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REGULARITY CRITERIA FOR THE WAVE MAP AND RELATED SYSTEMS

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Abstract. We obtain some regularity criteria for the wave map, a liquid crystals model, and the Hall-MHD with ion-slip effect.

1. Introduction

First, we consider the nD wave maps $d : \mathbb{R}^{1+n} \to \mathbb{S}^m \subset \mathbb{R}^{1+m}$ which obey the nonlinear wave equation

$$\partial_t^2 d - \Delta d = d(|\nabla d|^2 - |\partial_t d|^2) \tag{1.1}$$

with the initial conditions

$$(d, \partial_t d)(\cdot, 0) = (d_0, d_1),\ d_0 \in \mathbb{S}^m, \quad d_0 \cdot d_1 = 0. \tag{1.2}$$

Wave maps have wide applications in physics from the harmonic gauge in general relativity to the nonlinear σ-models in particle physics.

Local well-posedness of (1.1) (1.2) has been proved by Tao [20]. Shatah [19] showed that solutions to the Cauchy problem for wave maps may blow up in finite time. However, some smallness assumption on the initial data or integrability condition on the solution itself are sufficient to guarantee the regularity. Fan and Ozawa [11] obtained the regularity criterion

$$\nabla d, \partial_t d \in L^1(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)) \tag{1.3}$$

when $n = 2$.

The first aim of this article is to prove a following regularity criterion when $n \geq 3$.

Theorem 1.1. *Let $n \geq 3$ and $(\nabla d_0, d_1) \in H^{1+s}(\mathbb{R}^n)$ with $s > \frac{n}{2}$, $|d_0| = 1$, $d_0 \cdot d_1 = 0$ and d be a smooth solution of (1.1), (1.2). If (1.3) and $\partial_t d \in L^\infty(0, T; L^n(\mathbb{R}^n))$ hold true with $0 < T < \infty$, then the solution d can be extended beyond $T > 0$.*

Next, we consider the liquid crystals model [2, 3, 4, 16]:

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = -\nabla \cdot (\nabla d \odot \nabla d), \tag{1.4}$$

$$\partial_t d + u \cdot \nabla d - \Delta d = d|\nabla d|^2, \quad |d| = 1, \tag{1.5}$$

$$\operatorname{div} u = 0, \tag{1.6}$$

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$$(u,d)(\cdot,0)=(u_0,d_0) \quad \text{in } \mathbb{R}^n,\ |d_0|=1. \tag{1.7}$$

Here u is the velocity, π is the pressure, d is the direction vector, and $(\nabla d \odot \nabla d)_{i,j} := \sum_k \partial_i d_k \partial_j d_k$, and hence

$$\nabla\cdot(\nabla d\odot\nabla d)=\sum_k \Delta d_k \nabla d_k+\frac{1}{2}\nabla|\nabla d|^2.$$

If $u=0$, then (1.5) is the harmonic heat flow.

Fan-Gao-Guo [9] proved the blow-up criterion

$$u,\nabla d\in L^2(0,T;\dot{B}^0_{\infty,\infty}) \tag{1.8}$$

when $n=3$. One can find other related results in [8, 24] and references therein. We will prove the following theorem.

Theorem 1.2. *Let $n\ge 3$ and $s>\frac{n}{2}$ be an integer. Let u_0 and d_0 satisfy $u_0,\nabla d_0\in H^s, \operatorname{div} u_0=0$, and $|d_0|=1$ in $\mathbb{R}^n$. Let (u,d) be a local strong solution to the problem (1.4)-(1.7). If ∇u and $\nabla^2 d$ satisfy*

$$\nabla u,\nabla^2 d\in L^{\frac{2}{2-\alpha}}(0,T;\dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^n)) \tag{1.9}$$

with $0<\alpha<1$ and $0<T<\infty$, then the solution (u,d) can be extended beyond $T>0$.

Also we consider the incompressible MHD with the Hall or ion-slip system

$$\partial_t u+u\cdot\nabla u+\nabla\Big(\pi+\frac{1}{2}|b|^2\Big)-\Delta u=b\cdot\nabla b, \tag{1.10}$$

$$\partial_t b+u\cdot\nabla b-b\cdot\nabla u+h\operatorname{rot}(\operatorname{rot} b\times b)-\gamma\operatorname{rot}[(\operatorname{rot} b\times b)\times b]=\Delta b, \tag{1.11}$$

$$\operatorname{div} u=\operatorname{div} b=0, \tag{1.12}$$

$$(u,b)(\cdot,0)=(u_0,b_0) \quad \text{in } \mathbb{R}^3. \tag{1.13}$$

Here b is the magnetic field. h is the Hall effect coefficient, and $\gamma\ge 0$ the ion-slip effect coefficient, respectively.

Applications of the Hall-MHD system cover a very wide range of physical subjects, such as, magnetic reconnection in space plasmas, star formation, neutron stars, and geo-dynamos.

Very recently, Zhang [23] obtained the regularity criterion

$$u\in L^{\frac{2}{1-\alpha}}(0,T;\dot{B}^{-\alpha}_{\infty,\infty}),\quad \nabla b\in L^{\frac{2}{1-\beta}}(0,T;\dot{B}^{-\beta}_{\infty,\infty}) \tag{1.14}$$

with $-1<\alpha<1$ and $0<\beta<1$ when $h=1$ and $\gamma=0$.

Local well-posedness of strong solutions to (1.10)-(1.13) has been proved by Fan, Jia, Nakamura and Zhou [10], they also obtained the regularity criterion

$$u\in L^{\frac{2q}{q-3}}(0,T;L^q),\quad b\in L^\infty(0,T;L^\infty),\quad \nabla b\in L^{\frac{2p}{p-3}}(0,T;L^p) \tag{1.15}$$

with $3<p,q\le\infty$. For standard Hall-MHD system we refer to [1, 5, 6, 7, 13, 21, 22] and references therein.

By the method in [23], we will refine (1.15) as follows.

Theorem 1.3. *Let $u_0,b_0\in H^2$ with $\operatorname{div} u_0=\operatorname{div} b_0=0$ in $\mathbb{R}^3$. Let (u,b) be a local strong solution to the problem (1.10)-(1.13). If u and b satisfy (1.14) and $b\in L^\infty(0,T;L^\infty)$ with $0<T<\infty$, then the solution (u,b) can be extended beyond $T>0$.*

In the following proofs, we use the logarithmic Sobolev inequality [15]:

$$\|\nabla d\|_{L^\infty} \le C(1 + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|\nabla d\|_{H^{1+s}})), \tag{1.16}$$

$$\|\partial_t d\|_{L^\infty} \le C(1 + \|\partial_t d\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|\partial_t d\|_{H^{1+s}})) \tag{1.17}$$

for $s > \frac{n}{2} - 1$, and the bilinear product and commutator estimates due to Kato-Ponce [14]:

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|\Lambda^s g\|_{L^{q_2}}), \tag{1.18}$$

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}), \tag{1.19}$$

with $s > 0$, $\Lambda := (-\Delta)^{\frac{1}{2}}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

We also use the Gagliardo-Nirenberg inequalities

$$\|\nabla d\|^2_{L^{2p}} \le C\|d\|_{L^\infty} \|\nabla^2 d\|_{L^p}, \tag{1.20}$$

$$\|\nabla^2 d\|_{L^p} \le C\|\nabla d\|^{1-\theta}_{L^\infty} \|\Lambda^{2+s} d\|^\theta_{L^2}, \tag{1.21}$$

$$\|\Lambda^{1+s} d\|_{L^{\frac{2p}{p-2}}} \le C\|\nabla d\|^\theta_{L^\infty} \|\Lambda^{2+s} d\|^{1-\theta}_{L^2} \tag{1.22}$$

with $p := 2s + 2$ and $\theta := 1/(1 + s)$.

We also use the improved Gagliardo-Nirenberg inequalities [12, 17, 18]:

$$\|\nabla u\|_{L^{q_1}} \le C\|\nabla u\|^{1-\theta_1}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|u\|^{\theta_1}_{\dot{H}^{s+\alpha}}, \tag{1.23}$$

$$\|\Lambda^s u\|_{L^{\frac{2q_1}{q_1-2}}} \le C\|\nabla u\|^{\theta_1}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|u\|^{1-\theta_1}_{\dot{H}^{s+\alpha}}, \tag{1.24}$$

with $q_1 := \frac{2(s-1+2\alpha)}{\alpha}$ and $\theta_1 := 2/q_1$, and

$$\|\nabla d\|_{L^{q_2}} \le C\|\nabla d\|^{1-\theta_2}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|\nabla d\|^{\theta_2}_{\dot{H}^{s+\alpha}}, \tag{1.25}$$

$$\|\Lambda^s \nabla d\|_{L^{\frac{2q_2}{q_2-2}}} \le C\|\nabla d\|^{\theta_2}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|\nabla d\|^{1-\theta_2}_{\dot{H}^{s+\alpha}}, \tag{1.26}$$

with $q_2 := \frac{2(s+2\alpha)}{\alpha}$ and $\theta_2 := \frac{2}{q_2}$,

$$\|\nabla d\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \le C\|d\|^{\frac{1}{2-\alpha}}_{L^\infty} \|\nabla^2 d\|^{\frac{1-\alpha}{2-\alpha}}_{\dot{B}^{-\alpha}_{\infty,\infty}}, \tag{1.27}$$

and

$$\|D^k u\|_{L^{p_k}} \le C\|\nabla u\|^{1-\tilde\theta_k}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|u\|^{1-\tilde\theta_k}_{\dot{H}^{s+\alpha}}, \tag{1.28}$$

$$\|D^{s+2-k} d\|_{L^{\frac{2p_k}{p_k-2}}} \le C\|\nabla^2 d\|^{\tilde\theta_k}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|\nabla d\|^{1-\tilde\theta_k}_{\dot{H}^{s+\alpha}}, \tag{1.29}$$

with $p_k := \frac{2}{\tilde\theta_k}$ and $\tilde\theta_k := \frac{k+\alpha-1}{s+2\alpha-1}$, and

$$\|\nabla u\|^3_{L^3} \le C\|u\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \|u\|^2_{\dot{H}^{\frac{3+\alpha}{2}}} \quad \text{with} \quad -1 < \alpha < 1, \tag{1.30}$$

$$\|u\|_{\dot{H}^{\frac{3+\alpha}{2}}} \le C\|\nabla u\|^{\frac{1-\alpha}{2}}_{L^2} \|\Delta u\|^{\frac{1+\alpha}{2}}_{L^2} \quad \text{with} \quad -1 < \alpha < 1, \tag{1.31}$$

and

$$\|\nabla b\|^2_{L^4} \le C\|\nabla b\|_{\dot{B}^{-\beta}_{\infty,\infty}} \|b\|_{\dot{H}^{1+\beta}} \quad \text{with } 0 < \beta < 1, \tag{1.32}$$

$$\|b\|_{\dot{H}^{1+\beta}} \le C\|\nabla b\|^{1-\beta}_{L^2} \|\Delta b\|^\beta_{L^2} \quad \text{with } 0 < \beta < 1. \tag{1.33}$$

2. Proof of Theorem 1.1

Testing (1.1) by $\partial_t d$ and using $|d|=1$ and $d\cdot\partial_t d=0$, we easily get the conservation of the energy:

$$\frac{d}{dt}\int(|\partial_t d|^2+|\nabla d|^2)dx=0. \tag{2.1}$$

Applying the operator Λ^{1+s} to equation (1.1), testing by $\Lambda^{1+s}\partial_t d$, using (1.18), (1.16), (1.17), (1.20), (1.21) and (1.22), we reach

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(|\Lambda^{1+s}\partial_t d|^2+|\Lambda^{2+s}d|^2)dx\\
&=\int\Lambda^{1+s}(d|\nabla d|^2-d|\partial_t d|^2)\Lambda^{1+s}\partial_t d dx\\
&\le(\|\Lambda^{1+s}(d|\nabla d|^2)\|_{L^2}+\|\Lambda^{1+s}(d|\partial_t d|^2)\|_{L^2})\|\Lambda^{1+s}\partial_t d\|_{L^2}\\
&\le C(\|d\|_{L^\infty}\|\Lambda^{1+s}(|\nabla d|^2)\|_{L^2}+\|\nabla d\|_{L^{2p}}^2\|\Lambda^{1+s}d\|_{L^{\frac{2p}{p-2}}})\|\Lambda^{1+s}\partial_t d\|_{L^2}\\
&\quad+C(\|d\|_{L^\infty}\|\Lambda^{1+s}(|\partial_t d|^2)\|_{L^2}+\|\partial_t d\|_{L^{2n}}^2\|\Lambda^{1+s}d\|_{L^{\frac{2n}{n-2}}})\|\Lambda^{1+s}\partial_t d\|_{L^2}\\
&\le C(\|\nabla d\|_{L^\infty}\|\Lambda^{2+s}d\|_{L^2}+\|\nabla^2 d\|_{L^p}\|\Lambda^{1+s}d\|_{L^{\frac{2p}{p-2}}})\|\Lambda^{1+s}\partial_t d\|_{L^2}\\
&\quad+C(\|\partial_t d\|_{L^\infty}\|\Lambda^{1+s}\partial_t d\|_{L^2}+\|\partial_t d\|_{L^n}\|\partial_t d\|_{L^\infty}\|\Lambda^{2+s}d\|_{L^2})\|\Lambda^{1+s}\partial_t d\|_{L^2}\\
&\le C\|\nabla d\|_{L^\infty}\|\Lambda^{2+s}d\|_{L^2}\|\Lambda^{1+s}\partial_t d\|_{L^2}\\
&\quad+C\|\partial_t d\|_{L^\infty}\|\Lambda^{1+s}\partial_t d\|_{L^2}^2+C\|\partial_t d\|_{L^\infty}\|\Lambda^{2+s}d\|_{L^2}\|\Lambda^{1+s}\partial_t d\|_{L^2}\\
&\le C(\|\nabla d\|_{L^\infty}+\|\partial_t d\|_{L^\infty})(\|\Lambda^{2+s}d\|_{L^2}^2+\|\Lambda^{1+s}\partial_t d\|_{L^2}^2)\\
&\le C(1+\|\nabla d\|_{\dot B^0_{\infty,\infty}}+\|\partial_t d\|_{\dot B^0_{\infty,\infty}})\log(e+y^2)y^2,
\end{aligned}$$

with $y^2:=\|\Lambda^{1+s}\partial_t d\|_{L^2}^2+\|\Lambda^{2+s}d\|_{L^2}^2$, which gives

$$\sup_{0\le t\le T}(\|\Lambda^{1+s}\partial_t d\|_{L^2}^2+\|\Lambda^{2+s}d\|_{L^2}^2)\le C.$$

This completes the proof.

3. Proof of Theorem 1.2

Since it is easy to prove that there are $T_0>0$ and a unique strong solution (u,π,d) to the problem (1.4)-(1.7) in $[0,T_0]$, we only need to prove a priori estimates. Testing (1.4) by u and using (1.6), we see that

$$\frac{1}{2}\frac{d}{dt}\int|u|^2dx+\int|\nabla u|^2dx=-\int(u\cdot\nabla)d\cdot\Delta d dx. \tag{3.1}$$

Testing (1.5) by $-\Delta d$, using $d\Delta d=-|\nabla d|^2$ and $|d|=1$, we find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int|\nabla d|^2dx+\int|\Delta d|^2dx&=\int(u\cdot\nabla)d\cdot\Delta d dx+\int(d\Delta d)^2dx\\
&\le\int(u\cdot\nabla)d\cdot\Delta d dx+\int|\Delta d|^2dx.
\end{aligned}\tag{3.2}$$

Summing up (3.1) and (3.2), we have

$$\int(|u|^2+|\nabla d|^2)dx\le\int(|u_0|^2+|\nabla d_0|^2)dx. \tag{3.3}$$

Applying D^s to (1.4), testing by $D^s u$, using (1.6), (1.18), (1.19), (1.23), (1.24), (1.25), (1.26) and (1.27), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |D^s u|^2 dx + \int |D^{1+s}u|^2 dx \\
&= -\int (D^s(u\cdot\nabla u) - u\nabla D^s u) D^s u dx + \int D^s(\nabla d\odot\nabla d) : \nabla D^s u dx \\
&\le C\|\nabla u\|_{L^{q_1}} \|D^s u\|_{L^{\frac{2q_1}{q_1-2}}} \|D^s u\|_{L^2} + C\|\nabla d\|_{L^{q_2}} \|D^s\nabla d\|_{L^{\frac{2q_2}{q_2-2}}} \|\nabla D^s u\|_{L^2} \\
&\le C\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}} \|D^{s+\alpha}u\|_{L^2}\|D^s u\|_{L^2} + C\|\nabla d\|_{\dot B^{-\alpha}_{\infty,\infty}} \|D^{s+\alpha}\nabla d\|_{L^2}\|D\Lambda^s u\|_{L^2} \\
&\le C\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}} \|D^s u\|_{L^2}^{2-\alpha}\|D^{1+s}u\|_{L^2}^{\alpha} \\
&\quad + C\|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{1-\alpha}{2-\alpha}} \|D^{s+1}d\|_{L^2}^{1-\alpha}\|D^{s+2}d\|_{L^2}^{\alpha}\|D^{1+s}u\|_{L^2} \\
&\le \frac{1}{8}\|D^{1+s}u\|_{L^2}^2 + \frac{1}{8}\|D^{s+2}d\|_{L^2}^2 + C\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}} \|D^s u\|_{L^2}^2 \\
&\quad + C\|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}} \|D^{s+1}d\|_{L^2}^2.
\end{aligned}
\tag{3.4}
$$

Applying D^{s+1} to (1.5), testing by $D^{s+1}d$ and using (1.6), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |D^{s+1}d|^2 dx + \int |D^{s+2}d|^2 dx \\
&= \int D^{s+1}(d|\nabla d|^2) D^{s+1} d dx \\
&\quad - \int (D^{s+1}(u\cdot\nabla d) - u\nabla D^{s+1}d) D^{s+1} d\, dx =: I_1 + I_2.
\end{aligned}
\tag{3.5}
$$

Using (1.18), $|d| = 1$, (1.25), (1.26), and (1.27), we bound I_1 as follows.

$$
\begin{aligned}
I_1 &\le \|D^{s+1}(d|\nabla d|^2)\|_{L^{\frac{2q_2}{q_2+2}}} \|D^{s+1}d\|_{L^{\frac{2q_2}{q_2-2}}} \\
&\le C(\|d\|_{L^\infty}\|D^{s+1}(|\nabla d|^2)\|_{L^{\frac{2q_2}{q_2+2}}} + \|\nabla d\|_{L^{q_2}}^2 \|D^{s+1}d\|_{L^{\frac{2q_2}{q_2-2}}}) \|D^{s+1}d\|_{L^{\frac{2q_2}{q_2-2}}} \\
&\le C(\|\nabla d\|_{L^{q_2}}\|D^{s+2}d\|_{L^2} + \|\nabla d\|_{L^{q_2}}^2 \|D^{s+1}d\|_{L^{\frac{2q_2}{q_2-2}}}) \|D^{s+1}d\|_{L^{\frac{2q_2}{q_2-2}}} \\
&\le C\|\nabla d\|_{L^{q_2}}^2 \|D^{s+1}d\|_{L^{\frac{2q_2}{q_2-2}}}^2 + \frac{1}{16}\|D^{s+2}d\|_{L^2}^2 \\
&\le C\|\nabla d\|_{\dot B^{-\alpha}_{\infty,\infty}}^2 \|\nabla d\|_{\dot H^{s+\alpha}}^2 + \frac{1}{16}\|D^{s+2}d\|_{L^2}^2 \\
&\le \frac{1}{8}\|D^{s+2}d\|_{L^2}^2 + C\|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}} \|D^{s+1}d\|_{L^2}^2.
\end{aligned}
$$

Using the Leibniz rule, we write I_2 as follows.

$$
\begin{aligned}
I_2 &= -\int (C_1 DuD^{s+1}d + \sum_{k=2}^{s} C_k D^k u D^{s+2-k}d + C_{s+1}D^{s+1}u\cdot\nabla d) D^{s+1} d dx \\
&=: I_2^1 + \sum_{k=2}^{s} I_2^k + I_2^{s+1}.
\end{aligned}
\tag{3.6}
$$

By the same calculations as that of I_1, we have

$$\begin{aligned} I_2^{s+1} &\le C\|\nabla d\|_{L^{q_2}}\|D^{s+1}d\|_{L^{\frac{2q_2}{q_2-2}}}\|D^{s+1}u\|_{L^2} \\ &\le \frac{1}{16}\|D^{s+1}u\|_{L^2}^2 + C\|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}\|D^{s+1}d\|_{L^2}^2. \end{aligned} \tag{3.7}$$

Using (1.23) and (1.24), we bound I_2^1 as follows.

$$\begin{aligned} I_2^1 &\le C\|\nabla u\|_{L^{q_1}}\|D^{s+1}d\|_{L^{\frac{2q_1}{q_1-2}}}\|D^{s+1}d\|_{L^2} \\ &\le C\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{1-\theta_1}\|u\|_{\dot H^{s+\alpha}}^{\theta_1}\cdot\|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\theta_1}\|\nabla d\|_{\dot H^{s+\alpha}}^{1-\theta_1}\|D^{s+1}d\|_{L^2} \\ &\le C(\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}} + \|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}})(\|u\|_{\dot H^{s+\alpha}} + \|\nabla d\|_{\dot H^{s+\alpha}})\|D^{s+1}d\|_{L^2} \\ &\le \frac{1}{16}\|D^{s+1}u\|_{L^2}^2 + \frac{1}{16}\|D^{s+2}d\|_{L^2}^2 \\ &\quad + C(\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}} + \|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}})(\|D^s u\|_{L^2}^2 + \|D^{s+1}d\|_{L^2}^2). \end{aligned} \tag{3.8}$$

Using (1.28) and (1.29), we bound $\sum_{k=2}^{s} I_2^k$ as follows.

$$\begin{aligned} \sum_{k=2}^{s} I_2^k &\le C\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{1-\tilde\theta_k}\|u\|_{\dot H^{s+\alpha}}^{\tilde\theta_k}\|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\tilde\theta_k}\|\nabla d\|_{\dot H^{s+\alpha}}^{1-\tilde\theta_k}\|D^{s+1}d\|_{L^2} \\ &\le C(\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}} + \|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}})(\|u\|_{\dot H^{s+\alpha}} + \|\nabla d\|_{\dot H^{s+\alpha}})\|D^{s+1}d\|_{L^2} \\ &\le C(\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}} + \|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}})(\|u\|_{\dot H^{s+\alpha}} + \|\nabla d\|_{\dot H^{s+\alpha}}) \\ &\quad \times (\|D^s u\|_{L^2} + \|D^{s+1}d\|_{L^2}) \\ &\le \frac{1}{16}\|D^{s+1}u\|_{L^2}^2 + \frac{1}{16}\|D^{s+2}d\|_{L^2}^2 \\ &\quad + C(\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}} + \|\nabla^2 d\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}})(\|D^s u\|_{L^2}^2 + \|D^{s+1}d\|_{L^2}^2). \end{aligned} \tag{3.9}$$

Inserting the above estimates into (3.5) and combining with (3.4) and using the Gronwall inequality, we arrive at

$$\|D^s u\|_{L^\infty(0,T;L^2)} + \|D^{s+1}d\|_{L^\infty(0,T;L^2)} \le C.$$

This completes the proof.

4. Proof of Theorem 1.3

We only need to show a priori estimates. For simplicity, we will take $h = \gamma = 1$. First, testing (1.10) by u and using (1.12), we see that

$$\frac{1}{2}\frac{d}{dt}\int |u|^2 dx + \int |\nabla u|^2 dx = \int (b\cdot\nabla)b\cdot u dx. \tag{4.1}$$

Testing (1.11) by b and using (1.12), we find that

$$\frac{1}{2}\frac{d}{dt}\int |b|^2 dx + \int |\nabla b|^2 dx + \int |b\times \operatorname{rot} b|^2 dx = \int (b\cdot\nabla)u\cdot b dx. \tag{4.2}$$

Summing up (4.1) and (4.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\int (|u|^2 + |b|^2) dx + \int (|\nabla u|^2 + |\nabla b|^2 + |b\times \operatorname{rot} b|^2) dx = 0. \tag{4.3}$$

Testing (1.10) by $-\Delta u$, using (1.12), (1.30) and (1.31), we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla u|^2dx+\int|\Delta u|^2dx\\
&=\int(u.\nabla)u\cdot\Delta udx-\int(b\cdot\nabla)b\cdot\Delta udx\\
&=-\sum_{i,j}\int\partial_ju_i\partial_iu\partial_judx-\int(b\cdot\nabla)b\cdot\Delta udx\\
&\le C\|\nabla u\|_{L^3}^3+\|b\|_{L^\infty}\|\nabla b\|_{L^2}\|\Delta u\|_{L^2}\\
&\le C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|u\|^2_{\dot H^{\frac{3+\alpha}{2}}}+C\|\nabla b\|_{L^2}\|\Delta u\|_{L^2}\\
&\le C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla u\|_{L^2}^{1-\alpha}\|\Delta u\|_{L^2}^{1+\alpha}+C\|\nabla b\|_{L^2}\|\Delta u\|_{L^2}\\
&\le\frac{1}{8}\|\Delta u\|_{L^2}^2+C\|u\|^{\frac{2}{1-\alpha}}_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla u\|_{L^2}^2+C\|\nabla b\|_{L^2}^2.
\end{aligned}\tag{4.4}$$

Testing (1.11) by $-\Delta b$ and using (1.12), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla b|^2dx+\int|\Delta b|^2dx\\
&=\int(u\cdot\nabla)b\cdot\Delta bdx-\int(b\cdot\nabla)u\cdot\Delta bdx\\
&\quad+\int(\operatorname{rot}b\times b)\operatorname{rot}\Delta bdx-\int[(\operatorname{rot}b\times b)\times b]\operatorname{rot}\Delta bdx\\
&=:\ell_1+\ell_2+\ell_3+\ell_4.
\end{aligned}\tag{4.5}$$

We bound ℓ_1 and ℓ_2 as follows.

$$\begin{aligned}
\ell_1&=\sum_{i,j}\int u_i\partial_ib\partial_j^2bdx=-\sum_{i,j}\int\partial_ju_i\partial_ib\partial_jbdx\le C\|\nabla u\|_{L^2}\|\nabla b\|_{L^4}^2\\
&\le C\|\nabla u\|_{L^2}\|b\|_{L^\infty}\|\Delta b\|_{L^2}\le C\|\nabla u\|_{L^2}\|\Delta b\|_{L^2}\le\frac{1}{16}\|\Delta b\|_{L^2}^2+C\|\nabla u\|_{L^2}^2.\\
\ell_2&\le\|b\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta b\|_{L^2}\le C\|\nabla u\|_{L^2}\|\Delta b\|_{L^2}\le\frac{1}{16}\|\Delta b\|_{L^2}^2+C\|\nabla u\|_{L^2}^2.
\end{aligned}$$

Using (1.32) and (1.33), we bound ℓ_3 and ℓ_4 as follows.

$$\begin{aligned}
\ell_3&=-\sum_i\int(\operatorname{rot}b\times\partial_ib)\partial_i\operatorname{rot}bdx\le C\|\nabla b\|_{L^4}^2\|\Delta b\|_{L^2}\\
&\le C\|\nabla b\|_{\dot B^{-\beta}_{\infty,\infty}}\|b\|_{\dot H^{1+\beta}}\|\Delta b\|_{L^2}\le C\|\nabla b\|_{\dot B^{-\beta}_{\infty,\infty}}\|\nabla b\|_{L^2}^{1-\beta}\|\Delta b\|_{L^2}^{1+\beta}\\
&\le\frac{1}{16}\|\Delta b\|_{L^2}^2+C\|\nabla b\|^{\frac{2}{1-\beta}}_{\dot B^{-\beta}_{\infty,\infty}}\|\nabla b\|_{L^2}^2.
\end{aligned}$$

$$\begin{aligned}
\ell_4&=\sum_i\int\partial_i[(\operatorname{rot}b\times b)\times b]\partial_i\operatorname{rot}bdx\le\sum_i\int[(\operatorname{rot}b\times\partial_ib)\times b]\partial_i\operatorname{rot}bdx\\
&\quad+\sum_i\int[(\operatorname{rot}b\times b)\times\partial_ib]\partial_i\operatorname{rot}bdx\le C\|b\|_{L^\infty}\|\nabla b\|_{L^4}^2\|\Delta b\|_{L^2}\\
&\le\frac{1}{16}\|\Delta b\|_{L^2}^2+C\|\nabla b\|^{\frac{2}{1-\beta}}_{\dot B^{-\beta}_{\infty,\infty}}\|\nabla b\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates into (4.5), and combining this with (4.4), and using the Gronwall inequality, we conclude that

$$\|\nabla u\|_{L^\infty(0,T;L^2)} + \|\nabla b\|_{L^\infty(0,T;L^2)} \le C. \tag{4.6}$$

This completes the proof by (1.15).

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