

BLOW-UP FOR PARABOLIC EQUATIONS IN NONLINEAR DIVERGENCE FORM WITH TIME-DEPENDENT COEFFICIENTS

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ABSTRACT. In this article, we study the blow-up of solutions to the nonlinear parabolic equation in divergence form,

$$\begin{aligned}(h(u))_t &= \sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} - k(t)f(u) \quad \text{in } \Omega \times (0, t^*), \\ \sum_{i,j=1}^n a^{ij}(x)u_{x_i}\nu_j &= g(u) \quad \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) &= u_0(x) \geq 0 \quad \text{in } \bar{\Omega},\end{aligned}$$

where Ω is a bounded convex domain in \mathbb{R}^n ($n \geq 2$) with smooth boundary $\partial\Omega$. By constructing suitable auxiliary functions and using a differential inequality technique, when $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), we establish conditions for the solution blow up at a finite time, and conditions for the solution to exist for all time. Also, we find an upper bound for the blow-up time. In addition, when $\Omega \subset \mathbb{R}^n$ with ($n \geq 3$), we use a Sobolev inequality to obtain a lower bound for the blow-up time.

1. INTRODUCTION

There are many results about the blow-up of solutions to nonlinear parabolic problems; see for example [4, 6, 16, 18, 19] and the references therein. A variety of methods have been used to study the blow-up phenomena of the solutions to parabolic equations in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). Authors often derive lower bounds for the blow-up time by restricting $\Omega \subset \mathbb{R}^3$ (see [12, 13, 14]). Recently, some studies determined lower bounds for the blow-up time when $\Omega \subset \mathbb{R}^n$ ($n \geq 3$), see [1, 3, 8, 9, 10].

In this article, we investigate the blow-up of solutions to the nonlinear parabolic equation in divergence form,

2010 *Mathematics Subject Classification*. 35K55, 35B44.

Key words and phrases. Nonlinear parabolic equation; blow-up; upper bound; lower bound.

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Submitted January 27, 2021. Published January 25, 2022.

$$\begin{aligned}
(h(u))_t &= \sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} - k(t)f(u) \quad \text{in } \Omega \times (0, t^*), \\
\sum_{i,j=1}^n a^{ij}(x)u_{x_i}\nu_j &= g(u) \quad \text{on } \partial\Omega \times (0, t^*), \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{in } \bar{\Omega},
\end{aligned} \tag{1.1}$$

where Ω is a bounded convex domain in \mathbb{R}^n ($n \geq 2$) with smooth boundary $\partial\Omega$, $(a^{ij}(x))_{n \times n}$ is a differentiable positive definite matrix, ν is the outward normal vector to $\partial\Omega$, $u_0(x)$ is the initial value, t^* is the maximal existence time of u , and $\bar{\Omega}$ is the closure of Ω . Set $\mathbb{R}_+ = (0, +\infty)$. We assume, in this paper, that h is a $C^2(\bar{\mathbb{R}}_+)$ function with $h'(s) > 0$ for all $s \geq 0$, k is a positive $C^1(\bar{\mathbb{R}}_+)$ function, g and f are nonnegative $C(\bar{\mathbb{R}}_+)$ functions, and u_0 is a nonnegative $C^1(\bar{\Omega})$ function.

The blow-up phenomena in parabolic equations with nonlinear boundary conditions have been studied in [2, 7, 11, 16]. Payne, Philippin and Vernier Piro [15] studied a special case of (1.1),

$$\begin{aligned}
u_t &= \Delta u - f(u) \quad \text{in } \Omega \times (0, t^*), \\
\frac{\partial u}{\partial \nu} &= g(u) \quad \text{on } \partial\Omega \times (0, t^*), \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{in } \bar{\Omega},
\end{aligned} \tag{1.2}$$

where Ω is a bounded convex domain in \mathbb{R}^n ($n \geq 2$) with smooth boundary $\partial\Omega$. When $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), some conditions on data were established to ensure that $u(x, t)$ exists for all time or blows up at some finite time. Moreover, they also derived an upper bound for blow-up time. In particular, when $\Omega \subset \mathbb{R}^3$, they obtained a lower bound for blow-up time under more appropriate hypotheses.

In [2, 11], the following special case of (1.1) has been discussed,

$$\begin{aligned}
u_t &= \sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} - f(u) \quad \text{in } \Omega \times (0, t^*), \\
\sum_{i,j=1}^n a^{ij}(x)u_{x_i}\nu_j &= g(u) \quad \text{on } \partial\Omega \times (0, t^*), \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{in } \bar{\Omega},
\end{aligned} \tag{1.3}$$

where Ω is a bounded convex domain in \mathbb{R}^n ($n \geq 2$) with smooth boundary $\partial\Omega$. Under certain conditions on data, Li and Li [11] showed that the solution blows up or remains global when $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). For $\Omega \subset \mathbb{R}^3$, a lower bound for blow-up time was also derived. By restricting $\Omega \subset \mathbb{R}^n$ ($n \geq 3$), Baghaei and Hesaaraki [2] derived a lower bound for blow-up time when blow-up occurs.

Motivated by above works, we study the more general problem (1.1). It seems that the auxiliary functions defined in [2, 11, 15] are no longer applicable for problem (1.1). By constructing suitable auxiliary functions and using a differential inequality technique, we establish conditions on the data for the solution $u(x, t)$ to blow up at a finite time, and for the solution to exist for all time when $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). Also, we obtain an upper bound for the blow-up time. When $\Omega \subset \mathbb{R}^n$ ($n \geq 3$), we use a Sobolev inequality to derive a lower bound for the blow-up time. Note that if $h(u) \equiv u$, $(a^{ij}(x))_{n \times n}$ is a unit matrix, and $k(t) \equiv 1$, problem (1.1) is the same as

problem (1.2); if $h(u) \equiv u$ and $k(t) \equiv 1$, problem (1.1) becomes problem (1.3). In the above two cases, our results derived in this paper still hold. Hence, our results can be regarded as an extension of the results in [2, 11, 15].

This article is organized as follows. In Section 2, we establish the conditions on the data sufficient to guarantee that the solution $u(x, t)$ exists for all time. In Section 3, we obtain an upper bound for the blow-up time under some appropriate assumptions. In Section 4, we obtain a lower bound for blow-up time. Section 5, we give two examples that illustrate the results obtained.

2. GLOBAL SOLUTION

In this section, we establish a sufficient condition for the existence of a global solution. We define the auxiliary functions

$$\Phi(t) = \int_{\Omega} H(u(x, t)) \, dx, \quad t \geq 0; \quad H(s) = 2 \int_0^s y h'(y) \, dy, \quad s \geq 0. \quad (2.1)$$

Since $(a^{ij}(x))_{n \times n}$ is a positive definite matrix, there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad (2.2)$$

for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

Theorem 2.1. *Let $u(x, t)$ be the nonnegative classical solution of problem (1.1). Suppose that functions f , g , h , and k satisfy*

$$f(s) \geq \gamma_1 s^p, \quad g(s) \leq \gamma_2 s^q, \quad h'(s) \leq \zeta_0, \quad s \geq 0, \quad (2.3)$$

$$k(t) \geq m, \quad t \geq 0, \quad (2.4)$$

where $p, q, \gamma_1, \gamma_2, \zeta_0$, and m are some positive constants, and

$$p > 1, \quad q > 1, \quad p + 1 > 2q. \quad (2.5)$$

Then $u(x, t)$ exists for all $t > 0$ in the measure $\Phi(t)$.

Proof. Using the divergence theorem and assumptions (2.2)–(2.4), we have

$$\begin{aligned} \Phi'(t) &= \int_{\Omega} H'(u(x, t)) u_t \, dx = 2 \int_{\Omega} u h'(u) u_t \, dx \\ &= 2 \int_{\Omega} u \left(\sum_{i,j=1}^n (a^{ij}(x) u_{x_i})_{x_j} - k(t) f(u) \right) \, dx \\ &= 2 \int_{\partial\Omega} u \sum_{i,j=1}^n a^{ij}(x) u_{x_i} \nu_j \, ds - 2 \int_{\Omega} \sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} \, dx \\ &\quad - 2k(t) \int_{\Omega} u f(u) \, dx \\ &\leq 2 \int_{\partial\Omega} u g(u) \, ds - 2\theta \int_{\Omega} |\nabla u|^2 \, dx - 2k(t) \int_{\Omega} u f(u) \, dx \\ &\leq 2\gamma_2 \int_{\partial\Omega} u^{q+1} \, ds - 2\theta \int_{\Omega} |\nabla u|^2 \, dx - 2m\gamma_1 \int_{\Omega} u^{p+1} \, dx. \end{aligned} \quad (2.6)$$

Owing to [11, Lemma 2.1], we deduce that

$$\int_{\partial\Omega} u^{q+1} ds \leq \frac{n}{\rho_0} \int_{\Omega} u^{q+1} dx + \frac{(q+1)d}{\rho_0} \int_{\Omega} u^q |\nabla u| dx, \quad (2.7)$$

where

$$\rho_0 = \min_{\partial\Omega} (x \cdot \nu), \quad d = \max_{\partial\Omega} |x|. \quad (2.8)$$

Substituting (2.7) into (2.6), we obtain

$$\begin{aligned} \Phi'(t) &\leq \frac{2n\gamma_2}{\rho_0} \int_{\Omega} u^{q+1} dx + \frac{2\gamma_2(q+1)d}{\rho_0} \int_{\Omega} u^q |\nabla u| dx - 2\theta \int_{\Omega} |\nabla u|^2 dx \\ &\quad - 2m\gamma_1 \int_{\Omega} u^{p+1} dx. \end{aligned} \quad (2.9)$$

Using Hölder's and Young's inequalities in the second term of (2.9), we have

$$\begin{aligned} \int_{\Omega} u^q |\nabla u| dx &\leq \left(\varepsilon \int_{\Omega} u^{2q} dx \right)^{1/2} \left(\frac{1}{\varepsilon} \int_{\Omega} |\nabla u|^2 dx \right)^{1/2} \\ &\leq \frac{\varepsilon}{2} \int_{\Omega} u^{2q} dx + \frac{1}{2\varepsilon} \int_{\Omega} |\nabla u|^2 dx, \end{aligned} \quad (2.10)$$

where

$$\varepsilon = \frac{\gamma_2(q+1)d}{2\rho_0\theta} > 0. \quad (2.11)$$

Inserting (2.10) into (2.9) and using (2.11), we can rewrite (2.9) as

$$\begin{aligned} \Phi'(t) &\leq \frac{2n\gamma_2}{\rho_0} \int_{\Omega} u^{q+1} dx + \frac{\varepsilon\gamma_2(q+1)d}{\rho_0} \int_{\Omega} u^{2q} dx \\ &\quad + \left(\frac{\gamma_2(q+1)d}{\rho_0\varepsilon} - 2\theta \right) \int_{\Omega} |\nabla u|^2 dx - 2m\gamma_1 \int_{\Omega} u^{p+1} dx \\ &= \frac{2n\gamma_2}{\rho_0} \int_{\Omega} u^{q+1} dx + 2\theta\varepsilon^2 \int_{\Omega} u^{2q} dx - 2m\gamma_1 \int_{\Omega} u^{p+1} dx. \end{aligned} \quad (2.12)$$

It follows from (2.5) that $0 < \frac{p+1-2q}{p-q} < 1$. We apply Hölder's and Young's inequalities to obtain

$$\begin{aligned} \int_{\Omega} u^{2q} dx &\leq \left(\int_{\Omega} u^{q+1} dx \right)^{\frac{p+1-2q}{p-q}} \left(\int_{\Omega} u^{p+1} dx \right)^{\frac{q-1}{p-q}} \\ &= \left(\sigma^{\frac{1-q}{p+1-2q}} \int_{\Omega} u^{q+1} dx \right)^{\frac{p+1-2q}{p-q}} \left(\sigma \int_{\Omega} u^{p+1} dx \right)^{\frac{q-1}{p-q}} \\ &\leq \frac{p+1-2q}{p-q} \sigma^{\frac{1-q}{p+1-2q}} \int_{\Omega} u^{q+1} dx + \frac{q-1}{p-q} \sigma \int_{\Omega} u^{p+1} dx, \end{aligned} \quad (2.13)$$

where

$$0 < \sigma < \frac{m\gamma_1(p-q)}{\varepsilon^2\theta(q-1)}. \quad (2.14)$$

Next, we substitute (2.13) into (2.12) to obtain

$$\Phi'(t) \leq I_1 \int_{\Omega} u^{q+1} dx - I_2 \int_{\Omega} u^{p+1} dx \quad (2.15)$$

with

$$I_1 = \frac{2n\gamma_2}{\rho_0} + \frac{2\theta\varepsilon^2(p+1-2q)}{p-q} \sigma^{\frac{1-q}{p+1-2q}}, \quad I_2 = 2m\gamma_1 - \frac{2\theta\varepsilon^2(q-1)}{p-q} \sigma.$$

In view of (2.14) and (2.5), we have $I_1, I_2 > 0$.

Using Hölder’s inequality, we have

$$\int_{\Omega} u^{q+1} dx \leq \left(\int_{\Omega} u^{p+1} dx \right)^{\frac{q+1}{p+1}} |\Omega|^{\frac{p-q}{p+1}}, \tag{2.16}$$

$$\int_{\Omega} u^2 dx \leq \left(\int_{\Omega} u^{p+1} dx \right)^{\frac{2}{p+1}} |\Omega|^{\frac{p-1}{p+1}}, \tag{2.17}$$

where $|\Omega|$ is the volume of Ω . Thanks to (2.3),

$$H(u) = 2 \int_0^u yh'(y) dy \leq 2\zeta_0 \int_0^u y dy = \zeta_0 u^2;$$

that is

$$u^2 \geq \frac{1}{\zeta_0} H(u). \tag{2.18}$$

Combining (2.15)-(2.18), we obtain

$$\begin{aligned} \Phi'(t) &\leq I_1 \left(\int_{\Omega} u^{p+1} dx \right)^{\frac{q+1}{p+1}} \left(|\Omega|^{\frac{p-q}{p+1}} - \frac{I_2}{I_1} \left(\int_{\Omega} u^{p+1} dx \right)^{\frac{p-q}{p+1}} \right) \\ &\leq I_1 \left(\int_{\Omega} u^{p+1} dx \right)^{\frac{q+1}{p+1}} \left(|\Omega|^{\frac{p-q}{p+1}} - \frac{I_2}{I_1} \left((|\Omega|^{\frac{1-p}{p+1}} \int_{\Omega} u^2 dx \right)^{\frac{p+1}{2}} \right)^{\frac{p-q}{p+1}} \right) \\ &\leq I_1 \left(\int_{\Omega} u^{p+1} dx \right)^{\frac{q+1}{p+1}} \left(|\Omega|^{\frac{p-q}{p+1}} \right. \\ &\quad \left. - \frac{I_2}{I_1} |\Omega|^{\frac{(p-q)(1-p)}{2(p+1)}} \left(\frac{1}{\zeta_0} \right)^{\frac{p-q}{2}} \left(\int_{\Omega} H(u) dx \right)^{\frac{p-q}{2}} \right) \\ &= I_1 \left(\int_{\Omega} u^{p+1} dx \right)^{\frac{q+1}{p+1}} \left(|\Omega|^{\frac{p-q}{p+1}} - \frac{I_2}{I_1} |\Omega|^{\frac{(p-q)(1-p)}{2(p+1)}} \left(\frac{1}{\zeta_0} \right)^{\frac{p-q}{2}} \Phi^{\frac{p-q}{2}}(t) \right). \end{aligned} \tag{2.19}$$

Thus, $u(x, t)$ cannot blow up in measure $\Phi(t)$ for all time $t > 0$. In fact, if $u(x, t)$ blows up at finite time t^* in measure $\Phi(t)$, by passing to the limit as $t \rightarrow t^{*-}$, we have $\lim_{t \rightarrow t^{*-}} \Phi(t) = +\infty$ and

$$\lim_{t \rightarrow t^{*-}} \left(|\Omega|^{\frac{p-q}{p+1}} - \frac{I_2}{I_1} |\Omega|^{\frac{(p-q)(1-p)}{2(p+1)}} \left(\frac{1}{\zeta_0} \right)^{\frac{p-q}{2}} \Phi^{\frac{p-q}{2}}(t) \right) = -\infty. \tag{2.20}$$

In view of (2.19) and (2.20), we deduce $\Phi'(t) < 0$ in some interval $[t_0, t^*)$. Hence, for any $t \in [t_0, t^*)$, we have $\Phi(t) \leq \Phi(t_0)$. Taking the limits as $t \rightarrow t^{*-}$, we obtain

$$+\infty = \lim_{t \rightarrow t^{*-}} \Phi(t) \leq \Phi(t_0)$$

which is a contradiction. The proof is complete. □

3. BLOW-UP SOLUTION

In this section, we establish conditions for the solution of (1.1) to blow up in finite time, and give an upper bound for the blow-up time. We set the following auxiliary functions:

$$F(s) = \int_0^s f(y) dy, \quad G(s) = \int_0^s g(y) dy, \quad s \geq 0, \tag{3.1}$$

$$\Psi(t) = 2 \int_{\partial\Omega} G(u) ds - \int_{\Omega} \sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} dx - 2k(t) \int_{\Omega} F(u) dx, \quad t \geq 0, \tag{3.2}$$

where u is the nonnegative classical solution of (1.1). We also use the auxiliary function $\Phi(t)$ defined by (2.1). Our main result reads as follows.

Theorem 3.1. *Let $u(x, t)$ be the nonnegative classical solution of problem (1.1). Suppose that functions $f, g, h,$ and k satisfy*

$$sf(s) \leq 2(1 + \alpha)F(s), \quad sg(s) \geq 2(1 + \beta)G(s), \quad h''(s) \leq 0, \quad s \geq 0, \quad (3.3)$$

$$k'(t) \leq 0, \quad t \geq 0, \quad (3.4)$$

where α and β are nonnegative constants with $0 \leq \alpha \leq \beta$. In addition, assume that the initial value u_0 satisfies

$$\begin{aligned} \Psi(0) &= 2 \int_{\partial\Omega} G(u_0) \, ds - \int_{\Omega} \sum_{i,j=1}^n a^{ij}(x) u_{0x_i} u_{0x_j} \, dx \\ &\quad - 2k(0) \int_{\Omega} F(u_0) \, dx > 0. \end{aligned} \quad (3.5)$$

Then $u(x, t)$ blows up at a finite time t^* in the measure $\Phi(t)$, and

$$t^* \leq \frac{\Phi(0)}{2\beta(1 + \beta)\Psi(0)}, \quad \beta > 0.$$

When $\beta = 0$, we have $t^* = \infty$.

Proof. Using the divergence theorem and (3.3), we have

$$\begin{aligned} \Phi'(t) &= \int_{\Omega} H'(u(x, t)) u_t \, dx = 2 \int_{\Omega} u h'(u) u_t \, dx \\ &= 2 \int_{\Omega} u \left(\sum_{i,j=1}^n (a^{ij}(x) u_{x_i})_{x_j} - k(t) f(u) \right) \, dx \\ &= 2 \int_{\partial\Omega} u \sum_{i,j=1}^n a^{ij}(x) u_{x_i} \nu_j \, ds - 2 \int_{\Omega} \sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} \, dx \\ &\quad - 2k(t) \int_{\Omega} u f(u) \, dx \\ &= 2 \int_{\partial\Omega} u g(u) \, ds - 2 \int_{\Omega} \sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} \, dx - 2k(t) \int_{\Omega} u f(u) \, dx \\ &\geq 4(1 + \beta) \int_{\partial\Omega} G(u) \, ds - 2 \int_{\Omega} \sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} \, dx \\ &\quad - 4(1 + \alpha)k(t) \int_{\Omega} F(u) \, dx \\ &= 2(1 + \beta) \left(2 \int_{\partial\Omega} G(u) \, ds - \frac{1}{1 + \beta} \int_{\Omega} \sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} \, dx \right. \\ &\quad \left. - \frac{2(1 + \alpha)}{1 + \beta} k(t) \int_{\Omega} F(u) \, dx \right) \\ &\geq 2(1 + \beta)\Psi(t). \end{aligned} \quad (3.6)$$

Furthermore, from (3.4) and the divergence theorem,

$$\begin{aligned}
\Psi'(t) &= 2 \int_{\partial\Omega} G'(u)u_t \, ds - \int_{\Omega} \left(\sum_{i,j=1}^n a^{ij}(x)u_{x_i}u_{x_j} \right)_t \, dx - 2k'(t) \int_{\Omega} F(u) \, dx \\
&\quad - 2k(t) \int_{\Omega} F'(u)u_t \, dx \\
&\geq 2 \int_{\partial\Omega} g(u)u_t \, ds - 2 \int_{\Omega} \sum_{i,j=1}^n a^{ij}(x)u_{x_i}(u_{x_j})_t \, dx - 2k(t) \int_{\Omega} f(u)u_t \, dx \\
&= 2 \int_{\partial\Omega} g(u)u_t \, ds - 2 \int_{\Omega} \sum_{j=1}^n \left(\sum_{i=1}^n a^{ij}(x)u_{x_i}u_t \right)_{x_j} \, dx \\
&\quad + 2 \int_{\Omega} u_t \sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} \, dx - 2k(t) \int_{\Omega} f(u)u_t \, dx \tag{3.7} \\
&= 2 \int_{\partial\Omega} g(u)u_t \, ds - 2 \int_{\partial\Omega} u_t \sum_{i,j=1}^n a^{ij}(x)u_{x_i}\nu_j \, ds \\
&\quad + 2 \int_{\Omega} u_t \sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} \, dx - 2k(t) \int_{\Omega} f(u)u_t \, dx \\
&= 2 \int_{\Omega} u_t \left(\sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} - k(t)f(u) \right) \, dx \\
&= 2 \int_{\Omega} h'(u)u_t^2 \, dx \geq 0.
\end{aligned}$$

Hence, $\Psi(t)$ is a nondecreasing function in t . By (3.5), we know that $\Psi(t) \geq \Psi(0) > 0$ for all $t \in (0, t^*)$. It follows from (3.6) that

$$\Phi'(t) > 0. \tag{3.8}$$

Employing Hölder's inequality, (3.6)–(3.8), and the fact that $h'(s) > 0$ for all $s \geq 0$, we have

$$\begin{aligned}
\Psi(t)\Phi'(t) &\leq \frac{1}{2(1+\beta)}(\Phi'(t))^2 = \frac{2}{1+\beta} \left(\int_{\Omega} uh'(u)u_t \, dx \right)^2 \\
&\leq \frac{2}{1+\beta} \int_{\Omega} h'(u)u_t^2 \, dx \int_{\Omega} h'(u)u^2 \, dx \tag{3.9} \\
&\leq \frac{1}{1+\beta} \Psi'(t) \int_{\Omega} h'(u)u^2 \, dx.
\end{aligned}$$

Using assumption (3.3) and integration by parts, we obtain

$$\begin{aligned}
H(u) &= 2 \int_0^u yh'(y) \, dy = \int_0^u h'(y) \, dy^2 \\
&= h'(u)u^2 - \int_0^u y^2 h''(y) \, dy \geq h'(u)u^2.
\end{aligned} \tag{3.10}$$

Combining this and (3.9), we have

$$\Psi(t)\Phi'(t) \leq \frac{1}{1+\beta} \Psi'(t) \int_{\Omega} H(u) \, dx = \frac{1}{1+\beta} \Psi'(t)\Phi(t).$$

Multiplying the above inequality by $\Phi^{-2-\beta}(t)$, we obtain

$$(\Psi(t)\Phi^{-1-\beta}(t))' \geq 0. \quad (3.11)$$

We integrate (3.11) from 0 to t to obtain

$$\Psi(t)\Phi^{-1-\beta}(t) \geq \Psi(0)\Phi^{-1-\beta}(0) = M > 0. \quad (3.12)$$

Now (3.6) and (3.12) imply

$$\Phi'(t) \geq 2(1 + \beta)\Psi(t) \geq 2M(1 + \beta)\Phi^{1+\beta}(t). \quad (3.13)$$

If $\beta > 0$, it follows from (3.13) that

$$(\Phi^{-\beta}(t))' = -\beta\Phi^{-1-\beta}(t)\Phi'(t) \leq -2M\beta(1 + \beta). \quad (3.14)$$

Integrating (3.14) over $[0, t]$, we obtain

$$\Phi^{-\beta}(t) \leq \Phi^{-\beta}(0) - 2M\beta(1 + \beta)t. \quad (3.15)$$

It is obvious that (3.15) cannot hold for all time t . Consequently, $u(x, t)$ blows up at some finite time t^* in the measure $\Phi(t)$ and

$$t^* \leq \frac{\Phi(0)}{2\beta(1 + \beta)\Psi(0)}.$$

For $\beta = 0$, we have $\alpha = 0$. It follows from (3.13) that $\Phi(t) \geq \Phi(0)e^{2Mt}$, which implies that $t^* = \infty$. The proof is complete. \square

4. LOWER BOUND FOR THE BLOW-UP TIME

In this section, we consider $\Omega \subset \mathbb{R}^n$ ($n \geq 3$), and assume that f, g, h , and k to satisfy the following conditions:

$$f(s) \leq \gamma_1 s^p, \quad g(s) \leq \gamma_2 s^q, \quad h'(s) \geq \zeta, \quad s \geq 0, \quad (4.1)$$

$$k(t) \geq m, \quad \frac{k'(t)}{k(t)} \leq \eta, \quad t \geq 0, \quad (4.2)$$

where $p, q, \gamma_1, \gamma_2, \zeta$, and m are positive constants, and η is a nonnegative constant. Moreover, we assume that

$$p > 1, \quad q > 1, \quad 2q > p + 1. \quad (4.3)$$

We define the auxiliary functions

$$A(t) = k^{\frac{2r}{p-1}}(t) \int_{\Omega} B(u) \, dx, \quad t \geq 0, \quad B(s) = 2r \int_0^s h'(y)y^{2r-1} \, dy, \quad s \geq 0, \quad (4.4)$$

where r is a constant such that

$$r > \max\left\{1, \frac{1}{2}n(q-1)\right\}. \quad (4.5)$$

In this section, we need the Sobolev inequality

$$\left(\int_{\Omega} (u^r)^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{2n}} \leq c \left(\int_{\Omega} u^{2r} \, dx + \int_{\Omega} |\nabla u^r|^2 \, dx\right)^{1/2}, \quad (4.6)$$

where $c = c(n, \Omega)$ is the best Sobolev constant depending on n ($n \geq 3$) and Ω . For the more details we refer reader to [5, Corollary 9.14]. We state our result as follows.

Theorem 4.1. *Let $u(x, t)$ be the nonnegative classical solution of (1.1). Assume that (4.1)–(4.3) and (4.5) hold, and $u(x, t)$ becomes unbounded in the measure $A(t)$ at a finite time t^* . Then*

$$t^* \geq \int_{A(0)}^{\infty} \frac{d\tau}{J_1\tau + J_2\tau^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}}},$$

where

$$J_1 = \frac{2r\eta}{p-1} + \frac{r\gamma_2}{\zeta}, \quad (4.7)$$

$$J_2 = c_1 \left(\frac{(2q-p-1)[2r-n(q-1)]}{2r(2q-p-1) + n(q-1)(p-1)} c_2^{-\frac{2n(q-1)^2}{(2q-p-1)[2r-n(q-1)]}} \right. \\ \left. + \frac{2r-n(q-1)}{2r} c_3^{-\frac{n(q-1)}{2r-n(q-1)}} \right) \left(\frac{1}{\zeta} \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}}, \quad (4.8)$$

$$c_1 = r\gamma_2 m^{-\frac{2(q-1)}{p-1}} c^{\frac{n(q-1)}{r}} 2^{\frac{n(q-1)}{2r}} \left(\left(\frac{n}{\rho_0} \right)^2 + \frac{\gamma_2(2r+q-1)^2 d^2}{(2r-1)\theta\rho_0^2} \right), \quad (4.9)$$

$$c_2 = \frac{r\gamma_1 |\Omega|^{-\frac{p-1}{2r}} [2r(2q-p-1) + n(q-1)(p-1)]}{n(q-1)^2 c_1}, \quad c_3 = \frac{2(2r-1)\theta}{n(q-1)c_1}, \quad (4.10)$$

and ρ_0 and d are defined by (2.8).

Proof. By the divergence theorem and assumptions (2.2), (4.1), (4.2), and (4.5), we obtain

$$\begin{aligned} & A'(t) \\ &= \frac{2r}{p-1} k^{\frac{2r}{p-1}-1}(t) k'(t) \int_{\Omega} B(u) \, dx + k^{\frac{2r}{p-1}}(t) \int_{\Omega} B'(u) u_t \, dx \\ &= \frac{2r}{p-1} \frac{k'(t)}{k(t)} k^{\frac{2r}{p-1}}(t) \int_{\Omega} B(u) \, dx + 2rk^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r-1} h'(u) u_t \, dx \\ &\leq \frac{2r\eta}{p-1} k^{\frac{2r}{p-1}}(t) \int_{\Omega} B(u) \, dx \\ &\quad + 2rk^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r-1} \left(\sum_{i,j=1}^n (a^{ij}(x) u_{x_i})_{x_j} - k(t) f(u) \right) \, dx \\ &\leq \frac{2r\eta}{p-1} A(t) + 2rk^{\frac{2r}{p-1}}(t) \int_{\partial\Omega} u^{2r-1} \sum_{i,j=1}^n a^{ij}(x) u_{x_i} \nu_j \, ds \\ &\quad - 2r(2r-1)\theta k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r-2} |\nabla u|^2 \, dx - 2rk^{\frac{2r+p-1}{p-1}}(t) \int_{\Omega} u^{2r-1} f(u) \, dx \\ &= \frac{2r\eta}{p-1} A(t) + 2rk^{\frac{2r}{p-1}}(t) \int_{\partial\Omega} u^{2r-1} g(u) \, ds - 2r(2r-1)\theta k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r-2} |\nabla u|^2 \, dx \\ &\quad - 2rk^{\frac{2r+p-1}{p-1}}(t) \int_{\Omega} u^{2r-1} f(u) \, dx \\ &\leq \frac{2r\eta}{p-1} A(t) + 2r\gamma_2 k^{\frac{2r}{p-1}}(t) \int_{\partial\Omega} u^{2r+q-1} \, ds - \frac{2(2r-1)\theta}{r} k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 \, dx \\ &\quad - 2r\gamma_1 k^{\frac{2r+p-1}{p-1}}(t) \int_{\Omega} u^{2r+p-1} \, dx. \end{aligned} \quad (4.11)$$

It follows from (4.3) and (4.5) that $0 < \frac{2r}{2r+p-1} < 1$. Applying Hölder's inequality, we deduce that

$$k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} \, dx \leq \left(k^{\frac{2r+p-1}{p-1}}(t) \int_{\Omega} u^{2r+p-1} \, dx \right)^{\frac{2r}{2r+p-1}} |\Omega|^{\frac{p-1}{2r+p-1}},$$

or equivalently,

$$k^{\frac{2r+p-1}{p-1}}(t) \int_{\Omega} u^{2r+p-1} \, dx \geq |\Omega|^{-\frac{p-1}{2r}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} \, dx \right)^{\frac{2r+p-1}{2r}}. \quad (4.12)$$

Then we substitute (4.12) into (4.11) to obtain

$$\begin{aligned} A'(t) &\leq \frac{2r\eta}{p-1} A(t) + 2r\gamma_2 k^{\frac{2r}{p-1}}(t) \int_{\partial\Omega} u^{2r+q-1} \, ds \\ &\quad - \frac{2(2r-1)\theta}{r} k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 \, dx \\ &\quad - 2r\gamma_1 |\Omega|^{-\frac{p-1}{2r}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} \, dx \right)^{\frac{2r+p-1}{2r}}. \end{aligned} \quad (4.13)$$

Now, we deal with the second term on the right-hand side of (4.13). Using (4.5) and [11, Lemma 2.1], we obtain

$$\begin{aligned} &k^{\frac{2r}{p-1}}(t) \int_{\partial\Omega} u^{2r+q-1} \, ds \\ &\leq \frac{n}{\rho_0} k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r+q-1} \, dx + \frac{(2r+q-1)d}{\rho_0} k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r+q-2} |\nabla u| \, dx. \end{aligned} \quad (4.14)$$

Utilizing Hölder's inequality and Young's inequality, we have

$$\begin{aligned} \frac{n}{\rho_0} \int_{\Omega} u^{2r+q-1} \, dx &\leq \left(\frac{n^2}{\rho_0^2} \int_{\Omega} u^{2r+2q-2} \, dx \right)^{1/2} \left(\int_{\Omega} u^{2r} \, dx \right)^{1/2} \\ &\leq \frac{1}{2} \left(\frac{n}{\rho_0} \right)^2 \int_{\Omega} u^{2r+2q-2} \, dx + \frac{1}{2} \int_{\Omega} u^{2r} \, dx \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} &\frac{(2r+q-1)d}{\rho_0} \int_{\Omega} u^{2r+q-2} |\nabla u| \, dx \\ &\leq \frac{(2r+q-1)d}{\rho_0} \left(\frac{1}{r^2} \int_{\Omega} |\nabla u^r|^2 \, dx \right)^{1/2} \left(\int_{\Omega} u^{2r+2q-2} \, dx \right)^{1/2} \\ &= \left(\frac{\varepsilon_1}{r^2} \int_{\Omega} |\nabla u^r|^2 \, dx \right)^{1/2} \left(\frac{(2r+q-1)^2 d^2}{\rho_0^2 \varepsilon_1} \int_{\Omega} u^{2r+2q-2} \, dx \right)^{1/2} \\ &\leq \frac{\varepsilon_1}{2r^2} \int_{\Omega} |\nabla u^r|^2 \, dx + \frac{(2r+q-1)^2 d^2}{2\rho_0^2 \varepsilon_1} \int_{\Omega} u^{2r+2q-2} \, dx, \end{aligned} \quad (4.16)$$

where

$$\varepsilon_1 = \frac{(2r-1)\theta}{\gamma_2} > 0.$$

Inserting (4.15) and (4.16) into (4.14), we have

$$\begin{aligned} & k^{\frac{2r}{p-1}}(t) \int_{\partial\Omega} u^{2r+q-1} \, ds \\ & \leq \frac{1}{2} k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} \, dx \\ & \quad + \frac{1}{2} \left(\left(\frac{n}{\rho_0} \right)^2 + \frac{(2r+q-1)^2 d^2}{\rho_0^2 \varepsilon_1} \right) k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r+2q-2} \, dx \\ & \quad + \frac{\varepsilon_1}{2r^2} k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 \, dx. \end{aligned} \quad (4.17)$$

From this and (4.13), we deduce

$$\begin{aligned} A'(t) & \leq \frac{2r\eta}{p-1} A(t) + r\gamma_2 k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} \, dx \\ & \quad + r\gamma_2 \left(\left(\frac{n}{\rho_0} \right)^2 + \frac{(2r+q-1)^2 d^2}{\rho_0^2 \varepsilon_1} \right) k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r+2q-2} \, dx \\ & \quad + \left(\frac{-2(2r-1)\theta}{r} + \frac{\varepsilon_1 \gamma_2}{r} \right) k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 \, dx \\ & \quad - 2r\gamma_1 |\Omega|^{-\frac{p-1}{2r}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} \, dx \right)^{\frac{2r+p-1}{2r}}. \end{aligned} \quad (4.18)$$

Then, we pay our attention to the integral $k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r+2q-2} \, dx$ in (4.18). Noticing that $0 < \frac{(q-1)(n-2)}{2r} < 1$ in view of (4.5), and using Hölder's inequality, we obtain

$$\begin{aligned} & k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r+2q-2} \, dx \\ & \leq \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} (u^r)^{\frac{2n}{n-2}} \, dx \right)^{\frac{(q-1)(n-2)}{2r}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} \, dx \right)^{\frac{2r-(q-1)(n-2)}{2r}}. \end{aligned} \quad (4.19)$$

Owing to the Sobolev inequality given in (4.6), we obtain

$$\int_{\Omega} (u^r)^{\frac{2n}{n-2}} \, dx \leq c^{\frac{2n}{n-2}} \left(\int_{\Omega} u^{2r} \, dx + \int_{\Omega} |\nabla u^r|^2 \, dx \right)^{\frac{n}{n-2}}. \quad (4.20)$$

Substituting (4.20) into (4.19) and using (4.2) and the inequality

$$(a+b)^\mu \leq 2^\mu (a^\mu + b^\mu), \quad a, b, \mu > 0,$$

we derive

$$\begin{aligned} & k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r+2q-2} \, dx \\ & \leq \left(k^{\frac{2r}{p-1}}(t) c^{\frac{2n}{n-2}} \left(\int_{\Omega} u^{2r} \, dx + \int_{\Omega} |\nabla u^r|^2 \, dx \right)^{\frac{n}{n-2}} \right)^{\frac{(q-1)(n-2)}{2r}} \\ & \quad \times \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} \, dx \right)^{\frac{2r-(q-1)(n-2)}{2r}} \\ & \leq k^{\frac{(q-1)(n-2)}{p-1}}(t) c^{\frac{n(q-1)}{r}} 2^{\frac{n(q-1)}{2r}} \left(\left(\int_{\Omega} u^{2r} \, dx \right)^{\frac{n(q-1)}{2r}} + \left(\int_{\Omega} |\nabla u^r|^2 \, dx \right)^{\frac{n(q-1)}{2r}} \right) \\ & \quad \times \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} \, dx \right)^{\frac{2r-(q-1)(n-2)}{2r}} \end{aligned}$$

$$\begin{aligned}
&= k^{-\frac{2(q-1)}{p-1}}(t) c^{\frac{n(q-1)}{r}} 2^{\frac{n(q-1)}{2r}} \left(\left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{n(q-1)}{2r}} \right. \\
&\quad \left. + \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 dx \right)^{\frac{n(q-1)}{2r}} \right) \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r}} \\
&\leq m^{-\frac{2(q-1)}{p-1}} c^{\frac{n(q-1)}{r}} 2^{\frac{n(q-1)}{2r}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{r+q-1}{r}} + m^{-\frac{2(q-1)}{p-1}} c^{\frac{n(q-1)}{r}} 2^{\frac{n(q-1)}{2r}} \\
&\quad \times \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 dx \right)^{\frac{n(q-1)}{2r}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r}}. \tag{4.21}
\end{aligned}$$

From (4.5), we can easily see that $0 < \frac{n(q-1)}{2r} < 1$. It follows from Young's inequality that

$$\begin{aligned}
&\left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 dx \right)^{\frac{n(q-1)}{2r}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r}} \\
&= \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 dx \right)^{\frac{n(q-1)}{2r}} \left(\left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}} \right)^{\frac{2r-n(q-1)}{2r}} \\
&= \left(c_3 k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 dx \right)^{\frac{n(q-1)}{2r}} \\
&\quad \times \left(c_3^{-\frac{n(q-1)}{2r-n(q-1)}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}} \right)^{\frac{2r-n(q-1)}{2r}} \\
&\leq \frac{n(q-1)c_3}{2r} k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 dx \\
&\quad + \frac{2r-n(q-1)}{2r} c_3^{-\frac{n(q-1)}{2r-n(q-1)}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}}, \tag{4.22}
\end{aligned}$$

where c_3 is defined in (4.10). Combining (4.21) and (4.22) with (4.18), we have

$$\begin{aligned}
A'(t) &\leq \frac{2r\eta}{p-1} A(t) + r\gamma_2 k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx + r\gamma_2 m^{-\frac{2(q-1)}{p-1}} c^{\frac{n(q-1)}{r}} 2^{\frac{n(q-1)}{2r}} \\
&\quad \times \left(\left(\frac{n}{\rho_0} \right)^2 + \frac{(2r+q-1)^2 d^2}{\rho_0^2 \varepsilon_1} \right) \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{r+q-1}{r}} \\
&\quad + r\gamma_2 m^{-\frac{2(q-1)}{p-1}} c^{\frac{n(q-1)}{r}} 2^{\frac{n(q-1)}{2r}} \left(\left(\frac{n}{\rho_0} \right)^2 + \frac{(2r+q-1)^2 d^2}{\rho_0^2 \varepsilon_1} \right) \\
&\quad \times c_3^{-\frac{n(q-1)}{2r-n(q-1)}} \frac{2r-n(q-1)}{2r} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}} \\
&\quad + \left[-\frac{2(2r-1)\theta}{r} + \frac{\varepsilon_1 \gamma_2}{r} + r\gamma_2 m^{-\frac{2(q-1)}{p-1}} c^{\frac{n(q-1)}{r}} 2^{\frac{n(q-1)}{2r}} \right. \\
&\quad \times \left. \left(\left(\frac{n}{\rho_0} \right)^2 + \frac{(2r+q-1)^2 d^2}{\rho_0^2 \varepsilon_1} \right) \frac{n(q-1)}{2r} c_3 \right] k^{\frac{2r}{p-1}}(t) \int_{\Omega} |\nabla u^r|^2 dx \\
&\quad - 2r\gamma_1 |\Omega|^{-\frac{p-1}{2r}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r+p-1}{2r}} \\
&= \frac{2r\eta}{p-1} A(t) + r\gamma_2 k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx + c_1 \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{r+q-1}{r}}
\end{aligned}$$

$$\begin{aligned}
& + \frac{2r - n(q-1)}{2r} c_1 c_3^{-\frac{n(q-1)}{2r-n(q-1)}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}} \\
& - 2r\gamma_1 |\Omega|^{-\frac{p-1}{2r}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r+p-1}{2r}}, \tag{4.23}
\end{aligned}$$

where c_1 is given in (4.9). In view of (4.3) and (4.5), we have

$$0 < \frac{(2q-p-1)[2r-n(q-1)]}{2r(2q-p-1)+n(q-1)(p-1)} < 1.$$

Then Young's inequality implies that

$$\begin{aligned}
& \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{r+q-1}{r}} \\
& = \left(c_2^{-\frac{2n(q-1)^2}{(2q-p-1)[2r-n(q-1)]}} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}} \right)^{\frac{(2q-p-1)[2r-n(q-1)]}{2r(2q-p-1)+n(q-1)(p-1)}} \\
& \quad \times \left(c_2 \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r+p-1}{2r}} \right)^{\frac{2n(q-1)^2}{2r(2q-p-1)+n(q-1)(p-1)}} \tag{4.24} \\
& \leq \frac{(2q-p-1)[2r-n(q-1)]}{2r(2q-p-1)+n(q-1)(p-1)} c_2^{-\frac{2n(q-1)^2}{(2q-p-1)[2r-n(q-1)]}} \\
& \quad \times \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}} \\
& \quad + \frac{2n(q-1)^2 c_2}{2r(2q-p-1)+n(q-1)(p-1)} \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r+p-1}{2r}},
\end{aligned}$$

where c_2 is defined by (4.10).

Finally, we insert (4.24) into (4.23) to obtain

$$\begin{aligned}
A'(t) & \leq \frac{2r\eta}{p-1} A(t) + r\gamma_2 k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \\
& \quad + c_1 \left(\frac{(2q-p-1)[2r-n(q-1)]}{2r(2q-p-1)+n(q-1)(p-1)} c_2^{-\frac{2n(q-1)^2}{(2q-p-1)[2r-n(q-1)]}} \right. \tag{4.25} \\
& \quad \left. + \frac{2r-n(q-1)}{2r} c_3^{-\frac{n(q-1)}{2r-n(q-1)}} \right) \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} u^{2r} dx \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}}.
\end{aligned}$$

From (4.1), we obtain

$$B(u) = 2r \int_0^u h'(y) y^{2r-1} dy \geq 2r\zeta \int_0^u y^{2r-1} dy = \zeta u^{2r};$$

that is

$$u^{2r} \leq \frac{1}{\zeta} B(u). \tag{4.26}$$

By (4.26), we rewrite (4.25) as

$$\begin{aligned}
 A'(t) &\leq \frac{2r\eta}{p-1}A(t) + \frac{r\gamma_2}{\zeta}k^{\frac{2r}{p-1}}(t) \int_{\Omega} B(u) \, dx \\
 &\quad + c_1 \left(\frac{(2q-p-1)[2r-n(q-1)]}{2r(2q-p-1) + n(q-1)(p-1)} \right)^{-\frac{2n(q-1)^2}{(2q-p-1)[2r-n(q-1)]}} c_2 \\
 &\quad + \frac{2r-n(q-1)}{2r} c_3^{-\frac{n(q-1)}{2r-n(q-1)}} \left(\frac{1}{\zeta} \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}} \\
 &\quad \times \left(k^{\frac{2r}{p-1}}(t) \int_{\Omega} B(u) \, dx \right)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}} \\
 &= J_1 A(t) + J_2 A(t)^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}},
 \end{aligned} \tag{4.27}$$

where J_1 and J_2 are defined in (4.7) and (4.8), respectively. The integration of (4.27) from 0 to t results in

$$\int_{A(0)}^{A(t)} \frac{d\tau}{J_1 \tau + J_2 \tau^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}}} \leq t.$$

Since $u(x, t)$ blows up in measure $A(t)$ at finite time t^* , we pass to the limits as $t \rightarrow t^{*-}$ to obtain a lower bound

$$t^* \geq \int_{A(0)}^{\infty} \frac{d\tau}{J_1 \tau + J_2 \tau^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}}}.$$

The proof is complete. □

5. APPLICATIONS

We provide two applications of Theorems 2.1, 3.1, and 4.1.

Example 5.1. Let $u(x, t)$ be a nonnegative classical solution of

$$\begin{aligned}
 (u + \ln(1 + u))_t &= \sum_{i=1}^3 \left(\left(\frac{31}{16} + |x|^2 \right) u_{x_i} \right)_{x_i} - (1 + e^{-t})u^2 \quad \text{in } \Omega \times (0, t^*), \\
 \sum_{i=1}^3 \left(\frac{31}{16} + |x|^2 \right) u_{x_i} \nu_i &= u^2 \quad \text{on } \partial\Omega \times (0, t^*), \\
 u(x, 0) &= \frac{15}{16} + |x|^2 \quad \text{in } \bar{\Omega},
 \end{aligned}$$

where $\Omega = \{x = (x_1, x_2, x_3) : |x|^2 = \sum_{i=1}^3 x_i^2 < 1/16\}$ a ball of \mathbb{R}^3 . Now we have

$$\begin{aligned}
 (a^{ij}(x))_{3 \times 3} &= \begin{pmatrix} \frac{31}{16} + |x|^2 & 0 & 0 \\ 0 & \frac{31}{16} + |x|^2 & 0 \\ 0 & 0 & \frac{31}{16} + |x|^2 \end{pmatrix}, \quad h(u) = u + \ln(1 + u), \\
 k(t) &= 1 + e^{-t}, \quad f(u) = u^2, \quad g(u) = u^2, \quad u_0(x) = \frac{15}{16} + |x|^2, \quad n = 3.
 \end{aligned}$$

From (2.1), (3.1) and (3.2), it follows that

$$\begin{aligned}
 F(u) &= \int_0^u f(y) \, dy = \int_0^u y^2 \, dy = \frac{1}{3}u^3, \\
 G(u) &= \int_0^u g(y) \, dy = \int_0^u y^2 \, dy = \frac{1}{3}u^3,
 \end{aligned}$$

$$\begin{aligned}
H(u) &= 2 \int_0^u yh'(y) \, dy = 2 \int_0^u y \left(1 + \frac{1}{1+y}\right) \, dy = u^2 + 2u - 2 \ln(1+u), \\
\Phi(t) &= \int_{\Omega} H(u) \, dx = \int_{\Omega} \left(u^2 + 2u - 2 \ln(1+u)\right) \, dx,
\end{aligned}$$

and

$$\begin{aligned}
\Psi(t) &= 2 \int_{\partial\Omega} G(u) \, ds - \int_{\Omega} \sum_{i=1}^3 a^{ii}(x) (u_{x_i})^2 \, dx - 2k(t) \int_{\Omega} F(u) \, dx \\
&= \frac{2}{3} \int_{\partial\Omega} u^3 \, ds - \int_{\Omega} \sum_{i=1}^3 \left(\frac{31}{16} + |x|^2\right) (u_{x_i})^2 \, dx - \frac{2}{3} (1 + e^{-t}) \int_{\Omega} u^3 \, dx.
\end{aligned}$$

By choosing $\alpha = \beta = 1/2$, it is easy to check that (3.3) and (3.4) hold. We then calculate

$$\begin{aligned}
\Phi(0) &= \int_{\Omega} (u_0^2 + 2u_0 - 2 \ln(1+u_0)) \, dx \\
&= \int_{\Omega} \left(\left(\frac{15}{16} + |x|^2\right)^2 + 2\left(\frac{15}{16} + |x|^2\right) - 2 \ln\left(\frac{31}{16} + |x|^2\right) \right) \, dx = 0.1008
\end{aligned}$$

and

$$\begin{aligned}
\Psi(0) &= \frac{2}{3} \int_{\partial\Omega} u_0^3 \, ds - \int_{\Omega} \sum_{i=1}^3 \left(\frac{31}{16} + |x|^2\right) (u_{0x_i})^2 \, dx - \frac{4}{3} \int_{\Omega} u_0^3 \, dx \\
&= \frac{2}{3} \int_{\partial\Omega} \left(\frac{15}{16} + |x|^2\right)^3 \, ds - 4 \int_{\Omega} \left(\frac{31}{16} + |x|^2\right) |x|^2 \, dx - \frac{4}{3} \int_{\Omega} \left(\frac{15}{16} + |x|^2\right)^3 \, dx \\
&= 0.4232.
\end{aligned}$$

It follows from Theorem 3.1 that $u(x, t)$ blows up at t^* in measure $\Phi(t)$, and

$$t^* \leq \frac{\Phi(0)}{2\beta(1+\beta)\Psi(0)} = 0.1588. \quad (5.1)$$

To use Theorem 4.1 in obtaining a lower bound for the blow-up time t^* , we select $\gamma_1 = \gamma_2 = 1$, $p = q = 2$, $\zeta = 1$, $m = 1$, $\eta = 0$, $\theta = 31/16$, and $r = 3$. Here $|\Omega| = \pi/48$, $\rho_0 = 1/4$, and $|d| = 1/4$. It is easy to verify that (4.1)–(4.3) and (4.5) hold. The best Sobolev's constant $c = 3^{-1/2}4^{1/3}\pi^{-2/3}$ is given in [17]. Inserting the above parameters into (4.7)–(4.10), we obtain $c_1 = 270.2244$, $c_2 = 0.0525$, $c_3 = 0.0239$, $J_1 = 3$, and $J_2 = 3.8333 \times 10^4$. By (4.4), we have

$$\begin{aligned}
B(u) &= 2r \int_0^u h'(y)y^{2r-1} \, dy = 6 \int_0^u y^5 \left(1 + \frac{1}{1+y}\right) \, dy \\
&= u^6 + \frac{6}{5}u^5 - \frac{3}{2}u^4 + 2u^3 - 3u^2 + 6u - 6 \ln(1+u), \\
A(t) &= k^{\frac{2r}{p-1}}(t) \int_{\Omega} B(u) \, dx \\
&= (1 + e^{-t})^6 \int_{\Omega} \left(u^6 + \frac{6}{5}u^5 - \frac{3}{2}u^4 + 2u^3 - 3u^2 + 6u - 6 \ln(1+u)\right) \, dx,
\end{aligned}$$

$$\begin{aligned}
A(0) &= 64 \int_{\Omega} \left(u_0^6 + \frac{6}{5} u_0^5 - \frac{3}{2} u_0^4 + 2u_0^3 - 3u_0^2 + 6u_0 - 6 \ln(1 + u_0) \right) dx \\
&= 64 \int_{\Omega} \left(\left(\frac{15}{16} + |x|^2 \right)^6 + \frac{6}{5} \left(\frac{15}{16} + |x|^2 \right)^5 - \frac{3}{2} \left(\frac{15}{16} + |x|^2 \right)^4 \right. \\
&\quad \left. + 2 \left(\frac{15}{16} + |x|^2 \right)^3 - 3 \left(\frac{15}{16} + |x|^2 \right)^2 + 6 \left(\frac{15}{16} + |x|^2 \right) \right. \\
&\quad \left. - 6 \ln \left(\frac{31}{16} + |x|^2 \right) \right) dx = 5.5901.
\end{aligned}$$

Since $u(x, t)$ blows up in measure $\Phi(t)$ at finite time t^* , $u(x, t)$ must blow up in measure $A(t)$ at t^* . From Theorem 4.1, we obtain a lower bound

$$\begin{aligned}
t^* &\geq \int_{A(0)}^{\infty} \frac{d\tau}{J_1 \tau + J_2 \tau^{\frac{2r-(q-1)(n-2)}{2r-n(q-1)}}} \\
&= \int_{5.5901}^{\infty} \frac{d\tau}{3\tau + 3.8333 \times 10^4 \tau^{5/3}} = 1.2423 \times 10^{-5}.
\end{aligned}$$

Combining this with (5.1), we obtain

$$1.2423 \times 10^{-5} \leq t^* \leq 0.1588.$$

Example 5.2. Let $u(x, t)$ be a nonnegative classical solution of

$$\begin{aligned}
(u + \ln(1 + u))_t &= \sum_{i=1}^3 \left((1 + |x|^2) u_{x_i} \right)_{x_i} - e^t u^4 \quad \text{in } \Omega \times (0, t^*), \\
\sum_{i=1}^3 (1 + |x|^2) u_{x_i} \nu_i &= u^2 \quad \text{on } \partial\Omega \times (0, t^*), \\
u(x, 0) &= 1 + |x|^2 \quad \text{in } \bar{\Omega},
\end{aligned}$$

where $\Omega = \{x = (x_1, x_2, x_3) : |x|^2 = \sum_{i=1}^3 x_i^2 < 1\}$, a ball of \mathbb{R}^3 . Now

$$\begin{aligned}
(a^{ij}(x))_{3 \times 3} &= \begin{pmatrix} 1 + |x|^2 & 0 & 0 \\ 0 & 1 + |x|^2 & 0 \\ 0 & 0 & 1 + |x|^2 \end{pmatrix}, \quad h(u) = u + \ln(1 + u), \\
k(t) &= e^t, \quad f(u) = u^4, \quad g(u) = u^2, \quad u_0(x) = 1 + |x|^2, \quad n = 3.
\end{aligned}$$

Here we choose $\gamma_1 = \gamma_2 = 1$, $p = 4$, $q = 2$, $\zeta_0 = 2$, $\theta = 1$, and $m = 1$. It is easy to see that (2.2)–(2.5) are valid. Consequently, by Theorem 2.1, $u(x, t)$ exists for all time $t > 0$ in measure $\Phi(t)$ with

$$\Phi(t) = \int_{\Omega} H(u) dx = \int_{\Omega} (u^2 + 2u - 2 \ln(1 + u)) dx.$$

Acknowledgements. This work was supported by the National Natural Science Foundation of China (No. 61473180).

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