicity results with supercritical growth via the Lusternik-Schnirelman category theory.

Our main result for problem (1.1) is the following.

Theorem 1.1. *Suppose that the function V satisfies* (1.2)-(1.3)*. Then, for any $\delta > 0$, there exists $\overline{\epsilon} = \overline{\epsilon}(\delta) > 0$ and $\lambda_0 > 0$ such that* (1.1) *has at least* $\operatorname{cat}_{M_\delta} M$ *positive solutions for all $\epsilon \in (0, \overline{\epsilon})$ and for all $\lambda \in [0, \lambda_0]$. Moreover, if u_ϵ is a positive solution of* (1.1) *and $\eta_\epsilon \in \mathbb{R}^N$ a global maximum point of u_ϵ, then*

$$\lim_{\epsilon\to 0} V(\eta_\epsilon) = V_0.$$

To solve problem (1.1), we first consider a truncated problem which involves only a subcritical Sobolev exponent. We show that any positive solution of truncated problem is a positive solution of (1.1).

Hereafter, we will work with the following problem equivalent to (1.1), which is obtained under change of variable $z = \epsilon x$

$$\begin{gathered} -\Delta_p u + V(\epsilon x)|u|^{p-2}u = |u|^{q-2}u + \lambda|u|^{s-2}u \quad \text{in } \mathbb{R}^N \\ u \in W^{1,p}(\mathbb{R}^N) \quad \text{with } 1 < p < N \\ u(x) > 0, \quad \forall x \in \mathbb{R}^N. \end{gathered} \tag{1.7}$$

2. Truncated Problem

First of all, we have to note that because f has supercritical growth we cannot use directly variational techniques because of the lack of compactness of the Sobolev immersions.

So we construct a suitable truncation of f in order to use variational methods or more precisely, the Mountain Pass Theorem. This truncation was used in [16] (see also [7] and [12]).

Let $K > 0$, be a constant to be determined later, and $\widehat{f}_K : \mathbb{R} \to \mathbb{R}$ given by

$$\widehat{f}_K(t) = \begin{cases} 0 & \text{if } t < 0 \\ t^{q-1} + \lambda t^{s-1} & \text{if } 0 \le t < K \\ (1 + \lambda K^{s-q})t^{q-1} & \text{if } t \ge K. \end{cases}$$

Consider $\alpha, \gamma \in \mathbb{R}$ such that $\alpha < 1 < \gamma$ and $\eta \in C^1([\alpha K, \gamma K])$ with α and γ independent of K and η satisfying

$$\begin{gathered} \eta(t) \le \widehat{f}_K(t) \quad \text{for all } t \in [\alpha K, \gamma K], \\ \eta(\alpha K) = \widehat{f}_K(\alpha K), \quad \eta(\gamma K) = \widehat{f}_K(\gamma K), \\ \eta'(\alpha K) = \widehat{f}'_K(\alpha K), \quad \eta'(\gamma K) = \widehat{f}'_K(\gamma K), \\ t \mapsto \frac{\eta(t)}{t^{p-1}} \text{ is increasing for all } t \in [\alpha K, \gamma K]. \end{gathered}$$

Now using the functions η and $\widehat{f}_K$, we define

$$f_K(t) = \begin{cases} \eta(t) & \text{if } t \in [\alpha K, \gamma K], \\ \widehat{f}_K(t) & \text{if } t \notin [\alpha K, \gamma K] \end{cases}$$

and the truncated problem

$$\begin{gathered} -\Delta_p u + V(\epsilon x)|u|^{p-2}u = f_K(u) \\ u \in W^{1,p}(\mathbb{R}), \quad u > 0 \quad \text{in } \mathbb{R}^N. \end{gathered} \tag{2.1}$$

It is easy to check that $f_K \in C^1(\mathbb{R})$, and that

$$\begin{gathered} f_K(t) = 0, \quad \text{for all } t < 0, \\ f_K(t) \le (1 + \lambda K^{s-q})t^{q-1} \quad \text{for all } t \ge 0, \\ F_K(t) \le \frac{1}{q}(1 + \lambda K^{s-q})t^q \quad \text{for all } t \ge 0, \quad F_K(t) = \int_0^t f_K(\xi)d\xi, \end{gathered}$$

there exists $\theta \in \mathbb{R}$ such that $p < \theta$ and

$$0 < \theta F_K(t) \leq f_K(t)t \quad \text{for all } t > 0, \tag{2.2}$$

the function

$$t \mapsto \frac{f_K(t)}{t^{p-1}} \text{ is increasing for all } t > 0, \tag{2.3}$$

$$f_K'(t)t^2 - (p-1)f_K(t)t \geq (q-p)t^q. \tag{2.4}$$

Remark 2.1. Note that if $u_{\epsilon,\lambda}$ is a positive solution of (2.1) such that there exists $K_0 > 0$, where for each $K \geq K_0$, there exists $\lambda_0(K) > 0$ such that $|u_{\epsilon,\lambda}|_{L^\infty(\mathbb{R}^N)} \leq \alpha K$ for all $\epsilon \in (0, \bar{\epsilon})$ and for all $\lambda \in [0, \lambda_0]$, then $u_{\epsilon,\lambda}$ is a positive solution of (1.7).

3. Multiplicity and Concentration of positive solutions for Truncated Problem

The result below is related to the multiplicity and concentration of solutions for (2.1) and its proof can be found in [2, Theorem 1.1] or [12].

Theorem 3.1. *Suppose that V verify* (1.2)(1.3)*. Then, for any $\delta > 0$, there exists $\bar{\epsilon} = \bar{\epsilon}(\delta, \lambda, K) > 0$ such that (T_λ) has at least $\operatorname{cat}_{M_\delta} M$ positive solutions for all $\epsilon \in (0, \bar{\epsilon})$ and for each $\lambda > 0$. Moreover, if $u_{\epsilon,\lambda}$ is a positive solution of* (2.1) *and $\eta_\epsilon \in \mathbb{R}^N$ a global maximum point of $u_{\epsilon,\lambda}$, then*

$$\lim_{\epsilon \to 0} V(\eta_\epsilon) = V_0.$$

4. Multiplicity of positive solutions for (1.7)

We recall that the weak solutions of (2.1) are the critical points of the functional

$$I_{\epsilon,\lambda}(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon x)|u|^p - \int_{\mathbb{R}^N} F_K(u),$$

which is well defined for $u \in W_\epsilon$, where

$$W_\epsilon = \{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)|u|^p < \infty\}$$

endowed with the norm

$$\|u\|_\epsilon^p = \int_{\mathbb{R}^N} |\nabla u|^p + \int_{\mathbb{R}^N} V(\epsilon x)|u|^p.$$

Let us also denote by $E_{V_0,\lambda}$ the energy functional associated to the problem

$$\begin{gathered} -\Delta_p u + V_0|u|^{p-2}u = f_K(u) \\ u \in W^{1,p}(\mathbb{R}), \quad u > 0 \textit{ in } \mathbb{R}^N, \end{gathered} \tag{4.1}$$

that is,

$$E_{V_0,\lambda}(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + \frac{1}{p}\int_{\mathbb{R}^N} V_0|u|^p - \int_{\mathbb{R}^N} F_K(u),$$

Here we will establish a preliminary estimative for $\|u_{\epsilon,\lambda}\|_\epsilon$.

Lemma 4.1. *For any solution $u_{\epsilon,\lambda}$ of* (2.1)*, there exists $\bar{C} > 0$, such that*

$$\|u_{\epsilon,\lambda}\|_\epsilon \leq \bar{C},$$

for $\epsilon > 0$ sufficiently small and uniformly in λ.

Proof. By [2, Theorem 1.1] (see [12] too), we have that all solutions $u_{\epsilon,\lambda}$ from (2.1) verify the inequality

$$I_{\epsilon,\lambda}(u_{\epsilon,\lambda}) \leq c_{V_0,\lambda} + h_\lambda(\epsilon),$$

where $c_{V_0,\lambda}$ is the level Mountain Pass related of functional $E_{V_0,\lambda}$ and $h_\lambda(\epsilon) \to 0$ as $\epsilon \to 0$ for each $\lambda \geq 0$. In this case, we may suppose that

$$I_{\epsilon,\lambda}(u_{\epsilon,\lambda}) \leq c_{V_0,\lambda} + 1,$$

for all $\epsilon \in (0, \bar{\epsilon}(K,\lambda))$. Since $c_{V_0,\lambda} \leq c_{V_0,0}$, we have

$$I_{\epsilon,\lambda}(u_{\epsilon,\lambda}) \leq c_{V_0,0} + 1, \tag{4.2}$$

for all $\epsilon \in (0, \bar{\epsilon}(K,\lambda))$ and for all $\lambda \geq 0$. Moreover,

$$\begin{aligned} I_{\epsilon,\lambda}(u_{\epsilon,\lambda}) &= I_{\epsilon,\lambda}(u_{\epsilon,\lambda}) - \frac{1}{\theta} I'_{\epsilon,\lambda}(u_{\epsilon,\lambda})u_{\epsilon,\lambda} \\ &= \big(\frac{1}{p} - \frac{1}{\theta}\big)\|u_{\epsilon,\lambda}\|_\epsilon^p + \int_{\mathbb{R}^N} \big[\frac{1}{\theta} f_K(u_{\epsilon,\lambda})u_{\epsilon,\lambda} - F_K(u_{\epsilon,\lambda})\big]. \end{aligned}$$

By (2.2),

$$I_{\epsilon,\lambda}(u_\epsilon, \lambda) \geq \big(\frac{1}{p} - \frac{1}{\theta}\big)\|u_{\epsilon,\lambda}\|_\epsilon^p$$

Therefore, by (4.2), $\|u_{\epsilon,\lambda}\|_\epsilon \leq \bar{C}$, for $\epsilon \in (0, \bar{\epsilon}(K,\lambda))$ and for all $\lambda \geq 0$, where

$$\bar{C} = \Big[(c_{V_0,0} + 1)\big(\frac{\theta p}{\theta - p}\big)\Big]^{1/p}.$$

□

Now, we use the Moser iteration technique [15] (see also [7]) to prove that each solution found of (2.1) is a solution of (1.7)

Proof of Theorem 1.1. We use the notation $u_{\epsilon,\lambda} := u$. For each $L > 0$, we define

$$u_L = \begin{cases} u & \text{if } u \leq L, \\ L & \text{if } u \geq L, \end{cases}$$

$$z_L = u_L^{p(\beta-1)}u \quad \text{and} \quad w_L = uu_L^{\beta-1}$$

with $\beta > 1$ to be determined later. Taking z_L as a test function, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} u_L^{p(\beta-1)}|\nabla u|^p &= -p(\beta-1)\int_{\mathbb{R}^N} u_L^{p\beta-p-1}u|\nabla u|^{p-2}\nabla u \nabla u_L \\ &\quad + \int_{\mathbb{R}^N} f_K(u)uu_L^{p(\beta-1)} - \int_{\mathbb{R}^N} V(\epsilon x)|u|^p u_L^{p(\beta-1)}. \end{aligned}$$

By (2),

$$\int_{\mathbb{R}^N} u_L^{p(\beta-1)}|\nabla u|^p \leq C_{\lambda,K}\int_{\mathbb{R}^N} u^q u_L^{p(\beta-1)}, \tag{4.3}$$

where $C_{\lambda,K} = (1+\lambda K^{s-q})$. From Sobolev imbedding, Hölder inequalities and (4.3),

$$|w_L|_{p^*}^p \leq C_1\beta^p C_{\lambda,K}\Big(\int_{\mathbb{R}^N} u^{p^*}\Big)^{(q-p)/p^*}\Big(\int_{\mathbb{R}^N} w_L^{pp^*/[p^*-(q-p)]}\Big)^{[p^*-(q-p)]/p^*},$$

where $p < \frac{pp^*}{p^*-(q-p)} < p^*$. Recalling that $\|u_{\epsilon,\lambda}\|_\epsilon \leq \bar{C}$, we have

$$|w_L|_{p^*}^p \leq C_2\beta^p C_{\lambda,K}\bar{C}^{(q-p)/p^*}|w_L|_{\alpha^*}^p$$

where $\alpha^*=\frac{pp^*}{p^*-(q-p)}$. Note that if $u^\beta\in L^{\alpha^*}(\mathbb{R}^N)$, using the definition of w_L and the fact that $u_L\le u$, we obtain

$$\Big(\int_{\mathbb{R}^N}\mid uu_L^{\beta-1}\mid^{p^*}\Big)^{p/p^*}\le C_3\beta^pC_{\lambda,K}\Big(\int_{\mathbb{R}^N}u^{\beta\alpha^*}\Big)^{p/\alpha^*}<+\infty.$$

By Fatou's Lemma on the variable L, we get

$$|u|_{\beta p^*}\le(C_4C_{\lambda,K})^{1/\beta}\beta^{1/\beta}|u|_{\beta\alpha^*}. \tag{4.4}$$

The assertion is obtained by iteration of estimative (4.4). Namely, let $\chi=\frac{p^*}{\alpha^*}$; i.e., $p^*=\chi\alpha^*$. Then

$$|u|_{\chi^{(m+1)}\alpha^*}\le C_5(C_4C_{\lambda,K})^{\sum_{i=1}^m\frac{\chi^{-i}}{p}}\chi^{\sum_{i=1}^m i\chi^{-i}}\bar{C}.$$

Passing to the limit as $m\to\infty$, we have

$$|u|_{L^\infty(\mathbb{R}^N)}\le C_5(C_4C_{\lambda,K})^{\sigma_1}\chi^{\sigma_2}\bar{C},$$

where $\sigma_1=\sum_{i=1}^\infty\frac{\chi^{-i}}{p}$ and $\sigma_2=\sum_{i=1}^\infty i\chi^{-i}$. To choose λ_0, we consider the inequality

$$\Big[C_4(1+\lambda K^{s-q})\Big]^{\sigma_1}\chi^{\sigma_2}C_5\bar{C}\le\alpha K.$$

We conclude that

$$(1+\lambda K^{s-q})^{\sigma_1}\le\frac{\alpha KC_6}{C_4^{\sigma_1}\chi^{\sigma_2}\bar{C}}.$$

We choose λ_0 verifying the inequality

$$\lambda_0\le\Big[\frac{(\alpha KC_6)^{\frac{1}{\sigma_1}}}{C_4\chi^{\frac{\sigma_2}{\sigma_1}}\bar{C}^{1/\sigma_1}}-1\Big]\frac{1}{K^{s-q}}$$

and fixing K such that

$$\Big[\frac{(\alpha KC_6)^{1/\sigma_1}}{C_4\chi^{\frac{\sigma_2}{\sigma_1}}\bar{C}^{1/\sigma_1}}-1\Big]>0,$$

we have $|u_{\lambda,\epsilon}|_{L^\infty(\mathbb{R}^N)}\le\alpha K$ for all $\epsilon\in(0,\bar{\epsilon}(K,\lambda))$ and all $\lambda\in[0,\lambda_0]$. The result follows from Remark 2.1. $\square$

Acknowledgements. We would like to thank to Professor C. O. Alves for his help and encouragement, to the anonymous referee for several suggestions that improved this article.

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