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TRANSITION FRONTS OF KPP-TYPE LATTICE RANDOM EQUATIONS

FENG CAO, LU GAO

ABSTRACT. In this article, we investigate the existence and stability of random transition fronts of KPP-type lattice equations in random media, and explore the influence of the media and randomness on the wave profiles and wave speeds of such solutions. We first establish comparison principle for sub-solutions and super-solutions of KPP type lattice random equations and prove the stability of positive constant equilibrium solution. Next, by constructing appropriate sub-solutions and super-solutions, we show the existence of random transition fronts. Finally, we prove the stability of random transition fronts of KPP-type lattice random equations.

1. Introduction

This article studies the existence and stability of transition fronts for the KPP-type lattice random equation

$$\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + a(\theta_t\omega)u_i(t)(1 - u_i(t)), \quad i \in \mathbb{Z}, \tag{1.1}$$

where $\omega \in \Omega$, $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space, θ_t is an ergodic metric dynamical system on Ω, $a : \Omega \to (0, \infty)$ is measurable, and $a^\omega(t) := a(\theta_t\omega)$ is locally Hölder continuous in $t \in \mathbb{R}$ for every $\omega \in \Omega$.

Equation (1.1) is used to model the population dynamics of species living in patchy environments in biology and ecology (see, for example, [43, 44]). It is a spatial-discrete counterpart of the reaction diffusion equation

$$\partial_t u = u_{xx} + a(\theta_t\omega)u(1 - u), \quad x \in \mathbb{R}. \tag{1.2}$$

Equation (1.2) is widely used to model the population dynamics of species when the movement or internal dispersal of the organisms occurs between adjacent locations randomly in spatially continuous media. The study of traveling wave solutions of (1.2) traces back to Fisher [16] and Kolmogorov, Petrovsky and Piskunov [24] in the special case $a(\theta_t\omega) \equiv 1$. They investigated the existence of traveling wave solutions, that is, solutions of the form $u(x, t) = \phi(x - ct)$ with $\phi(-\infty) = 1$, $\phi(+\infty) = 0$. Fisher in [16] proved that (1.2) with $a(\theta_t\omega) \equiv 1$ admits traveling wave solutions if the wave speed $c \geq 2$ and showed that there are no such traveling wave solutions of slower speed. Kolmogorov, Petrovsky, and Piskunov in [24] proved that for any nonnegative solution $u(x, t)$ of (1.2) with $a(\theta_t\omega) \equiv 1$, if at time $t = 0$, u is 1 near

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$-\infty$ and 0 near ∞, then $\lim_{t\to\infty} u(t, ct)$ is 0 if $c > 2$ and 1 if $c < 2$. $c_* := 2$ is therefore the minimal wave speed and is also called the spreading speed of (1.2) with $a(\theta_t\omega) \equiv 1$. The spreading property was extended to more general monostable nonlinearities by Aronson and Weinberger [2].

Since then, traveling wave solutions of Fisher or KPP type evolution equations in spatially and temporally homogeneous media or spatially and/or temporally periodic media have been widely invetigated. The reader is referred to [1, 2, 5, 6, 7, 8, 14, 15, 17, 20, 23, 25, 26, 27, 28, 30, 34, 35, 36, 38, 40, 39, 45, 46] for the study of Fisher or KPP type reaction diffusion equations in homogeneous or periodic media. As for the study of Fisher or KPP type lattice equations in homogeneous or periodic media, the reader is referred to [11, 12, 13, 21, 29, 47, 48] for the existence and stability of traveling wave solutions in homogeneous media, and to [18, 19, 21] for the existence and stability of periodic traveling wave solutions in spatially periodic media. Recently, Cao and Shen [10] proved the existence and stability of periodic traveling wave solutions for Fisher or KPP type lattice equations in spatially and temporally periodic media.

The study of traveling wave solutions of general time and/or space dependent Fisher or KPP type equations is attracting more and more attention due to the presence of general time and space variations in real world problems. To study the front propagation dynamics of Fisher or KPP type equations with general time and/or space dependence, one first needs to properly extend the notion of traveling wave solutions in the classical sense. Some general extension has been introduced in the literature. For example, in [39, 41], notions of random traveling wave solutions and generalized traveling wave solutions are introduced for random Fisher or KPP type equations and quite general time dependent Fisher or KPP type equations, respectively. In [3, 4], a notion of generalized transition waves is introduced for Fisher or KPP type equations with general space and time dependence. Among others, the authors of [31, 32, 33] proved the existence of generalized transition waves of general time dependent and space periodic, or time independent and space almost periodic Fisher or KPP type reaction diffusion equations. Zlatos [49] established the existence of generalized transition waves of spatially inhomogeneous Fisher or KPP type reaction diffusion equations under some specific hypotheses. Shen [42] proved the stability of generalized transition waves of Fisher or KPP type reaction diffusion equations with quite general time and space dependence.

However, there is little study on the traveling wave solutions of Fisher or KPP type lattice equations with general time and/or space dependence. Since in nature, many systems are subject to irregular influences arisen from various kind of noise, it is also of great importance to study traveling wave solutions in random media. The purpose of this article is to investigate the existence and stability of traveling wave solutions for KPP-type lattice equations in random media under very general assumption (See (H1) below), and to understand the influence of the media and randomness on the wave profiles and wave speeds of such solutions. We note that the work [37] studied the existence and stability of random transition fronts for random KPP-type reaction diffusion equations.

It should be pointed out that Cao and Shen [9, 10] investigated the existence and stability of transition fronts for KPP-type lattice equations with general time dependence under some more restrictive assumptions. For KPP-type lattice equations in random media, although it's easy to get that the wave speed is stationary ergodic in t, but it is far from being obvious that the same is true for the random profile. Besides, when dealing with spatial-discrete equations, we need find another approach to get the existence of traveling wave solutions due to the lack of space regularity.

First we give notation and assumptions related to (1.1). Let

$$\underline{a}(\omega) = \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau := \lim_{r\to\infty}\inf_{t-s\geq r} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau,$$
$$\overline{a}(\omega) = \limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau := \lim_{r\to\infty}\sup_{t-s\geq r} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau.$$

We call $\underline{a}(\cdot)$ and $\overline{a}(\cdot)$ the least mean and the greatest mean of $a(\cdot)$, respectively. It's easy to show that

$$\underline{a}(\theta_t\omega) = \underline{a}(\omega), \quad \overline{a}(\theta_t\omega) = \overline{a}(\omega) \quad \forall t \in \mathbb{R},$$

and

$$\underline{a}(\omega) = \liminf_{t,s\in\mathbb{Q},t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau, \quad \overline{a}(\omega) = \limsup_{t,s\in\mathbb{Q},t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau.$$

Then $\underline{a}(\omega)$ and $\overline{a}(\omega)$ are measurable in ω. Throughout the paper, we assume that

(H1) $0 < \underline{a}(\omega) \leq \overline{a}(\omega) < \infty$ for a.e. $\omega \in \Omega$.

This implies that $\underline{a}(\cdot), a(\cdot), \overline{a}(\cdot) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ (see Lemma 2.4). Also (H1) and the ergodicity of the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ imply that, there are $\underline{a}, \overline{a} \in \mathbb{R}^+$ and a measurable subset $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\begin{gathered}\theta_t\Omega_0 = \Omega_0 \quad \forall t \in \mathbb{R}\\ \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau = \underline{a} \quad \forall \omega \in \Omega_0\\ \limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau = \overline{a} \quad \forall \omega \in \Omega_0.\end{gathered}$$

Let

$$l^\infty(\mathbb{Z}) = \{u = \{u_i\}_{i\in\mathbb{Z}} : \sup_{i\in\mathbb{Z}} |u_i| < \infty\}$$

with norm $\|u\| = \|u\|_\infty = \sup_{i\in\mathbb{Z}} |u_i|$. Since $a(\theta_t\omega)$ is locally Hölder continuous in $t \in \mathbb{R}$ for every $\omega \in \Omega$, for any given $u^0 \in l^\infty(\mathbb{Z})$, (1.1) has a unique (local) solution $u(t; u^0, \omega) = \{u_i(t; u^0, \omega)\}_{i\in\mathbb{Z}}$ with $u(0; u^0, \omega) = u^0$. Note that, if $u_i^0 \geq 0$ for all $i \in \mathbb{Z}$, then $u(t; u^0, \omega) = \{u_i(t; u^0, \omega)\}_{i\in\mathbb{Z}}$ exists for all $t \geq 0$ and $u_i(t; u^0, \omega) \geq 0$ for all $i \in \mathbb{Z}$ and $t \geq 0$ (see Proposition 2.1).

A solution $u(t; \omega) = \{u_i(t; \omega)\}_{i\in\mathbb{Z}}$ of (1.1) is called an *entire solution* if it is a solution of (1.1) for $t \in \mathbb{R}$.

Definition 1.1 (Transition front). A solution $u(t; \omega) = \{u_i(t; \omega)\}_{i\in\mathbb{Z}}$ is called a *random generalized traveling wave* or a *random transition front* of (1.1) connecting 1 and 0 if for a.e. $\omega \in \Omega$,

$$u_i(t; \omega) = \Phi\Big(i - \int_0^t c(s; \omega)ds, \theta_t\omega\Big)$$

for some $\Phi(x,\omega)$ ($x \in \mathbb{R}$) and $c(t;\omega)$, where $\Phi(x,\omega)$ and $c(t;\omega)$ are measurable in ω, and for a.e. $\omega \in \Omega$: $0 < \Phi(x,\omega) < 1$ and

$$\lim_{x\to-\infty} \Phi(x,\theta_t\omega) = 1, \quad \lim_{x\to\infty} \Phi(x,\theta_t\omega) = 0 \quad \text{uniformly in } t \in \mathbb{R}.$$

Suppose that $u(t;\omega) = \{u_i(t;\omega)\}_{i\in\mathbb{Z}}$ with $u_i(t;\omega) = \Phi(i - \int_0^t c(s;\omega)ds, \theta_t\omega)$ is a *random transition front* of (1.1). If $\Phi(x,\omega)$ is non-increasing in x for a.e. $\omega \in \Omega$ and all $x \in \mathbb{R}$, then $u(t;\omega)$ is said to be a *monotone random transition front*. If there is $\bar{c}_{\inf} \in \mathbb{R}$ such that for a.e. $\omega \in \Omega$,

$$\liminf_{t-s\to\infty} \frac{1}{t-s} \int_s^t c(\tau;\omega)d\tau = \bar{c}_{\inf},$$

then $\bar{c}_{\inf}$ is called its *least mean speed*.

For a given $\mu > 0$, let

$$c_0 := \inf_{\mu>0} \frac{e^{\mu} + e^{-\mu} - 2 + \underline{a}}{\mu}.$$

By [9, Lemma 5.1], there is a unique $\mu^* > 0$ such that

$$c_0 = \frac{e^{\mu^*} + e^{-\mu^*} - 2 + \underline{a}}{\mu^*}$$

and for any $\gamma > c_0$, the equation $\gamma = \frac{e^{\mu}+e^{-\mu}-2+\underline{a}}{\mu}$ has exactly two positive solutions for μ.

Now we are in a position to state the main results on the existence and stability of random transition fronts of KPP-type lattice random equations.

Theorem 1.2. *For any given $\gamma > c_0$, there is a monotone random transition front of* (1.1) *with least mean speed $\bar{c}_{\inf} = \gamma$. More precisely, for any given $\gamma > c_0$, let $0 < \mu < \mu^*$ be such that $\frac{e^{\mu}+e^{-\mu}-2+\underline{a}}{\mu} = \gamma$. Then* (1.1) *has a monotone random transition front $u(t;\omega) = \{u_i(t;\omega)\}_{i\in\mathbb{Z}}$ with $u_i(t;\omega) = \Phi(i - \int_0^t c(s;\omega,\mu)ds, \theta_t\omega)$, where $c(t;\omega,\mu) = \frac{e^{\mu}+e^{-\mu}-2+a(\theta_t\omega)}{\mu}$ and hence $\bar{c}_{\inf} = \frac{e^{\mu}+e^{-\mu}-2+\underline{a}}{\mu} = \gamma$. Moreover, for any $\omega \in \Omega_0$,*

$$\lim_{x\to-\infty} \Phi(x,\theta_t\omega) = 1, \quad \lim_{x\to\infty} \frac{\Phi(x,\theta_t\omega)}{e^{-\mu x}} = 1 \quad \textit{uniformly for } t \in \mathbb{R}.$$

Remark 1.3. (1) Let

$$c_*(\omega) = \sup\{c : \limsup_{t\to\infty} \sup_{s\in\mathbb{R}, i\in\mathbb{Z}, |i|\le ct} |u_i(t;u^0,\theta_s\omega) - 1| = 0 \text{ for all } u^0 \in l_0^{\infty}(\mathbb{Z})\},$$

where

$$l_0^{\infty}(\mathbb{Z}) = \{u = \{u_i\}_{i\in\mathbb{Z}} \in l^{\infty}(\mathbb{Z}) : u_i \ge 0 \text{ for all } i \in \mathbb{Z},\ u_i = 0 \text{ for } |i| \gg 1,\ \{u_i\} \ne 0\}.$$

Then by the similar arguments as proving [9, Theorem 1.3 (2)], we can get that for a.e. $\omega \in \Omega$, $c_*(\omega) = c_0$. If $u(t;\omega) = \{u_i(t;\omega)\}_{i\in\mathbb{Z}}$ with $u_i(t;\omega) = \Phi(i - \int_0^t c(s;\omega)ds, \theta_t\omega)$ is a random transition front of (1.1) connecting 1 and 0, then $\inf_{x\le z} \inf_{s\in\mathbb{R}} \Phi(x,\theta_s\omega) > 0$ for all $z \in \mathbb{R}$. Therefore, we can choose $u_\omega^0 \in l_0^{\infty}(\mathbb{Z})$ such that $u_\omega^0 \le \Phi(x,\theta_s\omega)$ for all $s \in \mathbb{R}$. Let $0 < \epsilon \ll 1$. Then by $c_*(\omega) = c_0$ and the comparison principle (see Proposition 2.1), we have

$$1 = \liminf_{t\to\infty} \inf_{s\in\mathbb{R}} u_{[(c_0-\epsilon)t]}(t;u_\omega^0,\theta_s\omega)$$

$$
\begin{aligned}
&\le \liminf_{t\to\infty} \inf_{s\in\mathbb{R}} u_{[(c_0-\epsilon)t]}(t;\Phi(\cdot,\theta_s\omega),\theta_s\omega)\\
&= \liminf_{t\to\infty} \inf_{s\in\mathbb{R}} \Phi\Big([(c_0-\epsilon)t] - \int_0^t c(\tau;\theta_s\omega)d\tau, \theta_{t+s}\omega\Big).
\end{aligned}
$$

Note that

$$
\int_0^{t+s} c(\tau;\omega)d\tau = \int_0^s c(\tau;\omega)d\tau + \int_0^t c(\tau;\theta_s\omega)d\tau.
$$

Then there is a constant $M(\omega)$ such that

$$
(c_0-\epsilon)t \le \int_0^{t+s} c(\tau;\omega)d\tau - \int_0^s c(\tau;\omega)d\tau + M(\omega)
$$

for all $t>0$, $s\in\mathbb{R}$. Hence,

$$
\overline{c}_{\rm inf} = \liminf_{t\to\infty} \inf_{s\in\mathbb{R}} \frac{\int_0^{t+s} c(\tau;\omega)d\tau - \int_0^s c(\tau;\omega)d\tau}{t} \ge c_0 - \epsilon.
$$

By the arbitrariness of $\epsilon>0$, we get $\overline{c}_{\rm inf} \ge c_0$. This implies that there is no random transition front of (1.1) with least mean speed less than c_0.

(2) As for the critical random transition front of (1.1), that is, random transition front of (1.1) with least mean speed $\overline{c}_{\rm inf} = c_0$. The approach used in [9] can't be applied as the stationary ergodic property of the critical random profile can't be guaranteed. We leave this as an question open.

Theorem 1.4. *For a given $\mu \in (0,\mu^*)$, the random transition front $u(t;\omega) = \{u_i(t;\omega)\}_{i\in\mathbb{Z}}$,*

$$
u_i(t;\omega) = \Phi(i - \int_0^t c(s;\omega,\mu)ds, \theta_t\omega)
$$

with $\lim_{i\to\infty} \frac{u_i(t;\omega)}{e^{-\mu(i-\int_0^t c(s;\omega,\mu)ds)}} = 1$ ($c(t;\omega,\mu) = \frac{e^{\mu}+e^{-\mu}-2+a(\theta_t\omega)}{\mu}$) is asymptotically stable, that is, for any $\omega\in\Omega_0$ and $u^0 \in l^\infty(\mathbb{Z})$ satisfying

$$
\inf_{i\le i_0} u_i^0 > 0 \quad \forall i_0\in\mathbb{Z}, \quad \lim_{i\to\infty} \frac{u_i^0}{u_i(0;\omega)} = 1,
$$

it holds

$$
\lim_{t\to\infty} \|\frac{u_\cdot(t;u^0,\omega)}{u_\cdot(t;\omega)} - 1\|_{l^\infty} = 0.
$$

The rest of this article is organized as follows. In Section 2, we establish the comparison principle for sub-solutions and super-solutions of KPP-type lattice random equations (1.1) and stability of the positive constant equilibrium solution. Also, we give in Section 2 some results including the technical lemmas for the use in later section. We investigate the existence and stability of random traveling waves for KPP-type lattice equations in random media and prove Theorem 1.2 and 1.4 in Section 3.

2. Preliminaries

We first present a comparison principle for sub-solutions and super-solutions of (1.1). Then we prove the stability of the positive constant equilibrium solution $u=1$ and the convergence of solutions on compact subsets. Finally we present some technical lemmas.

Consider now the following space continuous version of (1.1),

$$\partial_t v(x,t) = Hv(x,t) + a(\theta_t\omega)v(x,t)(1-v(x,t)), \quad x\in\mathbb{R},\ t\in\mathbb{R},\ \omega\in\Omega, \tag{2.1}$$

where

$$Hv(x,t) = v(x+1,t) + v(x-1,t) - 2v(x,t), \quad x\in\mathbb{R},\ t\in\mathbb{R}.$$

Recall that $l^\infty(\mathbb{Z}) = \{u:\mathbb{Z}\to\mathbb{R} : \sup_{x\in\mathbb{Z}}|u(x)|<\infty\}$. Let

$$l^\infty(\mathbb{R}) = \{u:\mathbb{R}\to\mathbb{R} : \sup_{x\in\mathbb{R}}|u(x)|<\infty\}$$

with norm $\|u\| = \sup_{x\in\mathbb{R}}|u(x)|$. Let

$$l^{\infty,+}(\mathbb{Z}) = \{u\in l^\infty(\mathbb{Z}) : \inf_{i\in\mathbb{Z}} u_i \ge 0\}, \quad l^{\infty,+}(\mathbb{R}) = \{u\in l^\infty(\mathbb{R}) : \inf_{x\in\mathbb{R}} u(x)\ge 0\}.$$

For any $u_0\in l^\infty(\mathbb{R})$, let $u(x,t;u_0,\omega)$ be the solution of (2.1) with $u(x,0;u_0,\omega) = u_0(x)$. Recall that for any $u^0\in l^\infty(\mathbb{Z})$, $u(t;u^0,\omega) = \{u_i(t;u^0,\omega)\}_{i\in\mathbb{Z}}$ is the solution of (1.1) with $u_i(0;u^0,\omega) = u_i^0$ for $i\in\mathbb{Z}$.

A function $v(x,t;\omega)$ on $\mathbb{R}\times[0,T)$ which is continuous in t is called a *super-solution* or *sub-solution* of (2.1) (resp. (1.1)) if for a.e. $\omega\in\Omega$ and any given $x\in\mathbb{R}$ (resp. $x\in\mathbb{Z}$), $v(x,t;\omega)$ is absolutely continuous in $t\in[0,T)$, and

$$v_t(x,t;\omega) \ge Hv(x,t;\omega) + a(\theta_t\omega)v(x,t;\omega)(1-v(x,t;\omega)) \quad \text{for } t\in[0,T)$$

or

$$v_t(x,t;\omega) \le Hv(x,t;\omega) + a(\theta_t\omega)v(x,t;\omega)(1-v(x,t;\omega)) \quad \text{for } t\in[0,T).$$

Proposition 2.1 (Comparison principle)**.**
(1) *If $u_1(x,t;\omega)$ and $u_2(x,t;\omega)$ are bounded sub-solution and super-solution of (2.1) (resp. (1.1)) on $[0,T)$, respectively, and $u_1(\cdot,0;\omega)\le u_2(\cdot,0;\omega)$, then $u_1(\cdot,t;\omega)\le u_2(\cdot,t;\omega)$ for $t\in[0,T)$.*

(2) *Suppose that $u_1(x,t;\omega)$, $u_2(x,t;\omega)$ are bounded and satisfy that for any given $x\in\mathbb{R}$ (resp. $x\in\mathbb{Z}$), $u_1(x,t;\omega)$ and $u_2(x,t;\omega)$ are absolutely continuous in $t\in[0,\infty)$, and*

$$\begin{aligned}&\partial_t u_2(x,t;\omega) - (Hu_2(x,t;\omega) + a(\theta_t\omega)u_2(x,t;\omega)(1-u_2(x,t;\omega)))\\ &> \partial_t u_1(x,t;\omega) - (Hu_1(x,t;\omega) + a(\theta_t\omega)u_1(x,t;\omega)(1-u_1(x,t;\omega)))\end{aligned}$$

for $t>0$. Moreover, suppose that $u_2(\cdot,0;\omega)\ge u_1(\cdot,0;\omega)$. Then $u_2(\cdot,t;\omega) > u_1(\cdot,t;\omega)$ for $t>0$.

(3) *If $u_0\in l^{\infty,+}(\mathbb{R})$ (resp. $u^0\in l^{\infty,+}(\mathbb{Z})$), then $u(x,t;u_0,\omega)$ (resp. $u(t;u^0,\omega)$) exists and $u(\cdot,t;u_0,\omega)\ge 0$ (resp. $u(t;u^0,\omega)\ge 0$) for all $t\ge 0$.*

Proof. We prove the proposition only for (2.1); it can be proved similarly for (1.1).

(1) This part is proved by we modifying the arguments in [22, Proposition 2.4]. Let $Q(x,t;\omega) = e^{ct}(u_2(x,t;\omega) - u_1(x,t;\omega))$, where $c := c(\omega)$ is to be determined later. Then there is a measurable subset $\bar\Omega$ of Ω with $\mathbb{P}(\bar\Omega) = 0$ such that for any

$\omega \in \Omega \setminus \bar{\Omega}$, we have

$$\begin{aligned}
&\partial_t Q(x,t;\omega)\\
&= e^{ct}(\partial_t u_2(x,t;\omega) - \partial_t u_1(x,t;\omega)) + ce^{ct}(u_2(x,t;\omega) - u_1(x,t;\omega))\\
&\geq e^{ct}(Hu_2(x,t;\omega) - Hu_1(x,t;\omega) + a(\theta_t\omega)u_2(x,t;\omega)(1-u_2(x,t;\omega))\\
&\quad - a(\theta_t\omega)u_1(x,t;\omega)(1-u_1(x,t;\omega))) + cQ(x,t;\omega)\\
&= HQ(x,t;\omega) + e^{ct}a(\theta_t\omega)(u_2(x,t;\omega) - u_1(x,t;\omega))(1-u_2(x,t;\omega))\\
&\quad - e^{ct}a(\theta_t\omega)(u_2(x,t;\omega) - u_1(x,t;\omega))u_1(x,t;\omega) + cQ(x,t;\omega)\\
&= Q(x+1,t;\omega) + Q(x-1,t;\omega) + (b(x,t;\omega) - 2 + c)Q(x,t;\omega)
\end{aligned} \tag{2.2}$$

for $x \in \mathbb{R}$ and $t \in [0,T]$, where

$$b(x,t;\omega) = a(\theta_t\omega)\big(1 - u_1(x,t;\omega) - u_2(x,t;\omega)\big) \quad \text{for } x \in \mathbb{R},\ t \in [0,T].$$

Let $p(x,t;\omega) = b(x,t;\omega) - 2 + c$. By the boundedness of u_1 and u_2, we can choose $c = c(\omega) > 0$ such that

$$\inf_{(x,t)\in\mathbb{R}\times[0,T]} p(x,t;\omega) > 0.$$

We claim that $Q(x,t;\omega) \geq 0$ for $x \in \mathbb{R}$ and $t \in [0,T]$.

Let $p_0(\omega) = \sup_{(x,t)\in\mathbb{R}\times[0,T]} p(x,t;\omega)$. It suffices to prove the claim for $x \in \mathbb{R}$ and $t \in (0,T_0]$ with $T_0 = \min\{T, \frac{1}{p_0(\omega)+2}\}$. Assume that there are $\tilde{x} \in \mathbb{R}$ and $\tilde{t} \in (0,T_0]$ such that $Q(\tilde{x},\tilde{t};\omega) < 0$. Then there is $t^0 \in (0,T_0)$ such that

$$Q_{\inf}(\omega) := \inf_{(x,t)\in\mathbb{R}\times[0,t^0]} Q(x,t;\omega) < 0.$$

Observe that there are $x_n \in \mathbb{R}$ and $t_n \in (0,t^0]$ such that

$$Q(x_n,t_n;\omega) \to Q_{\inf}(\omega) \quad \text{as } n \to \infty.$$

By (2.2) and the fundamental theorem of calculus for Lebesgue integrals, we obtain

$$\begin{aligned}
&Q(x_n,t_n;\omega) - Q(x_n,0;\omega)\\
&\geq \int_0^{t_n} [Q(x_n+1,t;\omega) + Q(x_n-1,t;\omega) + p(x_n,t;\omega)Q(x_n,t;\omega)]dt\\
&\geq \int_0^{t_n} [2Q_{\inf}(\omega) + p(x_n,t;\omega)Q_{\inf}(\omega)]dt\\
&\geq t^0(2+p_0(\omega))Q_{\inf}(\omega) \quad \text{for } n \geq 1.
\end{aligned}$$

Note that $Q(x_n,0;\omega) \geq 0$, we then have

$$Q(x_n,t_n;\omega) \geq t^0(2+p_0(\omega))Q_{\inf}(\omega) \qquad \text{for } n \geq 1.$$

Letting $n \to \infty$, we obtain

$$Q_{\inf}(\omega) \geq t^0(2+p_0(\omega))Q_{\inf}(\omega) > Q_{\inf}(\omega).$$

A contradiction. Hence the claim is true and $u_1(x,t;\omega) \leq u_2(x,t;\omega)$ for $\omega \in \Omega \setminus \bar{\Omega}$, $x \in \mathbb{R}$ and $t \in [0,T]$.

(2) For $\omega \in \Omega \setminus \bar{\Omega}$, by the similar arguments as for getting (2.2), we can find $c(\omega)$, $\mu(\omega) > 0$ such that

$$\partial_t Q(x,t;\omega) > Q(x+1,t;\omega) + Q(x-1,t;\omega) + \mu(\omega)Q(x,t;\omega) \quad \text{for } x \in \mathbb{R},\ t > s,$$

where $Q(x,t;\omega) = e^{c(\omega)t}(u_2(x,t;\omega) - u_1(x,t;\omega))$. Thus we have that for $x \in \mathbb{R}$,

$$Q(x,t;\omega) > Q(x,0;\omega) + \int_0^t \big(Q(x+1,\tau;\omega) + Q(x-1,\tau;\omega) + \mu(\omega)Q(x,\tau;\omega)\big)d\tau.$$

By the arguments in (1), $Q(x,t;\omega) \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$. It then follows that $Q(x,t;\omega) > Q(x,0;\omega) \geq 0$ and hence $u_2(x,t;\omega) > u_1(x,t;\omega)$ for $\omega \in \Omega \setminus \bar{\Omega}$, $x \in \mathbb{R}$ and $t > 0$.

(3) By (1), for any $u_0 \in l^{\infty,+}(\mathbb{R})$, $0 \leq u(\cdot,t;u_0,\omega) \leq \max\{\|u_0\|, 1\}$ for all $t > 0$ in the existence interval of $u(\cdot,t;u_0,\omega)$. It then follows that $u(\cdot,t;u_0,\omega)$ exists and $u(\cdot,t;u_0,\omega) \geq 0$ for all $t \geq 0$. □

We have the following proposition on the stability of the constant equilibrium solution $u = 1$.

Proposition 2.2. *For every $u_0 \in l^\infty(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} u_0(x) > 0$ and for every $\omega \in \Omega$, we have that*

$$\|u(x,t;u_0,\omega) - 1\|_\infty \to 0 \quad \textit{as } t \to \infty.$$

Proof. The proof is similar to that of [37, Theorem 2.1 (1)]. We give the details for completeness. For $u_0 \in l^\infty(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} u_0(x) > 0$ and $\omega \in \Omega$. Let $\underline{u}_0 := \min\{1, \inf_{x\in\mathbb{R}} u_0(x)\}$ and $\overline{u}_0 := \max\{1, \sup_{x\in\mathbb{R}} u_0(x)\}$. It follows from Proposition 2.1 that

$$\underline{u}_0 \leq u(x,t;\underline{u}_0,\omega) \leq \min\{1, u(x,t;u_0,\omega)\}, \quad \forall x \in \mathbb{R},\ t \geq 0, \tag{2.3}$$

$$\max\{1, u(x,t;u_0,\omega)\} \leq u(x,t;\overline{u}_0,\omega) \leq \overline{u}_0, \quad \forall x \in \mathbb{R},\ t \geq 0. \tag{2.4}$$

Note that $\underline{u}_0$ and $\overline{u}_0$ are constants. Then by the uniqueness of solution of (2.1) with respect to the initial value, we obtain

$$u(x,t;\underline{u}_0,\omega) = u(0,t;\underline{u}_0,\omega) \quad \text{and} \quad u(x,t;\overline{u}_0,\omega) = u(0,t;\overline{u}_0,\omega), \quad \forall x \in \mathbb{R},\ t \geq 0.$$

Since

$$\underline{u}(t) = \Big(\frac{1}{u(0,t;\underline{u}_0,\omega)} - 1\Big)e^{\int_0^t a(\theta_s\omega)ds} \quad \text{and} \quad \overline{u}(t) = \Big(1 - \frac{1}{u(0,t;\overline{u}_0,\omega)}\Big)e^{\int_0^t a(\theta_s\omega)ds}$$

satisfy

$$\frac{d}{dt}\underline{u} = \frac{d}{dt}\overline{u} = 0, \quad t > 0,$$

it follows that

$$\underline{u}(t) = \underline{u}(0) \quad \text{and} \quad \overline{u}(t) = \overline{u}(0), \quad \forall t \geq 0.$$

Therefore,

$$1 - u(x,t;\underline{u}_0,\omega) = \underline{u}(0)u(x,t;\underline{u}_0,\omega)e^{-\int_0^t a(\theta_s\omega)ds}, \tag{2.5}$$

$$u(x,t;\overline{u}_0,\omega) - 1 = \overline{u}(0)u(x,t;\overline{u}_0,\omega)e^{-\int_0^t a(\theta_s\omega)ds}. \tag{2.6}$$

By (2.3) and (2.4), we have

$$0 < \underline{u}_0 \leq u(x,t;\underline{u}_0,\omega) \leq u(x,t;\overline{u}_0,\omega) \leq \overline{u}_0, \quad \forall x \in \mathbb{R},\ t \geq 0.$$

It then follows from (2.3), (2.4), (2.5) and (2.6) that

$$|u(x,t;u_0,\omega) - 1| \leq \overline{u}_0 \max\{\overline{u}(0), \underline{u}(0)\}e^{-\int_0^t a(\theta_s\omega)ds}, \quad \forall x \in \mathbb{R},\ t \geq 0.$$

The Proposition thus follows. □

Proposition 2.3. *Suppose that $u_{0n}, u_0 \in l^{\infty,+}(\mathbb{R})$ $(n = 1, 2, \cdots)$ with $\{\|u_{0n}\|\}$ being bounded. If $u_{0n}(x) \to u_0(x)$ as $n \to \infty$ uniformly in x on bounded sets, then for each $t > 0$, $u(x, t; u_{0n}, \theta_{t_0}\omega) - u(x, t; u_0, \theta_{t_0}\omega) \to 0$ as $n \to \infty$ uniformly in x on bounded sets and $t_0 \in \mathbb{R}$.*

Proof. This is proved by the similar arguments in [9, Proposition 2.2]. Fix any $\omega \in \Omega$. Let $v^n(x, t; \theta_{t_0}\omega) = u(x, t; u_{0n}, \theta_{t_0}\omega) - u(x, t; u_0, \theta_{t_0}\omega)$. Then $v^n(x, t; \theta_{t_0}\omega)$ satisfies

$$v^n_t(x, t; \theta_{t_0}\omega) = Hv^n(x, t; \theta_{t_0}\omega) + b_n(x, t; \theta_{t_0}\omega)v^n(x, t; \theta_{t_0}\omega),$$

where $b_n(x, t; \theta_{t_0}\omega) = a(\theta_{t+t_0}\omega)(1 - u(x, t; u_{0n}, \theta_{t_0}\omega) - u(x, t; u_0, \theta_{t_0}\omega))$. Observe that $\{b_n(x, t; \theta_{t_0}\omega)\}_n$ is uniformly bounded. Take $\lambda > 0$, and let

$$X(\lambda) = \{u : \mathbb{R} \to \mathbb{R} \mid u(\cdot)e^{-\lambda|\cdot|} \in l^\infty(\mathbb{R})\}$$

with norm $\|u\|_\lambda = \|u(\cdot)e^{-\lambda|\cdot|}\|_{l^\infty(\mathbb{R})}$. Note that $H : X(\lambda) \to X(\lambda)$ generates an analytic semigroup, and there are $M > 0$ and $\alpha > 0$ such that

$$\|e^{Ht}\|_{X(\lambda)} \le Me^{\alpha t}, \quad \forall t \ge 0.$$

Hence,

$$v^n(\cdot, t; \theta_{t_0}\omega) = e^{Ht}v^n(\cdot, 0; \theta_{t_0}\omega) + \int_0^t e^{H(t-\tau)}b_n(\cdot, \tau; \theta_{t_0}\omega)v^n(\cdot, \tau; \theta_{t_0}\omega)d\tau$$

and then

$$\begin{aligned}
&\|v^n(\cdot, t; \theta_{t_0}\omega)\|_{X(\lambda)} \\
&\le Me^{\alpha t}\|v^n(\cdot, 0; \theta_{t_0}\omega)\|_{X(\lambda)} \\
&\quad + M \sup_{t_0\in\mathbb{R}, \tau\in[0,t], x\in\mathbb{R}} |b_n(x, \tau; \theta_{t_0}\omega)| \int_0^t e^{\alpha(t-\tau)}\|v^n(\cdot, \tau; \theta_{t_0}\omega)\|_{X(\lambda)}d\tau.
\end{aligned}$$

By Gronwall's inequality,

$$\|v^n(\cdot, t; \theta_{t_0}\omega)\|_{X(\lambda)} \le e^{(\alpha + M\sup_{t_0\in\mathbb{R}, \tau\in[0,t], x\in\mathbb{R}} |b_n(x,\tau;\theta_{t_0}\omega)|)t}(M\|v^n(\cdot, 0; \theta_{t_0}\omega)\|_{X(\lambda)}).$$

Note that $\|v^n(\cdot, 0; \theta_{t_0}\omega)\|_{X(\lambda)} \to 0$ uniformly in $t_0 \in \mathbb{R}$. It then follows that

$$\|v^n(\cdot, t; \theta_{t_0}\omega)\|_{X(\lambda)} \to 0 \quad \text{as } n \to \infty$$

uniformly in $t_0 \in \mathbb{R}$ and then

$$u(x, t; u_{0n}, \theta_{t_0}\omega) - u(x, t; u_0, \theta_{t_0}\omega) \to 0 \quad \text{as } n \to \infty$$

uniformly in x on bounded sets and $t_0 \in \mathbb{R}$. □

Now we present some lemmas including technical results.

Lemma 2.4. *We have $\underline{a}(\cdot), a(\cdot), \overline{a}(\cdot) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\underline{a}(\omega)$ and $\overline{a}(\omega)$ are independent of ω for a.e. $\omega \in \Omega$.*

The proof of the above lemma follows from [37, Lemma 2.1].

Lemma 2.5. *Suppose that for $\omega \in \Omega$, $a^\omega(t) = a(\theta_t\omega) \in C(\mathbb{R}, (0, \infty))$. Then for a.e. $\omega \in \Omega$,*

$$\underline{a} = \sup_{A\in W^{1,\infty}_{loc}(\mathbb{R})\cap L^\infty(\mathbb{R})} \operatorname{ess\,inf}_{t\in\mathbb{R}}(A' + a^\omega)(t).$$

The proof of the above lemma follows from [37, Lemma 2.2] and Lemma 2.4.

Lemma 2.6. *Let $\omega \in \Omega_0$. Then for any μ, $\tilde{\mu}$ with $0 < \mu < \tilde{\mu} < \min\{2\mu, \mu^*\}$, there exist $\{t_k\}_{k\in\mathbb{Z}}$ with $t_k < t_{k+1}$ and $\lim_{k\to\pm\infty} t_k = \pm\infty$, $A_\omega \in W^{1,\infty}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $A_\omega(\cdot) \in C^1((t_k, t_{k+1}))$ for $k \in \mathbb{Z}$, and $d_\omega > 0$ such that for any $d \geq d_\omega$ the function*

$$\tilde{v}^{\mu,d,A_\omega}(x,t,\omega) := e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)} - de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}(x-\int_0^t c(s;\omega,\mu)ds)}$$

satisfies

$$\partial_t \tilde{v}^{\mu,d,A_\omega} \leq H\tilde{v}^{\mu,d,A_\omega} + a(\theta_t\omega)\tilde{v}^{\mu,d,A_\omega}(1-\tilde{v}^{\mu,d,A_\omega})$$

for $t \in (t_k, t_{k+1})$, $x \geq \int_0^t c(s;\omega,\mu)ds + \frac{\ln d}{\tilde{\mu}-\mu} + \frac{A_\omega(t)}{\mu}$, $k \in \mathbb{Z}$.

Proof. For given $\omega \in \Omega_0$ and $0 < \mu < \tilde{\mu} < \min\{2\mu, \mu^*\}$, by the arguments in the proof of [9, Lemma 5.1] we can get that $\frac{e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2+\underline{a}}{\tilde{\mu}} < \frac{e^{\mu}+e^{-\mu}-2+\underline{a}}{\mu}$, and hence $\underline{a} > \frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu}$. Let $0 < \delta \ll 1$ be such that $(1-\delta)\underline{a} > \frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu}$. It then follows from Lemma 2.5 that there exist $T > 0$ and $A_\omega \in W^{1,\infty}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $A_\omega(\cdot) \in C^1((t_k, t_{k+1}))$ with $t_k = kT$ for $k \in \mathbb{Z}$, and

$$(1-\delta)a(\theta_t\omega) + A'_\omega(t) \geq \frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu} \tag{2.7}$$

for all $t \in (t_k, t_{k+1})$, $k \in \mathbb{Z}$.

Now fix $\delta > 0$ and $A_\omega(t)$ chosen in the above inequality. Let $\xi(x,t;\omega) = x - \int_0^t c(s;\omega,\mu)ds$, and $\tilde{v}^{\mu,d,A_\omega}(x,t,\omega) := e^{-\mu\xi(x,t;\omega)} - de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}$ with $d > 1$ to be determined later. Note that $c(t;\omega,\mu) = \frac{e^{\mu}+e^{-\mu}-2+a(\theta_t\omega)}{\mu}$. Then we have

$$\begin{aligned}
&\partial_t \tilde{v}^{\mu,d,A_\omega} - (H\tilde{v}^{\mu,d,A_\omega} + a(\theta_t\omega)\tilde{v}^{\mu,d,A_\omega}(1-\tilde{v}^{\mu,d,A_\omega}))\\
&= \mu c(t;\omega,\mu)e^{-\mu\xi(x,t;\omega)} + d(-(\frac{\tilde{\mu}}{\mu}-1)A'_\omega(t) - \tilde{\mu}c(t;\omega,\mu))e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}\\
&\quad - \Big((e^{\mu}+e^{-\mu}-2)e^{-\mu\xi(x,t;\omega)} - d(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}\Big)\\
&\quad - a(\theta_t\omega)(e^{-\mu\xi(x,t;\omega)} - de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)})\\
&\quad \times \Big(1 - (e^{-\mu\xi(x,t;\omega)} - de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)})\Big)\\
&= d(-(\frac{\tilde{\mu}}{\mu}-1)A'_\omega(t) - \tilde{\mu}c(t;\omega,\mu) + e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2+a(\theta_t\omega))e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}\\
&\quad + a(\theta_t\omega)(e^{-\mu\xi(x,t;\omega)} - de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)})^2\\
&= d(\frac{\tilde{\mu}}{\mu}-1)(-A'_\omega(t) + \frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu} - a(\theta_t\omega))\\
&\quad \times e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)} + a(\theta_t\omega)e^{-2\mu\xi(x,t;\omega)}\\
&\quad - d(2e^{-\mu\xi(x,t;\omega)} - de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)})e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}a(\theta_t\omega)\\
&= d(\frac{\tilde{\mu}}{\mu}-1)(\frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu} - (1-\delta)a(\theta_t\omega) - A'_\omega(t))\\
&\quad \times e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)} + (e^{-(2\mu-\tilde{\mu})\xi(x,t;\omega)}\\
&\quad - d\delta(\frac{\tilde{\mu}}{\mu}-1)e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)})a(\theta_t\omega)e^{-\tilde{\mu}\xi(x,t;\omega)}
\end{aligned}$$

$$+ d\big(- 2e^{-\mu\xi(x,t;\omega)} + de^{(\frac{\tilde\mu}{\mu}-1)A_\omega(t)-\tilde\mu\xi(x,t;\omega)}\big)e^{(\frac{\tilde\mu}{\mu}-1)A_\omega(t)-\tilde\mu\xi(x,t;\omega)}a(\theta_t\omega) \tag{2.8}$$

for $t \in (t_k, t_{k+1})$.

Let $d_\omega \geq \max\big\{\frac{e^{-(\frac{\tilde\mu}{\mu}-1)\|A_\omega\|_\infty}}{\delta(\frac{\tilde\mu}{\mu}-1)}, e^{(\frac{\tilde\mu}{\mu}-1)\|A_\omega\|_\infty}\big\}$. Then we have

$$d\delta(\frac{\tilde\mu}{\mu}-1)e^{(\frac{\tilde\mu}{\mu}-1)A_\omega(t)} \geq 1, \quad \forall d \geq d_\omega.$$

Note that if $x \geq \int_0^t c(s;\omega,\mu)ds + \frac{\ln d}{\tilde\mu-\mu} + \frac{A_\omega(t)}{\mu}$, then $\xi(x,t;\omega) = x - \int_0^t c(s;\omega,\mu)ds \geq 0$ and $\tilde v^{\mu,d,A_\omega}(x,t,\omega) \geq 0$. From (2.7), we obtain that every term on the right-hand side of (2.8) is less than or equal to zero. $\square$

3. Random transition fronts

In this section, we study the existence and stability of random transition fronts, and prove Theorem 1.2 and 1.4.

3.1. Existence of random transition fronts.

For any $\gamma > c_0$, let $0 < \mu < \mu^*$ be such that $\frac{e^\mu + e^{-\mu} - 2 + \underline{a}}{\mu} = \gamma$. Then for every $\omega \in \Omega$, let $c(t;\omega,\mu) = \frac{e^\mu + e^{-\mu} - 2 + a(\theta_t\omega)}{\mu}$ and $\hat v^\mu(x,t;\omega) = e^{-\mu(x - \int_0^t c(s;\omega,\mu)ds)}$. Then $\hat v^\mu(x,t;\omega)$ satisfies

$$\begin{aligned}
&\partial_t \hat v^\mu(x,t;\omega) - H\hat v^\mu(x,t;\omega) - a(\theta_t\omega)\hat v^\mu(x,t;\omega)\\
&= \hat v^\mu(x,t;\omega)[\mu c(t;\omega,\mu) - (e^\mu + e^{-\mu} - 2 + a(\theta_t\omega))] = 0, \quad \text{for } x \in \mathbb{R},\ t \in \mathbb{R}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\partial_t \hat v^\mu(x,t;\omega) &= H\hat v^\mu(x,t;\omega) + a(\theta_t\omega)\hat v^\mu(x,t;\omega)\\
&\geq H\hat v^\mu(x,t;\omega) + a(\theta_t\omega)\hat v^\mu(x,t;\omega)(1 - \hat v^\mu(x,t;\omega)),
\end{aligned}$$

for $x \in \mathbb{R}$ and $t \in \mathbb{R}$. Hence, $\hat v^\mu(x,t;\omega) = e^{-\mu(x - \int_0^t c(s;\omega,\mu)ds)}$ is a super-solution of (2.1). Let

$$\overline{v}^\mu(x,t;\omega) = \min\{1, \hat v^\mu(x,t;\omega)\}.$$

Lemma 3.1. *For $\omega \in \Omega_0$, we have*

$$u(x, t - t_0; \overline{v}^\mu(\cdot, t_0;\omega), \theta_{t_0}\omega) \leq \overline{v}^\mu(x,t;\omega), \quad \forall x \in \mathbb{R},\ t \geq t_0,\ t_0 \in \mathbb{R}.$$

Proof. For any constant C, the function $\hat u(x,t;\omega) := e^{Ct}\hat v^\mu(x,t;\omega)$ satisfies

$$\begin{aligned}
\partial_t \hat u(x,t;\omega) &= (\partial_t \hat v^\mu(x,t;\omega) + C\hat v^\mu(x,t;\omega))e^{Ct}\\
&\geq H\hat u(x,t;\omega) + C\hat u(x,t;\omega) + a(\theta_t\omega)\hat u(x,t;\omega)(1 - \hat v^\mu(x,t;\omega)),
\end{aligned}$$

hence,

$$\begin{aligned}
\hat u(x,t;\omega) \geq \hat u(x,t_0;\omega) &+ \int_{t_0}^t \Big(H\hat u(x,\tau;\omega) + C\hat u(x,\tau;\omega)\\
&+ a(\theta_\tau\omega)\hat u(x,\tau;\omega)(1 - \hat v^\mu(x,\tau;\omega))\Big)d\tau.
\end{aligned}$$

Let $\overline{u}(x,t;\omega) := e^{Ct}\overline{v}^\mu(x,t;\omega)$. Then we also have

$$\begin{aligned}
\overline{u}(x,t;\omega) \geq \overline{u}(x,t_0;\omega) &+ \int_{t_0}^t \Big(H\overline{u}(x,\tau;\omega) + C\overline{u}(x,\tau;\omega)\\
&+ a(\theta_\tau\omega)\overline{u}(x,\tau;\omega)(1 - \overline{v}^\mu(x,\tau;\omega))\Big)d\tau.
\end{aligned}$$

Let $Q(x,t;\omega)=e^{Ct}(\overline{v}^{\mu}(x,t;\omega)-u(x,t-t_0;\overline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega))$. Then

$$Q(x,t;\omega)\geq Q(x,t_0;\omega)+\int_{t_0}^{t}(HQ(x,\tau;\omega)+(C+b(x,\tau;\omega))Q(x,\tau;\omega))d\tau,$$

where

$$b(x,t;\omega)=a(\theta_t\omega)(1-\overline{v}^{\mu}(x,t;\omega)-u(x,t-t_0;\overline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega)).$$

Choose $C>0$ such that $b(x,t;\omega)-2+C>0$ for all $t\geq t_0$, $x\in\mathbb{R}$ and a.e. $\omega\in\Omega$. By the arguments of Proposition 2.1, we have

$$Q(x,t;\omega)\geq Q(x,t_0;\omega)=0,$$

and hence for $\omega\in\Omega_0$, we have

$$u(x,t-t_0;\overline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega)\leq\overline{v}^{\mu}(x,t;\omega),\quad\forall x\in\mathbb{R},\ ,t\geq t_0,\ t_0\in\mathbb{R}.$$

□

Next, we construct a sub-solution of (2.1). Let $\tilde{\mu}>0$ be such that $\mu<\tilde{\mu}<\min\{2\mu,\mu^*\}$ and $\omega\in\Omega_0$. Let A_ω and d_ω be given by Lemma 2.6, and let

$$x_\omega(t)=\int_0^t c(s;\omega,\mu)ds+\frac{\ln d_\omega+\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}+\frac{A_\omega(t)}{\mu}.$$

Recall that

$$\tilde{v}^{\mu,d,A_\omega}(x,t,\omega)=e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)}-de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}(x-\int_0^t c(s;\omega,\mu)ds)}$$

By calculation we have that for any given $t\in\mathbb{R}$,

$$\begin{aligned}\tilde{v}^{\mu,d_\omega,A_\omega}(x_\omega(t),t,\omega)&=\sup_{x\in\mathbb{R}}\tilde{v}^{\mu,d_\omega,A_\omega}(x,t,\omega)\\&=e^{-\mu(\frac{\ln d_\omega}{\tilde{\mu}-\mu}+\frac{A_\omega(t)}{\mu})}e^{-\mu\frac{\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}}(1-\frac{\mu}{\tilde{\mu}}).\end{aligned}\tag{3.1}$$

Define

$$\underline{v}^{\mu}(x,t;\theta_{t_0}\omega)=\begin{cases}\tilde{v}^{\mu,d_\omega,A_\omega}(x,t+t_0,\omega), & \text{if } x\geq x_\omega(t+t_0),\\ \tilde{v}^{\mu,d_\omega,A_\omega}(x_\omega(t+t_0),t+t_0,\omega), & \text{if } x\leq x_\omega(t+t_0).\end{cases}$$

It is clear that

$$0<\underline{v}^{\mu}(x,t;\theta_{t_0}\omega)<\overline{v}^{\mu}(x,t;\theta_{t_0}\omega)\leq 1,\quad\forall t\in\mathbb{R},\ x\in\mathbb{R},\ t_0\in\mathbb{R}.$$

and

$$\lim_{x\to\infty}\sup_{t\in\mathbb{R},t_0\in\mathbb{R}}\frac{\underline{v}^{\mu}(x,t;\theta_{t_0}\omega)}{\overline{v}^{\mu}(x,t;\theta_{t_0}\omega)}=1.\tag{3.2}$$

Note that by the similar arguments as in Lemma 3.1, we can prove that

$$u(x,t-t_0;\underline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega)\geq\underline{v}^{\mu}(x,t;\omega)$$

for $x\in\mathbb{R}$, $t\geq t_0$ and a.e. $\omega\in\Omega$.

Proof of Theorem 1.2. By Lemma 3.1,

$$u(x,t-t_0;\overline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega)\leq\overline{v}^{\mu}(x,t;\omega),\quad\forall x\in\mathbb{R},\ t\geq t_0,\ t_0\in\mathbb{R}.$$

It then follows that

$$u(x,\tau_2-\tau_1;\overline{v}^{\mu}(\cdot,-\tau_2;\omega),\theta_{-\tau_2}\omega)\leq\overline{v}^{\mu}(x,-\tau_1;\omega),\quad\forall x\in\mathbb{R},\ \ \tau_2>\tau_1\,.$$

Then

$$u(x,t+\tau_1;u(\cdot,\tau_2-\tau_1;\overline{v}^{\mu}(\cdot,-\tau_2;\omega),\theta_{-\tau_2}\omega),\theta_{-\tau_1}\omega)$$

$$\le u(x,t+\tau_1;\overline{v}^\mu(\cdot,-\tau_1;\omega),\theta_{-\tau_1}\omega)$$

for $x\in\mathbb{R}$, $t\ge -\tau_1$, $\tau_2>\tau_1$, and hence

$$u(x,t+\tau_2;\overline{v}^\mu(\cdot,-\tau_2;\omega),\theta_{-\tau_2}\omega)\le u(x,t+\tau_1;\overline{v}^\mu(\cdot,-\tau_1;\omega),\theta_{-\tau_1}\omega)$$

for $x\in\mathbb{R}$, $t\ge -\tau_1$, $\tau_2>\tau_1$. Therefore $\lim_{\tau\to\infty}u(x,t+\tau;\overline{v}^\mu(\cdot,-\tau;\omega),\theta_{-\tau}\omega)$ exists. We define

$$V(x,t;\omega)=\lim_{\tau\to\infty}u(x,t+\tau;\overline{v}^\mu(\cdot,-\tau;\omega),\theta_{-\tau}\omega) \tag{3.3}$$

for $x\in\mathbb{R}$, $t\in\mathbb{R}$, $\omega\in\Omega_0$. Then $V(x,t;\omega)$ is non-increasing in $x\in\mathbb{R}$ and by dominated convergence theorem we know that $V(x,t;\omega)$ is a solution of (2.1).

We claim that, for every $\omega\in\Omega_0$,

$$\lim_{x\to-\infty}V(x+\int_0^t c(s;\omega,\mu)ds,t;\omega)=1\quad\text{uniformly for } t\in\mathbb{R}. \tag{3.4}$$

In fact, for any $\omega\in\Omega_0$, letting $\hat{x}_\omega=\frac{\ln d_\omega+\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}-\frac{\|A_\omega\|_\infty}{\mu}$, it follows from $\underline{v}^\mu(x,t;\omega)\le V(x,t;\omega)$ and (3.1) that

$$0<(1-\frac{\mu}{\tilde{\mu}})e^{-\mu(\frac{\ln d_\omega+\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}+\frac{\|A_\omega\|_\infty}{\mu})}\le\inf_{t\in\mathbb{R}}V(\hat{x}_\omega+\int_0^t c(s;\omega,\mu)ds,t;\omega).$$

Let $u_0(x)\equiv u_0:=\inf_{t\in\mathbb{R}}V(\hat{x}_\omega+\int_0^t c(s;\omega,\mu)ds,t;\omega)$, and $\tilde{u}_0(x)$ be uniformly continuous such that $\tilde{u}_0(x)=u_0(x)$ for $x<\hat{x}_\omega-1$ and $\tilde{u}_0(x)=0$ for $x\ge\hat{x}_\omega$. Then $\lim_{n\to\infty}\tilde{u}_0(x-n)=u_0(x)$ locally uniformly in $x\in\mathbb{R}$. Note that by the proof of Proposition 2.2, we have

$$\lim_{t\to\infty}u(x,t;u_0,\theta_{t_0}\omega)=1$$

uniformly in $t_0\in\mathbb{R}$ and $x\in\mathbb{R}$. Then for any $\epsilon>0$, there is $T:=T(\epsilon)>0$ such that

$$1>u(x,T;u_0,\theta_{t_0}\omega)>1-\epsilon,\quad\forall t_0\in\mathbb{R},\ x\in\mathbb{R}.$$

Therefore, by (H1) and the definition of $c(t,\omega,\mu)$ we derive,

$$1>u(x+\int_0^T c(s;\theta_{t_0}\omega,\mu)ds,T;u_0,\theta_{t_0}\omega)>1-\epsilon,\quad\forall t_0\in\mathbb{R},\ x\in\mathbb{R}.$$

By Proposition 2.3, there is $N:=N(\epsilon)>1$ such that

$$1>u\Big(\int_0^T c(s;\theta_{t_0}\omega,\mu)ds,T;\tilde{u}_0(\cdot-N),\theta_{t_0}\omega\Big)>1-2\varepsilon,\quad\forall t_0\in\mathbb{R}.$$

That is,

$$1>u\Big(\int_0^T c(s;\theta_{t_0}\omega,\mu)ds-N,T;\tilde{u}_0(\cdot),\theta_{t_0}\omega\Big)>1-2\varepsilon,\quad\forall t_0\in\mathbb{R}.$$

Note that

$$V\Big(x+\int_0^{t-T}c(s;\omega,\mu)ds,t-T;\omega\Big)\ge\tilde{u}_0(x),\quad\forall t\in\mathbb{R},\ x\in\mathbb{R}.$$

and

$$\int_0^t c(s;\omega,\mu)ds=\int_0^T c(s;\theta_{t-T}\omega,\mu)ds+\int_0^{t-T}c(s;\omega,\mu)ds.$$

Then

$$1>V\Big(x+\int_0^t c(s;\omega,\mu)ds,t;\omega\Big)$$

$$
\begin{aligned}
&= u\Big(x + \int_0^T c(s;\theta_{t-T}\omega,\mu)ds, T; V(\cdot + \int_0^{t-T} c(s;\omega,\mu)ds, t-T;\omega), \theta_{t-T}\omega\Big) \\
&> 1 - 2\epsilon, \quad \forall t \in \mathbb{R},\ x \le -N.
\end{aligned}
$$

Thus (3.4) follows.

Note that by (3.2), for every $\omega \in \Omega_0$, we have

$$
\lim_{x\to\infty} \sup_{t\in\mathbb{R}} \frac{V(x + \int_0^t c(s;\omega,\mu)ds, t;\omega)}{e^{-\mu x}} = 1.
$$

Set

$$
\tilde{\Phi}(x,t;\omega) = V\big(x + \int_0^t c(s;\omega,\mu)ds, t;\omega\big), \quad \Phi(x,\omega) = \tilde{\Phi}(x,0;\omega).
$$

We claim that $\tilde{\Phi}(x,t;\omega)$ is stationary ergodic in t, that is, for a.e. $\omega \in \Omega$,

$$
\tilde{\Phi}(x,t;\omega) = \tilde{\Phi}(x,0;\theta_t\omega).
$$

In fact, note that for $\omega \in \Omega$,

$$
\begin{aligned}
\int_{-\tau}^t c(s;\omega,\mu)ds &= \int_{-\tau}^t \frac{e^\mu + e^{-\mu} - 2 + a(\theta_s\omega)}{\mu} ds \\
&= \frac{e^\mu + e^{-\mu} - 2}{\mu}(t+\tau) + \int_{-\tau}^t \frac{a(\theta_s\omega)}{\mu} ds
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
\int_{-(t+\tau)}^0 c(s;\theta_t\omega,\mu)ds &= \int_{-(t+\tau)}^0 \frac{e^\mu + e^{-\mu} - 2 + a(\theta_s \circ \theta_t\omega)}{\mu} ds \\
&= \frac{e^\mu + e^{-\mu} - 2}{\mu}(t+\tau) + \int_{-(t+\tau)}^0 \frac{a(\theta_{s+t}\omega)}{\mu} ds \\
&= \frac{e^\mu + e^{-\mu} - 2}{\mu}(t+\tau) + \int_{-\tau}^t \frac{a(\theta_s\omega)}{\mu} ds.
\end{aligned} \tag{3.6}
$$

Combining (3.5) with (3.6), we obtain $\int_{-\tau}^t c(s;\omega,\mu)ds = \int_{-(t+\tau)}^0 c(s;\theta_t\omega,\mu)ds$ for $\tau \ge 0$ and $t \in \mathbb{R}$. Recall that

$$
\overline{v}^\mu(x,t;\omega) = \min\big\{1, e^{-\mu(x - \int_0^t c(s;\omega,\mu)ds)}\big\}.
$$

Then

$$
\begin{aligned}
\tilde{\Phi}(x,t;\omega) &= \lim_{\tau\to\infty} u\Big(x + \int_0^t c(s;\omega,\mu)ds, t+\tau; \overline{v}^\mu(\cdot,-\tau;\omega), \theta_{-\tau}\omega\Big) \\
&= \lim_{\tau\to\infty} u\Big(x, t+\tau; \overline{v}^\mu\Big(\cdot + \int_0^t c(s;\omega,\mu)ds, -\tau;\omega\Big), \theta_{-\tau}\omega\Big) \\
&= \lim_{\tau\to\infty} u\Big(x, t+\tau; \overline{v}^\mu(\cdot, -(t+\tau);\theta_t\omega), \theta_{-\tau}\omega\Big) \\
&= \lim_{\tau\to\infty} u\Big(x, t+\tau; \overline{v}^\mu(\cdot, -(t+\tau);\theta_t\omega), \theta_{t-(t+\tau)}\omega\Big) \\
&= \lim_{\tau\to\infty} u\Big(x, \tau; \overline{v}^\mu(\cdot, -\tau;\theta_t\omega), \theta_{t-\tau}\omega\Big) \\
&= \tilde{\Phi}(x,0;\theta_t\omega).
\end{aligned}
$$

The claim thus follows and we obtain the desired random profile $\Phi(x,\omega)$. □

3.2. **Stability of random transition fronts.**

Proof of Theorem 1.4. We prove it by modifying the arguments of [37, Theorem 4.1]. For any $\omega \in \Omega_0$ and given $\mu \in (0, \mu^*)$, $u(t;\omega) = \{u_i(t;\omega)\}_{i\in\mathbb{Z}}$ with $u_i(t;w) = \Phi(i - \int_0^t c(s;\omega,\mu)ds, \theta_t\omega)$ is a random transition front of (1.1). Let $u^0 \in l^\infty(\mathbb{Z})$, $u^0 = \{u_i^0\}_{i\in\mathbb{Z}}$ satisfy

$$\inf_{i\le i_0} u_i^0 > 0 \ \forall i_0 \in \mathbb{Z}, \quad \lim_{i\to\infty} \frac{u_i^0}{u_i(0;\omega)} = 1.$$

Then there is $\alpha \ge 1$ such that

$$\frac{1}{\alpha} \le \frac{u_i^0}{u_i(0;\omega)} \le \alpha, \quad \forall i \in \mathbb{Z}.$$

By the comparison principle we have

$$u_i(t;u^0,\omega) \le u_i(t;\alpha u.(0;\omega),\omega), \quad \forall i \in \mathbb{Z},\ t \ge 0,$$

and

$$u_i(t;\omega) \le u_i(t;\alpha u^0,\omega), \quad \forall i \in \mathbb{Z},\ t \ge 0. \tag{3.7}$$

Also, we have

$$\frac{d}{dt}(\alpha u_i(t;u^0,\omega)) \ge H(\alpha u_i(t;u^0,\omega)) + a(\theta_t\omega)\alpha u_i(t;u^0,\omega)(1 - \alpha u_i(t;u^0,\omega)).$$

Again by the comparison principle and (3.7) we have

$$u_i(t;\omega) \le u_i(t;\alpha u^0,\omega) \le \alpha u_i(t;u^0,\omega), \quad \forall i \in \mathbb{Z},\ t \ge 0.$$

Similarly,

$$u_i(t;u^0,\omega) \le \alpha u_i(t;\omega), \quad \forall i \in \mathbb{Z},\ t \ge 0.$$

Thus for every $t \ge 0$, we can define $\alpha(t) \ge 1$ as

$$\alpha(t) = \inf\{\alpha \ge 1 : \frac{1}{\alpha} \le \frac{u_i(t;u^0,\omega)}{u_i(t;\omega)} \le \alpha \text{ for any } i \in \mathbb{Z}\}. \tag{3.8}$$

It is easy to see that $\alpha(t_2) \le \alpha(t_1)$ for every $0 \le t_1 \le t_2$. Therefore

$$\alpha_\infty := \inf\{\alpha(t) : t \ge 0\} = \lim_{t\to\infty} \alpha(t)$$

exists. Then to prove Theorem 1.4, it is sufficient to prove taht $\alpha_\infty = 1$.

Suppose by contradiction that $\alpha_\infty > 1$. Let $1 < \alpha < \alpha_\infty$ be fixed, we first prove that there is $I_\alpha \gg 1$ such that

$$\frac{1}{\alpha} \le \frac{u_i(t;u^0,\omega)}{u_i(t;\omega)} \le \alpha, \quad \forall i \ge I_\alpha + \int_0^t c(s;\omega,\mu)ds,\ t \ge 0. \tag{3.9}$$

To this end, we only need to prove that

$$\lim_{i\to\infty} \frac{u_i(t;u^0,\omega)}{e^{-\mu(i-\int_0^t c(s;\omega,\mu)ds)}} = 1 \quad \text{uniformly for } t \ge 0. \tag{3.10}$$

In fact, since for every $\epsilon > 0$, there is $J_{\epsilon,\omega} \gg 1$ such that

$$1 - \epsilon \le \frac{u_i^0}{u_i(0;\omega)} \le 1 + \epsilon, \quad \forall i \ge J_{\epsilon,\omega}.$$

Let $A_\omega(t)$ be as in Lemma 2.6. Since

$$e^{-\mu(i-\int_0^t c(s;\omega,\mu)ds)} - d_\omega e^{A_\omega(t)-\tilde{\mu}(i-\int_0^t c(s;\omega,\mu)ds)} \le u_i(t;\omega) \le e^{-\mu(i-\int_0^t c(s;\omega,\mu)ds)},$$

it follows that

$$(1-\epsilon)e^{-\mu i}-(1-\epsilon)d_\omega e^{A_\omega(0)-\tilde{\mu}i}\le u_i^0\le(1+\epsilon)e^{-\mu i},\quad \forall i\ge J_{\epsilon,\omega}. \tag{3.11}$$

We claim that there is $d\gg 1$ such that

$$(1-\epsilon)e^{-\mu i}-de^{A_\omega(0)-\tilde{\mu}i}\le u_i^0\le(1+\epsilon)e^{-\mu i}+de^{A_\omega(0)-\tilde{\mu}i},\quad \forall i\in\mathbb{Z}. \tag{3.12}$$

Indeed, note that

$$\|u^0\|_\infty e^{\tilde{\mu}J_{\epsilon,\omega}+|A_\omega(0)|}e^{A_\omega(0)-\tilde{\mu}i}\ge\|u^0\|_\infty e^{(\tilde{\mu}-\tilde{\mu})J_{\epsilon,\omega}}\ge u_i^0,\quad \forall i\le J_{\epsilon,\omega}.$$

Hence

$$u_i^0\le d_{\epsilon,\omega}e^{A_\omega(0)-\tilde{\mu}i}\le(1+\epsilon)e^{-\mu i}+d_{\epsilon,\omega}e^{A_\omega(0)-\tilde{\mu}i},\quad \forall i\le J_{\epsilon,\omega},$$

where $d_{\epsilon,\omega}=:\|u^0\|_\infty e^{\tilde{\mu}J_{\epsilon,\omega}+|A_\omega(0)|}$. Combining this with (3.11), we obtain

$$u_i^0\le(1+\epsilon)e^{-\mu i}+d_{\epsilon,\omega}e^{A_\omega(0)-\tilde{\mu}i},\quad \forall i\in\mathbb{Z}. \tag{3.13}$$

On the other hand, for every $d>1$, the function $\mathbb{Z}\ni i\mapsto(1-\epsilon)e^{-\mu i}-de^{A_\omega(0)-\tilde{\mu}i}$ attains its maximum value at

$$J_d:=\Big[\frac{\ln\big(\frac{d\tilde{\mu}e^{A_\omega(0)}}{(1-\epsilon)\mu}\big)}{\tilde{\mu}-\mu}\Big]\quad\text{or}\quad\Big[\frac{\ln\big(\frac{d\tilde{\mu}e^{A_\omega(0)}}{(1-\epsilon)\mu}\big)}{\tilde{\mu}-\mu}\Big]+1.$$

Note that $\lim_{d\to\infty}J_d=\infty$ and

$$\lim_{d\to\infty}((1-\epsilon)e^{-\mu J_d}-de^{A_\omega(0)-\tilde{\mu}J_d})=0.$$

Then there is $\tilde{d}_{\epsilon,\omega}\gg(1-\varepsilon)d_\omega$ such that $J_{\tilde{d}_{\epsilon,\omega}}\ge J_{\epsilon,\omega}$ and

$$(1-\epsilon)e^{-\mu J_{\tilde{d}_{\epsilon,\omega}}}-\tilde{d}_{\epsilon,\omega}e^{A_\omega(0)-\tilde{\mu}J_{\tilde{d}_{\epsilon,\omega}}}\le\inf_{i\le J_{\epsilon,\omega}}u_i^0.$$

Together with (3.11), it follows that

$$(1-\epsilon)e^{-\mu i}-de^{A_\omega(0)-\tilde{\mu}i}\le u_i^0,\quad \forall i\in\mathbb{Z},\ d\ge\tilde{d}_{\epsilon,\omega}. \tag{3.14}$$

By (3.13) and (3.14) we drive that claim (3.12) holds for $d\ge\max\{\tilde{d}_{\epsilon,\omega},d_{\epsilon,\omega}\}$. Thus by similar arguments as in Lemma 2.6, we can get that for $d\gg 1$,

$$\dot{\tilde{u}}_i(t,\omega)\le H\tilde{u}_i(t,\omega)+a(\theta_t\omega)\tilde{u}_i(t,\omega)(1-\tilde{u}_i(t,\omega))$$

on the set $D_\epsilon:=\{(i,t)\in\mathbb{Z}\times\mathbb{R}^+|\tilde{u}_i(t,\omega)\ge 0\}$, where

$$\tilde{u}_i(t,\omega)=(1-\epsilon)e^{-\mu(i-\int_0^t c(s;\omega,\mu)ds)}-de^{A_\omega(t)-\tilde{\mu}(i-\int_0^t c(s;\omega,\mu)ds)}.$$

Then by the comparison principle we obtain

$$(1-\epsilon)e^{-\mu(i-\int_0^t c(s;\omega,\mu)ds)}-de^{A_\omega(t)-\tilde{\mu}(i-\int_0^t c(s;\omega,\mu)ds)}\le u_i(t;u^0,\omega) \tag{3.15}$$

for $i\in\mathbb{Z}$, $t\ge 0$, $d\gg 1$. Similarly, we can obtain

$$u_i(t;u^0,\omega)\le(1+\epsilon)e^{-\mu(i-\int_0^t c(s;\omega,\mu)ds)}+de^{A_\omega(t)-\tilde{\mu}(i-\int_0^t c(s;\omega,\mu)ds)}$$

for $i\in\mathbb{Z}$, $t\ge 0$, $d\gg 1$. Then (3.10) and (3.9) follow form the last two inequalities and the arbitrariness of $\epsilon>0$.

Next, let I_α be given by (3.9) and set

$$m_\alpha:=\frac{1}{\alpha_0}\inf\big\{u_i(t;\omega):t\ge 0,\ i-\int_0^t c(s;\omega,\mu)ds\le I_\alpha\big\},$$

where $\alpha_0 = \alpha(0) = \sup_{t\ge 0}\alpha(t)$. From (3.8) it follows that

$$m_\alpha \le \min\{u_i(t;\omega), u_i(t;u^0,\omega)\}, \quad \forall i \le I_\alpha + \int_0^t c(s;\omega,\mu)ds,\ t\ge 0.$$

By (H1) there is $T = T(\omega) \ge 1$ such that

$$0 < \frac{\underline{a}T}{2} < \int_\tau^{\tau+T} a(\theta_s\omega)ds < 2\overline{a}T < \infty, \quad \forall \tau\in\mathbb{R}. \tag{3.16}$$

Let $0<\delta\ll 1$ satisfy

$$\alpha < e^{-2\delta T\overline{a}}\alpha_\infty \quad \text{and} \quad ((\alpha_\infty - 1) - \alpha_0(1-e^{-2\delta T\overline{a}}))m_\alpha > \delta. \tag{3.17}$$

We claim that

$$\alpha((k+1)T) \le e^{-\delta\int_{kT}^{(k+1)T} a(\theta_s\omega)ds}\alpha(kT), \quad \forall k\ge 0. \tag{3.18}$$

In fact, setting $a_k(t) = a(\theta_{t+kT}\omega)$, $\alpha_k = \alpha(kT)$, $W_k(i,t;\omega) = e^{\delta\int_{kT}^{t+kT} a(\theta_s\omega)ds}u_i(t+kT;u^0,\omega)$ and $V_k(i,t;\omega) = u_i(t;u_\cdot(0;\theta_{kT}\omega),\theta_{kT}\omega)$, it follows from (3.16) that

$$\begin{aligned}
&\frac{d}{dt}W_k\\
&= \delta a_k(t)W_k + HW_k + a_k(t)W_k(1-u_i(t+kT;u^0,\omega))\\
&= HW_k + a_k(t)W_k(1-W_k) + a_k(t)W_k((1-e^{-\delta\int_{kT}^{t+kT}a(\theta_s\omega)ds})W_k+\delta)\\
&\le HW_k + a_k(t)W_k(1-W_k) + a_k(t)W_k((1-e^{-2\delta T\overline{a}})W_k+\delta)
\end{aligned} \tag{3.19}$$

for all $t\in(0,T)$, $i\in\mathbb{Z}$ and $k\ge 0$. Also, it follows from (3.17) and $\alpha_\infty\le\alpha_k\le\alpha_0$ that

$$\begin{aligned}
&\frac{d}{dt}(\alpha_k V_k) - H(\alpha_k V_k)\\
&= a_k(t)(\alpha_k V_k)(1-V_k)\\
&= a_k(t)(\alpha_k V_k)(1-\alpha_k V_k) + a_k(t)(\alpha_k V_k)((1-e^{-2\delta T\bar{a}})(\alpha_k V_k)+\delta)\\
&\quad + a_k(t)(\alpha_k V_k)(((\alpha_k-1)-(1-e^{-2\delta T\bar{a}})\alpha_k)V_k-\delta)\\
&\ge a_k(t)(\alpha_k V_k)(1-\alpha_k V_k) + a_k(t)(\alpha_k V_k)((1-e^{-2\delta T\bar{a}})(\alpha_k V_k)+\delta)\\
&\quad + a_k(t)(\alpha_k V_k)(((\alpha_\infty-1)-(1-e^{-2\delta T\bar{a}})\alpha_0)m_\alpha-\delta)\\
&\ge a_k(t)(\alpha_k V_k)(1-\alpha_k V_k) + a_k(t)(\alpha_k V_k)((1-e^{-2\delta T\bar{a}})(\alpha_k V_k)+\delta)
\end{aligned} \tag{3.20}$$

for $i\le I_\alpha + \int_0^{t+kT} c(s;\omega,\mu)ds$, $0\le t\le T$ and $k\ge 0$. Therefore, from (3.8), (3.9), $e^{\delta\int_{kT}^{(k+1)T}a(\theta_s\omega)ds}\alpha\le\alpha_\infty\le\alpha_k$, and the comparison principle it follows that

$$e^{\delta\int_{kT}^{t+kT}a(\theta_s\omega)ds}u_i(t+kT;u^0,\omega)\le\alpha_k u_i(t+kT;\omega)$$

for $i\le I_\alpha+\int_0^{t+kT}c(s;\omega,\mu)ds$, $t\in[0,T]$ and $k\ge 0$. That is

$$u_i(t+kT;u^0,\omega)\le e^{-\delta\int_{kT}^{t+kT}a(\theta_s\omega)ds}\alpha_k u_i(t+kT;\omega)$$

for $i\le I_\alpha+\int_0^{t+kT}c(s;\omega,\mu)ds$, $t\in[0,T]$ and $k\ge 0$. Note that

$$\alpha\le e^{-\delta\int_{kT}^{(k+1)T}a(\theta_s\omega)ds}\alpha_\infty\le e^{-\delta\int_{kT}^{(k+1)T}a(\theta_s\omega)ds}\alpha_k.$$

Then by (3.9) we have

$$u_i(t+kT;u^0,\omega)\le e^{-\delta\int_{kT}^{t+kT}a(\theta_s\omega)ds}\alpha_k u_i(t+kT;\omega)$$

for $i \geq I_\alpha + \int_0^{t+kT} c(s;\omega,\mu)ds$, $t \in [0,T]$ and $k \geq 0$. Therefore,

$$u_i(t+kT;u^0,\omega) \leq e^{-\delta \int_{kT}^{t+kT} a(\theta_s\omega)ds}\alpha_k u_i(t+kT;\omega) \tag{3.21}$$

for $i \in \mathbb{Z}$, $t \in [0,T]$ and $k \geq 0$. By interchanging W_k and V_k in (3.19) and (3.20), we can also obtain

$$u_i(t+kT;\omega) \leq e^{-\delta \int_{kT}^{t+kT} a(\theta_s\omega)ds}\alpha_k u_i(t+kT;u^0,\omega) \tag{3.22}$$

for $i \in \mathbb{Z}$, $t \in [0,T]$ and $k \geq 0$. Then the claim (3.18) follows from (3.21) and (3.22).

From (3.18) it follows that

$$\alpha_\infty \leq \alpha((k+1)T) \leq e^{-\delta \sum_{i=0}^{k} \int_{iT}^{(i+1)T} a(\theta_s\omega)ds}\alpha(0) = e^{-\delta \int_0^{(k+1)T} a(\theta_s\omega)ds}\alpha_0 \tag{3.23}$$

for any $k \geq 0$. Note that $\int_0^\infty a(\theta_s\omega)ds = \infty$ for $\omega \in \Omega_0$. Then by letting $k \to \infty$ in (3.23), we get that $\alpha_\infty \leq 0$, a contradiction. So we get that $\alpha_\infty = 1$, which leads to the asymptotic stability of the random transition fronts. $\square$

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Feng Cao
Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, China
Email address: fcao@nuaa.edu.cn

Lu Gao
Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, China
Email address: gaolunuaa@163.com