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POSITIVE SOLUTION TO QUASILINEAR SCHRÖDINGER EQUATIONS VIA ORLICZ SPACE FRAMEWORK

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Abstract. This article concerns the existence of solutions for the generalized quasilinear Schrödinger equation

$$-\operatorname{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2+V(x)u=f(x,u),\quad x\in\mathbb{R}^N.$$

We obtain a positive solution by using a change of variables and a minimax theorem in an Orlicz space framework.

1. Introduction

We are concerned with the existence of positive solutions for the quasilinear Schrödinger equation

$$-\operatorname{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2+V(x)u=f(x,u),\quad x\in\mathbb{R}^N, \tag{1.1}$$

where $N\geq 3$, $g:\mathbb{R}\to\mathbb{R}^+$ is an even differential function and $g'(s)\geq 0$ for all $s\geq 0$, $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}^+$ is a continuous function, and $V(x)$ is a positive potential. There is a large number of publications discussing the existence of solutions for the generalized nonlinear Schrödinger equations arising in various backgrounds, see for example [18, 22, 23, 39, 35]. It a research hot spot in nonlinear analysis to study the existence of standing wave solutions for the quasilinear Schrödinger equation

$$i\partial_t z=-\Delta z+Wz-f(|z|^2)z-\kappa z l'(|z|^2)\Delta l(|z|^2), \tag{1.2}$$

where $W(x), x\in\mathbb{R}^N$ is a given potential, κ is a real constant and f,l are real functions of essentially pure power forms. The semilinear case corresponding to $\kappa=0$ has been studied extensively in recent years. We would like to point out that the quasilinear equation of the form (1.2) arises in various branches of mathematical physics and has been derived as models of several physical phenomena corresponding to various types of nonlinear term l. For instance, the case of $l(t)=t$ was used for the superfluid film equation in plasma physics by Kurihara in [15]. In the case $l(t)=(1+t)^{1/2}$, equation (1.2) models the self-channeling of a high-power ultrashort lasers in matter, see [2, 6, 30] and the references in [3]. Equation (1.2) also appears in plasma physics and fluid mechanics [16, 20, 25], in the theory of Heisenberg ferromagnets and magnons [1, 17, 28], in dissipative quantum mechanics [13], and in condensed matter theory [24].

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Without loss of generality we assume $\kappa = 1$. Setting $z(x,t) = \exp(-iEt)u(x)$, with $E \in \mathbb{R}$ and u being a real function, (1.2) can be reduced to the corresponding equation of elliptic type

$$-\Delta u + V(x)u - ul'(u^2)\Delta l(u^2) = f(x,u), \ x \in \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - E$. If we take

$$g^2(u) = 1 + \frac{(l(u^2)')^2}{2}, \tag{1.4}$$

equation (1.3) turns into quasilinear elliptic equation (1.1).

If we set $g^2(u) = 1 + 2u^2$, i.e., $l(t) = t$ in (1.4), we obtain the superfluid film equation in plasma physics

$$-\Delta u + V(x)u - u\Delta(u^2) = f(x,u), \quad x \in \mathbb{R}. \tag{1.5}$$

If we set $g^2(u) = 1 + \frac{u^2}{2(1+u^2)}$, i.e., $l(t) = (1+t)^{1/2}$ in (1.4), we obtain the equation

$$-\Delta u + V(x)u - \frac{u}{2(1+u^2)^{1/2}}[\Delta(1+u^2)^{1/2}] = f(x,u), \quad x \in \mathbb{R}^N,$$

which models the self-channeling of a high-power ultrashort laser in matter.

For equation (1.5), to the best of our knowledge, the first result for the existence of solutions was proved by Poppenberg, Schmitt and Wang in [27]. The idea in [27] is a constrained minimization argument. Subsequently, a general existence result for (1.5) was derived by Liu, Wang and Wang [21]. The main existence results were obtained by making a change of variables and reducing the quasilinear problem (1.5) to a semilinear one. It is worthy to be noticed that an Orlicz space framework was used to prove the existence of a positive solutions via Mountain Pass Theorem. The same method of changing variables was also used by Colin and Jeanjean in [8], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was chosen as the working space. We refer the readers to [5, 9, 22, 31, 37, 38, 40] for more results.

Recently, Shen and Wang in [32] studied the equation (1.1) by introducing the change of variable

$$G(s) = \int_0^s g(t)dt. \tag{1.6}$$

The authors obtained the positive solutions for (1.1) with a general function $l(t)$ when f is superlinear and subcritical. Later on, by using the same change of variable as (1.6), the problem has been studied extensively in recent years, see [4, 18, 19, 34, 39]. Several authors proposed the critical problem as follows

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = |u|^{\alpha 2^*-2}u + f(x,u), \quad x \in \mathbb{R}^N. \tag{1.7}$$

For instance, Shen, Wang in [33], Deng et al. in [10, 11], and Cheng, Shen in [7] obtained the positive solutions of (1.7).

However, to the best of our knowledge, there is no one considering equation (1.1) in an Orlicz space framework based on the idea from Liu, Wang and Wang [21]. This paper will make some contribution to this research field.

This article is organized as follows. In Section 2, we introduce the variational framework to restate the problem in an equivalent form and give the main result of this paper; in Section 3, we prove the main theorem of this paper.

We will use the following notation frequently.

- C, C_0, C_1, ... denote positive (possibly different) constants;

- $L^p(\mathbb{R}^N)$ denotes Lebesgue space with the norm $|\cdot|_p$;
- X^* denotes the conjugate Banach space of X;
- $\langle\cdot,\cdot\rangle$ is the dual pairing on the space X^* and X;
- The weak convergence is denoted by $\rightharpoonup$, and the strong convergence by $\to$;
- Abbreviate $\int_{\mathbb{R}^N} f(x)dx$ to $\int f$.

2. Reformulation of the problem

Next, for ease reference we state our assumptions in a more precise way. We assume following on the potential V:

(A1) The function $V:\mathbb{R}^N\to\mathbb{R}$ is continuous and uniformly positive, that is,
$$0<V_0\le V(x) \text{ for all } x\in\mathbb{R}^N;$$
(A2) V is radially symmetric, i.e., $V(x)=V(|x|)$;
(A3) $\nabla V(x)x\le 0$ for all $x\in\mathbb{R}^N$.

We assume following on $f(x,t)$:

(A4) $f\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R}^+)$ satisfies $f(x,t)=0$ for all $x\in\mathbb{R}^N$ and $t<0$;
(A5) There exist $C>0$ and $2<p<2^*:=2N/(N-2)$ such that
$$|f(x,t)|\le C(1+|t|^{2p-1}) \quad \text{for all } (x,t)\in(\mathbb{R}^N\times\mathbb{R});$$
(A6) As $|t|\to 0, f(x,t)=o(t)$ uniformly in $x\in\mathbb{R}^N$;
(A7) $F(x,t)/t^4\to+\infty$ uniformly in x as $t\to+\infty$ where $F(x,t)=\int_0^t f(x,t)dt$.

The classical Ambrosetti-Rabinowitz type condition
$$0<\mu F(x,t)\le f(x,t)t \text{ for some } \mu>4 \text{ and all } (x,t)\in\mathbb{R}^N\times\mathbb{R}^+, \tag{2.1}$$
plays an important role in proving existence results for variational problems. In fact, if $f(x,t)$ satisfies (2.1), we obtain
$$F(x,t)\ge F(x,1)t^\mu \text{ for } t>1,$$
which implies that (A7) holds.

Choose $f(x,t)=t^3\ln(1+t)$ when $t\ge 0$ and $f=0$ when $t<0$. Then f satisfies the assumptions (A4)–(A7). But it does not satisfy the classical Ambrosetti-Rabinowitz type condition (2.1).

We assume that $g(t)$ satisfies the following conditions:

(A8) $g\in C^1(\mathbb{R})$ is an even positive function and $g'(t)\ge 0$ for all $t\ge 0$, $g(0)=1$;
(A9) There exists a constant $\beta>0$ satisfying $\lim_{t\to+\infty}\frac{g(t)}{t}=\beta$;
(A10) $0<\frac{tg'(t)}{g(t)}\le 1$ for all $t\in(0,+\infty)$.

We note that conditions (A8)–(A10) are satisfied by many functions. In particular, if let $l(t)=t$, i.e., $g^2(u)=1+2u^2$, then g satisfies the above conditions.

By a direct computation, we observe that (1.1) is the Euler-Lagrange equation associated with the energy functional
$$J(u)=\frac12\int g^2(u)|\nabla u|^2+\frac12\int V(x)u^2-\int F(x,u). \tag{2.2}$$
But this functional J may be not be well defined in $H^1(\mathbb{R}^N)$. We employ a change of variable developed by Shen and Wang in [32] to overcome this difficulty. That is
$$v=G(u)=\int_0^u g(t)dt. \tag{2.3}$$

We give out the properties of G in the following lemma for the readers convenience.

Lemma 2.1. *The function G defined above satisfies the following properties:*

(1) $G(t)$ *and* $G^{-1}(s)$ *are odd and* C^2;
(2) $\lim_{t\to\infty} g(t) = +\infty$;
(3) $0 < \frac{1}{g(G^{-1}(s))} \le 1$, *for any* $s \in \mathbb{R}$;
(4) $|G^{-1}(s)| \le |s|$, *for any* $s \in \mathbb{R}$;
(5) $\left|\frac{G^{-1}(s)}{g(G^{-1}(s))}\right| < \frac{1}{\beta}$, *for any* $s \in \mathbb{R}$;
(6) $tg(t) \le 2G(t) \le 2tg(t)$, *for any* $t > 0$;
(7) *For* $s \ge 0$, $\frac{G^{-1}(s)}{s}$ *is nonincreasing and* $\lim_{s\to 0} \frac{G^{-1}(s)}{s} = 1$;
(8) *For* $s \ge 0$, $\frac{|G^{-1}(s)|^2}{s}$ *is nondecreasing and* $\lim_{s\to+\infty} \frac{|G^{-1}(s)|^2}{s} = \frac{2}{\beta}$;
(9) $|G^{-1}(s)| \le (\frac{2}{\beta})^{1/2}|s|^{1/2}$, *for any* $s \in \mathbb{R}$;
(10) *There exists a positive constant C such that*
$$\widehat{G}(s) = |G^{-1}(s)|^2 \sim \begin{cases} s^2, & |s| \ll 1; \\ C|s|, & |s| \gg 1; \end{cases}$$
(11) *There exists a positive constant C_0 such that* $\widehat{G}(2s) \le C_0\widehat{G}(s)$;
(12) $\widehat{G}''(s) \ge 0$, *i.e.,* $\widehat{G}(s)$ *is convex.*

Proof. Conclusions (1)–(5) and the right hand side of the inequality (6) are trivial. Let
$$c(t) = 2G(t) - tg(t).$$
Note $c(0) = 0$ and $c'(t) \ge 0$ from (A10). Then the left hand side of the inequality (6) is proved.

It is easy to obtain (7) and (8) from (6). By L'Hospital's rule,
$$\lim_{s\to+\infty} \frac{|G^{-1}(s)|^2}{s} = \lim_{t\to+\infty} \frac{t^2}{G(t)} = \lim_{t\to+\infty} \frac{2t}{g(t)} = \frac{2}{\beta}.$$

We can obtain (9) by (8), (10) by (7) and (8). The inequality (11) is trivial. For (12), we can see
$$\widehat{G}'(s) = \frac{2G^{-1}(s)}{g(G^{-1}(s))} \quad \text{and} \quad \widehat{G}''(s) = \frac{2\big(1 - \frac{g'(G^{-1}(s))G^{-1}(s)}{g(G^{-1}(s))}\big)}{(g(G^{-1}(s)))^2} \ge 0.$$
Then conclusion of (12) holds. □

After changing of variable by (2.3) we can rewrite the functional $J(u)$ as
$$\Phi(v) = J(G^{-1}(v)) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int V(x)|G^{-1}(v)|^2 - \int F(x, G^{-1}(v)),$$
which is well defined in the Orlicz space
$$E := \big\{v \in H^1_{rad}(\mathbb{R}^N) | \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx < \infty\big\}.$$
E is a Banach space (Proposition 2.2) endowed with the norm
$$\|v\| = |\nabla v|_2 + \inf_{\xi>0} \frac{1}{\xi}\Big[1 + \int_{\mathbb{R}^N} V(x)|G^{-1}(\xi v)|^2 dx\Big] \tag{2.4}$$
where $H^1_{rad}(\mathbb{R}^N) = \{v \in H^1(\mathbb{R}^N) | v(x) = v(|x|)\}$. (see Subsection 2.2, and for more details on Orlicz spaces we refer for instance to [29]).

We collect some facts on the space E which are crucial in our argument. The proof is analogous to the references [21, 26], just by changing the function f therein to G^{-1}.

Proposition 2.2. (1) *E is a Banach space with respect to the norm given in* (2.4)*;*

(2) *There exists a positive constant C such that for all $v \in E$*
$$\frac{\int V(x)|G^{-1}(v)|^2}{1+\left(\int V(x)|G^{-1}(v)|^2\right)^{1/2}} \le C\|v\|;$$

(3) *If $v_n \to v$ in E then*
$$\int V(x)\left||G^{-1}(v_n)|^2-|G^{-1}(v)|^2\right| \to 0, \quad \int V(x)\left|G^{-1}(v_n)-G^{-1}(v)\right|^2 \to 0;$$

(4) *If $v_n \to v$ almost everywhere and*
$$\int V(x)|G^{-1}(v_n)|^2 \to \int V(x)|G^{-1}(v)|^2,$$
then
$$\inf_{\xi>0}\frac{1}{\xi}\Big[1+\int_{\mathbb{R}^N} V(x)|G^{-1}(\xi(v_n-v))|^2dx\Big] \to 0.$$

Proposition 2.3. *We denote*
$$X := \left\{v \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} V(x)v^2dx < \infty\right\}$$
with the norm
$$\|v\|_X = \Big[\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)v^2dx\Big]^{1/2},$$
and
$$\widetilde{E} := \left\{v \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2dx < \infty\right\}$$
with the norm defined in (2.4)*. Then*

(1) *The embedding $X \hookrightarrow \widetilde{E}$ is continuous;*
(2) *The embedding $\widetilde{E} \hookrightarrow H^1(\mathbb{R}^N)$ is continuous.*

Proposition 2.4. *The map $v \mapsto G^{-1}(v)$ from E into $L^s(\mathbb{R}^N)$ is continuous for $2 \le s \le 2\cdot 2^*$. Moreover, under the assumption (A2), the above map is compact for $2 < s < 2\cdot 2^*$.*

From condition (A3) we have $V(x) \le V(0) < +\infty$. Together with Proposition 2.3, we obtain $\|\cdot\|$ and $\|\cdot\|_X$ are a pair of equivalent norms in E and $\Phi \in C^1(E,\mathbb{R})$. Moreover, if v is a critical point for the functional Φ, then it should satisfy
$$\int \nabla v\nabla w + \int V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}w = \int \frac{f(x,G^{-1}(v))}{g(G^{-1}(v))}w, \quad w \in E. \tag{2.5}$$
Therefore, v is a solution for the equation
$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{f(x,G^{-1}(v))}{g(G^{-1}(v))},\ x \in \mathbb{R}^N. \tag{2.6}$$
By setting $v = G(u)$, it is easy to see that equation (2.6) is equivalent to problem (1.1), which takes $u = G^{-1}(v)$ as its solution.

Motivated by the above, we give the following definition of the weak solution for (1.1).

Definition 2.5. We say u is a weak solution of problem (1.1), if $v = G(u) \in E$ is a critical point of the following functional corresponding to problem (2.6)

$$\Phi(v) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int V(x)|G^{-1}(v)|^2 - \int F(x, G^{-1}(v)).$$

Now we state our main result of this article.

Theorem 2.6. *Let* (A1)–(A10) *be satisfied. Then* (1.1) *has at least one positive solution in the sense of Definition 2.5.*

Remark 2.7. Indeed, we can find that any critical point v of Φ is nonnegative. In fact, denoting $v^{\pm} := \pm \max\{\pm v, 0\}$ and taking $w = v^-$ in (2.5), we can obtain

$$\int |\nabla v^-|^2 + \int V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}v^- = 0.$$

Consequently, from the definition of G, we obtain

$$\int |\nabla v^-|^2 = 0, \int V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}v^- = 0.$$

Hence we have $v^- = 0$ almost everywhere in $\mathbb{R}^N$ and therefore $v = v^+ \geq 0$. Then by the elliptic regularity theory and the maximum principle [12], we know $v > 0$.

3. Proof of the main theorem

To prove Theorem 2.6, we use the following minimax theorem due to Jeanjean [14] to obtain a (PS) sequence with some fine properties.

Definition 3.1. Let X be a Banach space. Let $\Phi \in C^1(X, \mathbb{R})$, we say $\{v_n\}$ a (PS) sequence if $\Phi(v_n)$ is bounded and $\Phi'(v_n) \to 0$ in X^* as $n \to \infty$.

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a Banach space and $\mathbb{I} \subset \mathbb{R}_+$ be an interval. Consider the following family of C^1-functionals on X*

$$I_\lambda(v) = A(v) - \lambda B(v), \ \lambda \in \mathbb{I}$$

with B nonnegative and either $A(v) \to +\infty$ or $B(v) \to +\infty$ as $\|v\| \to \infty$. We assume there are two points v_1, v_2 in X such that

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\} \quad \text{for all } \lambda \in \mathbb{I},$$

where

$$\Gamma_\lambda = \{\gamma \in C([0,1], X) | \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then for almost every $\lambda \in \mathbb{I}$ there is a sequence $\{v_n\} \subset X$ such that

(1) *$\{v_n\}$ is bounded;*
(2) *$I_\lambda(v_n) \to c_\lambda$;*
(3) *$I'_\lambda(v_n) \to 0$ in the dual X^* of X.*

Moreover, the map $\lambda \mapsto c_\lambda$ is non-increasing and continuous from the left.

Fix $\lambda \in [1/2, 1]$. We define the energy functional

$$\Phi_\lambda(v) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int V(x)|G^{-1}(v)|^2 - \lambda \int F(x, G^{-1}(v)),$$

and set

$$A(v) := \int |\nabla v|^2 + \int V(x)|G^{-1}(v)|^2, \quad B(v) := \int F(x, G^{-1}(v)).$$

Next, we prove that the functional Φ_λ exhibits the mountain-pass geometry. For that purpose, let us first consider the set $S_\rho = \{v \in E | A(v) = \rho^2\}$. Since the functional $A(v)$ is continuous, then S_ρ is a closed subset which disconnects the space E.

Lemma 3.3. *There exist ρ, $\delta > 0$ such that $\Phi_\lambda(v) \geq \delta$ for any $v \in S_\rho$.*

Proof. From assumptions (A5) and (A6), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int F(x, G^{-1}(v))dx \leq \varepsilon \int |G^{-1}(v)|^2 + C_\varepsilon \int |G^{-1}(v)|^{2p}.$$

Clearly we have $\int |G^{-1}(v)|^2 \leq C\rho^2$. And taking $0 < \tau < 1$ such that $p = \tau + (1-\tau)2^*$, by Hölder inequality, Sobolev inequality $|v|_{2^*} \leq S|\nabla v|_2$ and (9) of Lemma 2.1 we obtain

$$\begin{aligned}
\int |G^{-1}(v)|^{2p} &\leq \Big(\int |G^{-1}(v)|^2\Big)^{\tau} \Big(\int |G^{-1}(v)|^{2\cdot 2^*}\Big)^{1-\tau} \\
&\leq \Big(\frac{2}{\beta}\Big)^{2^*(1-\tau)} \Big(\int |G^{-1}(v)|^2\Big)^{\tau} \Big(\int |v|^{2^*}\Big)^{1-\tau} \\
&\leq C^\tau \Big(\frac{2}{\beta}\Big)^{2^*(1-\tau)} S^{2^*(1-\tau)} \rho^{2\tau} \Big(\int |\nabla v|^2\Big)^{2^*(1-\tau)/2} \\
&\leq C^\tau \Big(\frac{2}{\beta}\Big)^{2^*(1-\tau)} S^{2^*(1-\tau)} \rho^{2\tau + (1-\tau)2^*}.
\end{aligned}$$

From the above inequalities, we know that

$$\Phi_\lambda(v) \geq (\frac{1}{2} - C\lambda\varepsilon)\rho^2 - \lambda C_\varepsilon C^\tau \Big(\frac{2}{\beta}\Big)^{2^*(1-\tau)} S^{2^*(1-\tau)} \rho^{2\tau+(1-\tau)2^*}$$

for every $v \in S_\rho$. Since $2\tau + (1-\tau)2^* > 2$, choosing ε small enough, we conclude that there exist δ, $\rho > 0$ such that $\Phi_\lambda|_{S_\rho} \geq \delta > 0$. □

Lemma 3.4. *There exists $v_0 \in E$ such that $\Phi_\lambda(v_0) < 0$.*

Proof. For any $v > 0$, we want to prove $\Phi_\lambda(tv) < 0$ as $t \to +\infty$. Suppose by contradiction that there exists a sequence $t_n \to +\infty$ such that

$$\int |\nabla (t_n v)|^2 + V(x)|G^{-1}(t_n v)|^2 \to +\infty, \quad \text{as } n \to \infty$$

and $\Phi_\lambda(t_n v) \geq 0$ for all n. Set $w = v/\|v\|$. Then

$$\begin{aligned} 0 &\leq \frac{\Phi_\lambda(t_n v)}{\int |\nabla(t_n v)|^2 + V(x)|G^{-1}(t_n v)|^2} \\ &= \frac{1}{2} - \lambda \int \frac{F(x, G^{-1}(t_n v))}{|t_n v|^2} \frac{|t_n v|^2}{\int |\nabla(t_n v)|^2 + V(x)|G^{-1}(t_n v)|^2} \\ &\leq \frac{1}{2} - C\lambda \int \frac{F(x, G^{-1}(t_n v))}{|G^{-1}(t_n v)|^4} \frac{|G^{-1}(t_n v)|^4}{|t_n v|^2} |w|^2. \end{aligned} \tag{3.1}$$

Since $v > 0$, $t_n v(x) \to +\infty$, from (A7) and (8) of Lemma 2.1, applying Fatou's lemma, we obtain

$$\int \frac{F(x, G^{-1}(t_n v))}{|G^{-1}(t_n v)|^4} \frac{|G^{-1}(t_n v)|^4}{|t_n v|^2} |w|^2 \to +\infty \text{ as } n \to \infty.$$

This is a contradiction to (3.1). $\square$

Lemma 3.5. *Assume that $\{v_n(\lambda)\} \subset E$ is a bounded (PS) sequence of the functional Φ_λ for $\lambda \in [1/2, 1]$. Then there exists a convergent subsequence of $\{v_n(\lambda)\}$ in E.*

Proof. It is clear that $\{v_n(\lambda)\}$ is bounded in $H^1_{rad}(\mathbb{R}^N)$. Up to a subsequence, for some $v \in H^1_{rad}(\mathbb{R}^N)$, we have $v_n \rightharpoonup v$ in $H^1_{rad}(\mathbb{R}^N)$, $v_n \rightharpoonup v$ in $L^s(\mathbb{R}^N)$ for all $2 \leq s \leq 2^*$ and $v_n \to v$ a.e. in $\mathbb{R}^N$. From Proposition 2.4 we have $G^{-1}(v_n) \to G^{-1}(v)$ in $L^s(\mathbb{R}^N)$ for all $2 < s < 2 \cdot 2^*$. Then, since $\frac{|G^{-1}(v)|^{2p-1}}{g(G^{-1}(v))} \in L^{2N/(N+2)}(\mathbb{R}^N)$ and $v_n \rightharpoonup v$ in $L^{2^*}(\mathbb{R}^N)$, we have

$$\int \frac{|G^{-1}(v)|^{2p-1}}{g(G^{-1}(v))} (v_n - v) \to 0. \tag{3.2}$$

On the other hand, the Lebesgue dominated convergence theorem implies that

$$\frac{|G^{-1}(v_n)|^{2p-1}}{g(G^{-1}(v_n))} \to \frac{|G^{-1}(v)|^{2p-1}}{g(G^{-1}(v))}, \quad \text{in } L^{2N/(N+2)}(\mathbb{R}^N).$$

By the Hölder inequality and $|v_n - v|_{2^*} \leq C$, it follows that

$$\int \Big[\frac{|G^{-1}(v_n)|^{2p-1}}{g(G^{-1}(v_n))} - \frac{|G^{-1}(v)|^{2p-1}}{g(G^{-1}(v))} \Big] (v_n - v) \to 0. \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\int \frac{|G^{-1}(v_n)|^{2p-1}}{g(G^{-1}(v_n))} (v_n - v) \to 0.$$

Since $\widehat{G}(s)$ is convex, the functional $A(u)$ is convex and

$$\begin{aligned} &\frac{1}{2} A(v) - \frac{1}{2} A(v_n) \geq \frac{1}{2} \langle A'(v_n), v - v_n \rangle \\ &\qquad = \int \nabla v_n \nabla (v - v_n) + \int V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} (v - v_n). \end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\int\big[|\nabla v|^2+V(x)|G^{-1}(v)|^2\big]-\frac{1}{2}\int\big[|\nabla v_n|^2+V(x)|G^{-1}(v_n)|^2\big]\\&\geq\langle\Phi'_\lambda(v_n),v-v_n\rangle+\lambda\int\frac{f(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}(v-v_n).\end{aligned}\tag{3.4}$$

From

$$|\langle\Phi'_\lambda(v_n),v-v_n\rangle|\leqslant C\|\Phi'_\lambda(v_n)\|_{E^*}\to 0$$

and

$$\begin{aligned}&\Big|\int\frac{f(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}(v-v_n)\Big|\\&\leq\varepsilon\int\frac{|G^{-1}(v_n)|}{g(G^{-1}(v_n))}|v-v_n|+C_\varepsilon\int\frac{|G^{-1}(v_n)|^{2p-1}}{g(G^{-1}(v_n))}|v-v_n|\\&\leq\varepsilon|v_n|_2|v-v_n|_2+o(1)C_\varepsilon,\end{aligned}$$

taking the limit in (3.4), we obtain

$$\liminf_{n\to\infty}\int\big[|\nabla v_n|^2+V(x)|G^{-1}(v_n)|^2\big]\leq\int\big[|\nabla v|^2+V(x)|G^{-1}(v)|^2\big].$$

Combining the semicontinuity of the norm and Fatou's lemma, we have

$$\liminf_{n\to\infty}\int|\nabla v_n|^2=\int|\nabla v|^2,$$
$$\liminf_{n\to\infty}\int V(x)|G^{-1}(v_n)|^2=\int V(x)|G^{-1}(v)|^2.$$

Using (4) of Proposition 2.2, we obtain

$$\inf_{\xi>0}\frac{1}{\xi}\Big[1+\int_{\mathbb{R}^N}V(x)|G^{-1}(\xi(v_n-v))|^2dx\Big]\to 0,$$

which implies that $v_n\to v$ in E. □

Using an argument as in [36, Theorem B.1], we obtain the following Pohožaev identity.

Lemma 3.6. *If $v\in E$ be a critical point of Φ_λ, then v satisfies*

$$\begin{aligned}&\frac{N-2}{2}\int|\nabla v|^2+\frac{N}{2}\int V(x)|G^{-1}(v)|^2+\frac{1}{2}\int\nabla V(x)x|G^{-1}(v)|^2\\&=\lambda N\int F(x,G^{-1}(v)).\end{aligned}$$

Lemma 3.7. *Assume that $\Phi_{\lambda_j}(v_j)=c_j$ and $\Phi'_{\lambda_j}(v_j)=0$ for $\lambda\in[1/2,1]$, $c_{\lambda_j}\leq c_{1/2}$. Then sequence $\{v_j\}$ is bounded in E.*

Proof. Since $\Phi_{\lambda_j}(v_j)=c_j$, by (A3) and Lemma 3.6 we have

$$\begin{aligned}c_{1/2}&\geq c_{\lambda_j}\\&=\frac{1}{2}\int|\nabla v_j|^2+\frac{1}{2}\int V(x)|G^{-1}(v_j)|^2-\lambda_j\int F(x,G^{-1}(v_j))\\&=\frac{1}{2}\int|\nabla v_j|^2+\frac{1}{2}\int V(x)|G^{-1}(v_j)|^2\end{aligned}$$

$$
\begin{aligned}
&- \Big(\frac{N-2}{2N}\int |\nabla v_j|^2 + \frac{1}{2}\int V(x)|G^{-1}(v_j)|^2 + \frac{1}{2N}\int \nabla V(x)x|G^{-1}(v_j)|^2\Big)\\
&= \frac{1}{N}\int |\nabla v_j|^2 - \frac{1}{2N}\int \nabla V(x)x|G^{-1}(v_j)|^2\\
&\geq \frac{1}{N}\int |\nabla v_j|^2.
\end{aligned}
$$

Choose $w_j = G^{-1}(v_j)g(G^{-1}(v_j))$, and note that

$$
|w_j|_2 \leq 2|v_j|_2, \quad |\nabla w_j|_2 \leq 2|\nabla v_j|_2, \ \|w_j\| \leq C\|v_j\|.
$$

Then we obtain

$$
\begin{aligned}
0 &= \langle \Phi'_{\lambda_j}(v_j), w_j\rangle\\
&= \int \Big(1 + \frac{g'(G^{-1}(v_j))G^{-1}(v_j)}{g(G^{-1}(v_j))}\Big)|\nabla v_j|^2\\
&\quad + \int V(x)|G^{-1}(v_j)|^2 - \lambda_j \int f(x, G^{-1}(v_j))G^{-1}(v_j).
\end{aligned}
$$

By (A5), (A6), Sobolev inequality $|v|_{2^*} \leq S|\nabla v|_2$, and (9) of Lemma 2.1, it follows that

$$
\begin{aligned}
\int V(x)|G^{-1}(v_j)|^2 &= \lambda_j \int f(x, G^{-1}(v_j))G^{-1}(v_j)\\
&\quad - \int \Big(1 + \frac{g'(G^{-1}(v_j))G^{-1}(v_j)}{g(G^{-1}(v_j))}\Big)|\nabla v_j|^2\\
&\leq \varepsilon \int |G^{-1}(v_j)|^2 + C_\varepsilon \int |G^{-1}(v_j)|^{2\cdot 2^*}\\
&\leq \varepsilon \int |G^{-1}(v_j)|^2 + C_\varepsilon \Big(\frac{2}{\beta}\Big)^{2^*} S^{2^*}\Big(\int |\nabla v_j|^2\Big)^{2^*/2}.
\end{aligned}
$$

So we have

$$
\int V(x)|G^{-1}(v_j)|^2 \leq C_1 \int (V(x)-\varepsilon)|G^{-1}(v_j)|^2 \leq C_1 C_\varepsilon \Big(\frac{2}{\beta}\Big)^{2^*} S^{2^*}\Big(\int |\nabla v_j|^2\Big)^{2^*/2},
$$

which implies the result. □

Proof of Theorem 2.6. Take $\mathbb{I} = [1/2, 1]$. It is easy to see that $B(v) \geq 0$ for all $v \in E$. Since

$$
\begin{aligned}
\|v\| &= |\nabla v|_2 + \inf_{\xi>0} \frac{1}{\xi}\Big[1 + \int_{\mathbb{R}^N} V(x)|G^{-1}(\xi v)|^2 dx\Big]\\
&\leq |\nabla v|_2 + 1 + \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx := C(v),
\end{aligned}
\tag{3.5}
$$

and

$$
A(v) - C(v) = \int |\nabla v|^2 - \Big(\int |\nabla v|^2\Big)^{1/2} - 1 \geq -\frac{5}{4},
$$

we deduce that $A(v) \to \infty$, as $\|v\| \to \infty$. And by Lemma 3.3 and 3.4 we have $c_\lambda > 0 = \max\{\Phi_\lambda(0), \Phi_\lambda(v_0)\}$ for $\lambda \in \mathbb{I}$. Therefore, by Theorem 3.2, it is easy to know that for almost all $\lambda \in \mathbb{I}$, there exists a sequence $\{v_n(\lambda)\} \subset E$ such that

(1) $\{v_n(\lambda)\}$ is bounded in E;
(2) $\Phi_\lambda(v_n(\lambda)) \to c_\lambda$;

(3) $\Phi'_\lambda(v_n(\lambda)) \to 0$ in E^*;
(4) $0 < c_\lambda \le c_{1/2}$ for $\lambda \in \mathbb{I}$.

Therefore, by Lemma 3.5 we can choose a sequence $\{\lambda_j\} \in [1/2, 1]$ and $v_j = v(\lambda_j)$ such that $\lambda_j \to 1$, $\Phi_{\lambda_j}(v_j) = c_j$ and $\Phi'_{\lambda_j}(v_j) = 0$. We can deduce that v is a solution to (2.6) if we show that there exists a convergent subsequence of $\{v_j\} \in E$(still denoted by $\{v_j\}$) such that $v_j \to v$ in E. To prove this, in view of Lemma 3.5, we need to check that $\{v_j\}$ is a bounded (PS) sequence of Φ. Indeed, the boundedness of $\{v_j\}$ in E follows from Lemma 3.7. We now show that $\{v_j\}$ is a (PS) sequence. It is easy to verify that $G^{-1}(v_j)$ is bounded in $L^s(\mathbb{R}^N)$ for $2 \le s \le 2\cdot 2^*$ by Proposition 2.4. Therefore

$$\lim_{j\to\infty}(1-\lambda_j)\int F(x, G^{-1}(v_j)) \le \lim_{j\to\infty} C(1-\lambda_j)\big[|G^{-1}(v_j)|_2^2 + |G^{-1}(v_j)|_{2\cdot 2^*}^{2\cdot 2^*}\big] = 0.$$

Since $\Phi_{\lambda_j}(v_j) = c_{\lambda_j}$, we have

$$\lim_{j\to\infty}\Phi(v_j) = \lim_{j\to\infty}\Phi_{\lambda_j}(v_j) - \lim_{j\to\infty}(1-\lambda_j)\int F(x, G^{-1}(v_j)) = \lim_{j\to\infty} c_{\lambda_j}.$$

We note that $0 < c_{\lambda_j} \le c_{1/2}$. Therefore there exists a constant $M > 0$ such that $|\Phi(v_j)| \le M$. Similarly, we can verify that $\langle \Phi'(v_j), w\rangle \to 0$ for any $w \in E$. Then from Lemma 3.5, we deduce that there exists a function $v \in E$ such that $v_j \to v$ in E, i.e., $\langle \Phi'(v), w\rangle = 0$ for any $w \in E$, which implies that $u = G^{-1}(v)$ is a solution to (1.1). $\square$

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