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Radially Symmetric Solutions for a Class of Critical Exponent Elliptic Problems in $\mathbb{R}^N$ *

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Abstract

We give a method for obtaining radially symmetric solutions for the critical exponent problem

$$\begin{cases} -\Delta u + a(x)u = \lambda u^q + u^{2^*-1} \text{ in } \mathbb{R}^N \\ u > 0 \text{ and } \int_{\mathbb{R}^N} |\nabla u|^2 < \infty \end{cases}$$

where, outside a ball centered at the origin, the non-negative function a is bounded from below by a positive constant $a_o > 0$. We remark that, differently from the literature, we do not require any conditions on a at infinity.

1 Introduction

Our purpose in this paper is to solutions for the semi-linear elliptic problem:

$$\begin{cases} -\Delta u + a(x)u = \lambda u^q + u^{2^*-1} \text{ in } \mathbb{R}^N \\ u > 0 \text{ and } \int_{\mathbb{R}^N} |\nabla u|^2 < \infty. \end{cases} \tag{1}$$

where $a : \mathbb{R}^N \to \mathbb{R}$ is a non-negative radially symmetric C^1 function, $2* = 2N/(N-2)$; $1 < q < 2^* - 1$, $\lambda > 0$, and $N \geq 3$.

Several researchers have studied variants of problem (1). Among others, we can cite the article by Brèzis & Nirenberg [8] which treats the case $a \equiv 0$ in bounded domains. Azorero & Alonzo in [3] and [4] generalize some similar results for the $p-$Laplacian operator in bounded domains. Egnell [11] also generalizes some results in [9]. In the case of unbounded domains, Rabinowitz [21] considers a more general non-linearity, but he does not treat the Sobolev critical exponent case. Benci & Cerami [5] consider the problem (1) when $\lambda = 0$, and [2] deals with the case where λ is replaced by an integrable function. In

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[10], a variation of this problem, with a constant, was solved for the biharmonic operator. To finish citations we list the following works: [1] Alves & Gonçalves, [14] Gonçalves & Miyagaki, [15] Jianfu, [16] Jianfu & Xiping. All the last results in unbounded domains are obtained under the crucial hypothesis that a is a coercive function or that $\lim_{|x|\to+\infty} a(x)$ exists.

We improve their results, relaxing the coerciveness of a and the existence of the above limit. As in [21], we shall use variational method to solve problem (1). To describe precisely our results, we present below the hypotheses on the function a:

(A_o) $a \in C^1(\mathbb{R}^N)$ is a radially symmetric function and there are a_o, $R > 0$ such that $a(x) \geq a_o$, for all $|x| \geq R$.

Let us consider the following $W^{1,2}(\mathbb{R}^N)$ Hilbert subspace:

$$H^1_{\mathrm{rad}}(\mathbb{R}^N) = \{u \in W^{1,2}(\mathbb{R}^N) : u \text{ radially symmetric } \}.$$

Our main result is the following.

Theorem 1 *If (A_o) is satisfied, then problem (1) possesses a nontrivial classical solution $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$, for all $\lambda > 0$ and $1 < q < 2^* - 1$ when $N \geq 4$. In the case $N = 3$ the same result is valid if $3 < q < 6$.*

Remark 1 *When λ is large enough, (1) possesses a nontrivial classical solution. Later we shall justify this remark.*

Employing the same techniques used to prove the above theorem, we improve the results obtained in the subcritical exponent case due to Rabinowitz (see [21]), where he considers the problem

$$-\Delta u + a(x)u = f(x, u) \text{ in } \mathbb{R}^N \tag{2}$$

for a given C^1-function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ with a coercive.

Results related to this kind of problem can be found in [6], [21], among others.

In [6], H. Berestycki and P. L. Lions obtained positive solution of problem (1) when the non-linearity f does not depend on x. They obtained the solution as a limit of positive solutions of the problem restricted to bounded domains. In their paper they basically made use of H^1-estimates.

Our second result is a global version on $\mathbb{R}^N$ of a well known result for bounded domain due to Rabinowitz (theorem 2.15 in [20]):

Theorem 2 *Suppose that $a \in C^1(\mathbb{R}^N)$ satisfies (A_o) and f satisfies:*

(f_o) *The function f is a C^1, radially symmetric function in x, i.e., $f(x, s) = f(r, s)$ where $r = |x|$, for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$.*

(f_1) *For each* $\varepsilon > 0$, *there is a constant* $a_1 > 0$, *such that*

$$|f(x,s)| \le \varepsilon|s| + a_1|s|^p, \text{ for all } x \in \mathbb{R}^N, \quad s \in \mathbb{R},$$

where $1 \le p < 2^* - 1$.

(f_2) *There is* $\mu > 2$, *such that*

$$0 < \mu F(x,s) \dot{\le} s f(x,s), \text{ for all } x \in \mathbb{R}^N, \ s \in \mathbb{R}\backslash\{0\},$$

where $F(x,s) = \int_o^s f(x,t)dt$.

Then (2) possesses a nontrivial classical solution $u \in W^{1,2}(\mathbb{R}^N)$.

2 Proof of Theorem 1

First, let us formulate a proper framework to solve problem (1). Define the Hilbert space

$$E = \{u \in H^1_{\text{rad}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)u^2 < \infty\},$$

endowed with the inner product $\langle u, v\rangle =: \int_{\mathbb{R}^N}(\nabla u \nabla v + a(x)uv)$ and the norm $||u||^2 =: \int_{\mathbb{R}^N}(|\nabla u|^2 + a(x)u^2$

Now we present two lemmas that will be used in the proof of the Theorem 1.

Lemma 1 *Let* w *be a* $W^{1,s}_{\text{loc}}(\mathbb{R}^N)$ *function satisfying*

$$-\Delta w = h, \tag{3}$$

in $\mathbb{R}^N\backslash\{0\}$ *in the weak sense, where* h *is a* $L^1_{\text{loc}}(\mathbb{R}^N)$ *function and* $s \ge \frac{N}{N-1}$. *Then (3) is weakly satisfied in the whole* $\mathbb{R}^N$.

Proof: In order to prove this result, consider $\varphi \in C^\infty(\mathbb{R}^N)$ such that $\varphi(x) = 0$ in $|x| \le 1$ and $\varphi(x) = 1$ in $|x| \ge 2$. For each $\varepsilon > 0$, define $\psi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon})$. Fix a function $\phi \in C_c^\infty(\mathbb{R}^N)$. As $\psi_\varepsilon\phi \in C_c^\infty(\mathbb{R}^N\backslash\{0\})$ we have that

$$\int_{\mathbb{R}^N} \nabla w \nabla(\psi_\varepsilon \phi) = \int_{\mathbb{R}^N} h(x)(\psi_\varepsilon\phi),$$

and then

$$\int_{\mathbb{R}^N} \psi_\varepsilon \nabla w \nabla \phi + \int_{\mathbb{R}^N} \phi \nabla w \nabla \psi_\varepsilon = \int_{\mathbb{R}^N} h(x)(\psi_\varepsilon\phi). \tag{4}$$

Using the dominated convergence theorem, we obtain the limits

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \psi_\varepsilon \nabla w \nabla\phi &= \int_{\mathbb{R}^N} \nabla w \nabla \phi \\ \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} h(x)(\psi_\varepsilon\phi) &= \int_{\mathbb{R}^N} h(x)\phi. \end{aligned} \tag{5}$$

We claim that the limit of the second term on the left side of (4) is zero. In fact,

$$\left|\int_{\mathbb{R}^N} \phi\nabla w\nabla\psi_\varepsilon\right| \le ||\phi||_{L^\infty(\mathbb{R}^N)}\int_{|x|\le 2\varepsilon}|\nabla w||\nabla\psi_\varepsilon|.$$

Using Hölder's inequality in the above inequality with $\frac{1}{s}+\frac{1}{q}=1$, we obtain that

$$\left|\int_{\mathbb{R}^N} \phi\nabla w\nabla\psi_\varepsilon\right| \le ||\phi||_{L^\infty(\mathbb{R}^N)}\left(\int_{|x|\le 2\varepsilon}|\nabla w|^s\right)^{1/s}\left(\int_{|x|\le 2\varepsilon}|\nabla\psi_\varepsilon|^q\right)^{1/q}$$

and then

$$\left|\int_{\mathbb{R}^N} \phi\nabla w\nabla\psi_\varepsilon\right| \le ||\phi||_{L^\infty(\mathbb{R}^N)}||\nabla\varphi||_{L^q(\mathbb{R}^N)}\left(\int_{|x|\le 2\varepsilon}|\nabla w|^s\right)^{1/s}\varepsilon^{\frac{N-q}{q}}.$$

Observe that $N\ge q$ and passing to the limit in this last inequality we prove the claim.

Finally using the claim and the limits (5) in (4) we have that

$$\int_{\mathbb{R}^N}\nabla w\nabla\phi=\int_{\mathbb{R}^N}h(x)\phi.$$

Remark 2 *The above result is not valid for $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ functions. The function $w=|x|^{2-N}$ (if $N\ge 3$, or $w=\log|x|$, if $N=2$) belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$, satisfies $-\Delta w=0$ in $\mathbb{R}^N\backslash\{0\}$, but if v is a radially symmetric function in $C^\infty_c(\mathbb{R}^N)$ such that $v(0)\ne 0$, we have that*

$$\int_{\mathbb{R}^N}\nabla w\nabla v=\frac{\omega_N}{2-N}\int_o^\infty r^{N-1}r^{1-N}v'(r)dr=\frac{\omega_N}{2-N}v(0)\ne 0,\ \textit{if } N\ge 3$$

or

$$\int_{\mathbb{R}^N}\nabla w\nabla v=2\pi v(0)\ne 0,\ \textit{if } N=2.$$

Lemma 2 *Let $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ be a C^1 function satisfying (f_o) such that*

$$|f(x,s)|\le c|s|+|s|^{2^*-1}\ \textit{for all } x\in\mathbb{R}^N,\ s\in\mathbb{R};$$

and let a be a radially symmetric function. Suppose that $u\in E$ satisfies

$$\int_{\mathbb{R}^N}(\nabla u\nabla v+a(x)uv)=\int_{\mathbb{R}^N}f(x,u)v,\ \textit{for all } v\in E\,.$$

Then $u\in C^2(\mathbb{R}^N)$ and $-\Delta u(x)+a(x)u(x)=f(x,u(x))$ for all $x\in\mathbb{R}^N$.

Proof: Since a and f are radially symmetric we rewrite the above expression as

$$\int_o^\infty r^{N-1}(u'v'+a(r)uv)dr = \int_o^\infty r^{N-1}f(r,u)vdr, \tag{6}$$

for all $v \in E$. We have that $h(r) := -a(r)u(r) + f(r,u(r))$ is in $C^{o,\alpha}(\mathbb{R}^N\backslash\{0\})$, since $H^1_{\rm rad}(\mathbb{R}^N)$ is contained in $C^{o,\alpha}(\mathbb{R}^N\backslash\{0\})$. Hence

$$\int_o^\infty r^{N-1}u'\psi' dr = \int_o^\infty r^{N-1}h(r)\psi dr, \text{ for all } \psi \in C_c^\infty(0,+\infty),$$

and

$$\int_o^\infty u'(r^{N-1}\psi)'dr = \int_o^\infty [\frac{N-1}{r}u' + h(r)](r^{N-1}\psi)dr, \tag{7}$$

for all $\psi \in C_c^\infty(0,+\infty)$. For each $\varphi \in C_c^\infty(0,+\infty)$, considering $\psi = r^{1-N}\varphi$ in (7) we conclude that u is a weak solution of

$$-u'' = \frac{N-1}{r}u' + h(r), \text{ for } r > 0.$$

Since $u' \in L^2_{\rm loc}(0,+\infty)$, it follows that $u \in H^2_{\rm loc}(0,+\infty)$, $u' \in H^1_{\rm loc}(0,+\infty)$, and

$$u \in H^1_{\rm rad}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N\backslash\{0\})\,.$$

Moreover for $|x| > 0$, the function u satisfies (1) in the classical sense. ♢

Proof of Theorem 1 . This proof consists of using variational methods to get critical points of the Euler-Lagrange functional associated to (1) and defined on E:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + a(x)u^2) - \frac{\lambda}{q+1}\int_{\mathbb{R}^N}(u^+)^{q+1} - \frac{1}{2^*}\int_{\mathbb{R}^N}(u^+)^{2^*}$$

where $u^+(x) = \max\{u(x),0\}$ and $u^+(x) = \min\{-u(x),0\}$.

The critical points of I are precisely the weak solutions of (1). These solutions may be regularized.

The Hilbert space E is immersed continuously in $W^{1,2}(\mathbb{R}^N)$. This assertion comes from (A_o) and the following inequalities

$$\begin{aligned} \left(\int_{|x|\le R} u^2\right)^{1/2} &\le c_1\left(\int_{|x|\le R}|u|^{2^*}\right)^{1/2^*} \\ &\le c_1\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{1/2^*} \\ &\le c_2\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{1/2}. \end{aligned}$$

We also have that $H^1_{\mathrm{rad}}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ continuously if $2 \le p \le 2^*$ and compactly if $2 \le p < 2^*$(see [17]). Using these results one has the following lemma:

Lemma 3 *The Banach space E is continuously immersed in $L^p(\mathbb{R}^N)$ if $2 \le p \le 2^*$ and compactly if $2 \le p < 2^*$.*

Using lemma 3 we verify that I is a well-defined $C^1(E)$ functional - see [22]. It is easy to verify that

$$\frac{\lambda}{q+1}\int_{\mathbb{R}^N}(u^+)^{q+1} + \frac{1}{2^*}\int_{\mathbb{R}^N}(u^+)^{2^*} = o(||u||^2) \text{ as } u \to 0, \tag{8}$$

and hence that I has a local minimum at the origin. This is not a global minimum. If $u \in E\backslash\{0\}$, $u \ge 0$, we have that

$$I(tu) = \frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + a(x)u^2) - \frac{\lambda t^{q+1}}{q+1}\int_{\mathbb{R}^N}(u^+)^{q+1} - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}(u^+)^{2^*}.$$

Since $\int_{\mathbb{R}^N}(u^+)^{2^*} \neq 0$, we conclude that $I(tu) \to -\infty$ as $t \to \infty$. So, we have just seen that I has the *Mountain Pass Theorem Geometry.*

Let $e \in E$ such that $I(e) < 0$, and

$$\Gamma = \{g : [0,1] \to E : g(0) = 0 \text{ and } g(1) = e\}$$

and

$$c = \inf_{g\in\Gamma}\max_{0\le t\le 1} I(g(t)).$$

Thus c is the mountain pass minimax value associated to I. At this moment, it is important to notice that c is not the minimax value associated to the Euler Lagrange functional of problem (1) defined in the whole $W^{1,2}(\mathbb{R}^N)$. Assertion (8) implies $c > 0$. Using an application of the Ekeland Variational Principle (Theorem 4.3 of [19]), there exists a sequence $\{u_m\} \subset E$ such that

$$I(u_m) \to c,\ I'(u_m) \to 0. \tag{9}$$

Lemma 4 *The above sequence $\{u_m\}$ is bounded.*

Proof: Notice that

$$I(u_m) - \frac{1}{q+1}I'(u_m)u_m = \left(\frac{1}{2} - \frac{1}{q+1}\right)||u_m||^2 + \left(\frac{1}{q+1} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N}\left(u_m^+\right)^{2^*},$$

then

$$I(u_m) - \frac{1}{q+1}I'(u_m)u_m \ge \left(\frac{1}{2} - \frac{1}{q+1}\right)||u_m||^2.$$

Combining this last inequality with

$$I(u_m) - \frac{1}{q+1} I'(u_m)u_m \leq 1 + c + ||u_m||$$

for large m, we conclude the proof. $\diamondsuit$

The following lemma shows that we can choose a vector $e \in E\backslash\{0\}$ in the definition of Γ, such that $I(e) < 0$ and

$$0 < c < \frac{1}{N} S^{N/2}, \tag{10}$$

where S is the best constant of the Sobolev immersion $W^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$, this is

$$S = \inf\{\int_{\mathbb{R}^N} |\nabla u|^2;\ u \in W^{1,2}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |u|^{2^*} = 1\}.$$

Using the above facts and arguments due to Brèzis & Nirenberg [9], we will show that the choice in (10) applies in obtaining a non-trivial solution of (1).

Lemma 5 *Suppose that $\lambda > 0$ and one of the following conditions is satisfied:*

(i) $N \geq 4$;

(ii) $N = 3$ *and* $3 < q < 6$.

Then, there is a vector $e \in E\backslash\{0\}$, $e \geq 0$, $I(e) < 0$ such that

$$\sup_{t \geq 0} I(te) < \frac{1}{N} S^{N/2}, \tag{11}$$

Proof: For each $\varepsilon > 0$, consider the function

$$\phi_\varepsilon(x) = \frac{[N(N-2)\varepsilon]^{(N-2)/4}}{(\varepsilon + |x|^2)^{(N-2)/2}}.$$

The functions ϕ_ε satisfy the problem

$$\begin{cases} -\Delta u = u^{2^*-1}, \text{ in } \mathbb{R}^N \\ u > 0,\ \int_{\mathbb{R}^N} |\nabla u|^2 < \infty \end{cases}$$

and

$$\int_{\mathbb{R}^N} |\nabla \phi_\varepsilon|^2 = \int_{\mathbb{R}^N} |\phi_\varepsilon|^{2^*} = S^{N/2}$$

(see [23], lemma 2 - pp. 364). Now, consider $v_\varepsilon = \varphi\phi_\varepsilon$ where $\varphi \in C_o^\infty(\mathbb{R}^N)$, $0 \leq \varphi(x) \leq 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in B_1 \\ 0 & \text{if } x \notin B_1 \end{cases}.$$

Using arguments due to [18] there is $\varepsilon > 0$ such that

$$\sup_{t\geq 0} I(tv_\varepsilon) < \frac{1}{N} S^{N/2}.$$

If $t_\varepsilon > 0$ is such that $I(t_\varepsilon v_\varepsilon) < 0$, we choose $e = t_\varepsilon v_\varepsilon$ and the proof is complete. $\diamondsuit$

In order to complete the proof of Theorem 1, let us consider $e \in E\backslash\{0\}$ given by lemma 5. Let $\{u_m\}$ be the sequence in E satisfying (9). From Lemmas 3 and 4, we may assume that

$$\begin{gathered} u_m \rightharpoonup u \text{ in } E \\ u_m \to u \text{ in } L^s(\mathbb{R}^N),\ 2 \leq s < 2^* \\ u_m(x) \to u(x) \text{ a.e. in } \mathbb{R}^N . \end{gathered}$$

The above limits with an observation in Brèzis & Lieb [7] yield that u must be a critical point of I in E, that is,

$$I'(u) = 0.$$

We claim that $u \neq 0$. In fact, if $u \equiv 0$ and taking $l \geq 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u_m|^2 \to l,$$

then

$$\int_{\mathbb{R}^N} \left(u_m^+\right)^{2^*} \to l$$

for the reason that $I'(u_m) \to 0$ and $E \subset L^{q+1}(\mathbb{R}^N)$ compactly. Since $I(u_m) \to c$, we get

$$Nc = l. \tag{12}$$

From the definition of S,

$$\int_{\mathbb{R}^N} |\nabla u_m|^2 \geq S\left(\int_{\mathbb{R}^N} |u_m|^{2^*}\right)^{\frac{2}{2^*}} \geq S\left(\int_{\mathbb{R}^N} \left(u_m^+\right)^{2^*}\right)^{\frac{2}{2^*}}.$$

Taking the limit in the last inequalities, we achieve that

$$l \geq S l^{2/2^*}$$

and by (12) that

$$c \geq \frac{1}{N} S^{N/2} > c$$

which contradicts the above choice of e, and thus the claim is proved.

Observe that $I'(u)u^- = 0$ implies $\int_{\mathbb{R}^N} |\nabla u^-|^2 + a(x)\left(u^-\right)^2 = 0$ and then $u^- \equiv 0$ which implies $u \geq 0$. Notice that at this moment we do not know if u satisfies (1) in the $W^{1,2}(\mathbb{R}^N)$ sense but, thanks to lemma 2, u is a nontrivial classical solution of (1) with $u \geq 0$. The Hopf maximum principle assures that $u > 0$. Theorem 1 is proved. $\diamondsuit$

We conclude this section by justifying Remark 1 in the beginning of Section 1. The argument we are going to use is due to Azorero & Alonzo [4].

Justification of Remark 1. Fix $\varphi \in C_o^\infty(\mathbb{R}^N \backslash \{0\})$, $\varphi(x) \geq 0$. Notice that the real function $I(t\varphi)$ possesses a positive maximum value . Suppose that this maximum value is assumed for $t = t_\lambda$. Thus

$$\frac{d}{dt}\, I(t\varphi)|_{t=t_\lambda} = 0$$

then

$$||\varphi||^2 = t_\lambda^{q-1}\lambda \int_{\mathbb{R}^N} \varphi^{q+1} + t_\lambda^{2^*-2} \int_{\mathbb{R}^N} \varphi^{2^*} \geq t_\lambda^{q-1}\lambda \int_{\mathbb{R}^N} \varphi^{q+1}.$$

From the last inequality we have that $t_\lambda \to 0$ as $\lambda \to \infty$. On the other hand

$$\sup_{t\geq 0} I(t\varphi) \leq \frac{t_\lambda}{2} \int_{\mathbb{R}^N} |\nabla\varphi|^2$$

and for large enough $\lambda > 0$ we get

$$\sup_{t\geq 0} I(t\varphi) < \frac{1}{N} S^{\frac{N}{2}}.$$

Using the same arguments employed in the proof of Theorem 1 we conclude the justification.

We have just finished the proof of Theorem 1. Our next step is the proof of Theorem 2

3 Proof of Theorem 2

Let $(E, ||\cdot||)$ be the same defined in the proof of Theorem 1 and consider

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2) - \int_{\mathbb{R}^N} F(x,u) \tag{13}$$

defined in E, as the associated Euler-Lagrange functional to problem (2), which is C^1- see [22] . Under hypothesis (f_1), it is easy to verify that

$$\int_{\mathbb{R}^N} F(x,u) = o(||u||^2) \text{ as } u \to 0, \tag{14}$$

and hence that I has a local minimum at the origin. Hypothesis (f_2) implies that

$$F(x,s) \geq a_2|s|^\mu \tag{15}$$

for large $|s|$. Then, by (14) and (15), I has the *Mountain Pass Theorem Geometry.* Let

$$\Gamma = \{g : [0,1] \to E : g(0) = 0 \text{ and } I(g(1)) \leq 0\}$$

and

$$c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I(g(t)).$$

As in the proof of Theorem 1, $c > 0$ and there is a sequence $\{u_m\} \subset E$ satisfying (9). Using standard arguments, (f_2) implies that $||u_m||$ is a bounded sequence. Therefore, along a subsequence, u_m converges weakly in E and strongly in $L^p(\mathbb{R}^N)$, $2 \leq p < \frac{2N}{N-2}$, to a function $u \in E$ which is a weak solution of (2). We claim that $u \neq 0$. In fact, for large m,

$$\frac{c}{2} \leq I(u_m) - \frac{1}{2} I'(u_m) u_m = \int_{\mathbb{R}^N} [\frac{1}{2} f(x, u_m) u_m - F(x, u_m)].$$

Taking $m \to \infty$, in the above expression we obtain that

$$\int_{\mathbb{R}^N} [\frac{1}{2} f(x, u) u - F(x, u)] \geq \frac{c}{2}$$

contradicting a possible vanishing of u. Then the claim is proved.

We have that $u \in E \subset H^1_{\text{rad}}(\mathbb{R}^N)$ is a non-zero function satisfying

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + b(x) uv) = \int_{\mathbb{R}^N} f(x, u) v, \text{ for all } v \in E\,.$$

As in the proof of Theorem 1, using Lemma 2 we have u is a classical solution of (2). ♢

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