Electronic Journal of Differential Equations, Vol. 2004(2004), No. 54, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

SOLVABILITY AND THE NUMBER OF SOLUTIONS OF HAMMERSTEIN EQUATIONS

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Abstract. We study the solvability and the number of solutions to Hammerstein operator equations in Banach spaces using a projection like method and degree theory for corresponsing vector fields. The linear part is assumed to be either selfadjoint or non-seladjoint. We present also applications to Hammerstein integral equations.

1. Introduction

In this paper, we study the solvability and the number of solutions to the Hammerstein operator equation

$$x - KFx = f \tag{1.1}$$

where K is linear and F is a nonlinear map. We consider (1.1) in a general setting between two Banach spaces. To that end, we use two approaches. One is based on using the degree theory for ϕ-condensing maps or applying the Brouwer degree theory directly to the finite dimensional approximations of the map $I - KF$ in conjunction with the (pseudo) A-proper mapping approach. The other one is based on splitting first the map K as a product of two suitable maps and then using again these degree theories. The linear part K is assumed to be either selfadjoint or non-selfadjoint. In the second case, we assume that K is either positive in the sense of Krasnoselskii, potentially positive, P-positive (i.e., angle- bounded) or that it is P-quasi-positive, which means that its selfadjoint part has at most a finite number of negative eigenvalues of finite multiplicity. The nonlinear part is assumed to be such that either $I - KF$ is A-proper or KF is ϕ-condensing, or that the corresponding map in an equivalent reformulation of (1.1) is a k-ball contractive perturbation of a strongly monotone map and is therefore A-proper.

We begin with proving some continuation results on the number of solutions of general nonlinear operator equations. Then we use them to establish various results on the number of solutions of (1.1) assuming different conditions on the nonlinearity F that imply a priori estimates on the solution set. In particular, depending on the structure of the linear part K, we assume that either F has a linear growth, and/or F satisfies a side estimate of the form $(Fx, x) \leq a(x)$ for a suitable functional a. Unlike earlier studies, we also study (1.1) with nonlinearities that are the sum

2000 *Mathematics Subject Classification.* 47H30, 35L70, 35L75, 35J40.
Key words and phrases. Approximation solvability, Hammerstein equations, A-proper maps.

Submitted December 12, 2003. Published April 8, 2004.

of a strongly monotone and k-ball condensing maps. The last part of the paper is devoted to applications of these abstract results to Hammerstein integral equations. This work is a continuation of our study of these equations in [15, 19]. There is an extensive literature on Hammerstein equations and we refer to [5, 6, 23]. In particular, for the unique (approximation) solvability of these equations we refer to [23, 1, 15, 19].

2. Some preliminaries on A-proper maps

Let $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of Banach spaces X and Y respectively such that $\dim X_n = \dim Y_n$ for each n and $\mathrm{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$. Let $P_n : X \to X_n$ and $Q_n : Y \to Y_n$ be linear projections onto X_n and Y_n respectively such that $P_n x \to x$ for each $x \in X$ and $\delta = \max \|Q_n\| < \infty$. Then $\Gamma = \{X_n, P_n; Y_n, Q_n\}$ is a projection scheme for (X, Y).

Definition 2.1. A map $T : D \subset X \to Y$ is said to be *approximation-proper* (*A-proper* for short) with respect to Γ if (i) $Q_n T : D \cap X_n \to Y_n$ is continuous for each n and (ii) whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $\|Q_{n_k} T x_{n_k} - Q_{n_k} f\| \to 0$ for some $f \in Y$, then a subsequence $x_{n_{k(i)}} \to x$ and $Tx = f$. T is said to be *pseudo A-proper* with respect to Γ if in (ii) above we do not require that a subsequence of $\{x_{n_k}\}$ converges to x for which $Tx = f$. If (ii) holds for a given f, we say that T is (pseudo) A-proper at f.

For the developments of the (pseudo) A-proper mapping theory and applications to differential equations, we refer to [11, 18] and [21]. To demonstrate the generality and the unifying nature of the (pseudo) A-proper mapping theory, we state now a number of examples of A-proper and pseudo A-proper maps.

To look at ϕ-condensing maps, we recall that the *set measure of noncompactness* of a bounded set $D \subset X$ is defined as $\gamma(D) = \inf\{d > 0 : D$ has a finite covering by sets of diameter less than $d\}$. The *ball-measure of noncompactness* of D is defined as $\chi(D) = \inf\{r > 0 | D \subset \cup_{i=1}^{n} B(x_i, r),\ x \in X,\ n \in N\}$. Let ϕ denote either the set or the ball-measure of noncompactness. Then a map $N : D \subset X \to X$ is said to be $k - \phi$ *contractive* (*ϕ-condensing*) if $\phi(N(Q)) \le k\phi(Q)$ (respectively $\phi(N(Q)) < \phi(Q)$) whenever $Q \subset D$ (with $\phi(Q) \neq 0$).

Recall that $N : X \to Y$ is K-monotone for some $K : X \to Y^*$ if $(Nx - Ny, K(x - y)) \ge 0$ for all $x,\ y \in X$. It is said to be generalized pseudo-K-monotone (of type (KM)) if whenever $x_n \rightharpoonup x$ and $\limsup (Nx_n, K(x_n - x)) \le 0$ then $(Nx_n, K(x_n - x)) \to 0$ and $Nx_n \rightharpoonup Nx$ (then $Nx_n \rightharpoonup Nx$). Recall that N is said to be of type (KS_+) if $x_n \rightharpoonup x$ and $\limsup (Nx_n, K(x_n - x)) \le 0$ imply that $x_n \to x$. If $x_n \rightharpoonup x$ implies that $\limsup (Nx_n, K(x_n - x)) \ge 0$, N is said to be of type (KP). If $Y = X^*$ and K is the identity map, then these maps are called monotone, generalized pseudo monotone, of type (M) and (S_+) respectively. If $Y = X$ and $K = J$ the duality map, then J-monotone maps are called accretive. It is known that bounded monotone maps are of type (M). We say that N is demicontinuous if $x_n \to x$ in X implies that $Nx_n \rightharpoonup Nx$. It is well known that $I - N$ is A-proper if N is ball-condensing and that K-monotone like maps are pseudo A-proper under some conditions on N and K. Moreover, their perturbations by Fredholm or hyperbolic like maps are A-proper or pseudo A-proper (see [11, 12, 13, 16, 17, 18]).

The following result states that ball-condensing perturbations of stable A-proper maps are also A-proper.

Theorem 2.1 ([7]). *Let $D \subset X$ be closed, $T : X \to Y$ be continuous and A-proper with respect to a projectional scheme Γ and a-stable, i.e. for some $c > 0$ and n_0*

$$\|Q_nTx - Q_nTy\| \geq c\|x - y\| \quad \text{for } x, y \in X_n \text{ and } n \geq n_0$$

and $F : D \to Y$ be continuous. Then $T + F : D \to Y$ is A-proper with respect to Γ if F is k-ball contractive with $k\delta < c$, or it is ball-condensing if $\delta = c = 1$.

Remark 2.2. The A-properness of T in Theorem 2.1 is equivalent to T being surjective. In particular, as T we can take a c-strongly K- monotone map for a suitable $K : X \to Y^*$, i.e., $(Tx - Ty, K(x - y)) \geq c\|x - y\|^2$ for all $x, y \in X$. In particular, since c-strongly accretive maps are surjective, we have the following important special case [7].

Corollary 2.3. *Let X be a π_1 space, $D \subset X$ be closed, $T : X \to X$ be continuous and c-strongly accretive and $F : D \to X$ be continuous and either k-ball contractive with $k < c$, or it is ball-condensing if $c = 1$. Then $T + F : D \to X$ is A-proper with respect to Γ.*

3. On the number of solutions of Hammerstein equations

In this section, we shall study the solvability and the number of solutions of (1.1) imposing various types of conditions on K and F. Our results will be based on Theorems 3.1–3.3 below. We shall study (1.1) directly as well as using splitting techniques for the map K.

We say that a map $T : X \to Y$ satisfies condition (+) if whenever $Tx_n \to f$ in Y then $\{x_n\}$ is bounded in X. T is locally injective at $x_0 \in X$ if there is a neighborhood $U(x_0)$ of x_0 such that T is injective on $U(x_0)$. T is locally injective on X if it is locally injective at each point $x_0 \in X$. A continuous map $T : X \to Y$ is said to be locally invertible at $x_0 \in X$ if there are a neighborhood $U(x_0)$ and a neighborhood $U(T(x_0))$ of $T(x_0)$ such that T is a homeomorphism of $U(x_0)$ onto $U(T(x_0))$. It is locally invertible on X if it is locally invertible at each point $x_0 \in X$.

Let Σ be the set of all points $x \in X$ where T is not locally invertible and let $\operatorname{card} T^{-1}(\{f\})$ be the cardinal number of the set $T^{-1}(\{f\})$.

We need the following basic theorem on the number of solutions of nonlinear equations for A-proper maps (see [17]).

Theorem 3.1. *Let $T : X \to Y$ be a continuous A-proper map that satisfies condition (+). Then*

- (a) *The set $T^{-1}(\{f\})$ is compact (possibly empty) for each $f \in Y$.*
- (b) *The range $R(T)$ of T is closed and connected.*
- (c) *Σ and $T(\Sigma)$ are closed subsets of X and Y, respectively, and $T(X \setminus \Sigma)$ is open in Y.*
- (d) *$\operatorname{card} T^{-1}(\{f\})$ is constant and finite (it may be 0) on each connected component of the open set $Y \setminus T(\Sigma)$.*

We need the following homotopy version of Theorem 3.1.

Theorem 3.2. *Let $H : [0, 1] \times X \to Y$ be an A-proper homotopy with respect to Γ and satisfy condition (+), i.e. if $H(t_n, x_n) \to f$ then $\{x_n\}$ is bounded in X. Let, for each $f \in Y$, the numbers $r_f > 0$ and $n_f \geq 1$ be such that*

$$\deg(P_nH_0, B(0, r_f) \cap X_n, 0) \neq 0 \quad \text{for all } n \geq n_f.$$

Then the equation $H(1,x) = f$ is approximation solvable with respect to Γ for each $f \in Y$. Moreover, if $\Sigma = \{x \in X : H_1$ is not invertible at $x\}$ and H_1 is continuous, then $H_1^{-1}(\{f\})$ is compact for each $f \in Y$ and the cardinal number card$(H_1^{-1}(\{f\}))$ is constant, finite and positive on each connected component of the set $Y \setminus H_1(\Sigma)$.

Proof. The condition (+) implies that for each $f \in Y$ there is an $r > R$ and $\gamma > 0$ such that

$$\|H(t,x) - tf\| \geq \gamma \quad \text{for all } t \in [0,1],\ x \in \partial B(0,r).$$

Indeed, if this were not the case, there would exist $t_n \in [0,1]$ and $x_n \in X$ such that $t_n \to t$ and $\|x_n\| \to \infty$ and $H(t_n, x_n) - t_n f \to 0$ as $n \to \infty$. Hence, $H(t_n, x_n) \to tf$ and $\{x_n\}$ is unbounded, in contradiction to condition (+). Since H_t is an A-proper homotopy, this implies that there is an $n_0 \geq 1$ such that

$$P_n H(t,x) \neq t P_n f \quad \text{for all } t \in [0,1],\ x \in \partial B(0,r) \cap X_n,\ n \geq n_0.$$

By the Brouwer degree properties and the A-properness of H_1, there is an $x \in X$ such that $H(1,x) = f$. The other conclusions follow from Theorem 3.1. $\square$

Next, we have the following homotopy theorem for ϕ-condensing maps.

Theorem 3.3. *Let $F : [0,1] \times X \to X$ be a ϕ-condensing homotopy and $H = I - F$ satisfy condition (+). Let, for each $f \in X$, there be an $r_f > 0$ such that*

$$\deg(H_0, B(0, r_f), 0) \neq 0.$$

Then the equation $H(1,x) = f$ is solvable for each $f \in X$. Moreover, if $\Sigma = \{x \in X : H_1$ is not invertible at $x\}$ and H_1 is continuous, then $H_1^{-1}(\{f\})$ is compact for each $f \in X$ and the cardinal number card$(H_1^{-1}(\{f\}))$ is constant, finite and positive on each connected component of the set $X \setminus H_1(\Sigma)$.

Proof. As before, condition (+) implies that for each $f \in X$ there is an $r > R$ and $\gamma > 0$ such that

$$\|H(t,x) - tf\| \geq \gamma \quad \text{for all } t \in [0,1],\ x \in \partial B(0,r).$$

By the ϕ -condensing degree properties [20], there is an $x \in X$ such that $H(1,x) = f$. The other conclusions follow from [22, Theorem 3.2] since its coercivity condition can be replaced by condition (+). $\square$

The existence part of the following result can be found in [15].

Theorem 3.4. *Let X and Y be Banach spaces, $K : Y \to X$ be linear and continuous and $F : X \to Y$ be nonlinear and such that there are some constants a and b such that $a\|K\| < 1$ and*

$$\|Fx\| \leq a\|x\| + b \quad \text{for all } \|x\| \geq R.$$

a) Let $H_t = I - tKF : X \to X$ be A-proper with respect to a projection scheme $\Gamma = \{X_n, P_n\}$ for X for each $t \in [0,1]$, or H_1 is A-proper with respect to Γ and $\delta a\|K\| < 1$, where $\delta = \max\|P_n\|$. Then (1.1) is approximation solvable for each $f \in X$. Moreover, if $\Sigma = \{x \in X : I - KF$ is not invertible at $x\}$ and $I - KF$ is continuous, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in X$, and the cardinal number card$(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I - KF)(\Sigma)$.
b) If $I - KF : X \to X$ is pseudo A-proper with respect to Γ and $\delta a\|K\| < 1$, then (1.1) is solvable for each $f \in X$.

Proof. (a) We shall show that the homotopy $H_t = I - tKF$ satisfies condition (+). Indeed, let $H(t_n, x_n) = x_n - t_n KFx_n \to f$ in Y for some $t_n \in [0,1]$ and $x_n \in X$. It follows that for some $M > 0$

$$\|x_n\| \le \|H(t_n, x_n)\| + \|K\| \, \|Fx\| \le M + \|K\|(a\|x_n\| + b).$$

Hence $\{x_n\}$ is bounded in X since $a\|K\| < 1$. Moreover, for each $r > 0$ and each $n \ge 1$, $\deg(P_n H_0, B(0,r) \cap X_n, 0) \ne 0$. Hence, the conclusions follow from Theorem 3.2. If only H_1 is A-proper, then it satisfies condition (+) as above and $x - KFx = f$ is approximation solvable for each $f \in X$ (see part b)). Hence, Theorem 3.1 applies.
(b) If $I - KF$ is pseudo A-proper, then condition

$$P_n H(t,x) \ne tP_n f \quad \text{for all } t \in [0,1],\ x \in \partial B(0,r) \cap X_n,\ n \ge n_0.$$

holds since $\delta a\|K\| < 1$ and the solvability follows from the pseudo A-propernes of $I - KF$. $\square$

Since a ball condensing perturbation of the identity map is an A-proper map, we have the following special case.

Corollary 3.5. *Let $K : Y \to X$ be linear and continuous and $F : X \to Y$ be nonlinear and such that KF is a continuous ϕ-condensing map and there are some constants a and b such that $a\|K\| < 1$, and*

$$\|Fx\| \le a\|x\| + b \quad \textit{for all } \|x\| \ge R.$$

Then (1.1) *is approximation solvable for each $f \in X$ with respect to a projection scheme $\Gamma = \{X_n, P_n\}$ for X with $\delta = max\|P_n\| = 1$ if $\phi = \chi$. It is solvable if $\phi = \gamma$. Moreover, if $\Sigma = \{x \in X : I - KF$ is not locally invertible at $x\}$, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in X$, and the cardinal number card$(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I - KF)(\Sigma)$.*

Next, we shall discuss other sets of conditions on K and F that imply the A-properness of an operator in an equivalent formulation of our equation. Recall that a map K acting in a Hilbert space H is called positive in the sense of Krasnoselski if there exists a number $\mu > 0$ for which

$$(Kx, Kx) \le \mu(Kx, x) \quad x \in H.$$

The infimum of all such numbers μ is called the positivity constant of K and is denoted by $\mu(K)$. The simplest example of a positive map is provided by any bounded selfadjoint positive definite map K on H. Then $\mu(K) = \|K\|$ for such maps. A compact normal map K in a Hilbert space is positive on H if and only if (cf. [4]) the number

$$[\inf_{\lambda \in \sigma(K), \lambda \ne 0} Re(\lambda^{-1})]^{-1}$$

is well defined and positive. In that case, it is equal to $\mu(K)$.

Let X be a reflexive embeddable Banach space, that is, there is a Hilbert space H such that $X \subset H \subset X^*$ so that $< y, x >= (y, x)$ for each $y \in H, x \in X$, where $<,>$ is the duality pairing of X and X^*. Let $K : X^* \to X$ be a positive semidefinite bounded selfadjoint map in the sense that $< Kx, y >=< x, Ky >$ for all $x, y \in X^*$. Then the positive semidefinite square root $K_H^{1/2}$ of the restriction K_H of K to H

can be extended to a bounded linear map $T : X^* \to H$ such that $K = T^*T$, where the adjoint map $T^* = K_H^{1/2}$ of T is a bounded map from H to X (see [23]).

We shall look at the following equivalent formulation of (1.1)

$$y - TFCy = h, \quad h \in H. \tag{3.1}$$

We need the following lemma (cf. [23]).

Lemma 3.6. *Equations* (1.1) *and* (3.1) *are equivalent with f restricted to $C(H)$; each solution y of* (3.1) *determines a solution $x = Cy$ of* (1.1) *and each solution x of* (1.1) *with $f \in C(H)$ determines a solution $y = TFx + h$ of* (3.1) *with $f = Ch$ and $x = Cy$. Moreover, the map $C : S(h) = (I - TFC)^{-1}(\{h\}) \to S = (I - KF)^{-1}(\{Ch\})$ is bijective.*

Proof. Let y_1 and y_2 be distinct solutions of (3.1). Applying C to $y_i - TFCy_i = h$ and using the fact that $K = CT$, we get that $x_1 = Cy_1$ and $x_2 = Cy_2$ are solutions of (1.1). They are distinct since

$$0 < \|y_1 - y_2\|^2 = (TFCy_1 - TFCy_2, y_1 - y_2) =$$
$$(FCy_1 - FCy_2, C(y_1 - y_2)) = (Fx_1 - Fx_2, x_1 - x_2).$$

Conversely, let $f \in C(H)$ and x_1 and x_2 be distinct solutions of (1.1). Let $f = Ch$ for some $h \in H$. Set $y_i = TFx_i + h$. Then $Cy_i = CTFx_i + h = KFx_i + f$ and so $x_i = Cy_i$. Hence, $y_i = TFCy_i + h$, i.e., y_i are solutions of (3.1). They are distinct since $y_1 = y_2$ implies that $x_1 = Cy_1 = Cy_2 = x_2$. These arguments show that $C : S(h) \to S$ is a bijection. □

Corollary 3.7. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a positive semidefinite bounded selfadjoint map, and $C = K_H^{1/2}$, where K_H is the restriction of K to H, $\mu(K) = \|C\|^2$ and $T : X^* \to H$ be a bounded linear extension of $K_H^{1/2}$. Let $F = F_1 + F_2 : X \to X^*$ be a nonlinear map, a and b be constants and c be the smallest number such that*

(i) *$(F_1x - F_1y, x - y) \le c\|x - y\|^2$ for all $x, y \in X$, and either*
(ii) *$a\|K\| < 1$ and $\|Fx\| \le a\|x\| + b$ for all $\|x\| \ge R$, or*
(iii) *$(a + c)\mu(K) < 1$ and $\|F_2x\| \le a\|x\| + b$ for all $\|x\| \ge R$,*

and TF_2C is a continuous k-ball contraction with $k < 1 - c\mu(K)$. Then (1.1) *is approximation solvable in X for each $f \in C(H) \subset X$ with respect to a projection scheme $\Gamma = \{X_n, P_n\}$ for X, δ =max $\|P_n\| = 1$. Moreover, if $\Sigma_H = \{h \in H :$ $I - TFC$ is not locally invertible at $h\}$, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in C(H)$, and the cardinal number card$(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (I - TFC)(\Sigma_H)$ intersected by $C(H)$.*

Proof. Equation (1.1) is equivalent to (3.1). Hence, we shall consider this more suitable formulation. We claim that the map $I - TF_1C : H \to H$ is $1 - c\mu(K)$-strongly monotone. Indeed, for $x, y \in H$, we have

$$\begin{aligned}(x - TF_1Cx - y + TF_1Cy, x - y) &= \|x - y\|^2 - (TF_1Cx - TF_1Cy, x - y)\\ &= \|x - y\|^2 - (F_1Cx - F_1Cy, Cx - Cy)\\ &\ge (1 - c\mu(K))\|x - y\|^2.\end{aligned}$$

Since TF_2C is k-ball contractive with $k < 1 - c\mu(K)$, we see that $I - tTFC$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$ for H by Corollary 2.3. Moreover, $I - tTFC$

satisfies condition (+). Indeed, let $H(t_n, x_n) = x_n - t_n TFCx_n \to g$. If (ii) holds, then

$$\|x_n\| \le \|H(t_n, x_n)\| + \|T\|(a\|Cx_n\| + b) \le M + a\|K\|\|x_n\|$$

for some constant M since $\|T\| = \|C\| = \|K\|^{1/2}$. It follows that $\{x_n\}$ is bounded. Next, let (iii) hold. Then

$$\begin{aligned}
&(H(t_n, x_n), x_n)\\
&= (x_n - t_n TFCx_n, x_n)\\
&= (x_n - t_n TF_1 Cx_n + t_n TF_1 0, x_n) - t_n(TF_1 0, x_n) - t_n(TF_2 Cx_n, x_n)\\
&\ge (1 - c\mu(K))\|x_n\|^2 - \|TF_1 0\|\,\|x_n\| - a\|Cx_n\|^2 - b\|Cx_n\|\\
&\ge (1 - (a + c)\mu(K))\|x_n\|^2 - (\|TF_1 0\| + b\|C\|)\|x_n\|.
\end{aligned}$$

It follows that $\{x_n\}$ is bounded, for otherwise dividing by $\|x_n\|^2$ and passing to the limit we get that $(a + c)\mu(K) \ge 1$, a contradiction. Hence, condition (+) holds in either case.

By Theorem 3.2, we have that the equation $y - TFCy = h$ is solvable for each $h \in H$, $S(h) = (I - TFC)^{-1}(\{h\}) \ne \emptyset$ and compact, and $\mathrm{card}S(h)$ is constant, positive and finite on each connected component of the open set $H\setminus(I-TFC)(\Sigma_H)$, where $\Sigma_H = \{h \in H : I - TFC$ is not locally invertible at $h\}$.

Next, applying C to $y - TFCy = h$ and using the fact that $K = CT$, we get that $x - KFx = f$ with $x = Cy \in X$. By Lemma 3.6, we get that $\mathrm{card}S = (I - KF)^{-1}(\{Ch\}) = \mathrm{card}S(h)$. Hence, $\mathrm{card}(I - KF)^{-1}(\{f\})$ is constant, finite and positive on each connected component of $H \setminus (I - TFC)(\Sigma_H)$ intersected by $C(H)$. ☐

Next, let us look at the case when K is not selfadjoint. We begin by describing the setting of the problem. Let X be an embeddable Banach space, $X \subset H \subset X^*$. Let $K : X^* \to X$ be a linear map and K_H be the restriction of K to H such that $K_H : H \to H$. Let $A = (K + K^*)/2$ denote the selfadjoint part of K and $B = (K - K^*)/2$ be the skew-adjoint part of K. Assume that A is positive definite. Under our assumptions on K, both A and B map X^* into X. We know that A can be represented in the form $A = CC^*$, where $C = A^{1/2}$ is the square root of A, $C : H \to X$, and the adjoint map $C^* : X^* \to H$.

As in [1] and [19], we say that K is P-positive if $C^{-1}K(C^*)^{-1}$ exists and is bounded in H. It is S-positive if $K(C^*)^{-1}$ exists and is bounded in H. Clearly, the P-positivity implies the S-positivity but not conversely. It is easy to see that K is P-positive if and only if $C^{-1}B(C^*)^{-1}$ is bounded in H, and is S-positive if and only if $B(C^*)^{-1}$ is bounded in H. Moreover, K is P-positive if and only if K is angle-bounded, i.e.,

$$|(Kx, y) - (y, Kx)| \le a(Kx, x)^{1/2}(Ky, y)^{1/2}, \quad x, y \in H.$$

Denote by M and N the closure of the maps $C^{-1}K(C^*)^{-1}$ and $K(C^*)^{-1}$, respectively, in H. Note that M and N are defined on the closure (in H) of the range of $C = A^{1/2}$ and suppose that their domains coincide with H. We require that the following decompositions hold

$$K = CMC^*, \quad K = NC^*.$$

Note that K, M and N are related as: $N = CM, N^* = M^*C^*$ and we have $(Mx, x) = \|x\|^2$ for all $x \in H$. Hence, both M and M^* have trivial nullspaces.

Denote by $\mu(K) = \|N\|^2$, which is the positivity constant of K in the sense of Krasnoselski.

Let $F : X \to X^*$ be a nonlinear map and consider the Hammerstein equation

$$x - KFx = f \tag{3.2}$$

For $f \in N(H)$, let $h \in H$ be a solution of

$$M^*h - N^*FNh = M^*k \tag{3.3}$$

where $f = Nk$ for some $k \in H$. Then $M^*(h - C^*FNh - k) = 0$ since $N = CM$ and $N^* = M^*C^*$. Hence, $h = C^*FNh + k$ since M^* is injective and therefore

$$Nh = NC^*FNh + Nk = KFNh + f$$

since $K = NC^*$. Thus $x = Nh$ is a solution of (3.2). So the solvability of (3.2) is reduced to the solvability of (3.3). Actually these two equations are equivalent.

Lemma 3.8. *Equations* (3.2) *and* (3.3) *are equivalent with f restricted to $N(H)$; each solution h of* (3.3) *determines a solution $x = Nh$ of* (3.2) *and each solution x of* (3.2) *with $f \in N(H)$ determines a solution $h = C^*Fx + k$ of* (3.3) *with $f = Nk$ and $x = Nh$. Moreover, the map $N : S(M^*k) = (M^* - N^*FN)^{-1}(\{M^*k\}) \to S = (I - KF)^{-1}(\{Nk\})$ is bijective.*

Proof. Let h_1 and h_2 be distinct solutions of (3.3). We have seen above that $x_1 = Nh_1$ and $x_2 = Nh_2$ are solutions of (3.2). They are distinct since

$$\begin{aligned} 0 < \|h_1 - h_2\|^2 &= (M(h_1 - h_2), h_1 - h_2) \\ &= (N^*FNh_1 - N^*FNh_2, h_1 - h_2) \\ &= (FNh_1 - FNh_2, N(h_1 - h_2)) \\ &= (Fx_1 - Fx_2, x_1 - x_2). \end{aligned}$$

Conversely, let $f \in N(H)$ and x_1 and x_2 be distinct solutions of (3.2). Let $f = Nk$ for some $k \in H$. Set $h_i = C^*Fx_i + k$. Then $Nh_i = NC^*Fx_i + Nk = KFx_i + f$ and so $x_i = Nh_i$. Hence, $M^*h_i = M^*C^*FNh_i + M^*k = N^*FNh_i + M^*k$, i.e., h_i are solutions of (3.3). They are distinct since $h_1 = h_2$ implies that $x_1 = Nh_1 = Nh_2 = x_2$. These arguments show that $N : S(M^*k) \to S$ is bijective. □

Corollary 3.9. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$) and $K : X^* \to X$ be a linear continuous P-positive map. Let $F = F_1 + F_2 : X \to X^*$ be a nonlinear map, N^*F_2N be continuous and k-ball contractive with $k < 1 - c\mu(K)$ and there are positive constants a and b, $R > 0$ and c be the smallest number such that*

- (i) $(F_1x - F_1y, x - y) \le c\|x - y\|^2$ *for all* $x, y \in X$, *and either*
- (ii) $a\|K\| < 1$ *and* $\|Fx\| \le a\|x\| + b$ *for all* $\|x\| \ge R$, *or*
- (iii) $(a + c)\mu(K) < 1$ *and* $\|F_2x\| \le a\|x\| + b$ *for all* $\|x\| \ge R$.

Then (1.1) *is solvable in X for each $f \in N(H) \subset X$. Moreover, if $\Sigma_H = \{h \in H : M^* - N^*FN$ is not locally invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in N(H)$, and the cardinal number card$(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.*

Proof. Define the homotopy $H(t,x) = M^*x - tN^*FNx$ on $[0,1] \times H$. It suffices to show that the map $H_t = M^* - tN^*FN : H \to H$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$ for each $t \in [0,1]$ and satisfies condition (+). Set $H_{1t} = M^* - tN^*F_1N : H \to H$. Then, for each $x, y \in H$ we have that

$$\begin{aligned}(H_1(t,x) - H_1(t,y), x-y) &= \|x-y)\|^2 - t(N^*(F_1Nx - F_1Ny), x-y)\\ &= \|x-y\|^2 - t(F_1Nx - F_1Ny, Nx - Ny)\\ &\geq (1 - c\mu(K))\|x-y\|^2.\end{aligned}$$

Since N^*F_2N is k-ball contraction, H_t is A-proper with respect to Γ by Corollary 2.3.

Next, let $f \in N(H) \subset X$, $f = Nk$, be fixed. We claim that $H(t,x) - tM^*h$ satisfies condition (+). If not, then there would exist $x_n \in H$, $t_n \in [0,1]$ such that $\|x_n\| \to \infty$ and

$$y_n = H(t_n, x_n) - t_nM^*k \to g$$

as $n \to \infty$. Let (ii) hold. Then

$$M^*x_n = y_n + t_nN^*FNx_n - t_nM^*k$$

and

$$\begin{aligned}\|x_n\|^2 &= (M^*x_n, x_n) = (y_n, x_n) + t_n(FNx_n, Nx_n) - t_n(M^*k, x_n)\\ &\leq (\|y_n\| + \|M^*k\| + b\|N\|)\|x_n\| + a\mu(K)\ \|x_n\|^2.\end{aligned}$$

Dividing by $\|x_n\|^2$ and passing to the limit, we get that $1 \leq a\mu(K)$, a cotradiction. Hence, condition (+) holds.

Next, let (iii) hold. Then, as above,

$$\begin{aligned}&(H(t_n, x_n), x_n)\\ &= (M^*x_n - t_nN^*FNx_n, x_n)\\ &= (M^*x_n - t_nN^*F_1Nx_n + t_nN^*F_10, x_n) - t_n(N^*F_10, x_n) - t_n(N^*F_2Nx_n, x_n)\\ &\geq (1 - c\mu(K))\|x_n\|^2 - \|N^*F_10\|\ \|x_n\| - a\|Nx_n\|^2 - b\|Nx_n\|\\ &\geq (1 - (a+c)\mu(K))\|x_n\|^2 - (\|N^*F_10\| + b\|N\|)\|x_n\|.\end{aligned}$$

It follows that $\{x_n\}$ is bounded, for otherwise dividing by $\|x_n\|^2$ and passing to the limit we get that $(a+c)\mu(K) \geq 1$, a contradiction. Hence, condition (+) holds in either case.

This and the A-properness of $M^* - N^*FN$ imply that $M^*h - N^*FNh = M^*k$ for some $h \in H$ by Theorem 3.2. As before, we get that $Nh = NC^*FNh + Nk = KFNh + f$ since $K = NC^*$. Thus, $x - KFx = f$ with $x = Nh \in X$. Next, we have that $Y = N(H)$ is a Banach subspace of X and $I - KF : Y \to Y$, since $N : H \to X$ is continuous and therefore it is closed. Moreover, $S(M^*k)$ is nonempty and compact, and $\operatorname{card} S(M^*k)$ is constant and finite on each connected component of the open set $H \setminus (M^* - N^*FN)(\Sigma_H)$ by Theorem 3.2. By Lemma 3.8, we get $\operatorname{card} S = (I - KF)^{-1}(\{f\}) = \operatorname{card} S(M^*k)$ with $f = Nk$. Hence, $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, positive and finite on each connected component of $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$. $\square$

Next, we shall look at the case when the selfadjoint part A of K is not positive definite. Suppose that A is quasi-positive definite in H, i.e., it has at most a finite number of negative eigenvalues of finite multiplicity. Let U be the subspace spanned by the eigenvectors of A corresponding to these negative eigenvalues of A and $P : H \to U$ be the orthogonal projection onto U. Then P commutes with A, but not necessarily with B, and acts both in X and X^*. The operator $|A| = (I - 2P)A$ is easily seen to be positive definite. Hence, we have the decomposition $|A| = DD^*$, where $D = |A|^{1/2} : H \to X$ and $D^* : X^* \to H$.

Following [1, 19], we call the map K P-quasi-positive if the map $D^{-1}K(D^*)^{-1}$ exists and is bounded in H, and S-quasi-positive if the map $K(D^*)^{-1}$ exists and is bounded in H. Let M and N denote the closure in H of the the bounded maps $D^{-1}K(D^*)^{-1}$ and $K(D^*)^{-1}$ respectively. Assume that they are both defined on the whole space H. We assume that we have the following decompositions

$$K = DMD^*, \quad K = ND^*.$$

Then we have $N = DM$, $N^* = M^*D^*$, and $\langle Mh, h\rangle = \|h\|^2 - 2\|Ph\|^2$ for all $h \in H$. Define the number

$$\nu(K) = \sup\{\nu : \nu > 0,\ \|Nh\| \geq (\nu)^{1/2}\|Ph\|,\ h \in H\}.$$

Note that for a selfadjoint map K, $\nu(K)$ is the absolute value of the largest negative eigenvalue of K.

Lemma 3.10. *Equations* (3.2) *and* (3.3) *are equivalent with f restricted to $N(H)$; each solution h of* (3.3) *determines a solution $x = Nh$ of* (3.2) *and each solution x of* (3.2) *with $f \in N(H)$ determines a solution $h = D^*Fx + k$ of* (3.3) *with $f = Nk$ and $x = Nh$. Moreover, the map $N : S(M^*k) \to S = (I - KF)^{-1}(\{Nk\})$ is bijective.*

Proof. Let h_1 and h_2 be distinct solutions of (3.3). Since $N = DM$, $K = ND^*$ and M is injective, we get as before that $x_1 = Nh_1$ and $x_2 = Nh_2$ are solutions of (3.2). They are distinct since

$$\begin{aligned} 0 &\neq \|h_1 - h_2\|^2 - 2\|P(h_1 - h_2)\|^2 \\ &= (M(h_1 - h_2), h_1 - h_2) = (N^*FNh_1 - N^*FNh_2, h_1 - h_2) \\ &= (FNh_1 - FNh_2, N(h_1 - h_2)) = (Fx_1 - Fx_2, x_1 - x_2). \end{aligned}$$

Conversely, let $f \in N(H)$ and x_1 and x_2 be distinct solutions of (3.2). Let $f = Nk$ for some $k \in H$. Set $h_i = D^*Fx_i + k$. Then $Nh_i = ND^*Fx_i + Nk = KFx_i + f$ and so $x_i = Nh_i$. Hence, $M^*h_i = M^*D^*FNh_i + M^*k = N^*FNh_i + M^*k$, i.e., h_i are solutions of (3.3). They are distinct since $h_1 = h_2$ implies that $x_1 = Nh_1 = Nh_2 = x_2$. These arguments show that $N : S(M^*k) \to S$ is bijective. □

We have the following result when K is P-quasi-positive.

Corollary 3.11. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$) and $K : X^* \to X$ be a linear continuous P-quasi-positive map with $c\nu(K) < -1$. Let $F = F_1 + F_2 : X \to X^*$ be a nonlinear map, N^*F_2N be continuous and k-ball contractive with $k < -(1 + c\nu(K))$ and there are positive constants a and b, $R > 0$ and let c be the smallest number such that $1 + (a + c)\nu(K) < 0$ and*

- (i) $(F_1x - F_1y, x - y) \leq c\|x - y\|^2$ *for all* $x, y \in X$
- (ii) $\|F_2x\| \leq a\|x\| + b$ *for all* $\|x\| \geq R$,

Then (1.1) *is solvable in X for each $f \in N(H) \subset X$. Moreover, if $\Sigma_H = \{h \in H : M^* - N^*FN$ is not invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in N(H)$, and the cardinal number* $\operatorname{card}(I - KF)^{-1}(\{f\})$ *is constant, finite, and* *positive on each connected component of the set $H\setminus(M^*-N^*FN)(\Sigma_H)$ intersected by $N(H)$.*

Proof. Define the homotopy $H(t,x)=M^*x-tN^*FNx$ on $[0,1]\times H$. Again, it suffices to show that he map $H_t=M^*-tN^*FN : H\to H$ is A-proper with respect to $\Gamma=\{H_n,P_n\}$ for each $t\in[0,1]$ and satisfies condition (+). Set $H_{1t}=M^*-tN^*F_1N : H\to H$. Then, for each $x,y\in H$ we have that

$$\begin{aligned}
&(H_1(t,x)-H_1(t,y),x-y)=(M^*x-M^*y,x-y)-t(N^*(F_1Nx-F_1Ny),x-y)\\
&=\|x-y\|^2-2\|P(x-y)\|^2-t(F_1Nx-F_1Ny,Nx-Ny)\\
&\ge\|x-y\|^2-2\|P(x-y)\|^2-tc\|Nx-Ny\|^2)\\
&\ge\|x-y\|^2-2\|P(x-y)\|^2-c\nu(K)\|P(x-y)\|^2\\
&\ge\|x-y\|^2-(2+c\nu(K))\|P(x-y)\|^2=-(1+c\nu(K))\|x-y\|^2.
\end{aligned}$$

Since N^*F_2N is a k-ball contraction, H_t is A-proper with respect to Γ by Corollary 2.3.

Next, let $f\in N(H)\subset X$, $f=Nk$, be fixed. We claim that $H(t,x)-tM^*h$ satisfies condition (+). If not, then there would exist $x_n\in H$, $t_n\in[0,1]$ such that $\|x_n\|\to\infty$ and

$$y_n=H(t_n,x_n)-t_nM^*k\to g$$

as $n\to\infty$. Then, as above,

$$\begin{aligned}
(H(t_n,x_n),x_n)&=(M^*x_n-t_nN^*FNx_n,x_n)\\
&=(M^*x_n,x_n)-t_n(N^*F_1Nx_n,x_n)-t_n(N^*F_2Nx_n,x_n)\\
&=\|x_n\|^2-2\|Px_n\|^2-t_n(N^*F_1Nx_n+t_nN^*F_10,x_n)\\
&\quad-t_n(F_10,Nx_n)-t_n(F_2Nx_n,Nx_n)\\
&\ge-(1+c\nu(K))\|x_n\|^2-\|N^*F_10\|\,\|x_n\|-a\|Nx_n\|^2-b\|Nx_n\|\\
&\ge-(1+(a+c)\nu(K))\|x_n\|^2-\|N^*F_10\|\,\|x_n\|-b\nu(K)^{1/2}\|Px_n\|)\\
&\ge-(1+(a+c)\nu(K))\|x_n\|^2-(\|N^*F_10\|-b\nu(K)^{1/2})\|x_n\|.
\end{aligned}$$

Since

$$(H(t_n,x_n),x_n)=(y_n,x_n)+t_n(M^*k,x_n)\le C\|x_n\|$$

for some constant C, we get that

$$-(1+(a+c)\nu(K))\|x_n\|^2-(\|N^*F_10\|-b\nu(K)^{1/2})\|x_n\|\le C\|x_n\|.$$

It follows that $\{x_n\}$ is bounded, for otherwise dividing by $\|x_n\|^2$ and passing to the limit we get that $1+(a+c)\nu(K)\ge 0$, a contradiction. Hence, condition (+) holds in either case. □

Next, we shall continue our study of (1.1) assuming that the nonlinearity has a one sided estimate and the linear map K is either positive or P-(quasi)-positive.

Theorem 3.12. *Let X be a reflexive embeddable Banach space ($X\subset H\subset X^*$), $K : X^*\to X$ be a map such that the restriction of K to H, K_H, is selfadjoint and positive semidefinite and $F : X\to X^*$ be such that $I-tTFC$ is A-proper in H for each $t\in[0,1]$, and for some constants a, b, d, $R>0$ and $\gamma\in(0,2]$,*

$$(Fx,x)\le a\|x\|^2+b\|x\|^{2-\gamma}+d,\quad x\in X\setminus B(0,R)$$

and $a\lambda < 1$, where λ is the leading eigenvalue of K. Then (1.1) *is solvable for each $f \in C(H) \subset X$. Further, if $\Sigma_H = \{h \in H : I - TFC$ is not locally invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in C(H)$, and the cardinal number* $\operatorname{card}(I-KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set $H \setminus (I - TFC)(\Sigma_H)$ intersected by $C(H)$.*

Proof. The map K can be represented as $K = CC^*$, where $C : H \to X$ and $C^* : X^* \to H$. The restriction of C to H coincides with the selfadjoint positive semidefinite square root of K. Moreover, $\|C\| = \lambda^{1/2}$ when considered as a map in H. Consider the homotopy $H(t,x) = x - tTFCx$ on $[0,1] \times H$. We claim that $H(t,x) - tf$ satisfies condition (+). Indeed, let (t_n, x_n) be such that $H(t_n, x_n) - t_n f \to g$. If $\|x_n\| \to \infty$, then for some M,

$$
\begin{aligned}
\|x_n\|^2 &= (H(t_n, x_n) - t_n f, x_n) + t(TFCx, x) \\
&\le M\|x_n\| + (TFCx, x) \\
&\le M\|x_n\| + (FCx, Cx) \\
&\le M\|x_n\| + a\|Cx_n\|^2 + b\|x_n\|^{2-\gamma} + d \le M\|x_n\| + a\lambda\|x_n\|^2 + b\|x_n\|^{2-\gamma} + d.
\end{aligned}
$$

Dividing by $\|x_n\|^2$, we get

$$1 \le a\lambda + M\|x_n\|^{-1} + b\|x_n\|^{-\gamma} + d\|x_n\|^{-2}.$$

Passing to the limit, we get that $1 \le a\lambda$, a contradiction. Hence, $\{x_n\}$ is bounded and condition (+) holds. By Theorem 3.2, we get a solution y of $y - C^*FCy = h$ for each $h \in H$ and $x = Cy$ is a solution of $x - KFx = f$. The other conclusions follow as in Corollary 3.7. □

Remark 3.13. The one sided condition on F in Theorem 3.12, as well as in other results below where it appears, can be replaced by

$$(Fx, x) \le a(x) \quad \text{for all } x \in X \setminus B(0,R)$$

for a suitable function $a : X \to R^+$.

An easy consequence of Theorems 3.3 and 3.12 is the following result.

Corollary 3.14. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a map such that the restriction of K to H, K_H, is selfadjoint and positive semidefinite and $F : X \to X^*$ be such that TFC is ϕ-condensing, and for some constants a, b, d and $\gamma \in (0,2]$*

$$(Fx, x) \le a\|x\|^2 + b\|x\|^{2-\gamma} + d, \quad x \in X \setminus B(0,R)$$

and $a\lambda < 1$, where λ is the leading eigenvalue of K. Then (1.1) *is solvable for each $f \in C(H) \subset X$. Further, if $\Sigma_H = \{h \in H : I - TFC$ is not locally invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in C(H)$, and the cardinal number* $\operatorname{card}(I-KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set $H \setminus (I - TFC)(\Sigma_H)$ intersected by $C(H)$.*

Corollary 3.15. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a positive semidefinite bounded selfadjoint map in H, and $C = K_H^{1/2}$, where K_H is the restriction of K to H, $\mu(K) = \|C\|^2$ and $T : X^* \to H$ be a bounded linear extension of $K_H^{1/2}$. Let $F = F_1 + F_2 : X \to X^*$ be a nonlinear map,* *a b, d and $\gamma \in (0,2]$ be constants such that $a\lambda < 1$, $R > 0$ and c be the smallest number such that*

$$(F_1x - F_1y, x - y) \le c\|x - y\|^2 \quad \text{for all } x, y \in X$$
$$(Fx, x) \le a\|x\|^2 + b\|x\|^{2-\gamma} + d, \quad x \in X \setminus B(0, R)$$

and TF_2C is a continuous k-ball contraction with $k < 1 - c\mu(K)$. Then (1.1) is approximation solvable in X for each $f \in C(H) \subset X$ with respect to a projection scheme $\Gamma = \{X_n, P_n\}$ for X, $\delta = \max \|P_n\| = 1$. Moreover, if $\Sigma_H = \{h \in H : I - TFC$ is not locally invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in C(H)$, and the cardinal number $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (I - TFC)(\Sigma_H)$ intersected by $C(H)$.

Proof. We have shown before that $I - tTFC : H \to H$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$, $t \in [0, 1]$. Then the conclusions follow from Theorem 3.12. $\square$

Next, we shall give an extension of Theorem 3.12 to non-selfadjoint K.

Theorem 3.16. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a linear continuous P-positive map and $F : X \to X^*$ be such that $M^* - tN^*FN$ is A-proper with respect to Γ for each $t \in [0, 1]$, and for some constants a, b, d, $\gamma \in (0, 2]$ and $R > 0$,*

$$(Fx, x) \le a\|x\|^2 + b\|x\|^{2-\gamma} + d, \quad x \in X \setminus B(0, R)$$

and $a\mu(K) < 1$. Then (1.1) is solvable for each $f \in N(H) \subset X$. Moreover, if $\Sigma_H = \{h \in H : M^* - N^*FN$ is not locally invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in N(H)$, and the cardinal number $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.

Proof. The homotopy $H(t, x) = M^*x - tN^*FNx$ on $[0, 1] \times H$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$. By Theorem 3.2, it is left to show that it satisfies condition (+). Let $f \in N(H) \subset X$, $f = Nk$, be fixed. We claim that $H(t, x) - tM^*h$ satisfies condition (+). If not, then there would exist $x_n \in H$, $t_n \in [0, 1]$ such that $\|x_n\| \to \infty$ and

$$y_n = H(t_n, x_n) - t_nM^*k \to g$$

as $n \to \infty$. Then

$$M^*x_n = y_n + t_nN^*FNx_n - t_nM^*k$$

and

$$\begin{aligned}\|x_n\|^2 &= (M^*x_n, x_n)\\ &= (y_n, x_n) + t_n(FNx_n, Nx_n) - t_n(M^*k, x_n)\\ &\le (\|y_n\| + \|M^*k\|)\|x_n\| + b(\|N\|\,\|x_n\|)^{2-\gamma} + d + a\mu(K)\,\|x_n\|^2.\end{aligned}$$

Dividing by $\|x_n\|^2$ and passing to the limit, we get that $1 \le a\mu(K)$, a contradiction. Hence, condition (+) holds. This and the A-properness of $M^* - N^*FN$ imply that $M^*h - N^*FNh = M^*k$ for some $h \in H$. The rest of the proof follows as in Corollary 3.9. $\square$

Corollary 3.17. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a linear P-positive map and $F : X \to X^*$ be such that N^*FN is ball condensing, and for some constants a, b, d, $\gamma \in (0, 2]$ and $R > 0$*

$$(Fx, x) \le a\|x\|^2 + b\|x\|^{2-\gamma} + d, \quad x \in X \setminus B(0, R)$$

and $a\mu(K) < 1$. Then (1.1) is solvable for each $f \in N(H) \subset X$. Moreover, if $\Sigma_H = \{h \in H : M^* - N^*FN$ is not invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in N(H)$, and the cardinal number $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.

Proof. It suffices to observe that $M^* - N^*FN$ is A-proper with respect to Γ by Corollary 2.3. $\square$

Corollary 3.18. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$) and $K : X^* \to X$ be a linear continuous P-positive map. Let $F = F_1 + F_2 : X \to X^*$ be a nonlinear map, N^*F_2N be continuous and k-ball contraction with $k < 1 - c\mu(K)$ and there are positive constants a, b, d, $\gamma \in (0, 2]$ and $R > 0$ with $a\mu(K) < 1$, and let c be the smallest number such that*

(i) $(F_1x - F_1y, x - y) \le c\|x - y\|^2 \quad x, y \in X$
(ii) $(Fx, x) \le a\|x\|^2 + b\|x\|^{2-\gamma} + d, \quad x \in X \setminus B(0, R)$

Then (1.1) is solvable in X for each $f \in N(H) \subset X$. Moreover, if $\Sigma_H = \{h \in H : M^* - N^*FN$ is not invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in N(H)$, and the cardinal number $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.

Proof. As in the proof of Corollary 3.11, we have that $M^* - tN^*FN$ is A-proper with respect to Γ for each $t \in [0, 1]$. Hence, the conclusions follow from Theorem 3.16. $\square$

For P-quasi-positive K we have the following statement.

Theorem 3.19. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a linear continuous P-quasi-positive map and $F : X \to X^*$ be such that $M^* - tN^*FN$ is A-proper with respect to Γ for each $t \in [0, 1]$, and for some constants a, b, d, $\gamma \in (0, 2]$ and $R > 0$*

$$(Fx, x) \le a\|x\|^2 + b\|x\|^{2-\gamma} + d, \quad x \in X \setminus B(0, R)$$

and $-(1 + a\nu(K)) > 0$. Then (1.1) is solvable for each $f \in N(H) \subset X$. Moreover, if $\Sigma_H = \{h \in H : M^* - N^*FN$ is not invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in N(H)$, and the cardinal number $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.

Proof. The homotopy $H(t, x) = M^*x - tN^*FNx$ on $[0, 1] \times H$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$. By Theorem 3.2, it is left to show that it satisfies condition (+). Let $f \in N(H) \subset X$, $f = Nk$, be fixed. We claim that $H(t, x) - tM^*h$ satisfies condition (+). If not, then there would exist $x_n \in H$, $t_n \in [0, 1]$ such that $\|x_n\| \to \infty$ and

$$y_n = H(t_n, x_n) - t_nM^*k \to g$$

as $n \to \infty$. Then

$$M^*x_n = y_n + t_nN^*FNx_n - t_nM^*k$$

and

$$\begin{aligned}
&(y_n, x_n)\\
&= (M^*x_n, x_n) - t_n(FNx_n, Nx_n) + t_n(M^*k, x_n)\\
&\geq \|x_n\|^2 - 2\|Px_n\|^2 - t_na\|Nx_n\|^2 - t_nb\|Nx_n\|^{2-\gamma} - t_nd - t_n\|M^*k\|\,\|x_n\|\\
&\geq \|x_n\|^2 - (2 + a\nu(K))\|Px_n\|^2 - b(\nu(K)^{1/2}\|P_nx_n\|)^{2-\gamma} - \|M^*k\|\,\|x_n\| - d\\
&\geq -(1 + a\nu(K))\|x_n\|^2 - b(\nu(K)^{1/2}\|x_n\|)^{2-\gamma} - \|M^*k\|\,\|x_n\| - d.
\end{aligned}$$

Dividing by $\|x_n\|^2$ and passing to the limit, we get that $1 + a\nu(K) \geq 0$, a contradiction. Hence, condition (+) holds.

This and the A-properness of $M^* - N^*FN$ imply that $M^*h - N^*FNh = M^*k$ for some $h \in H$. The rest of the proof follows as in Corollary 3.9. $\square$

Corollary 3.20. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a linear P-quasi-positive map and $F : X \to X^*$ be such that N^*FN is ball condensing, and for some constants a, b, d, $\gamma \in (0, 2]$ and $R > 0$*

$$(Fx, x) \leq a\|x\|^2 + b\|x\|^{2-\gamma} + d, \quad x \in X \setminus B(0, R)$$

and $-(1 + a\nu(K)) > 0$. Then (1.1) is solvable for each $f \in N(H) \subset X$. Moreover, if $\Sigma_H = \{h \in H : M^* - N^*FN$ is not invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in N(H)$, and the cardinal number $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.

Proof. We have that $M^* - tN^*FN + 2P$ is A-proper with respect to Γ for each $t \in [0, 1]$ by Corollary 2.3 since $(M^*x + 2Px, x) = \|x\|^2$. But, P is a compact map and therefore the map $M^* - tN^*FN$ is A-proper as a compact perturbation of an A-proper map. Hence, the conclusions follow by Theorem 3.19. $\square$

Corollary 3.21. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$) and $K : X^* \to X$ be a linear continuous P-quasi-positive map. Let $F = F_1 + F_2 : X \to X^*$ be a nonlinear map, N^*F_2N be continuous and k-ball contractive with $k < 1 - c\mu(K)$ and there are positive constants a, b, d, $\gamma \in (0, 2]$ and $R > 0$ with $-(1 + a\nu(K)) < 0$, and let c be the smallest number such that*

(i) $(F_1x - F_1y, x - y) \leq c\|x - y\|^2 \;\; x, y \in X$

(ii) $(Fx, x) \leq a\|x\|^2 + b\|x\|^{2-\gamma} + d, \;\; x \in X \setminus B(0, R)$.

Then (1.1) is solvable in X for each $f \in N(H) \subset X$. Moreover, if $\Sigma_H = \{h \in H : M^* - N^*FN$ is not invertible at $h\}$ then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in N(H)$, and the cardinal number $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (M^* - N^*FN)(\Sigma_H)$ intersected by $N(H)$.

Proof. As in the proof of Corollary 3.11, we have that $M^* - tN^*FN$ is A-proper with respect to Γ for each $t \in [0, 1]$. Hence, the conclusions follow by Theorem 3.19. $\square$

For our next result, assume that a Hilbert space H is in a duality with a Banach space Y with $H \subset Y$ and $K : H \to H$ is positive. Then its selfadjoint part

$A = 1/2(K + K^*)$ is positive semidefinite and therefore its square root $C = A^{1/2}$: $H \to H$ is also positive and semidefinite. Assume that K has a decomposition of the form $K = NC : H \to Y$ for some continuous linear map $N : H \to Y$.

Lemma 3.22. *Equations* (1.1) *and* $y - FKy = h$ *with* $h \in H$ *and* $f \in K(H)$ *are equivalent; each solution* y *of* $y - FKy = h$ *determines a solution* $x = Ky$ *of* (1.1) *and each solution* x *of* (1.1) *with* $f \in K(H)$ *determines a solution* $y = Fx + h$ *of* $y - FKy = h$ *with* $f = Kh$ *and* $x = Kh$. *Moreover, the map* $K : S(h) = (I - FK)^{-1}(\{h\}) \to S = (I - KF)^{-1}(\{Kh\})$ *is bijective.*

Proof. Let y_1 and y_2 be distinct solutions of $y - FKy = h$. Applying K to $y_i - FKy_i = h$, we get that $x_1 = Ky_1$ and $x_2 = Ky_2$ are solutions of (1.1). They are distinct since

$$0 < \|y_1 - y_2\|^2 = (FKy_1 - FKy_2, y_1 - y_2)$$

implies that $FKy_1 \neq FKy_2$ and therefore $x_1 = Ky_1 \neq x_2 = Ky_2$. Conversely, let $f \in K(H)$ and x_1 and x_2 be distinct solutions of (1.1). Let $f = Kh$ for some $h \in H$. Set $y_i = Fx_i + h$. Then $Ky_i = KFx_i + f$ and so $x_i = Ky_i$. Hence, $y_i = FKy_i + h$, i.e., y_i are solutions of $y - FKy = h$. They are distinct since $y_1 = y_2$ implies that $x_1 = Ky_1 = Ky_2 = x_2$. These arguments show that $K : S(h) \to S$ is a bijection. □

Theorem 3.23. *Let a Hilbert space* H *be in a duality with a Banach space* Y *with* $H \subset Y$ *and* $K : H \to H$ *be positive and* $K = NC : H \to Y$ *for some continuous linear map* $N : H \to Y$. *Let* $F : Y \to H$ *be a bounded nonlinear map such that* $I - FK : H \to H$ *is A-proper and*

$$(Fx, x) \leq a\|x\|^2 + b\|x\|^{2-\gamma} + d, \quad x \in Y \setminus B(0, R)$$

for some constants $a, b, d, \gamma \in (0, 2], R > 0$ *and* $a\mu(K) < 1$. *Then* (1.1) *is solvable for each* $f \in K(H)$. *Moreover,if* $\Sigma = \{x \in H : I - FK$ *is not invertible at* $x\}$ *and* $I - FK$ *is continuous, then* $(I - KF)^{-1}(\{f\})$ *is compact for each* $f \in K(H)$, *and the cardinal number* $\operatorname{card}(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set* $H \setminus (I - FK)(\Sigma)$ *intersected by* $K(H)$.

Proof. Let $h \in H$ and and $f = Kh$. Consider the homotopy $H(t, y) = y - tFKy$ on $[0, 1] \times H$. We claim that $H(t, y) - th$ satisfies condition (+) for each $h \in H$. Let $t_n \in [0, 1]$ and $y_n \in H$ be such that $u_n = y_n - t_nFKy_n - t_nh \to g$. Then the positivity of K implies

$$\begin{aligned} 0 \leq (Ky_n, y_n) &\leq (u_n, Ky_n) + t_n(FKy_n + h, Ky_n) \\ &\leq \|u_n\| \, \|Ky_n\| + a\|Ky_n\|^2 + b\|Ky_n\|^{2-\gamma} + \|h\| \, \|Ky_n\| + d \\ &\leq a\mu(K)(y_n, Ky_n) + b\|Ky_n\|^{2-\gamma} + (\|h\| + |u_n\|)\|Ky_n\| + d. \end{aligned}$$

Hence,

$$(Ky_n, y_n) \leq (1 - a\mu(K))^{-1}(b\|Ky_n\|^{2-\gamma} + (\|h\| + \|u_n\|)\|Ky_n\| + d).$$

Moreover,

$$\begin{aligned} (Ky_n, y_n) &= (Ay_n, y_n) = (Cy_n, Cy_n) \\ &\leq (1 - a\mu(K))^{-1}(b\|Ky_n\|^{2-\gamma} + (\|h\| + \|u_n\|)\|Ky_n\| + d). \end{aligned}$$

But, $K = NC$ and therefore,

$$\|Ky_n\| \leq \|N\| \, \|Cy_n\| \leq c_1\|Ky_n\|^{1-\gamma/2} + c_2\|Ky_n\|^{1/2} + c_3$$

for some constants c_1, c_2 and c_3. Since the real function $f(t) = t - c_1 t^{1-\gamma/2} - c_2 t^{1/2}$ tends to infinity as $t \to \infty$, and for each n

$$\|Ky_n\| - c_1\|Ky_n|^{1-\gamma/2} - c_2\|Ky_n\|^{1/2} \le c_3$$

it follows that $\{\|Ky_n\| : n = 1, 2, \dots\}$ is a bounded set. Thus

$$\|y_n\| \le \|u_n\| + \|FKy_n\| + \|h\| \le c_4$$

for some constant c_4 and all n by the boundedness of F. This shows that H_t satisfies condition (+). By Theorem 3.2, we have that the equation $y - FKy = h$ is solvable for each $h \in H$, $S(h) = (I - FK)^{-1}(\{h\}) \neq \emptyset$ and compact, and $\operatorname{card} S(h)$ is constant and finite on each connected component of the open set $H \setminus (I - FK)(\Sigma)$.

Next, applying K to $y - FKy = h$ we get that $x - KFx = f$ with $x = Ky \in H$. By Lemma 3.22, we get that $\operatorname{card} S = (I - KF)^{-1}(\{Kh\}) = \operatorname{card} S(h)$. Hence, $\operatorname{card}(I - KF)^{-1}(\{f\})$ is constant, finite and positive on each connected component of $H \setminus (I - FK)(\Sigma)$ intersected by $K(H)$. ◻

Corollary 3.24. *Let a Hilbert space H be in a duality with a Banach space Y with $H \subset Y$ and $K : H \to H$ be positive and $K = NC : H \to Y$ for some continuous linear map $N : H \to Y$. Let $F : Y \to H$ be a bounded nonlinear map such that $FK : H \to H$ is continuous and ϕ-condensing, and*

$$(Fx, x) \le a\|x\|^2 + b\|x\|^{2-\gamma} + d, \quad x \in Y \setminus B(0, R)$$

for some constants $a, b, d, \gamma \in (0, 2], R > 0$ and $a\mu(K) < 1$. Then (1.1) *is solvable for each $f \in K(H)$. Moreover,if $\Sigma = \{x \in H : I - FK$ is not invertible at $x\}$, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in K(H)$, and the cardinal number* $\operatorname{card}(I - KF)^{-1}(\{f\})$ *is constant, finite, and positive on each connected component of the set $H \setminus (I - FK)(\Sigma)$ intersected by $K(H)$.*

Proof. If KF is ball condensing, Theorem 3.23 applies. If KF is set condensing, then arguing as in Theorem 3.23 we can prove this case again. ◻

Remark 3.25. If $K : H \to H$ is a positive, normal and compact map, then there is a map $N : H \to Y$ such that $K = NC$ (cf. [4]). In this case FK is compact and Corollary 3.24 is applicable.

Next, assuming only the positivity of K, we can still prove the solvability of (1.1) by requiring additionally that F has a linear growth.

Theorem 3.26. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a continuous map such that the restriction K_H of K to H is positive and $F : X \to X^*$ be such that $I - KF$ is A-proper,*

$$\|Fx\| \le a\|x\| + b, \quad x \in X$$
$$(Fx, x) \le c\|x\|^2 + d, \quad x \in X \setminus B(0, R)$$

for some constants $a, b, c, d, R > 0$ and $c\mu(K) < 1$. Then (1.1) *is approximation solvable for each $f \in X$. Moreover, if $\Sigma = \{x \in X : I - KF$ is not invertible at $x\}$ and $I - KF$ is continuous, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in X$, and the cardinal number card$(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I - KF)(\Sigma)$.*

Proof. Consider the homotopy $H(t,x) = x - tKFx$ on $[0,1] \times X$. We claim that $H(t,x) - tf$ satisfies condition (+) for each $f \in X$. Indeed, let $t_n \in [0,1]$ and $x_n \in X$ be such that $u_n = x_n - t_nKFx_n - t_nf \to g$. Then $Fx_n = F(u_n + t_nKFx_n + t_nf)$ and set $y_n = t_nFx_n$. Then $y_n = t_nF(u_n + Ky_n + t_nf)$ and

$$\begin{aligned}
&(y_n, Ky_n) \\
&= (t_nF(u_n + Ky_n + t_nf), Ky_n) \\
&= t_n(F(u_n + Ky_n + t_nf), u_n + Ky_n + t_nf) - (F(u_n + Ky_n + t_nf), u_n + t_nf) \\
&\le c\|u_n + Ky_n + t_nf\|^2 + \|F(u_n + Ky_n + t_nf)\| \, \|u_n + t_nf\| + d \\
&\le c\|Ky_n\|^2 + (a+2c)\|u_n + t_nf\| \, \|Ky_n\| + (a+c)\|u_n + t_nf\|^2 \\
&\quad + b\|u_n + t_nf\| + c_1).
\end{aligned}$$

Since $(y, Ky) \ge 1/\mu(K)\|Ky\|^2$ for all $y \in H$ and H is dense in X^*, we have that $(y, Ky) \ge 1/\mu(K)\|Ky\|^2$ for all $y \in X^*$. Hence,

$$\begin{aligned}
\|Ky_n\|^2 &\le \mu(K)(y_n, Ky_n) \\
&\le \mu(K)(c\|Ky_n\|^2 + (a+2c)\|u_n + t_nf\| \, \|Ky_n\| + (a+c)\|u_n + t_nf\|^2 \\
&\quad + b\|u_n + t_nf\| + c_1).
\end{aligned}$$

Next, we have that

$$\|x_n\| = \|u_n + Ky_n + t_nf\| \le \|u_n\| + \|Ky_n\| + \|f\| \le M + \|Ky_n\|$$

for some constant M. If $Ky_n \to 0$, it follows that $\{x_n\}$ is bounded. If $\{Ky_n\}$ does not converge to zero, then after dividing the above inequality by $\|Ky_n\|$ we get

$$\begin{aligned}
\|Ky_n\| &\le (1 - c\mu(K))^{-1}\mu(K)[(a+2c)\|u_n + t_nf\| \\
&\quad + \big((a+c)\|u_n + t_nf\|^2 + b\|u_n + t_nf\| + c_1\big)/\|Ky_n\|] \le M_1
\end{aligned}$$

for all n and some constant M_1. Hence, $\{x_n\}$ is bounded in either case and condition (+) holds. The conclusions now follow from Theorem 3.2 since

$$\deg(P_nH_0, B(0,r) \cap H_n, 0) = \deg(I, B(0,r) \cap H_n, 0) \ne 0$$

for all $n \ge n_0$. □

Corollary 3.27. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a map such that the restriction K_H of K to H is positive and $F : X \to X^*$ be nonlinear and such that KF is a continuous ϕ-condensing map and*

$$\begin{gathered}
\|Fx\| \le a\|x\| + b, \quad x \in X \\
(Fx, x) \le c\|x\|^2 + d, \quad x \in X \setminus B(0,R)
\end{gathered}$$

for some constants a, b, c, d and $c\mu(K) < 1$. Then (1.1) is solvable for each $f \in X$. Moreover, if $\Sigma = \{x \in X : I - KF$ is not invertible at $x\}$, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in X$, and the cardinal number card$(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I - KF)(\Sigma)$.

Proof. Consider the homotopy $H(t,x) = x - tKFx - tf$. Then the conclusions follow from Theorems 3.3 and 3.26. □

Next, we shall extend Theorem 3.26 to potentially positive maps. Recall that a map $K : H \to H$ is called potentially positive (from below) (cf. [4]) if there exists a $\lambda \in R$ such that the map $I - \lambda K$ is continuously invertible and the map $K_\lambda = (I - \lambda K)^{-1}K$ is positive on H. Clearly, any positive map is potentially positive. Moreover, a completely continuous selfadjoint map is potentially positive if and only if it has a finite number of negative eigenvalues.

Equation (1.1) can be written in the following equivalent form

$$x - K_\lambda F_\lambda x = (I - \lambda K)^{-1}(f)$$

where $F_\lambda = F - \lambda I$. Clearly,

$$S(f) = (I - KF)^{-1}(\{f\}) = S_\lambda((I - \lambda K)^{-1}f) = (I - K_\lambda F_\lambda)^{-1}(\{(I - \lambda K)^{-1}f\}).$$

Moreover, $I - KF : X \to X$ is locally invertible at $x_0 \in X$ if and only if $(I - \lambda K)^{-1}(I - KF) : X \to X$ is locally invertible at $x_0 \in X$ since $I - \lambda K : H \to H$ is a homeomorphism. Hence, $\Sigma = \{x \in X : I - KF \text{ is not invertible at } x\} = \Sigma_\lambda = \{x \in X : (I - \lambda K)^{-1}(I - KF) \text{ is not locally invertible at } x\} = \{x \in X : I - (I - \lambda K)^{-1}K(F - \lambda I) \text{ is not locally invertible at } x\}$. We have the following extension of Theorem 3.26 to potentially positive maps.

Theorem 3.28. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a map such that the restriction K_H of K to H is potentially positive and $F : X \to X^*$ be such that $I - K_\lambda F_\lambda$ is A-proper, and*

$$\|Fx\| \le a\|x\| + b, \quad x \in X$$
$$(Fx, x) \le c\|x\|^2 + c_1, \quad x \in X$$

for some constants a, b, c, c_1 and $(c - \lambda)\mu(K) < 1$. Then (1.1) is approximation solvable for each $f \in X$. Moreover, if $\Sigma = \{x \in X : I - KF$ is not invertible at $x\}$ and $I - KF$ is continuous, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in X$, and the cardinal number card$(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I - KF)(\Sigma)$.

Proof. Let $F_\lambda = F - \lambda I$ and consider the homotopy $H_\lambda(t, x) = x - tK_\lambda F_\lambda x - tf$. Then arguing as in Theorem 3.26, we get that $S(f) = (I - KF)^{-1}(\{f\}) = S_\lambda((I - \lambda K)^{-1}f)$ is not empty and compact, and card$S(f)$ is constant, finite and positive on each connected component U_i of the open set $X \setminus (I - K_\lambda F_\lambda)(\Sigma) = \cup_i U_i$. Since $(I - \lambda K)(X \setminus (I - K_\lambda F_\lambda)(\Sigma)) = X \setminus (I - \lambda K)(I - K_\lambda F_\lambda)(\Sigma) = X \setminus (I - KF)(\Sigma)$, we get that $X \setminus (I - KF)(\Sigma) = \cup_i (I - \lambda K)U_i$. Hence, $f \in (I - \lambda K)U_i$ if and only if $f = (I - \lambda K)g$ with $g \in U_i$. Therefore, card$(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I - KF)(\Sigma)$. $\square$

Corollary 3.29. *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a map such that the restriction K_H of K to H is potentially positive and $F : X \to X^*$ be such that $K_\lambda F_\lambda$ is ϕ-condensing, and*

$$\|Fx\| \le a\|x\| + b, \quad x \in X$$
$$(Fx, x) \le c\|x\|^2 + c_1, \quad x \in X$$

for some constants a, b, c, c_1 and $(c - \lambda)\mu(K) < 1$. Then (1.1) is solvable for each $f \in X$. Moreover, if $\Sigma = \{x \in X : I - KF$ is not invertible at $x\}$ and $I - KF$ is continuous, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in X$, and the cardinal *number card$(I-KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I-KF)(\Sigma)$.*

Proof. Let $F_\lambda = F - \lambda I$ and consider the homotopy $H_\lambda(t,x) = x - tK_\lambda F_\lambda x - tf$. Then the conclusions follow from Theorems 3.3 and 3.28. □

We say that T satisfies condition (++) if whenever $\{x_n\}$ is bounded and $Tx_n \to f$, then $Tx = f$ for some $x \in X$. Let $\sigma(K)$ denote the spectrum of K. Our next result involves a suitable Leray-Schauder type of condition.

Theorem 3.30. *Let $K : X \to X$ be a continuous linear map, $\lambda^{-1} \notin \sigma(K)$, $F : X \to X$ be nonlinear, $T_p = pI - (I-\lambda K)^{-1}K(F-\lambda I) : X \to X$ for $p \geq 1$, T_1 satisfy condition (+) and either F is odd or, for some $R > 0$,*

$$K(F-\lambda I)x \neq t(I-\lambda K)x \quad \text{for } \|x\| \geq R, t > 1. \tag{3.4}$$

a) If T_1 is A-proper with respect to Γ, then (1.1) is approximation solvable for each $f \in X$. Moreover, if $\Sigma = \{x \in X : I-KF \text{ is not invertible at } x\}$ and $I-KF$ is continuous, then $(I-KF)^{-1}(\{f\})$ is compact for each $f \in X$, and the cardinal number card$(I-KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I-KF)(\Sigma)$.
b) If T_p is A-proper with respect to Γ for each $p > 1$ and T_1 satisfies condition (++), then (1.1) is solvable for each $f \in X$.

Proof. Equation (1.1) is equivalent to

$$Ax - Nx = f \tag{3.5}$$

where $A = I - \lambda K$ and $N = K(F-\lambda I)$. It is easy to see that (3.4) implies that

$$Nx \neq tAx \ \text{ for } \|x\| \geq R, t > 1.$$

Hence, the (approximate) solvability of (1.1) follows from [9, Theorem 4.1]. Next, set $\Sigma_1 = \{x \in X : I - A^{-1}N \text{ is not invertible at } x\}$. Then $\{(I-A^{-1}N)^{-1}(\{h\})\}$ is compact for each $h \in X$ and the cardinal number card$(I-A^{-1}N)^{-1}(\{h\})$ is constant, finite and positive on each connected component of $X \setminus (I-A^{-1}N)(\Sigma_1)$ by Theorem 3.1. Since A is a homeomorphism and $\Sigma = \Sigma_1$, we have that card$((I-KF)^{-1}(\{f\}))$ =card$((I-A^{-1}N)^{-1}(\{A^{-1}(f)\})$ on each connected component U_i of $X \setminus (I-A^{-1}N)(\Sigma)$. As before, we get that card$(I-KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I-KF)(\Sigma)$. □

An easy consequence of Theorem 3.30 is the following result.

Corollary 3.31. *Let $K : X \to X$ be a continuous linear map, $\lambda^{-1} \notin \sigma(K)$, $F : X \to X$ be nonlinear, $T_p = pI - (I-\lambda K)^{-1}K(F-\lambda I) : X \to X$ for $p \geq 1$, and*

$$|F-\lambda I| = \limsup_{\|x\|\to\infty} \|Fx - \lambda x\|/\|x\| < \|(I-\lambda K)^{-1}K\|^{-1}. \tag{3.6}$$

a) If T_1 is A-proper with respect to Γ, then (1.1) is approximation solvable for each $f \in X$. Moreover, if $\Sigma = \{x \in X : I-KF \text{ is not invertible at } x\}$ and T_1 is continuous, then $(I-KF)^{-1}(\{f\})$ is compact for each $f \in X$, and the cardinal number card$(I-KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I-KF)(\Sigma)$.
b) If T_p is A-proper with respect to Γ for each $p > 1$ and T_1 satisfies condition (++), then (1.1) is solvable for each $f \in X$.

Corollary 3.32. *Let X be a uniformly convex space with a scheme $\Gamma = \{X_n, P_n\}$, $max\|P_n\| = 1$, $K : X \to X$ be a continuous linear map, $\lambda^{-1} \notin \sigma(K)$ and $F : X \to X$ be nonlinear such that $(I-\lambda K)^{-1}K(F-\lambda I) : X \to X$ is nonexpensive and* (3.6) *hold. Then* (1.1) *is solvable for each $f \in X$.*

Let us now look at some special cases.

Corollary 3.33. *Let $K : H \to H$ be a positive, compact and normal linear map, $\lambda^{-1} \notin \sigma(K)$, $F : X \to X$ be a nonlinear map such that*

$$(|F - \lambda I| + \lambda)\mu(K) < 1. \tag{3.7}$$

Then (1.1) *is approximation solvable for each $f \in H$. Moreover, if $\Sigma = \{x \in X : I - KF \text{ is not invertible at } x\}$, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in X$, and the cardinal number card$(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $X \setminus (I - KF)(\Sigma)$.*

Proof. Since K is compact, it suffices to show that (3.7) implies (3.6). We know that (3.7) is equivalent to

$$|F - \lambda I| + \lambda < \inf_{\gamma\in\sigma(K),\gamma\neq 0} \mathrm{Re}(\gamma^{-1})$$

and the spectrum is $\sigma(K_\lambda) = \{\gamma/(1 - \lambda\gamma) : \gamma \in \sigma(K)\}$. Hence

$$|F - \lambda I| < \inf_{\gamma\in\sigma(K),\gamma\neq 0} \mathrm{Re}(\gamma^{-1}) - \lambda = \inf_{\gamma\in\sigma(K),\ \gamma\neq 0} \mathrm{Re}((1 - \lambda\gamma)/\gamma).$$

Thus,

$$|F - \lambda I|^2 < \inf_{\gamma\in\sigma(K),\ \gamma\neq 0} \{[\mathrm{Re}(\gamma^{-1}) - \lambda]^2 + (Im\lambda^{-1})^2\}.$$

which is equivalent to $|F - \lambda I|\ \|K_\lambda\| < 1$. Hence, (3.6) holds. □

Let $\Sigma(K)$ be the set of characteristic values of K, i.e., $\Sigma(K) = \{\mu : 1/\mu \in \sigma(K)\}$.

Theorem 3.34. *Let $K : H \to H$ be a selfadjoint map, $\lambda \notin \Sigma(K)$, $F : H \to H$ be nonlinear and continuous and $T_p = pI - (I - \lambda K)^{-1}K(F - \lambda I) : H \to H$ for $p \geq 1$. Suppose that for some k with $k\delta < d = \mathrm{dist}(\lambda, \Sigma(K))$*

$$\limsup_{\|x\|\to\infty} \|Fx - \lambda x\|/\|x\| < k.$$

(a) If T_1 is A-proper with respect to Γ, then (1.1) *is approximation solvable for each $f \in H$. Moreover, if $\Sigma = \{x \in X : I - KF \text{ is not invertible at } x\}$ and T_1 is continuous, then $(I - KF)^{-1}(\{f\})$ is compact for each $f \in X$, and the cardinal number card$(I - KF)^{-1}(\{f\})$ is constant, finite, and positive on each connected component of the set $H \setminus (I - KF)(\Sigma)$.*
(b) If T_p is A-proper with respect to Γ for each $p > 1$ and T_1 satisfies condition (++), then (1.1) *is solvable for each $f \in X$.*

Proof. Equation (1.1) is equivalent to $x = (I - \lambda K)^{-1}K(F - \lambda)x + (I - \lambda K)^{-1}f$. Since $(I - \lambda K)^{-1}K = -1/\lambda + 1/\lambda(I - \lambda K)^{-1}$, we have that, [4],

$$\|(I - \lambda K)^{-1}K\| = \sup_{\mu\in\sigma(K)} | - 1/\lambda + 1/\lambda(1 - \lambda\mu)^{-1}| = \sup_{\mu\in\Sigma(K)} |(\mu - \lambda)^{-1}| = d^{-1}.$$

Then the conclusions follow from Corollary 3.31. □

4. Hammerstein integral equations

Let $Q \subset R^n$ be a bounded domain, $k(t,s) : Q \times Q \to R$ be measurable and $f(s,u) : Q \times R \to R$ be a Caratheodory function. We consider the problem of finding a solution $u \in L_2(Q)$ of the Hammerstein integral equation

$$u(t) = \int_Q k(t,s)f(s,u(s))ds + g(t) \tag{4.1}$$

where g is a measurable function. There is a vast literature on the solvability of (4.1) and we just mention the books by Krasnoselskii [5] and Vainberg [23]. Define the linear map

$$Ku(t) = \int_Q k(t,s)u(s)\,ds$$

in $H = L_2(Q)$. Define $Fu = f(s,u(s))$ and note that (4.1) can be written in the form $u - KFu = g$.

Theorem 4.1. *Let $K : H \to H$ be compact and selfadjoint, $\Sigma(K) = \{\lambda : \lambda^{-1} \in \sigma(K)\}$ and assume that either one of the following conditions holds*

(i) *Let $\lambda \notin \Sigma(K)$ and $a < \operatorname{dist}(\lambda, \Sigma(K))$ be such that for some $h \in L_2(Q)$,*

$$|f(s,u) - \lambda u| \le a|u| + h(s) \quad \textit{for all } s \in Q,\ u \in R,$$

(ii) *There are $\lambda, \mu \in \Sigma(K)$ such that $(\lambda,\mu) \cap \Sigma(K) = \emptyset$ and $\lambda < \alpha < \beta < \mu$ and $\epsilon > 0$ such that for $s \in Q$*

$$\alpha + \epsilon \le f_-(s) = \liminf_{|u|\to\infty}(f(s,u)/u) \le f_+(s) = \limsup_{|u|\to\infty}(f(s,u)/u) \le \beta - \epsilon.$$

(iii) *Let K be positive and compact normal map in H with*

$$\begin{gathered} |f(s,u)| \le c|u| + c(s), \quad s \in Q, u \in R, c(s) \in L_2(Q), \\ uf(s,u) \le ku^2 + b(s), \quad s \in Q, u \in R, b(s) \in L_1(Q), \\ \|(I - \gamma K)^{-1}K\|(c+k)/2 < 1\,. \end{gathered}$$

Then (4.1) is approximation solvable in L_2 for each $g \in L_2$ and the number of its solutions is constant and finite on each connected component of $L_2(Q)\backslash(I-KF)(\Sigma)$, where $\Sigma = \{u \in L_2(Q) : I - KF$ is not invertible at $u\}$.

Proof. We shall show first that (ii) implies (i). From (ii), we get that there is $R > 0$ such that

$$\alpha < f_-(s) - \epsilon \le f(s,u)/u \le f_+(s) + \epsilon < \beta, \quad \text{for all } s \in Q \text{ and } |u| \ge R.$$

Hence, for each $s \in Q$,

$$\begin{aligned} \Big|\frac{f(s,u)}{u} - \frac{\lambda+\mu}{2}\Big| &\le \min\Big(f_+(s) + \epsilon - \frac{\lambda+\mu}{2}, \frac{\lambda+\mu}{2} - f_-(s) + \epsilon\Big) \\ &\le \min\Big(\beta - \frac{\lambda+\mu}{2}, \frac{\lambda+\mu}{2} + \alpha\Big) = a \\ &< \frac{\mu-\lambda}{2} = \operatorname{dist}\Big(\frac{\lambda+\mu}{2}, \Sigma(K)\Big). \end{aligned}$$

Thus, (i) holds.

Next, we shall show that (iii) also implies (i).The inequalities in (iii) imply that

$$|f(s,u) - (k-c)/2u| \le (k+c)/2|u| + b_1(s), \quad b_1(s) \in L_2(Q).$$

Since $k > 0$ and $c > k$, we see that (i) holds with $\lambda = (k - c)/2$ and $a = (c + k)/2$. Hence, the conclusion holds by Theorem 3.34 and Corollary 3.31. $\square$

Let us now look at the case when K is not selfadjoint nor compact. Suppose first that K is P-positive in $H = L_2(Q)$. Suppose that K acts from L_q into L_p for $2 \le p \le \infty$ and $q = p/(p-1)$ with $q = 1$ if $p = \infty$. As before, let $A = 1/2(K + K^*)$ be the selfadjoint part of K and $B = 1/2(K - K^*)$ be the skew-adjoint part of K. They both act from L_q into L_p. Assume that A is positive definite. Then it can be represented in the form $A = CC^*$, where $C = A^{1/2} : L_2 \to L_p$ and the adjoint operator $C^* : L_q \to L_2$. Assume that K is P-positive operator in L_2. Denote by M and N the closure of the maps $C^{-1}K(C^*)^{-1}$ and $K(C^*)^{-1}$, respectively, in L_2. Note that M and N are defined on the closure (in L_2) of the range of $C = A^{1/2}$. This closure coincides with L_2 in our case. Since K is P-positive, the following decompositions hold (cf. [1])

$$K = CMC^*, \ K = NC^*.$$

Note that K, M and N are related as: $N = CM, N^* = M^*C^*$ and we have $(Mh, h) = \|h\|^2$ for all $h \in L_2$. Hence, both M and M^* have trivial nullspaces. Denote by $\mu(K) = \|N\|^2$, which is the positivity constant of K in the sense of Krasnoselski. Set $Fx(s) = f(s, x(s))$.

Theorem 4.2. *Suppose that K is P-positive in $L_2(Q)$, $f = f_1 + f_2$ satisfies the Caratheodory condition, $F : L_p \to L_q$, and there are constants a, b c and k such that $a\|K\| < 1$, $k < 1 - c\mu(K)$ and*

(i) $|f(s, u)| \le a|u| + b$ *($s \in Q,\ u \in \mathbb{R}$)*
(ii) $(f_1(s, u) - f_1(s, v), u - v) \le c|u - v|^2$ *($s \in Q,\ u, v \in \mathbb{R}$)*
(iii) $|f_2(s, u) - f_2(s, v)| \le k|u - v|$ $(s \in Q,\ u, v \in \mathbb{R})$.

Then (4.1) *is approximation solvable in L_2 for each $g \in N(L_2)$ and the number of its solutions is constant and finite on each connected component of $L_2(Q) \setminus (M^* - N^*FN)(\Sigma_{L_2})$ intersected by $N(L_2)$, where*
$\Sigma_{L_2} = \{u \in L_2 : M^* - N^*FN$ *is not invertible at* $u\}$.

The above theorem follows from Corollary 3.9.

Theorem 4.3. *Suppose that K is P-positive in $L_2(Q)$, $f = f_1 + f_2$ satisfies the Caratheodory condition, $F : L_p \to L_q$, and there are constants a, b, d, $\gamma \in (0, 2]$, c and k such that $a\mu(K) < 1$, $k < 1 - c\mu(K)$ and*

(i) $(f(s, u), u) \le a|u|^2 + b|u|^{2-\gamma} + d$ $(s \in Q,\ u \in \mathbb{R})$
(ii) $(f_1(s, u) - f_1(s, v), u - v) \le c|u - v|^2$ $(s \in Q,\ u, v \in \mathbb{R})$
(iii) $|f_2(s, u) - f_2(s, v)| \le k|u - v|$ $(s \in Q,\ u, v \in \mathbb{R})$.

Then (4.1) *is approximation solvable in L_2 for each $g \in N(L_2)$ and the number of its solutions is constant and finite on each connected component of $L_2(Q) \setminus (M^* - N^*FN)(\Sigma_{L_2})$ intersected by $N(L_2)$, where*
$\Sigma_{L_2} = \{u \in L_2 : M^* - N^*FN$ *is not invertible at* $u\}$.

The above theorem follows from Corollary 3.18.

Next, we shall look at the case when the selfadjoint part A of K is not positive definite. Suppose that A is quasi-positive definite in L_2, i.e., it has at most a finite number of negative eigenvalues of finite multiplicity. Let U be the subspace spanned by the eigenvectors of A corresponding to these negative eigenvalues of A and $P : L_2 \to U$ be the orthogonal projection onto U. Then P commutes with A, but not necessarily with B, and acts both in L_p and L_q. The operator $|A| = (I - 2P)A$ is easily seen to be positive definite. Hence, we have the decomposition $|A| = DD^*$, where $D = |A|^{1/2} : L_2 \to L_p$ and $D^* : L_q \to L_p$.

As before, the map K P-quasi-positive if the map $D^{-1}K(D^*)^{-1}$ exists and is bounded in L_2, and S-quasi-positive if the map $K(D^*)^{-1}$ exists and is bounded in H. Let M and N denote the closure in L_2 of the the bounded maps $D^{-1}K(D^*)^{-1}$ and $K(D^*)^{-1}$ respectively. They are both defined on the whole space L_2 (cf. [1]) and have the following decompositions

$$K = DMD^*, \quad K = ND^*.$$

Then we have $N = DM$, $N^* = M^*D^*$, and $\langle Mh, h\rangle = \|h\|^2 - 2\|Ph\|^2$ for all $h \in H$. Define the number

$$\nu(K) = \sup\{\nu : \nu > 0,\ \|Nh\| \geq (\nu)^{1/2}\|Ph\|,\ h \in H\}.$$

Note that for a selfadjoint map K, $\nu(K)$ is the absolute value of the largest negative eigenvalue of K.

We have the following result when K is P-quasi-positive.

Theorem 4.4. *Suppose that K is P-quasi-positive in $L_2(Q)$, $f = f_1 + f_2$ satisfies the Caratheodory condition, $F : L_p \to L_q$, and there are constants a, b, d, $\gamma \in (0, 2]$, c and k such that $a + c\nu(K) < -1$, $k < -(1 + c\nu(K))$ and*

(i) $|f(s, u)| \leq a|u| + b$ *($s \in Q$, $u \in \mathbb{R}$)*
(ii) $(f_1(s, u) - f_1(s, v), u - v) \leq c|u - v|^2$ *($s \in Q$, $u, v \in \mathbb{R}$)*
(iii) $|f_2(s, u) - f_2(s, v)| \leq k|u - v|$ *($s \in Q$, $u, v \in \mathbb{R}$).*

Then (4.1) is approximation solvable in L_2 for each $g \in N(L_2)$ and the number of its solutions is constant and finite on each connected component of $L_2(Q) \setminus (M^* - N^*FN)(\Sigma_{L_2})$ intersected by $N(L_2)$, where
$\Sigma_{L_2} = \{u \in L_2 : M^* - N^*FN$ *is not invertible at* $u\}$.

The above theorem follows from Corollary 3.11.

Theorem 4.5. *Suppose that K is P-quasi-positive in $L_2(Q)$, $f = f_1 + f_2$ satisfies the Caratheodory condition, $F : L_p \to L_q$, and there are constants a, b, d, $\gamma \in (0, 2]$, c and k such that $a + c\nu(K) < -1$, $k < -(1 + c\nu(K))$ and*

(i) $(f(s, u), u) \leq a|u|^2 + b|u|^{2-\gamma} + d$ *($s \in Q$, $u \in \mathbb{R}$)*
(ii) $(f_1(s, u) - f_1(s, v), u - v) \leq c|u - v|^2$ *($s \in Q$, $u, v \in \mathbb{R}$)*
(iii) $|f_2(s, u) - f_2(s, v)| \leq k|u - v|$ *($s \in Q$, $u, v \in \mathbb{R}$).*

Then (4.1) is approximation solvable in L_2 for each $g \in N(L_2)$ and the number of its solutions is constant and finite on each connected component of $L_2(Q) \setminus (M^* - N^*FN)(\Sigma_{L_2})$ intersected by $N(L_2)$, where
$\Sigma_{L_2} = \{u \in L_2 : M^* - N^*FN$ *is not invertible at* $u\}$.

The above theorem follows from Corollary 3.11.

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