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RESOLVENT KERNEL ON H-TYPE GROUPS AND A GREEN KERNEL FOR FRACTIONAL POWERS OF ITS SUB-LAPLACIAN

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Abstract. In this article, we give an integral representation of the resolvent kernel on H-type groups, then we derive an integral representation of Kaplan's fundamental solution on this groups. Also we obtain the Green kernel for fractional powers of its sub-Laplacian.

1. Introduction

H-type groups form an interesting class of Carnot groups of step two in connection with hypoellipticity questions. Such groups, which were introduced by Kaplan [11] around 1980 in the framework of his research about hypoelliptic partial differential equations, constitute a direct generalization of Heisenberg groups and are more complicated. This class suggests that this is the largest class of groups for which an elementary expression for the fundamental solution of the sub-Laplacian exists. Many interesting groups are H-type groups, including the two-step nilpotent group that appears in the Iwasawa decomposition of a rank-one semisimple Lie group. There has been subsequently a considerable amount of work in the study of such groups [5, 6, 16, 19].

In this article, we are interested in some complex spectral objects associated with the sub-Laplacian $\mathcal{L}$ on H-type groups $\mathbb{G}$. Namely, the heat, the resolvent and the green kernels are derived.

The first aim is to use the explicit formula for the heat kernel to derive an integral representation of the resolvent kernel. More precisely, one can use the well known formula connecting the resolvent $\mathcal{R}(\zeta, \mathcal{L}) = (\zeta - \mathcal{L})^{-1}$ and the heat $T(s) = e^{s\mathcal{L}}$ operators [7, p.56]

$$\mathcal{R}(\zeta, \mathcal{L}) = \int_0^\infty e^{-\zeta s} T(s)\, ds,$$

to find the resolvent kernel associated with the sub-Laplacian $\mathcal{L}$. We prove that its expression is given in terms of the Whittaker function $W_{\kappa,\mu}(z)$.

As applications of the obtained explicit formula for the resolvent kernel, we derive an integral representation of the Green function on H-type groups $\mathbb{G}$.

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The second aim is to prove that the Kaplan's fundamental solution obtained in [11] for the sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ can also be derived from the resolvent kernel of this sub-Laplacian. This provides us with a new integral representation for this fundamental solution.

The third purpose is to use the explicit formula for the resolvent kernel to give the Green kernel of the fractional power of the sub-Laplacian $\mathcal{L}$, i.e. $\mathcal{L}^\alpha$ for $\alpha \in]0,1[$, on the on H-type groups $\mathbb{G}$. We prove that its formula is given by a series expansion in terms of the generalized Laguerre polynomials.

An interesting relationship with special functions such as Gamma and Bessel functions will appear, showing the underlying harmony of this work.

The layout of this article is as follows. The aim of Section 2, is to provide the basic notation and definitions about H-type groups that we shall use throughout the paper. In Sections 3, we establish a new integral representation of the heat kernel obtained in [19]. In Section 4, we obtain an integral representation of the resolvent kernel on $\mathbb{G}$, that plays a major role in the following sections. We ends this section by establishing an integral representation of the Green function on H-type groups $\mathbb{G}$. In Section 5, we prove that the Kaplan's fundamental solution for the sub-Laplacian $\mathcal{L}$ can also be derived from the resolvent kernel of this sub-Laplacian. This provides us with a new integral representation for this fundamental solution. In the section 6, we give a formulas for the Green kernel for fractional powers of the sub-Laplacian $\mathcal{L}$.

This article extends the results in [2, 3, 14, 15] from the classical Heisenberg groups $\mathbb{C}\times\mathbb{R}$, $\mathbb{H}\times\mathbb{R}^3$ and $\mathbb{O}\times\mathbb{R}^7$ to the H-type groups $\mathbb{G}$ (Heisenberg groups with multi-dimensional center).

2. Notation and definitions

An H-type group $\mathbb{G}$ is characterized by being (canonically isomorphic to) $\mathbb{R}^{2n}\times\mathbb{R}^m$ with the group law

$$(x,u)\cdot(y,v) = \big(x+y, u+v+\frac{1}{2}\langle x, Uy\rangle\big),$$

with $x=(x_1,\dots,x_{2n})\in\mathbb{R}^{2n}$, $u=(u_1,\dots,x_m)\in\mathbb{R}^m$ and

$$\langle x, Uy\rangle = \big(\langle x, U^{(1)}y\rangle,\dots,\langle x, U^{(m)}y\rangle\big)\in\mathbb{R}^m,$$

where the $U^{(j)}$'s have the following properties:

(1) $U^{(j)}$ is an $m\times m$ skew-symmetric and orthogonal matrix for every $j\in\{1,\dots,m\}$,
(2) $U^{(i)}U^{(j)}+U^{(j)}U^{(i)}=0$, $1\le i\ne j\le m$.

It is clear that the point $e=(0,0)$ is the identity in $\mathbb{G}$ and the inverse operation is $(x,u)^{-1}=(-x,-u)$. The center of the group $\mathbb{G}$ is of dimension m and is given by $\mathcal{Z}(\mathbb{G})=\{(0,u):\, u\in\mathbb{R}^m\}$.

Let $U^{(j)}=\big(U^{(j)}_{k,l}\big)_{k,l\le 2n}$ $(1\le j\le m)$. The sub-Laplacian on $\mathbb{G}$ is the second-order differential operator $\mathcal{L}=\sum_{l=1}^{2n}X_l^2$, where $(X_l)_{1\le l\le 2n}$ are the left-invariant vector fields on $\mathbb{G}$ defined by

$$X_l = \frac{\partial}{\partial x_l}+\frac{1}{2}\sum_{j=1}^{m}\Big(\sum_{k=1}^{2n}x_kU^{(j)}_{k,l}\Big)\frac{\partial}{\partial u_j}.$$

Let

$$|x|^2 = \sum_{i=0}^{2n} x_i^2,\ |u|^2 = \sum_{j=1}^{m} u_j^2, \quad u \cdot v = \sum_{j=1}^{m} u_j v_j \quad \text{for } v \in \mathbb{R}^m.$$

We introduce on $\mathbb{G}$ the group $\{\delta_r : 0 < r < \infty\}$ of dilations, which is defined by

$$\delta_r(x,u) = (rx, r^2 u).$$

These dilations satisfy the distributive law

$$\delta_r\left((x,u).(y,v)\right) = \left(\delta_r(x,u)\right).\left(\delta_r(y,v)\right).$$

We also define the norm function on $\mathbb{G}$, which we will call the Kaplan distance, by

$$\rho(x,u) = \left(|x|^4 + 16|u|^2\right)^{1/4},$$

which satisfies

$$\rho(\delta_r(x,u)) = r\rho(x,u).$$

Note that, the Haar measure on $\mathbb{G}$ coincides with the Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}^m$ which is denoted by $dxdu$ and the homogeneous dimension of $\mathbb{G}$ is $Q = 2(n+m)$. We refer the reader to [5, 11] for further details.

3. Heat kernel on H-type groups

The heat kernel of the sub-Laplacian on an H-type group is given in [19].

Theorem 3.1. *On an H-type group $\mathbb{G} \simeq \mathbb{R}^{2n} \times \mathbb{R}^m$, the heat kernel $(p_t)_{t>0}$ has the form*

$$p_t(x,u) = (2\pi)^{-m}(4\pi)^{-n} \int_{\mathbb{R}^m} \Big(\frac{|\lambda|}{\sinh(|\lambda|t)}\Big)^n e^{-\frac{|\lambda||x|^2}{4}\coth(|\lambda|t) - i\lambda.u}\, d\lambda, \tag{3.1}$$

for every $t > 0$ and every (x,u) in $\mathbb{G}$.

Using polar coordinates, we establish a new integral representation of the heat kernel (3.1).

Proposition 3.2. *The heat kernel in* (3.1) *can be written as*

$$p_t(x,u) = (2\pi)^{-\frac{m}{2}}(4\pi)^{-n}|u|^{1-\frac{m}{2}} \int_0^\infty \frac{e^{-\frac{r|x|^2}{4}\coth(tr)}}{\sinh^n(tr)} J_{\frac{m}{2}-1}(|u|r) r^{n+\frac{m}{2}}\, dr, \tag{3.2}$$

where J_ν is the Bessel functions of the first kind.

Proof. We introduce polar coordinates for the λ-variable such that $\lambda = r\omega$, where $r = |\lambda|$ and $\omega = (\omega_1, \dots, \omega_m)$ is a point in the unit sphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$ with center at the origin. Then $dm(\lambda) = r^{m-1} dr d\sigma(\omega)$, where $d\sigma$ is the surface measure on $\mathbb{S}^{m-1}$. By Theorem 3.1,

$$\begin{aligned} p_t(x,u) &= (2\pi)^{-m}(4\pi)^{-n} \int_0^\infty \int_{\mathbb{S}^{m-1}} \Big(\frac{r}{\sinh(tr)}\Big)^n e^{-\frac{r|x|^2}{4}\coth(tr) - ir\omega.u} r^{m-1}\, dr\, d\sigma(\omega) \\ &= (2\pi)^{-m}(4\pi)^{-n} \int_0^\infty \frac{r^{n+m-1}}{\sinh^n(tr)} e^{-\frac{r|x|^2}{4}\coth(tr)} \mathcal{I}_u(r)\, dr, \end{aligned} \tag{3.3}$$

where

$$\mathcal{I}_u(r) = \int_{\mathbb{S}^{m-1}} e^{-ir\omega.u}\, d\sigma(\omega).$$

Using the identity [17, p.347]

$$\int_{\mathbb{S}^{m-1}} e^{i\langle a,\omega\rangle}\, d\sigma(\omega) = (2\pi)^{\nu+1}|a|^{-\nu} J_\nu(|a|), \quad \nu = \frac{m}{2} - 1,$$

for $a = -ru$, we obtain

$$\mathcal{I}_u(r) = (2\pi)^{\frac{m}{2}} |u|^{1-\frac{m}{2}} r^{1-\frac{m}{2}} J_{\frac{m}{2}-1}(|u|r). \tag{3.4}$$

Substituting (3.4) into the expression of the heat kernel in (3.3), we finally obtain

$$p_t(x,u) = (2\pi)^{-\frac{m}{2}} (4\pi)^{-n} |u|^{1-\frac{m}{2}} \int_0^\infty \frac{e^{-\frac{r|x|^2}{4}\coth(tr)}}{\sinh^n(tr)} J_{\frac{m}{2}-1}(|u|r) r^{n+\frac{m}{2}}\, dr,$$

as required. □

Remark 3.3. We can proof that, the solution of the Cauchy problem of heat type of $\mathcal{L}$ with initial-value is

$$\mathsf{p}_t\left((x,u),(y,v)\right) := p_t\left((x,u).(y,v)^{-1}\right) = p_t\big(x-y, u-v+\frac{1}{2}\langle x, Uy\rangle\big), \tag{3.5}$$

for all $(x,u),(y,v) \in \mathbb{G}$. On the other hand, it is more evident that p_t depends only on $|x|$ and $|u|$. This leads us (throughout this article) to the following notation:

$$\rho := |x-y| \quad \text{and} \quad \tau := \big|u - v + \frac{1}{2}\langle x, Uy\rangle\big|. \tag{3.6}$$

Hence, the heat kernel in (3.5) can be written as

$$\mathsf{p}_t\left((x,u),(y,v)\right) = \frac{(2\pi)^{-\frac{m}{2}}(4\pi)^{-n}}{\tau^{\frac{m}{2}-1}} \int_0^\infty \frac{e^{-\frac{r\rho^2}{4}\coth(tr)}}{\sinh^n(tr)} J_{\frac{m}{2}-1}(\tau r)\, r^{n+\frac{m}{2}}\, dr. \tag{3.7}$$

4. Resolvent kernel on H-type groups

The confluent hypergeometric function [10, p.204] is denoted by

$${}_1F_1(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{j=0}^\infty \frac{\Gamma(a+j)}{\Gamma(b+j)} \frac{z^j}{j!}. \tag{4.1}$$

As in [10, p.264], we define the Kummer's function of the second kind [1, p.505]

$$U(a,b;z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\, {}_1F_1(a,b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b}\, {}_1F_1(a-b+1, 2-b; z). \tag{4.2}$$

We denote the Whittaker function given by

$$W_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+\frac{1}{2}} U\big(\mu - \kappa + \frac{1}{2}, 1+2\mu; z\big). \tag{4.3}$$

Theorem 4.1. *Let $\zeta \in \mathbb{C}$ such that $\Re\zeta > 0$. Then, the resolvent kernel for an H-type group, $\mathbb{G}$, is*

$$\begin{aligned} &\mathcal{R}\left(\zeta;(x,u),(y,v)\right) \\ &= \frac{2^{\frac{n-2}{2}}(2\pi)^{-n-\frac{m}{2}}}{\rho^n \tau^{\frac{m-2}{2}}} \int_0^\infty \Gamma\big(\frac{\zeta}{2r} + \frac{n}{2}\big) J_{\frac{m}{2}-1}(\tau r) W_{-\frac{\zeta}{2r},\frac{n-1}{2}}\left(r\rho^2/2\right) r^{\frac{n+m-2}{2}}\, dr, \end{aligned} \tag{4.4}$$

where $\Gamma(\cdot)$ is Euler's Gamma-function.

Proof. We use the well known formula connecting the resolvent and the heat kernels

$$\mathcal{R}\big(\zeta;(x,u),(y,v)\big)=\int_0^\infty e^{-\zeta t}\,\mathfrak{p}_t\big((x,u),(y,v)\big)\,dt;\quad \Re e\zeta>0, \tag{4.5}$$

as well as the explicit formula for the heat kernel in (3.7), to obtain

$$\mathcal{R}\left(\zeta;(x,u),(y,v)\right)=(2\pi)^{-\frac{m}{2}}(4\pi)^{-n}\tau^{1-\frac{m}{2}}\int_0^\infty J_{\frac{m}{2}-1}(\tau r)\mathcal{J}_{\rho,\zeta}(r)r^{n+\frac{m}{2}}\,dr, \tag{4.6}$$

where

$$\mathcal{J}_{\rho,\zeta}(r)=\int_0^\infty e^{-\zeta t}e^{-\frac{r\rho^2}{4}\coth(rt)}\sinh^{-n}(rt)\,dt.$$

The change of variables $s=rt$ yields

$$\mathcal{J}_{\rho,\zeta}(r)=\frac{1}{r}\int_0^\infty e^{-\frac{\zeta}{r}s}e^{-\frac{r\rho^2}{4}\coth(s)}\sinh^{-n}(s)\,ds.$$

Next, using the integral representation

$$\begin{aligned}&\int_0^{+\infty}e^{-2\mu s}e^{-2\beta\coth(s)}\big(\sinh(s)\big)^{2\nu}\,ds\\&=\frac{1}{4}\beta^{\frac{1}{2}(\nu-1)}\Gamma(\mu-\nu)\big[W_{-\mu+\frac{1}{2},\nu}(4\beta)-(\mu-\nu)W_{-\mu-\frac{1}{2},\nu}(4\beta)\big],\end{aligned}$$

where $\Re e(\beta)>0$ and $\Re e(\mu)>\Re e(\nu)$ [8, p.358], we can write the integral $\mathcal{J}_{\rho,\zeta}(r)$ in terms of the Whittaker function $W_{\kappa,\mu}(z)$ given in (4.3). Hence for

$$\mu=\frac{\zeta}{2r},\quad \beta=\frac{r\rho^2}{8}\quad\text{and}\quad \nu=-\frac{n}{2},$$

we obtain

$$\begin{aligned}&\mathcal{J}_{\rho,\zeta}(r)\\&=\frac{8^{\frac{n+1}{2}}\Gamma(\frac{\zeta}{2r}+\frac{n}{2})}{2\rho^{n+1}r^{\frac{n+3}{2}}}\big[W_{-\frac{\zeta}{2r}+\frac{1}{2},-\frac{n}{2}}(r\rho^2/2)-\big(\frac{\zeta}{2r}+\frac{n}{2}\big)W_{-\frac{\zeta}{2r}-\frac{1}{2},-\frac{n}{2}}(r\rho^2/2)\big].\end{aligned} \tag{4.7}$$

Now, in view of the identity [1, p.507],

$$W_{\kappa+\frac{1}{2},\nu}(z)+(\kappa+\nu)W_{\kappa-\frac{1}{2},\nu}(z)=z^{1/2}W_{\kappa,\nu+\frac{1}{2}}(z),$$

we can rewrite (4.7) as

$$\begin{aligned}\mathcal{J}_{\rho,\zeta}(r)&=\frac{2^{\frac{3n-2}{2}}\Gamma\big(\frac{\zeta}{2r}+\frac{n}{2}\big)}{\rho^n r^{\frac{n}{2}+1}}W_{-\frac{\zeta}{2r},\frac{1-n}{2}}\big(r\rho^2/2\big)\\&=\frac{2^{\frac{3n-2}{2}}\Gamma\big(\frac{\zeta}{2r}+\frac{n}{2}\big)}{\rho^n r^{\frac{n}{2}+1}}W_{-\frac{\zeta}{2r},\frac{n-1}{2}}\big(r\rho^2/2\big).\end{aligned}$$

The above equality follows by using the identities on the Whittaker function [13, p.299]

$$W_{\kappa,\mu}(z)=W_{\kappa,-\mu}(z).$$

Hence, the resolvent kernel in (4.6) can be expressed as

$$\begin{aligned}&\mathcal{R}\left(\zeta;(x,u),(y,v)\right)\\&=\frac{2^{\frac{n-2}{2}}(2\pi)^{-n-\frac{m}{2}}}{\rho^n\tau^{\frac{m-2}{2}}}\int_0^\infty\Gamma\big(\frac{\zeta}{2r}+\frac{n}{2}\big)J_{\frac{m}{2}-1}(\tau r)W_{-\frac{\zeta}{2r},\frac{n-1}{2}}\big(r\rho^2/2\big)\,r^{\frac{n+m-2}{2}}\,dr.\end{aligned}$$

The proof is complete. ∎

Remark 4.2. Considering the limit value $\zeta = 0$ in (4.4), we obtain the kernel function

$$\begin{aligned} &\mathcal{R}\left(0;(x,u),(y,v)\right) \\ &= \frac{2^{\frac{n-2}{2}}\,(2\pi)^{-n-\frac{m}{2}}\,\Gamma\big(\frac{n}{2}\big)}{\rho^{n}\tau^{\frac{m-2}{2}}} \int_0^\infty J_{\frac{m}{2}-1}(\tau r) W_{0,\frac{n-1}{2}}\left(r\rho^2/2\right) r^{\frac{n+m-2}{2}}\,dr. \end{aligned} \tag{4.8}$$

Following [13, p.305], the Whittaker function $W_{0,\alpha}(z)$ can be expressed in terms of the modified Bessel function of the second kind $K_\alpha(z)$ as follows

$$W_{0,\alpha}(z) = \pi^{-1/2} z^{1/2} K_\alpha\big(\frac{z}{2}\big). \tag{4.9}$$

For the parameters $\alpha = \frac{n-1}{2}$ and $z = r\rho^2/2$, the integral in (4.8) takes the form

$$\begin{aligned} &\mathcal{R}\left(0;(x,u),(y,v)\right) \\ &= \frac{2^{\frac{n-2}{2}}\,(2\pi)^{-\frac{2n+m+1}{2}}\,\Gamma\big(\frac{n}{2}\big)}{\rho^{n-1}\tau^{\frac{m-2}{2}}} \int_0^\infty J_{\frac{m}{2}-1}(\tau r) K_{\frac{n-1}{2}}\left(r\rho^2/4\right) r^{\frac{n+m-1}{2}}\,dr, \end{aligned} \tag{4.10}$$

which corresponds to a right inverse of $\mathcal{L}$. That is,

$$\mathcal{L}^{-1} f(x,u) = \int_{\mathbb{G}} -\mathcal{R}\left(0;(x,u),(y,v)\right)\, f(y,v)\,dy\,dv.$$

In other words, $-\mathcal{R}\left(0;(x,u),(y,v)\right)$ is a Green kernel of $\mathcal{L}$.

5. An integral representation for Kaplan's fundamental solution

Kaplan [11] prove that the sub-Laplacian $\mathcal{L}$ admits a fundamental solution with source at $e = (0,0)$, the identity element of $\mathbb{G}$, of the form

$$\Phi_e(x,u) = c_Q\,\rho^{2-Q}(x,u), \quad (x,u) \in \mathbb{G}, \tag{5.1}$$

for a suitable constant $c_Q > 0$, where $Q = 2(n+m)$ is the homogeneous dimension of $\mathbb{G}$ and where ρ is the norm function on $\mathbb{G}$ given by

$$\rho(x,u) = \left(|x|^4 + 16|u|^2\right)^{1/4}.$$

In other words $\langle \mathcal{L}\varphi, \Phi_e\rangle = \varphi(e)$, for any function $\varphi \in C_0^\infty(\mathbb{G})$.

We prove that the Kaplan's fundamental solution for the sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ can also be derived form the resolvent kernel of this sub-Laplacian. This provides us with a new integral representation for this fundamental solution.

Proposition 5.1. *Kaplan's fundamental solution in* (5.1) *can also be expressed as*

$$\begin{aligned} &\Phi_e(x,u) \\ &= \frac{2^{\frac{n-2}{2}}\,(2\pi)^{-\frac{2n+m+1}{2}}\,\Gamma\big(\frac{n}{2}\big)}{|x|^{n-1}|u|^{\frac{m-2}{2}}} \int_0^\infty J_{\frac{m}{2}-1}(|u|r) K_{\frac{n-1}{2}}\left(r|x|^2/4\right) r^{\frac{n+m-1}{2}}\,dr. \end{aligned} \tag{5.2}$$

Proof. To prove (5.2), we recall first that the resolvent kernel of $\mathcal{L}$ has the form

$$\begin{aligned} &\mathcal{R}\left(\zeta;(x,u),(y,v)\right) \\ &= \frac{2^{\frac{n-2}{2}}\,(2\pi)^{-n-\frac{m}{2}}}{\rho^{n}\tau^{\frac{m-2}{2}}} \int_0^\infty \Gamma\Big(\frac{\zeta}{2r} + \frac{n}{2}\Big) J_{\frac{m}{2}-1}(\tau r) W_{-\frac{\zeta}{2r},\frac{n-1}{2}}\left(r\rho^2/2\right) r^{\frac{n+m-2}{2}}\,dr, \end{aligned} \tag{5.3}$$

In the limit as $\zeta \to 0$ in (5.3), we obtain the Green kernel $\mathcal{R}_0 := \mathcal{R}\left(0;(x,u),(y,v)\right)$ of $\mathcal{L}$ as pointed out in Remark 4.2. Now, to establish a connection between the integral kernel $\mathcal{R}_0$ and Kaplan's fundamental solution, we proceed by computing the integral

$$\mathcal{R}_0 = \frac{2^{\frac{n-2}{2}}(2\pi)^{-\frac{2n+m+1}{2}}\Gamma\big(\frac{n}{2}\big)}{\rho^{n-1}\tau^{\frac{m-2}{2}}}\int_0^\infty J_{\frac{m}{2}-1}(\tau r)K_{\frac{n-1}{2}}\left(r\rho^2/4\right)r^{\frac{n+m-1}{2}}\,dr. \tag{5.4}$$

We use the identity [9, p.684]

$$\begin{aligned}&\int_0^\infty r^{-\lambda}K_\mu(ar)J_\nu(br)\,dr\\&= \frac{b^\nu\Gamma\big(\frac{\nu+\mu-\lambda+1}{2}\big)\Gamma\big(\frac{\nu-\mu-\lambda+1}{2}\big)}{2^{\lambda+1}a^{\nu-\lambda+1}\Gamma(1+\nu)}\,{}_2F_1\Big(\frac{\nu+\mu-\lambda+1}{2},\frac{\nu-\mu-\lambda+1}{2};\nu+1;-\frac{b^2}{a^2}\Big),\end{aligned}$$

when $\Re(a\pm ib)>0$, $\Re(\nu-\lambda+1)>|\Re\mu|$ are fulfilled, and where

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^\infty \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k!},$$

denotes the hypergeometric function [13, p.37]. In our case $\lambda = -\frac{n+m-1}{2}$, $\mu=\frac{n-1}{2}$, $\nu=\frac{m}{2}-1$, $a=\rho^2/4$, and $b=\tau$, therefore

$$\begin{aligned}&\int_0^\infty J_{\frac{m}{2}-1}(\tau r)K_{\frac{n-1}{2}}\left(r\rho^2/4\right)r^{\frac{n+m-1}{2}}\,dr\\&= \frac{2^{\frac{3n+5m-5}{2}}\Gamma\big(\frac{n+m-1}{2}\big)}{\rho^{n+2m-1}\tau^{1-\frac{m}{2}}}\,{}_2F_1\Big(\frac{n+m-1}{2},\frac{m}{2};\frac{m}{2};-\frac{(4\tau)^2}{\rho^4}\Big).\end{aligned}$$

Returning to (5.4), we obtain that

$$\mathcal{R}_0 = \frac{2^{\frac{4n+5m-7}{2}}\Gamma\left(\frac{n+m-1}{2}\right)\Gamma\big(\frac{n}{2}\big)}{(2\pi)^{\frac{2n+m+1}{2}}\rho^{2(n+m+1)}}\,{}_2F_1\Big(\frac{n+m-1}{2},\frac{m}{2};\frac{m}{2};-\frac{(4\tau)^2}{\rho^4}\Big).$$

The hypergeometric function ${}_2F_1\big(\frac{n+m-1}{2},\frac{m}{2};\frac{m}{2};-\frac{(4\tau)^2}{\rho^4}\big)$ is an elementary function given by

$$\Gamma\Big(\frac{n+m-1}{2}\Big)\Big(1+\frac{(4\tau)^2}{\rho^4}\Big)^{-\frac{n+m-1}{2}}.$$

It follows that

$$\begin{aligned}\mathcal{R}_0 &= \frac{2^{\frac{4n+5m-7}{2}}\Gamma\big(\frac{n}{2}\big)\Gamma\left(\frac{n+m-1}{2}\right)}{(2\pi)^{\frac{2n+m+1}{2}}\rho^{2(n+m+1)}}\Big(1+\frac{(4\tau)^2}{\rho^4}\Big)^{-\frac{n+m-1}{2}}\\&= \frac{2^{\frac{4n+5m-7}{2}}\Gamma\left(\frac{n+m-1}{2}\right)\Gamma\big(\frac{n}{2}\big)}{(2\pi)^{\frac{2n+m+1}{2}}}\frac{1}{(\rho^4+16\tau^2)^{\frac{n+m-1}{2}}}.\end{aligned} \tag{5.5}$$

In particular, for $(y,v)=(0,0)$, keeping in mind the expression of ρ and τ given in (3.6), Equation (5.5) reduces to

$$\begin{aligned}\mathcal{R}_0 &= \frac{2^{\frac{4n+5m-7}{2}}\Gamma\left(\frac{n+m-1}{2}\right)\Gamma\big(\frac{n}{2}\big)}{(2\pi)^{\frac{2n+m+1}{2}}}\frac{1}{(|x|^4+16|u|^2)^{\frac{n+m-1}{2}}}\\&= \frac{2^{\frac{3Q-6}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{Q-2}{4}\right)}{(4\pi)^{\frac{Q+1}{2}}}\rho^{2-Q}(x,u),\end{aligned} \tag{5.6}$$

where $Q = 2(n+m)$ is the homogeneous dimension of $\mathbb{G}$ and where ρ is the norm function on $\mathbb{G}$ given by

$$\rho(x,u) = \left(|x|^4 + 16|u|^2\right)^{1/4}.$$

By combining (5.1) and (5.6), we obtain

$$\mathcal{R}_0 = \frac{2^{\frac{3Q-6}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{Q-2}{4}\right)}{(4\pi)^{\frac{Q+1}{2}}} c_Q^{-1}\Phi_e(x,u),$$

where the constant c_Q is as in (5.1) and then can be computed explicitly and it is given by

$$c_Q = \frac{2^{\frac{3Q-6}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{Q-2}{4}\right)}{(4\pi)^{\frac{Q+1}{2}}}.$$

The asserted formula is established. $\square$

Remark 5.2. The constant c_Q that appears in Kaplan's fundamental solution is given by [10]

$$c_Q^{-1} = \int_{\mathbb{G}} |x|^2 \left(1 + \rho(x,u)^4\right)^{-(Q+6)/4} \,dx\,du,$$

which can be also computed explicitly using polar coordinates on H-type groups in [10].

6. Green kernel for fractional powers of $\mathcal{L}$

For $0 < \alpha < 1$ one defines the (fractional) power $\mathcal{L}^\alpha$ by the usual functional calculus. It is still an unbounded self-adjoint operator. As application of the formula obtained for the resolvent kernel of $\mathcal{L}$, we give the Green kernel of the fractional power operator $\mathcal{L}^\alpha$ for $\alpha \in]0,1[$. More precisely, we have the following result.

Theorem 6.1. *Let $\alpha \in]0,1[$. Then the Green kernel of the fractional power operator $\mathcal{L}^\alpha$ is*

$$\begin{aligned} &\mathcal{G}_\alpha((x,u),(y,v)) \\ &= \frac{1}{2^\alpha (2\pi)^{\frac{2n+m}{2}} \tau^{\frac{m-2}{2}}} \int_0^\infty e^{r\rho^2/4} W_\alpha(r) J_{\frac{m}{2}-1}(\tau r) r^{\frac{2n+m-2\alpha}{2}}\,dr, \end{aligned} \tag{6.1}$$

where

$$W_\alpha(r) = \sum_{k=0}^\infty \left(k + \frac{n}{2}\right)^{-\alpha} L_k^{(n+1)}(r\rho^2/2).$$

Proof. Since $\mathcal{L}$ is a self-adjoint operator, its resolvent [12, p.21] satisfies

$$\|\mathcal{R}(s)\| \le \frac{1}{s}.$$

This estimate enables us to define the fractional powers $\mathcal{L}^\alpha$, $\alpha \in]0,1[$ according to the formula [12, p.127]

$$\mathcal{L}^\alpha g = \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^{\alpha-1}\mathcal{R}(s)\mathcal{L}g\,ds, \quad g \in D(\mathcal{L}). \tag{6.2}$$

Thanks to Kato's formula [12, p.124], the resolvent operator $\mathcal{R}_\alpha(\gamma) = (\gamma - \mathcal{L}^\alpha)^{-1}$, $\alpha\pi < |\arg\gamma| < \pi$, is given by

$$\mathcal{R}_\alpha(\gamma) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\lambda^\alpha \mathcal{R}(\lambda)}{\lambda^{2\alpha} - 2\lambda^\alpha\gamma\cos\pi\alpha + \gamma^2}\,d\lambda. \tag{6.3}$$

The action of $\mathcal{R}_\alpha(\gamma)$ on a function $f \in L^2(\mathbb{G})$ is

$$\mathcal{R}_\alpha(\gamma)f(x,u) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\lambda^\alpha \mathcal{R}(\lambda) f(x,u)}{\lambda^{2\alpha} - 2\lambda^\alpha\gamma\cos\pi\alpha + \gamma^2}d\lambda,$$

almost every where. Then the resolvent kernel of $\mathcal{L}^\alpha$ is

$$\mathcal{G}_\alpha(\gamma;(x,u),(y,v)) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\lambda^\alpha \mathcal{R}(\lambda;(x,u),(y,v))}{\lambda^{2\alpha} - 2\lambda^\alpha\gamma\cos\pi\alpha + \gamma^2}\,d\lambda. \tag{6.4}$$

The limit value $\gamma = 0$ in (6.4) gives a Green kernel of $\mathcal{L}^\alpha$:

$$\begin{aligned}\mathcal{G}_\alpha((x,u),(y,v)) &:= \mathcal{G}_\alpha(0;(x,u),(y,v)) \\ &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}\mathcal{R}(\lambda;(x,u),(y,v))\,d\lambda.\end{aligned} \tag{6.5}$$

Using expression in (4.4) and intertwining the integrals, we rewrite (6.5) as

$$\mathcal{G}_\alpha((x,u),(y,v)) = \frac{2^{n/2}\sin\pi\alpha}{(2\pi)^{\frac{2n+m+2}{2}}\rho^n\tau^{\frac{m-2}{2}}}\int_0^\infty N_\alpha(r)\,J_{\frac{m}{2}-1}(\tau r)\,r^{\frac{n+m-2}{2}}\,dr, \tag{6.6}$$

where

$$N_\alpha(r) = \int_0^\infty \lambda^{-\alpha}\Gamma\big(\frac{\lambda}{2r}+\frac{n}{2}\big)W_{-\frac{\lambda}{2r},\frac{n-1}{2}}\big(r\rho^2/2\big)\,d\lambda. \tag{6.7}$$

Next, using the integral representation [4, p.147],

$$\Gamma(\nu)W_{\frac12-\frac{p}{2}-\nu,-\frac{p}{2}}(z) = z^{1/2-p/2}e^{\frac{z}{2}}\int_0^\infty e^{-ps}(1-e^{-s})^{\nu-1}e^{-ze^s}\,ds; \quad \Re z,\ \Re\nu > 0.$$

In our case $z = r\rho^2/2$, $\nu = \frac{\lambda}{2r} + \frac{n}{2}$ and $p = 1-n$, and therefore (6.7) reads

$$N_\alpha(r) = \frac{r^{n/2}\rho^n e^{r\rho^2/4}}{2^{n/2}}\int_0^\infty e^{(n-1)s}(1-e^{-s})^{(n-2)/2}e^{-r|x|^2e^s/2}I_\alpha(s)\,ds, \tag{6.8}$$

where

$$I_\alpha(s) = \int_0^\infty \lambda^{-\alpha}(1-e^{-s})^{\lambda/2r}d\lambda = \int_0^\infty \lambda^{-\alpha}e^{-\frac{1}{2r}\log(\frac{e^s}{e^s-1})\lambda}d\lambda. \tag{6.9}$$

Hence, using [9, p.346],

$$\int_0^\infty \gamma^{\nu-1}e^{-\mu\gamma}\,d\gamma = \frac{\Gamma(\nu)}{\mu^\nu}; \quad \Re\mu > 0,\ \Re\nu > 0,$$

with $\mu = \frac{1}{2r}\log(\frac{e^s}{e^s-1})$ and $\nu = 1-\alpha$, we can write the right hand side in (6.9) as

$$I_\alpha(s) = \frac{2^{1-\alpha}r^{1-\alpha}\Gamma(1-\alpha)}{\log^{1-\alpha}(\frac{e^s}{e^s-1})}.$$

Then the integral in (6.8) reads

$$\begin{aligned}&N_\alpha(r)\\ &= \frac{\Gamma(1-\alpha)r^{\frac{n}{2}+1-\alpha}\rho^n e^{r\rho^2/4}}{2^{\frac{n}{2}+\alpha-1}}\int_0^\infty e^{(n-1)s}(1-e^{-s})^{\frac{n-2}{2}}e^{-r\rho^2e^s/2}\log^{\alpha-1}\Big(\frac{e^s}{e^s-1}\Big)\,ds.\end{aligned}$$

Making the change of variable $e^t = \frac{e^s}{e^s-1}$, the above equality becomes

$$N_\alpha(r) = \frac{\Gamma(1-\alpha) r^{\frac{n}{2}+1-\alpha}\rho^n e^{r\rho^2/4}}{2^{\frac{n}{2}+\alpha-1}} \int_0^\infty e^{-\frac{n}{2}t} t^{\alpha-1}(1-e^{-t})^{-n} e^{-\frac{r\rho^2 e^{-t}}{2(1-e^{-t})}}\, dt. \tag{6.10}$$

By using the identity [18, p.101],

$$(1-w)^{-\beta-1} e^{-\frac{zw}{1-w}} = \sum_{k=0}^\infty L_k^{(\beta)}(z)\, w^k; \quad \beta, z \in \mathbb{C},\ |w| < 1, \tag{6.11}$$

for $\beta = n+1$, $w = e^{-t}$ and $z = r\rho^2/2$, the integral $N_\alpha(r)$ may therefore be written as

$$N_\alpha(r) = \frac{\Gamma(1-\alpha) r^{\frac{n}{2}+1-\alpha}\rho^n e^{r\rho^2/4}}{2^{\frac{n}{2}+\alpha-1}} \sum_{k=0}^\infty L_k^{(n+1)}(r\rho^2/2) \int_0^\infty t^{\alpha-1} e^{-(\frac{n}{2}+k)t}\, dt. \tag{6.12}$$

Making the change variable $\delta = (\frac{n}{2}+k)t$ and using the integral representation of the Gamma function $\Gamma(\gamma) = \int_0^\infty s^{\gamma-1}e^{-s}\, ds$, we arrive at

$$\begin{aligned}
&N_\alpha(r)\\
&= \frac{\Gamma(1-\alpha) r^{\frac{n}{2}+1-\alpha}\rho^n e^{r\rho^2/4}}{2^{\frac{n}{2}+\alpha-1}} \sum_{k=0}^\infty \big(k+\frac{n}{2}\big)^{-\alpha} L_k^{(n+1)}(r\rho^2/2) \int_0^\infty \delta^{\alpha-1}e^{-\delta}\, d\delta\\
&= \frac{\Gamma(\alpha)\Gamma(1-\alpha) r^{\frac{n}{2}+1-\alpha}\rho^n e^{r\rho^2/4}}{2^{\frac{n}{2}+\alpha-1}} \sum_{k=0}^\infty \Big(k+\frac{n}{2}\Big)^{-\alpha} L_k^{(n+1)}(r\rho^2/2)\\
&= \frac{\pi r^{\frac{n}{2}+1-\alpha}\rho^n e^{r\rho^2/4}}{2^{\frac{n}{2}+\alpha-1}\sin\pi\alpha} \sum_{k=0}^\infty \big(k+\frac{n}{2}\big)^{-\alpha} L_k^{(n+1)}(r\rho^2/2).
\end{aligned} \tag{6.13}$$

The last equality follows using Euler's reflection formula [9, p.896]

$$\Gamma(\gamma)\Gamma(1-\gamma) = \frac{\pi}{\sin(\pi\gamma)}.$$

Substituting (6.13) into the expression of $\mathcal{G}_\alpha((x,u),(y,v))$ in (6.6), we obtain

$$\mathcal{G}_\alpha((x,u),(y,v)) = \frac{1}{2^\alpha (2\pi)^{\frac{2n+m}{2}} \tau^{\frac{m-2}{2}}} \int_0^\infty e^{r\rho^2/4}\, W_\alpha(r) J_{\frac{m}{2}-1}(\tau r)\, r^{\frac{2n+m-2\alpha}{2}}\, dr,$$

where

$$W_\alpha(r) = \sum_{k=0}^\infty \big(k+\frac{n}{2}\big)^{-\alpha} L_k^{(n+1)}(r\rho^2/2).$$

Hence we obtain the formula for the Green function, as asserted. □

Remark 6.2. When α approaches 1 in (6.1), we recover the expression of the Green function in Remark 4.2. We hope to return to the case $\alpha > 1$ in a future work.

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