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REACTION DIFFUSION EQUATIONS WITH BOUNDARY DEGENERACY

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Abstract. In this article, we consider the reaction diffusion equation

$$\frac{\partial u}{\partial t} = \Delta A(u), \quad (x,t) \in \Omega \times (0,T),$$

with the homogeneous boundary condition. Inspired by the Fichera-Oleĭnik theory, if the equation is not only strongly degenerate in the interior of Ω, but also degenerate on the boundary, we show that the solution of the equation is free from any limitation of the boundary condition.

1. Introduction

Consider the equation

$$\frac{\partial u}{\partial t} = \Delta A(u), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

with the homogeneous boundary condition, where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with the appropriately smooth boundary $\partial\Omega$, and

$$A(u) = \int_0^u a(s)ds, \quad a(s) \geq 0, \quad a(0) = 0. \tag{1.2}$$

One of particular cases of equation (1.1) is

$$\frac{\partial u}{\partial t} = \Delta u^m. \tag{1.3}$$

According to the degenerate parabolic equation theory, if there is no interior point in the set $\{s \in \mathbb{R} : a(s) = 0\}$, as usual we say that equation (1.1) is weakly degenerate; otherwise, we say that equation (1.1) is strongly degenerate.

For the Cauchy problem of equation (1.1), Vol'pert and Hudjave [14] investigated its solvability. Thereafter, much attention has dedicated to the study of its well-posedness [1, 2, 3, 10, 16, 17, 18].

When we consider the initial-boundary value problem of equation (1.1), usually one needs the initial condition as

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.4}$$

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However, can we impose the Dirichlet homogeneous boundary condition

$$u(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T), \tag{1.5}$$

into the problem?

Obviously, when both (1.2) and (1.5) hold, equation (1.1) is not only degenerate in the interior of Ω, but also on the boundary $\partial\Omega$. If it is weakly degenerate, we will show that equation (1.1) can be imposed by the boundary condition (1.5) actually. While, if it is in the strongly degenerate case, we will show that the solution of equation (1.1) is free from any limitation of the boundary condition. Let us give a brief review on the corresponding problems.

The memoir by Tricomi [13], as well as subsequent investigations of equations of mixed type, elicited interest in the general study of elliptic equations degenerating on the boundary of the domain. The paper by Keldyš [9] plays a significant role in the development of the theory. It was brought to light that in the case of elliptic equations degenerating on the boundary, under definite assumptions, a portion of the boundary may be free from the prescription of boundary conditions. Later, Fichera[6, 7] and Oleǐnik [11, 12] developed the general theory of second order equations with a nonnegative characteristic form, which, in particular contains those degenerating assumptions on the boundary. We can call the theory as the Fichera-Oleǐnik theory.

To study the boundary value problem of a linear degenerate elliptic equation:

$$\sum_{r,s=1}^{N+1} a^{rs}(x)\frac{\partial^2 u}{\partial x_r\partial x_s}+\sum_{r=1}^{N+1} b_r(x)\frac{\partial u}{\partial x_r}+c(x)u=f(x), x\in\widetilde{\Omega}\subset\mathbb{R}^{N+1}, \tag{1.6}$$

it needs and only needs the part boundary condition. In detail, let $\{n_s\}$ be the unit inner normal vector of $\partial\widetilde{\Omega}$ and denote

$$\begin{gathered}\Sigma_2=\{x\in\partial\widetilde{\Omega}: a^{rs}n_rn_s=0, (b_r-a^{rs}_{x_s})n_r<0\},\\ \Sigma_3=\{x\in\partial\widetilde{\Omega}: a^{rs}n_sn_r>0\}.\end{gathered} \tag{1.7}$$

Then, to ensure the well-posedness of equation (1.7), according to the Fichera-Oleinik theory, the suitable boundary condition is

$$u|_{\Sigma_2\cup\Sigma_3}=g(x). \tag{1.8}$$

In particular, if the matrix (a^{rs}) is definite positive, (1.8) is the regular Dirichlet boundary condition.

If A^{-1} exists, in other words, equation (1.1) is weakly degenerate, let $v=A(u)$ and $u=A^{-1}(v)$. Then it has

$$\Delta v-(A^{-1}(v))_t=0. \tag{1.9}$$

According to the Fichera-Oleinik theory, one can impose the Dirichlet homogeneous boundary condition (1.5).

But, if equation (1.1) is strongly degenerate, then A^{-1} does not exist, we can not deal with it as equation (1.9). We rewrite equation (1.1) as

$$\frac{\partial u}{\partial t}=a(u)\Delta u+a'(u)|\nabla u|^2, \quad (x,t)\in\Omega\times(0,T), \tag{1.10}$$

and let $t=x_{N+1}$. We regard the strongly degenerate parabolic equation (1.10) as the form of a "linear" degenerate elliptic equation as follows: when $i,j=1,2,\dots,N$,

$a^{ii}(x,t) = a(u(x,t))$, $a^{ij}(x,t) = 0$, $i \neq j$, then it has

$$(\widetilde{a}^{rs})_{(N+1)\times(N+1)} = \begin{pmatrix} a^{ij} & 0 \\ 0 & 0 \end{pmatrix}.$$

If $a(0) = 0$, then equation (1.10) is not only strongly degenerate in the interior of Ω, but also degenerate on the boundary $\partial\Omega$. We can see that Σ_3 is an empty set, while

$$\widetilde{b}_s(x,t) = \begin{cases} a'(u)\frac{\partial u}{\partial x_i}, & 1 \leq s \leq N, \\ -1, & s = N+1. \end{cases}$$

Under this observation, according to the Fichera-Oleinik theory, the initial condition (1.4) is always required. But on the lateral boundary $\partial\Omega \times (0,T)$, by $a(0) = 0$, the part of boundary in which we should give the boundary value is

$$\Sigma_p = \Big\{x \in \partial\Omega : \Big(a'(0)\frac{\partial u}{\partial x_i}\big|_{x\in\partial\Omega} - a'(0)\frac{\partial u}{\partial x_i}\big|_{x\in\partial\Omega}\Big)n_i < 0\Big\} = \emptyset, \tag{1.11}$$

where $\{n_i\}$ is the unit inner normal vector of $\partial\Omega$. This implies that no any boundary condition is necessary. In other words, the initial-boundary problem of equation (1.1) is actually free from the limitation of the boundary condition. Certainly, the above discussion is based on the assumption that there is a classical solution of equation (1.1). In fact, due to the strongly degenerate properties of $A(u)$, equation (1.1) generally only has a weak solution. So it remains to be clarified whether the solution of the equation is actually free from the limitation of the boundary condition or not?

2. Main results

For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)d\tau, \quad h_\eta(s) = \frac{2}{\eta}(1 - \frac{|s|}{\eta})_+. \tag{2.1}$$

Obviously, $h_\eta(s) \in C(\mathbb{R})$, and

$$\begin{gathered} h_\eta(s) \geq 0, \quad |sh_\eta(s)| \leq 1, \quad |S_\eta(s)| \leq 1, \\ \lim_{\eta\to 0} S_\eta(s) = \operatorname{sign} s, \quad \lim_{\eta\to 0} sS'_\eta(s) = 0. \end{gathered} \tag{2.2}$$

Definition 2.1. A function u is said to be the entropy solution of (1.1) with the initial condition (1.4), if

1. u satisfies

$$u \in BV(Q_T) \cap L^\infty(Q_T), \quad \frac{\partial}{\partial x_i}\int_0^u \sqrt{a(s)}ds \in L^2(Q_T). \tag{2.3}$$

2. For any $\varphi \in C_0^2(Q_T)$, $\varphi \geq 0$, $k \in \mathbb{R}$, with a small $\eta > 0$, u satisfies

$$\iint_{Q_T} \Big[I_\eta(u-k)\varphi_t + A_\eta(u,k)\Delta\varphi - S'_\eta(u-k)|\nabla \int_0^u \sqrt{a(s)}ds|^2\varphi\Big]\,dx\,dt \geq 0. \tag{2.4}$$

3. The initial condition is true in the sense that

$$\lim_{t\to 0}\int_\Omega |u(x,t) - u_0(x)|dx = 0. \tag{2.5}$$

One can see that if (1.1) has a classical solution u, by multiplying (1.1) by $\varphi_1 S_\eta(u-k)$ and integrating it over Q_T, we are able to show that u satisfies Definition 2.1. On the other hand, letting $\eta \to 0$ in (2.4), we have

$$\iint_{Q_T} [|u-k|\varphi_t + \operatorname{sign}(u-k)(A(u)-A(k))\Delta\varphi]\, dx\, dt \geq 0.$$

Thus if u is the entropy solution as in Definition 2.1, then u is a entropy solution as defined in [10, 14] et al.

Theorem 2.2. *Suppose that $A(s)$ is C^3 and $u_0(x) \in L^\infty(\Omega)$. Suppose that*

$$A'(0) = a(0) = 0. \tag{2.6}$$

Then (1.1) *with the initial condition* (1.4) *has a entropy solution in the sense of Definition 2.1.*

Theorem 2.3. *Suppose that $A(s)$ is C^2. Let u and v be solutions of* (1.1) *with the different initial values $u_0(x)$, $v_0(x) \in L^\infty(\Omega)$ respectively. Suppose that the distance function $d(x) = \operatorname{dist}(x, \Sigma) < \lambda$ satisfies*

$$|\Delta d| \leq c,\ \frac{1}{\lambda}\int_{\Omega_\lambda} dx \leq c, \tag{2.7}$$

where λ is a sufficiently small constant, and $\Omega_\lambda = \{x \in \Omega, d(x, \partial\Omega) < \lambda\}$. Then

$$\int_\Omega |u(x,t)-v(x,t)|dx \leq \int_\Omega |u_0 - v_0|dx + \operatorname{ess\,sup}_{(x,t)\in\partial\Omega\times(0,T)} |u(x,t)-v(x,t)|. \tag{2.8}$$

3. Proof of Theorem 2.2

Let Γ_u be the set of all jump points of $u \in BV(Q_T), v$ be the normal of Γ_u at $X = (x,t)$, $u^+(X)$ and $u^-(X)$ be the approximate limit of u at $X \in \Gamma_u$ with respect to $(v, Y-X) > 0$ and $(v, Y-X) < 0$ respectively. For the continuous function $p(u,x,t)$ and $u \in BV(Q_T)$, we define

$$\widehat{p}(u,x,t) = \int_0^1 p(\tau u^+ + (1-\tau)u^-, x, t)d\tau, \tag{3.1}$$

which is called the composite mean value of p.

For a given t, we denote $\Gamma_u^t,\ H^t, (v_1^t, \dots, v_N^t)$ and $u_\pm^t$ as all jump points of $u(\cdot,t)$, Housdorff measure of Γ_u^t, the unit normal vector of Γ_u^t, and the asymptotic limit of $u(\cdot,t)$ respectively. Moreover, if $f(s) \in C^1(\mathbb{R})$ and $u \in BV(Q_T)$, then $f(u) \in BV(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u)\frac{\partial u}{\partial x_i},\ i = 1, 2, \dots, N, N+1, \tag{3.2}$$

holds, where $x_{N+1} = t$.

Lemma 3.1. *Let u be a solution of* (1.1)*. Then*

$$a(s) = 0, \quad s \in I(u^+(x,t), u^-(x,t)) \quad \text{a.e. on } \Gamma_u, \tag{3.3}$$

where $I(\alpha,\beta)$ denote the closed interval with endpoints α and β, and (3.3) *is in the sense of Hausdorff measure $H_N(\Gamma_u)$.*

The proof of the above lemma is similar to the one in [18], so we omit it.

Lemma 3.2 ([4]). *Assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set and let $f_k, f \in L^q(\Omega)$, as $k \to \infty$, $f_k \rightharpoonup f$ weakly in $L^q(\Omega)$ and $1 \le q < \infty$. Then*

$$\liminf_{k\to\infty} \|f_k\|^q_{L^q(\Omega)} \ge \|f\|^q_{L^q(\Omega)}. \tag{3.4}$$

We now consider the regularized problem

$$\frac{\partial u}{\partial t} = \Delta A(u) + \varepsilon \Delta u, \ (x,t) \in \Omega \times (0,T), \tag{3.5}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{3.6}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{3.7}$$

It is well known that there are classical solutions $u_\varepsilon \in C^2(\overline{Q_T}) \cap C^3(Q_T)$ of this problem provided that $A(s)$ satisfies the assumptions in Theorem 2.2. One can refer to [15] or the eighth chapter of [8] for details.

We need to make some estimates for u_ε of (3.5). Firstly, since $u_0(x) \in L^\infty(\Omega)$ is sufficiently smooth, by the maximum principle we have

$$|u_\varepsilon| \le \|u_0\|_{L^\infty} \le M. \tag{3.8}$$

Secondly, let us make the BV estimates of u_ε. To the end, we begin with the local coordinates of the boundary $\partial\Omega$.

Let $\delta_0 > 0$ be small enough. Denote

$$E^{\delta_0} = \{x \in \bar{\Omega}; \mathrm{dist}(x,\Sigma) \le \delta_0\} \subset \cup_{\tau=1}^n V_\tau,$$

where V_τ is a region, and one can introduce local coordinates of V_τ,

$$y_k = F_\tau^k(x) \quad (k = 1,2,\dots,N), \quad y_N|_\Sigma = 0, \tag{3.9}$$

with F_τ^k appropriately smooth and $F_\tau^N = F_l^N$, such that the y_N-axes coincides with the inner normal vector.

Lemma 3.3 ([15]). *Let u_ε be the solution of equation (3.5) with (3.6), (3.7). If the assumptions of Theorem 2.2 are true, then*

$$\varepsilon \int_\Sigma |\frac{\partial u_\varepsilon}{\partial n}| d\sigma \le c_1 + c_2(|\nabla u_\varepsilon|_{L^1(\Omega)} + |\frac{\partial u_\varepsilon}{\partial t}|_{L^1(\Omega)}), \tag{3.10}$$

with constants c_i, $i = 1,2$ independent of ε.

We have the following important estimates of the solutions u_ε of equation (3.5) with (3.6), (3.7).

Theorem 3.4. *Let u_ε be the solution of equation (3.5) with (3.6), (3.7). If the assumptions of Theorem 2.2 are true, then*

$$|\operatorname{grad} u_\varepsilon|_{L^1(\Omega)} \le c, \tag{3.11}$$

where $|\operatorname{grad} u|^2 = \sum_{i=1}^N |\frac{\partial u}{\partial x_i}|^2 + |\frac{\partial u}{\partial t}|^2$, and c is independent of ε.

Proof. Differentiate (3.5) with respect to x_s, $s = 1,2,\cdot,N,N+1$, $x_{N+1} = t$, and sum up for s after multiplying the resulting relation by $u_{\varepsilon x_s} \frac{S_\eta(|\operatorname{grad} u_\varepsilon|)}{|\operatorname{grad} u_\varepsilon|}$. In what follows, we simply denote u_ε by u, denote $\partial\Omega$ by Σ, and denote $d\sigma$ by the surface integral unite on Σ.

Integrating it over Ω yields

$$\int_\Omega \frac{\partial u_{x_s}}{\partial t} u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|} dx = \int_\Omega \frac{\partial}{\partial t} \int_0^{|\operatorname{grad} u|} S_\eta(\tau) d\tau dx = \frac{d}{dt} \int_\Omega I_\eta(|\operatorname{grad} u| dx,$$

where pairs of the indices of s imply a summation from 1 to $N+1$, pairs of the indices of i, j imply a summation from 1 to N, and $\{n_i\}_{i=1}^N$ is the inner normal vector of Ω. So we have

$$\begin{aligned} &\int_\Omega \Delta(a(u) u_{x_s}) u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|} dx \\ &= \int_\Omega \frac{\partial}{\partial x_i} [a'(u) u_{x_i} u_{x_s} + a(u) u_{x_i x_s}] u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|} dx \\ &= \int_\Omega \frac{\partial}{\partial x_i} (a'(u) u_{x_i} u_{x_s}) u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|} dx \\ &\quad + \int_\Omega \frac{\partial}{\partial x_i} (a(u) u_{x_i x_s}) u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|} dx \,, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial x_i} (a'(u) u_{x_i} u_{x_s}) u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|} dx \\ &= \sum_{s=1}^{N+1} \int_\Omega \frac{\partial}{\partial x_i} (a'(u) u_{x_i}) u_{x_s}^2 \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|} dx \\ &\quad + \int_\Omega a'(u) u_{x_i} \frac{\partial}{\partial x_i} I_\eta(|\operatorname{grad} u|) dx \\ &= \int_\Omega \frac{\partial}{\partial x_i} (a'(u) u_{x_i}) |\operatorname{grad} u| S_\eta(|\operatorname{grad} u|) dx \\ &\quad - \int_\Sigma a'(u) u_{x_i} n_i I_\eta(|\operatorname{grad} u|) d\sigma - \int_\Omega I_\eta(|\operatorname{grad} u|) \frac{\partial}{\partial x_i} (a'(u) u_{x_i}) dx \\ &= \int_\Omega \frac{\partial}{\partial x_i} (a'(u) u_{x_i}) \left[|\operatorname{grad} u| S_\eta(|\operatorname{grad} u|) - I_\eta(|\operatorname{grad} u|)\right] dx \\ &\quad - \int_\Sigma a'(u) u_{x_i} n_i I_\eta(|\operatorname{grad} u|) d\sigma \,, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial x_i} (a(u) u_{x_i x_s}) u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|} dx \\ &= \int_\Omega \frac{\partial}{\partial x_i} (a(u) u_{x_i x_s}) \frac{\partial}{\partial \xi_s} I_\eta(|\operatorname{grad} u|) dx \\ &= - \int_\Sigma a(u) u_{x_i x_s} n_i \frac{\partial}{\partial \xi_s} I_\eta(|\operatorname{grad} u|) d\sigma \\ &\quad - \int_\Omega a(u) \frac{\partial^2 I_\eta(|\operatorname{grad} u|)}{\partial \xi_s \partial \xi_p} u_{x_s x_i} u_{x_p x_i} dx \,, \end{aligned} \tag{3.14}$$

where $\xi_s = u_{x_s}$.

$$\begin{aligned} &\varepsilon \int_\Omega \Delta u_{x_s} u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|} dx \\ &= -\varepsilon \int_\Sigma \frac{\partial I_\eta(|\operatorname{grad} u|)}{\partial x_i} n_i d\sigma - \varepsilon \int_\Omega \frac{\partial^2 I_\eta(|\operatorname{grad} u|)}{\partial \xi_s \partial \xi_p} u_{x_s x_i} u_{x_p x_i} dx. \end{aligned} \tag{3.15}$$

From (3.12)–(3.15), by the assumption $a(0)=0$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega I_\eta(|\operatorname{grad} u|dx\\
&=\int_\Omega \frac{\partial}{\partial x_i}(a'(u)u_{x_i})\,[|\operatorname{grad} u|S_\eta(|\operatorname{grad} u|)-I_\eta(|\operatorname{grad} u|)]\,dx\\
&\quad -\int_\Omega a(u)\frac{\partial^2 I_\eta(|\operatorname{grad} u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}\,dx\\
&\quad -\varepsilon\int_\Omega \frac{\partial^2 I_\eta(|\operatorname{grad} u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}\,dx\\
&\quad -\Big[\int_\Sigma a'(u)u_{x_i}n_iI_\eta(|\operatorname{grad} u|)d\sigma+\varepsilon\int_\Sigma \frac{\partial I_\eta(|\operatorname{grad} u|)}{\partial x_i}n_i d\sigma\Big].
\end{aligned}
\tag{3.16}
$$

Note that on Σ, we have

$$
0=\varepsilon\Delta u+\Delta A(u),\quad u=0,
\tag{3.17}
$$

then the surface integrals in (3.16) can be rewritten as

$$
\begin{aligned}
S&=-\Big[\varepsilon\int_\Sigma \frac{\partial I_\eta(|\operatorname{grad} u|)}{\partial x_i}n_i d\sigma+\int_\Sigma a'(u)u_{x_i}n_iI_\eta(|\operatorname{grad} u|)d\sigma\Big]\\
&=-\varepsilon\int_\Sigma\big[\frac{\partial I_\eta(|\operatorname{grad} u|)}{\partial x_i}n_i-\Delta u\frac{I_\eta(|\operatorname{grad} u|)}{\frac{\partial u}{\partial n}}\big]d\sigma\\
&\quad+\int_\Sigma a(u)\big[\frac{\partial I_\eta(|\operatorname{grad} u|)}{\partial x_i}n_i-\Delta u\frac{I_\eta(|\operatorname{grad} u|)}{\frac{\partial u}{\partial n}}\big]d\sigma\\
&=-\varepsilon\int_\Sigma\big[\frac{\partial I_\eta(|\operatorname{grad} u|)}{\partial x_i}n_i-\Delta u\frac{I_\eta(|\operatorname{grad} u|)}{\frac{\partial u}{\partial n}}\big]d\sigma.
\end{aligned}
$$

Since $u_{x_{N+1}}|_\Sigma=u_t|_\Sigma=0$, we have

$$
\lim_{\eta\to0}S=-\varepsilon\int_\Sigma \operatorname{sign}(\frac{\partial u}{\partial n})(u_{x_ix_j}n_jn_i-\Delta u)d\sigma.
\tag{3.18}
$$

Using the local coordinates on $V_\tau,\tau=1,2,\dots,n$, we have

$$
y_k=F_\tau^k(x),\quad k=1,2,\dots,N,\quad y_m|_\Sigma=0.
$$

By a direct computation (refer to [15]), on $\Sigma\cap V_\tau$ we obtain

$$
\begin{gathered}
u_{x_ix_j}=\sum_{k=1}^N u_{y_Ny_k}F_{x_i}^NF_{x_j}^k+\sum_{k=1}^{N-1}u_{y_Ny_k}F_{x_i}^NF_{x_j}^k+u_{y_m}F_{x_ix_j}^m\,,\\
u_{x_ix_j}n_jn_i=\frac{\sum_{k=1}^N u_{y_Ny_k}F_{x_i}^NF_{x_j}^kF_{x_j}^NF_{x_i}^N}{|\operatorname{grad} F^N|^2}+\sum_{k=1}^{N-1}u_{y_Ny_k}F_{x_i}^kF_{x_j}^N+\frac{u_{y_m}F_{x_ix_j}^mF_{x_j}^NF_{x_i}^N}{|\operatorname{grad} F^N|^2},
\end{gathered}
$$

in which $F^k=F_\tau^k$. since the inner normal vector is

$$
\vec n=-(\frac{\partial F^N}{\partial x_1},\dots,\frac{\partial F^N}{\partial x_N})=-\operatorname{grad} F^N,
$$

it follows that

$$
u_{x_ix_j}n_jn_i-\Delta u=u_{y_m}\Big(\frac{F_{x_ix_j}^mF_{x_j}^NF_{x_i}^N}{|\operatorname{grad} F^N|^2}-F_{x_ix_i}^m\Big).
$$

By Lemma 3.3, we see that $\lim_{\eta\to0}S$ can be estimated by $|\operatorname{grad} u|_{L^1(\Omega)}$.

By letting $\eta \to 0$, from

$$\lim_{\eta\to 0}[|\operatorname{grad} u| S_\eta(|\operatorname{grad} u|) - I_\eta(|\operatorname{grad} u|)] = 0,$$

we have

$$\frac{d}{dt}\int_\Omega |\operatorname{grad} u| dx \le c_1 + c_2 \int_\Omega |\operatorname{grad} u| dx.$$

Further, by Gronwall's Lemma we have

$$\int_\Omega |\operatorname{grad} u| dx \le c. \tag{3.19}$$

□

By (3.5) and (3.19), it is easy to show that

$$\iint_{Q_T} (a(u_\varepsilon)+\varepsilon)|\nabla u_\varepsilon|^2\,dx\,dt \le c. \tag{3.20}$$

Thus, there exists a subsequence $\{u_{\varepsilon_n}\}$ of u_ε and a function $u \in BV(Q_T) \cap L^\infty(Q_T)$ such that $u_{\varepsilon_n} \to u$ a.e. on Q_T.

Proof. We now prove that u is a generalized solution of equation (1.1) with the initial condition (1.4). For any $\varphi(x,t) \in C_0^1(Q_T)$, we have

$$\begin{aligned}
&\iint_{Q_T} [\frac{\partial}{\partial x_i}\int_0^{u_\varepsilon} \sqrt{a(s)}ds - \frac{\partial}{\partial x_i}\int_0^{u} \sqrt{a(s)}ds]\varphi(x,t)\,dx\,dt \\
&= -\iint_{Q_T} \Big[\int_0^{u_\varepsilon} \sqrt{a(s)}ds - \int_0^{u} \sqrt{a(s)}ds\Big]\varphi_{x_i}(x,t)\,dx\,dt.
\end{aligned}$$

By a limiting process, we know the above equality is also true for any $\varphi(x,t) \in L^2(Q_T)$. By (3.20), we have

$$\frac{\partial}{\partial x_i}\int_0^{u_\varepsilon} \sqrt{a(s)}ds \rightharpoonup \frac{\partial}{\partial x_i}\int_0^{u} \sqrt{a(s)}ds$$

weakly in $L^2(Q_T)$ for $i = 1, 2, \dots, N$. This implies

$$\frac{\partial}{\partial x_i}\int_0^{u} \sqrt{a(s)}ds \in L^2(Q_T), \quad i = 1, 2, \dots, N.$$

Thus u satisfies (2.3) in Definition 2.1.

Let $\varphi \in C_0^2(Q_T)$, $\varphi \ge 0$, and $\{n_i\}$ be the inner normal vector of Ω. Multiplying (3.5) by $\varphi S_\eta(u_\varepsilon - k)$, and integrating it over Q_T, we obtain

$$\begin{aligned}
&\iint_{Q_T} I_\eta(u_\varepsilon - k)\varphi_t\,dx\,dt + \iint_{Q_T} A_\eta(u_\varepsilon, k)\Delta\varphi\,dx\,dt \\
&- \varepsilon\iint_{Q_T} \nabla u_\varepsilon \cdot \nabla\varphi S_\eta(u_\varepsilon - k)\,dx\,dt - \varepsilon\iint_{Q_T} |\nabla u_\varepsilon|^2 S_\eta'(u_\varepsilon - k)\varphi\,dx\,dt \\
&- \iint_{Q_T} a(u_\varepsilon)|\nabla u_\varepsilon|^2 S_\eta'(u_\varepsilon - k)\varphi\,dx\,dt = 0.
\end{aligned} \tag{3.21}$$

By Lemma 3.2,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\iint_{Q_T} S_\eta'(u_\varepsilon - k)a(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_i}\varphi\,dx\,dt \\
&\ge \iint_{Q_T} S_\eta'(u - k)|\nabla\int_0^u \sqrt{a(s)}ds|^2\varphi\,dx\,dt.
\end{aligned} \tag{3.22}$$

Let $\varepsilon \to 0$ in (3.21). By (3.22), we get (2.4). Finally, we can prove equality (2.5) in a similar manner as that in [14] or [18], we omit the details. $\square$

4. Proof of Theorem 2.3

Proof. Let u and v be two entropy solutions of (1.1) in the sense of Definition 2.1. Suppose the initial values are

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x)\,. \tag{4.1}$$

By Definition 2.1, for any $\varphi \in C_0^2(Q_T)$, $\varphi \geq 0$, and $\eta > 0$, $k, l \in \mathbb{R}$, we have

$$\iint_{Q_T} \Big[I_\eta(u-k)\varphi_t + A_\eta(u,k)\Delta\varphi - S'_\eta(u-k)|\nabla \int_0^u \sqrt{a(s)}ds|^2\varphi \Big]\,dx\,dt \geq 0, \tag{4.2}$$

$$\iint_{Q_T} \Big[I_\eta(v-l)\varphi_\tau + A_\eta(v,l)\Delta\varphi - S'_\eta(v-l)|\nabla \int_0^v \sqrt{a(s)}ds|^2\varphi \Big]\,dy\,d\tau \geq 0. \tag{4.3}$$

Let $\psi(x,t,y,\tau) = \phi(x,t)j_h(x-y,t-\tau)$, where $\phi(x,t) \geq 0$, $\phi(x,t) \in C_0^\infty(Q_T)$, and

$$j_h(x-y,t-\tau) = \omega_h(t-\tau)\Pi_{i=1}^N \omega_h(x_i - y_i), \tag{4.4}$$

$$\begin{gathered} \omega_h(s) = \frac{1}{h}\omega(\frac{s}{h}), \quad \omega(s) \in C_0^\infty(R), \quad \omega(s) \geq 0, \quad \omega(s) = 0 \\ \text{if } |s| > 1, \quad \int_{-\infty}^{\infty} \omega(s)ds = 1. \end{gathered} \tag{4.5}$$

We choose $k = v(y,\tau)$, $l = u(x,t)$, $\varphi_1 = \psi(x,t,y,\tau)$ in (4.2)-(4.3), integrating over Q_T we obtain

$$\begin{aligned} &\iint_{Q_T}\iint_{Q_T} [I_\eta(u-v)(\psi_t + \psi_\tau) + A_\eta(u,v)\Delta_x\psi + A_\eta(v,u)\Delta_y\psi] \\ &- S'_\eta(u-v)\Big(|\nabla \int_0^u \sqrt{a(s)}ds|^2 + |\nabla \int_0^v \sqrt{a(s)}ds|^2\Big)\psi\,dx\,dt\,dy\,d\tau = 0. \end{aligned} \tag{4.6}$$

Clearly,

$$\begin{gathered} \frac{\partial j_h}{\partial t} + \frac{\partial j_h}{\partial \tau} = 0, \quad \frac{\partial j_h}{\partial x_i} + \frac{\partial j_h}{\partial y_i} = 0, \quad i = 1, \dots, N; \\ \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \tau} = \frac{\partial \phi}{\partial t} j_h, \quad \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} = \frac{\partial \phi}{\partial x_i} j_h. \end{gathered}$$

For the third and the fourth terms in (4.6), we have

$$\begin{aligned} &\iint_{Q_T} [A_\eta(u,v)\Delta_x\psi + A_\eta(v,u)\Delta_y\psi]\,dx\,dt\,dy\,d\tau \\ &= \iint_{Q_T}\iint_{Q_T} \{A_\eta(u,v)(\Delta_x\phi j_h + 2\phi_{x_i} j_{hx_i} + \phi\Delta j_h) + A_\eta(v,u)\phi\Delta_y j_h\}\,dx\,dt\,dy\,d\tau \\ &= \iint_{Q_T}\iint_{Q_T} \{A_\eta(u,v)\Delta_x\phi j_h + A_\eta(u,v)\phi_{x_i} j_{hx_i} + A_\eta(v,u)\phi_{x_i} j_{hy_i}\}\,dx\,dt\,dy\,d\tau \\ &\quad - \iint_{Q_T}\iint_{Q_T} \{a(u)\widehat{S_\eta(u} - v)\frac{\partial u}{\partial x_i} - \int_u^v a(s)\widehat{S'_\eta(s} - v)ds\frac{\partial u}{\partial x_i})\phi j_{hx_i}\}\,dx\,dt\,dy\,d\tau, \end{aligned}$$

where

$$a(u)\widehat{S_\eta(u} - v) = \int_0^1 a(su^+ + (1-s)u^-)S_\eta(su^+ + (1-s)u^- - v)ds,$$

$$\int_u^v a(s)\widehat{S'_\eta(s}-v)ds = \int_0^1\int_{su^++(1-s)u^-}^v a(\sigma)S_\eta(\sigma - su^+ - (1-s)u^-)d\sigma ds.$$

Since

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\Big(|\nabla_x\int_0^u\sqrt{a(s)}ds|^2 + |\nabla_y\int_0^v\sqrt{a(s)}ds|^2\Big)\psi\,dx\,dt\,dy\,d\tau\\
&= \iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\Big(|\nabla_x\int_0^u\sqrt{a(s)}ds| - |\nabla_y\int_0^v\sqrt{a(s)}ds|\Big)^2\psi\,dx\,dt\,dy\,d\tau\\
&\quad + 2\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\nabla_x\int_0^u\sqrt{a(s)}ds\cdot\nabla_y\int_0^v\sqrt{a(s)}ds\psi\,dx\,dt\,dy\,d\tau\,.
\end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\nabla_x\nabla_y\int_v^u\sqrt{a(\delta)}\int_\delta^v\sqrt{a(\sigma)}S'_\eta(\sigma-\delta)\,d\sigma\,d\delta\psi\,dx\,dt\,dy\,d\tau\\
&= \iint_{Q_T}\iint_{Q_T}\int_0^1\int_0^1\sqrt{a(su^++(1-s)u^-)}\sqrt{a(\sigma v^+ + (1-\sigma)v^-}\\
&\quad\times\times S'_\eta[\sigma v^+ + (1-\sigma)v^- - su^+ - (1-s)u^-]\,d\,d\sigma\nabla_x u\nabla_y v\,dx\,dt\,dy\,d\tau\\
&= \iint_{Q_T}\iint_{Q_T}\int_0^1\int_0^1 S'_\eta[\sigma v^+ + (1-\sigma)v^- - su^+ - (1-s)u^-]\,d\,d\sigma\\
&\quad\times\widehat{\sqrt{a(u)}}\nabla_x u\widehat{\sqrt{a(v)}}\nabla_y v\,dx\,dt\,dy\,d\tau\\
&= \iint_{Q_T}\iint_{Q_T}\int_0^1\int_0^1 S'_\eta(v-u)\nabla_x\int_0^u\sqrt{a(s)}ds\nabla_y\int_0^v\sqrt{a(s)}ds\,dx\,dt\,dy\,d\tau.
\end{aligned}$$

and

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\nabla_x\nabla_y\int_v^u\sqrt{a(\delta)}\int_\delta^v\sqrt{a(\sigma)}S'_\eta(\sigma-\delta)\,d\sigma\,d\delta\psi\,dx\,dt\,dy\,d\tau\\
&= \iint_{Q_T}\iint_{Q_T}\int_0^1\sqrt{a(su^++(1-s)u^-)}\\
&\quad\times\int_{su^++(1-s)u^-}^v\sqrt{a(\sigma)}S'_\eta(\sigma - su^+ - (1-s)u^-)d\sigma ds\frac{\partial u}{\partial x_i}j_{hx_i}\phi\,dx\,dt\,dy\,d\tau.
\end{aligned}$$

We further have

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}(a(u)\widehat{S_\eta(u}-v)\frac{\partial u}{\partial x_i} - \int_u^v a(s)\widehat{S'_\eta(s-u)ds}\frac{\partial u}{\partial x_i})j_{hx_i}\phi\,dx\,dt\,dy\,d\tau\\
&\quad + 2\iint_{Q_T}\iint_{Q_T}S'_\eta(u-v)\nabla_x\int_0^u\sqrt{a(s)}ds\cdot\nabla_y\int_0^v\sqrt{a(s)}ds\psi\,dx\,dt\,dy\,d\tau\\
&= \iint_{Q_T}\iint_{Q_T}\Big[\int_0^1 a(su^++(1-s)u^-)S_\eta(su^++(1-s)u^- - v)ds\\
&\quad - \int_0^1\int_{su^++(1-s)u^-}^v a(\sigma)S'_\eta(\sigma - su^+ - (1-s)u^-)d\sigma ds\\
&\quad + 2\int_0^1\sqrt{a(su^++(1-s)u^-)}\int_{su^++(1-s)u^-}^v\sqrt{a(\sigma)}S'_\eta(\sigma - su^+
\end{aligned}$$

$$
\begin{aligned}
&- (1-s)u^-)d\sigma ds\Big]\frac{\partial u}{\partial x_i}j_{hx_i}\phi\,dx\,dt\,dy\,d\tau\\
&= -\iint_{Q_T}\iint_{Q_T}\int_0^1\int_{su^++(1-s)u^-}^{v}[\sqrt{a(\sigma)}-\sqrt{a(su^++(1-s)u^-)}]\\
&\quad\times S'_\eta(\sigma - su^+ - (1-s)u^-)d\sigma ds\frac{\partial u}{\partial x_i}j_{hx_i}\phi\,dx\,dt\,dy\,d\tau \to 0,
\end{aligned}
$$

as $\eta\to 0$. Since

$$
\lim_{\eta\to 0}A_\eta(u,v)=\lim_{\eta\to 0}A_\eta(v,u)=\operatorname{sign}(u-v)[A(u)-A(v)],
$$

we have

$$
\lim_{\eta\to 0}[A_\eta(u,v)\phi_{x_i}j_{hx_i}+A_\eta(u,v)\phi_{y_i}j_{hy_i}]=0. \tag{4.7}
$$

By (4.6)–(4.7) and letting $\eta\to 0, h\to 0$ in (4.6), we obtain

$$
\iint_{Q_T}[|u(x,t)-v(x,t)|\phi_t+|A(u)-A(v)|\Delta\phi]\;dx\,dt\ge 0. \tag{4.8}
$$

Let δ_ε be the mollifier. For any given $\varepsilon>0$, $y=(y_1,\dots,y_N)$, $\delta_\varepsilon(y)$ is defined by

$$
\delta_\varepsilon(y)=\frac{1}{\varepsilon^N}\delta(\frac{y}{\varepsilon}),
$$

where

$$
\delta(y)=\begin{cases}\frac{1}{A}e^{\frac{1}{|y|^2-1}}, & \text{if } |y|<1,\\ 0, & \text{if } |y|\ge 1,\end{cases}
$$

with

$$
A=\int_{B_1(0)}e^{\frac{1}{|y|^2-1}}dx.
$$

Especially, we can choose ϕ in (4.8) by

$$
\phi(x,t)=\omega_{\lambda\varepsilon}(x)\eta(t),
$$

where $\eta(t)\in C_0^\infty(0,T)$, and $\omega_{\lambda\varepsilon}(x)$ is the mollified function of ω_λ. Let $\omega_\lambda(x)\in C_0^2(\Omega)$ be defined as follows: for any given small enough $0<\lambda$, $0\le\omega_\lambda\le 1$, $\omega|_{\partial\Omega}=0$ and

$$
\omega_\lambda(x)=1, if\;\; d(x)=\operatorname{dist}(x,\partial\Omega)\ge\lambda,
$$

where $0\le d(x)\le\lambda$ and

$$
\omega_\lambda(d(x))=1-\frac{(d(x)-\lambda)^2}{\lambda^2}.
$$

Then $\omega_{\lambda\varepsilon}=\omega_\lambda*\delta_\varepsilon(d)$,

$$
\begin{aligned}
\omega'_{\lambda\varepsilon}(d)&=-\int_{\{|s|<\varepsilon\}\cap\{0<d-s<\lambda\}}\omega'_\lambda(d-s)\delta_\varepsilon(s)ds\\
&=-\int_{\{|s|<\varepsilon\}\cap\{0<d-s<\lambda\}}\frac{2(d-s-\lambda)}{\lambda^2}\delta_\varepsilon(s)ds.
\end{aligned}
$$

We know that

$$
\begin{aligned}
\Delta\phi&=\eta(t)\Delta(\omega_{\lambda\varepsilon}(d(x)))\\
&=\eta(t)\nabla(\omega'_{\lambda\varepsilon}(d)\nabla d)\\
&=\eta(t)[\omega''_{\lambda\varepsilon}(d)|\nabla d|^2+\omega'_{\lambda\varepsilon}(d)\Delta d]
\end{aligned}
$$

$$= \eta(t)[-\frac{2}{\lambda^2}\int_{\{|s|<\varepsilon\}\cap\{0<d-s<\lambda\}} ds + \omega'_{\lambda\varepsilon}(d)\Delta d].$$

By using condition (2.7), and that $|\nabla d(x)| = 1$, a.e. $x \in \Omega$, from (4.8) we have

$$\int_{Q_T} |u(x,t) - v(x,t)|\phi_t \, dx \, dt + c\int_0^T \int_{\Omega_\lambda} \eta(t)|\omega'_{\lambda\varepsilon}(d)||u-v| \, dx \, dt \geq 0. \tag{4.9}$$

where $\Omega_\lambda = \{x \in \Omega, d(x, \partial\Omega) < \lambda\}$. Since $|\omega'_{\lambda\varepsilon}(d)| \leq \frac{c}{\lambda}$, let $\varepsilon \to 0$ in (4.9). We have

$$\lim_{\varepsilon\to 0}\int_0^T \int_{\Omega_\lambda} \eta(t)|\omega'_{\lambda\varepsilon}(d)||u-v| \, dx \, dt \leq \frac{c}{\lambda}\int_0^T \int_{\Omega_\lambda} \eta(t)|u-v| \, dx \, dt.$$

According to the the definition of the trace of the BV functions [4], we let $\varepsilon \to 0$ and $\lambda \to 0$. Then we have

$$c \operatorname{ess\,sup}_{\partial\Omega\times(0,T)} |u(x,t) - v(x,t)| + \int_{Q_T} |u(x,t) - v(x,t)|\eta'_t \, dx \, dt \geq 0. \tag{4.10}$$

Let $0 < s < \tau < T$, and

$$\eta(t) = \int_{\tau-t}^{s-t} \alpha_\varepsilon(\sigma)d\sigma, \quad \varepsilon < \min\{\tau, T-s\}.$$

Then it follows that

$$c \operatorname{ess\,sup}_{\partial\Omega\times(0,T)} |u(x,t) - v(x,t)| + \int_0^T [\alpha_\varepsilon(t-s) - \alpha_\varepsilon(t-\tau)]|u-v|_{L^1(\Omega)}dt \geq 0.$$

By letting $\varepsilon \to 0$, we obtain

$$|u(x,\tau) - v(x,\tau)|_{L^1(\Omega)} \leq |u(x,s) - v(x,s)|_{L^1(\Omega)} + c \operatorname{ess\,sup}_{\partial\Omega\times(0,T)} |u(x,t) - v(x,t)|.$$

Consequently, the desired result follows by letting $s \to 0$. $\square$

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References

[1] M. Bendahamane, K. H. Karlsen; Reharmonized entropy solutions for quasilinear anisotropic degenerate parabolic equations, *SIAM J.Math. Anal.*, **36(2)** (2004), 405-422.

[2] G. Q. Chen, B. Perthame; Well-Posedness for non-isotropic degenerate parabolic-hyperbolic equations, *Ann. I. H. Poincare-AN*, **20(4)** (2003), 645-668.

[3] B. Cockburn, G. Gripenberg; Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations, *J. of Diff. Equ.*, **151** (1999), 231-251.

[4] G. Enrico' Minimal Surfaces and Functions of Bounded Variation, *Birkhauser, Bosten. Basel. Stuttgart*, Switzerland,1984.

[5] L. C. Evans; Weak convergence methods for nonlinear partial differential equations, *Conference Board of the Mathematical Sciences, Regional Conferences Series in Mathematics*, Number 74, 1998.

[6] G. Fichera; Sulle equazioni differenziatli lineari ellittico-paraboliche del secondo ordine, *Atti Accd, Naz. Lincei. Mem, CI. Sci. Fis. Mat. Nat. Sez.1*, **5(8)**, (1956), 1-30. MR 19, 1432.

[7] G. Fichera; On a unified theory of boundary value problems for elliptic-parabolic equations of second order, in Boundary Problems, Differential Equations, *Univ. of Wisconsin Press, Madison, Wis.*, **9** (1960), 97-120. MR 22, 2789.

[8] L. Gu; Second order parabolic partial differential equations, *The publishing company of Xiamen University*, China, 2004.

[9] M. V. Keldyš; On certain cases of degeneration of elliptic type on the boundary of a domain, *Dokl. Akad. Aauk SSSR*, **77** (1951), 181-183.

[10] S. N. Kružkov; First order quasilinear equations in several independent variables, *Math. USSR-Sb.*, **10** (1970), 217-243.

[11] O. A. Oleinǐk; A problem of Fichera, *Dokl.Akad. Nauk SSSR*, **154** (1964), 1297-1300. *Soviet Math. Dokl.*, **5** (1964), 1129-1133. MA 30, 1293.
[12] O. A. Oleinǐk; Linear equations of second order with nonnegative characteristic form, *Math. Sb.*, **69** (1966), 111-140. English transl., *Amer. Math. Soc. Tranl.*, **65(2)** (1967), 167-199, MR 33, 1603.
[13] F. Tricomi; Sulle equazioni lineari alle derivate parziali di secondo ordine, di tipo misto, *Rend. Reale Accad. Lincei.*, **14(5)** (1923), 134-247.
[14] A. I. Vol'pert, S. I. Hudjaev; On the problem for quasilinear degenerate parabolic equations of second order (Russian), *Mat. Sb.*, **3** (1967), 374-396.
[15] Z. Wu, J. Zhao; The first boundary value problem for quasilinear degenerate parabolic equations of second order in several variables, *Chin.Ann. of Math.*, **4B(1)** (1983), 57-76.
[16] Z. Wu, J. Zhao, J. Yin, H. Li; Nonlinear diffusion equations, *Word Scientific Publishing*, 2001.
[17] H. Zhan; The study of the Cauchy problem of a second order quasilinear degenerate parabolic equation and the parallelism of a Riemannian manifold, *Doctor Thesis, Xiamen University*, 2004.
[18] J. Zhao, H. Zhan; Uniqueness and stability of solution for Cauchy problem of degenerate quasilinear parabolic equations, *Science in China Ser. A*, **48** (2005), 583-593.

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