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# HIGHER ORDER BRANCHING OF PERIODIC ORBITS FROM POLYNOMIAL ISOCHRONES

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Abstract. We discuss the higher order local bifurcations of limit cycles from polynomial isochrones (linearizable centers) when the linearizing transformation is explicitly known and yields a polynomial perturbation one-form. Using a method based on the relative cohomology decomposition of polynomial one-forms complemented with a step reduction process, we give an explicit formula for the overall upper bound of branch points of limit cycles in an arbitrary $n$ degree polynomial perturbation of the linear isochrone, and provide an algorithmic procedure to compute the upper bound at successive orders.

We derive a complete analysis of the nonlinear cubic Hamiltonian isochrone and show that at most nine branch points of limit cycles can bifurcate in a cubic polynomial perturbation. Moreover, perturbations with exactly two, three, four, six, and nine local families of limit cycles may be constructed.

## 1. Introduction

If a planar system with an annulus of periodic orbits is subjected to an autonomous polynomial perturbation, an interesting question is do any of the periodic orbits survive giving birth to limit cycles (isolated periodic orbits).

In this paper we address this problem in the case of an isochronous annulus of periodic orbits (all orbits have the same constant period), and the unperturbed system is explicitly linearizable by a birational transformation of Darboux form, i.e. involving polynomial maps and their complex powers [6]. The usual method for the perturbation is to use the Poincaré-Andronov-Melnikov integral of the perturbation one-form (divided if necessary by the integrating factor) along the closed orbits of the unperturbed system. In general such an integral is a transcendental function, and any question about its zeros is highly nontrivial.

The approach in this paper as in [10] is to apply an explicit linearizing transformation, and solve the perturbation problem in the new coordinates by reducing it to computing the integral of a rational one-form $R_1(u,v)du + R_2(u,v)dv$ over the family of concentric circles $u^2 + v^2 = r^2$. Using this idea a complete analysis at first order has been given in [10] for the linear isochrone under an arbitrary degree polynomial perturbation, and for the reduced Kukles system subjected to one-parameter arbitrary cubic polynomial perturbation. Here we discuss higher order perturbations, first for the linear isochrone at any order and then the more general case when the polynomial perturbation remains polynomial under the linearizing transformation. Our approach is based on the relative cohomology decomposition

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of polynomial one-forms [9]. As an application we give a complete analysis for cubic planar Hamiltonian systems with an isochronous center subjected to one-parameter arbitrary cubic polynomial perturbation.

More precisely, consider an autonomous polynomial perturbation $(p,q)$ of a plane vector field in the form

$$\mathcal{P}_\epsilon := (P(x,y)+\epsilon p(x,y))\frac{\partial}{\partial x} + (Q(x,y)+\epsilon q(x,y))\frac{\partial}{\partial y}, \tag{1-1}$$

where

$$P(x,y) = -y + \sum_{2\le i+j\le n} P_{ij}x^i y^j, \quad Q(x,y) = x + \sum_{2\le i+j\le n} Q_{ij}x^i y^j$$

$$p(x,y) = \sum_{i=1}^{n}\sum_{k=0}^{i} p_{i-k,k}x^{i-k}y^k, \quad q(x,y) = \sum_{i=1}^{n}\sum_{k=0}^{i} q_{i-k,k}x^{i-k}y^k,$$

with $\lambda^n = (P_{ij}, Q_{ij}, p_{ij}, q_{ij}, 1 \le i+j \le n)$ the set of system coefficients, and $\epsilon$ a small parameter. When $\epsilon = 0$, we assume further that the unperturbed vector field $(\mathcal{P}_0)$ has an *isochronous period annulus* $\mathbb{A}$.

For fixed $\lambda^n$, there is a neighborhood $U$ of the origin in $\mathbb{R}^2$ on which the flow associated with $(\mathcal{P}_\epsilon)$ exists for all initial values in $U$. Assume, furthermore, that $U$ is small enough so that a Poincaré return mapping $\delta(r,\epsilon,\lambda^n)$ is defined on $U$, with the distance coordinate $r$. The solution $\gamma_\epsilon(t)$ starting at $(r,0)$, $r>0$, intersects the positive $x-$axis for the first time at some point $(\delta(r,\epsilon,\lambda^n),0)$ after time $T(r,\epsilon)$. Let $\Sigma = \{(x,0) \in U, x>0\}$ denote the transversal or Poincaré section of $U$. By transversality and blowing up arguments the mapping $\delta$ is analytic. On $\Sigma$ we define the displacement function

$$d(r,\epsilon,\lambda^n) := \delta(r,\epsilon,\lambda^n) - r = \sum_{i=1}^{k} d_i(r,\lambda^n)\epsilon^i + O(\epsilon^{k+1}), \tag{1-2}$$

where $d_i(r,\lambda^n) = \frac{1}{i!}\frac{\partial^i d(r,\epsilon,\lambda^n)}{\partial \epsilon^i}|_{\epsilon=0}$. The isolated zeros of $d(r,\epsilon,\lambda^n)$ correspond to limit cycles (isolated periodic orbits) of $(\mathcal{P}_\epsilon)$ intersecting $\Sigma$. In the period annulus $\mathbb{A}$, $d(r,0,\lambda^n) \equiv 0$. We reduce the analysis to that of finding the roots of a suitable bifurcation function derived from the displacement function. For the higher order bifurcation analysis we need to determine $d_k(r,\lambda^n)$ under the assumptions that $d_j(r,\lambda^n) \equiv 0$ for $j<k$.

Below in section two we describe our improved *isochrone reduction* method introduced in [10] where we proved that to first order at most $\frac{n-1}{2}$ (resp. $\frac{n-2}{2}$) local families of limit cycles bifurcate from a polynomial perturbation of odd (resp. even) degree $n$ of the linear isochrone. In section three, using the relative cohomology decomposition of polynomial one-forms along with a so-called step-reduction process, we effectively compute the explicit formula for the maximum number of branch points of limit cycles in an arbitrary $n$ degree polynomial perturbation of the linear isochrone. This upper bound is three (resp. five) in a quadratic (resp. cubic) perturbation. An algorithmic construction for the upper bounds at successive orders is presented. Section four addresses the cubic Hamiltonian isochrones. We show that, from these isochrones, at most nine local families of limit cycles bifurcate in a cubic polynomial perturbation. Moreover, in all cases, one may construct in the usual way perturbations with the maximum number. As shown in [10] each limit cycle is asymptotic to a circle whose radius is a simple positive zero of the bifurcation function.

## 2. Isochrone reduction

The isochrone reduction technique has been introduced in [10]. We recall it for the sake of completeness, and present here a partially generalized version. Consider $r_* \in \Sigma$ a simple zero of $d_1(r, \lambda^n)$. Thus, by the Implicit Function Theorem, there exits a smooth function $r = r(\epsilon)$ defined in some neighborhood of $\epsilon = 0$ such that $r(0) = r_*$ and $d(r(\epsilon), \epsilon, \lambda^n) \equiv 0$. The curve $r = r(\epsilon)$ corresponds to a local family of limit cycles emerging from the periodic trajectory $\gamma(r_*)$ of the unperturbed system which meets $\Sigma$ at $r_*$. For $d_1(r, 0, \lambda^n) \equiv 0$, or if one of the zeros is not simple, then higher order derivatives must be computed. Actually, in $\mathbb{A}$, $\partial_r d(r, 0, \lambda^n) = 0$ for all values of $r$, and so we cannot apply the Implicit Function Theorem. However, from the perturbation of the Taylor series

$$d(r, \epsilon, \lambda^n) = \epsilon d_1(r, 0, \lambda^n) + O(\epsilon^2) = \epsilon(d_1(r, 0, \lambda^n) + O(\epsilon)) = \epsilon B_1^n(r, \lambda^n), \qquad (2\text{-}1)$$

with $B_1^n(r, \lambda^n) := d_1(r, 0, \lambda^n) + O(\epsilon)$, we define a reduced displacement function by

$$B_1^n(r, \lambda^n) := d_1(r, \lambda^n), \qquad (2\text{-}2)$$

for small real values of $\epsilon$. Clearly, if $B_1^n(r(\epsilon), \lambda^n) \equiv 0$ then $d(r(\epsilon), \epsilon, \lambda^n) \equiv 0$ and the Implicit Function Theorem does apply to $B_1^n$. A simple zero $r_*$ of $B_1^n$ is called *a first order branch point of periodic orbits* for the system $(\mathcal{P}_\epsilon)$. The corresponding periodic orbit $\gamma(r_*)$ is said to *survive* or to *persist* after perturbation.

If $r_*$ is a simple root of $B_1^n(r, \lambda^n)$ of order $k$, then the corresponding perturbation Taylor series

$$d(r, \epsilon, \lambda^n) = \epsilon^k(d_k(r, \lambda^n) + O(\epsilon)) := \epsilon^k B_k^n(r, \lambda^n) \qquad (2\text{-}3)$$

yields $B_k^n(r_*, \lambda^n) = 0$ and $\partial_r B_k^n(r_*, \lambda^n) \neq 0$. $B_k^n(r, \lambda^n)$ is the order $k$ bifurcation function. Similarly by the Implicit Function Theorem applied to $B_k^n$, there is a local family of limit cycles emerging from $\gamma(r_*)$, whereas there are at most $m$ such local families for a root $r_*$ of multiplicity $m$ following from the Weierstrass Preparation Theorem [7].

In the case of an isochronous period annulus the isochronal assumption is essential to our approach for determining the order $k$ bifurcation function $B_k^n$ under the assumptions $B_j^n(r, \lambda^n) \equiv 0$ for $j < k$. It is well known (see, e.g., [6]) that the origin of the unperturbed system $(\mathcal{P}_0)$ is isochronous if and only if there exists an analytic change of coordinates

$$(\mathcal{T}_l): \ (u(x, y), v(x, y)) = (x + o(|(x, y)|), y + o(|(x, y)|)) \qquad (2\text{-}4)$$

in its neighborhood, reducing the system to the linear isochrone $\mathcal{I}_0 = -y\partial_x + x\partial_y$. Once we know explicitly $(\mathcal{T}_l)$, we reduce the autonomous perturbation of the nonlinear isochrone to that of a linear one; we then derive a simple expression of the bifurcation function $B_k^n$. In fact through $(\mathcal{T}_l)$, $(\mathcal{P}_\epsilon)$ is simplified to the weakly linear system

$$\begin{aligned} \dot u &= -v + \epsilon\bar p(u, v) \\ \dot v &= u + \epsilon\bar q(u, v), \end{aligned} \qquad (\bar{\mathcal{P}}_\epsilon)$$

whose orbits are in correspondence with the solutions of the one-parameter family of differential one-forms on the plane

$$\bar\omega_\epsilon = dH + \epsilon\bar\omega, \qquad (2\text{-}5)$$

with $H(u, v) = \frac{1}{2}(u^2 + v^2)$, and $\bar\omega(u, v) = \bar q(u, v)du - \bar p(u, v)dv$.

The expression of the first order bifurcation function is recalled in the following theorem that we proved in [10].

**Theorem 2.1.** *Consider a weakly linear system in the form* $(\bar{\mathcal{P}}_\epsilon)$. *Assume the unperturbed system has a period annulus parametrized by* $r$. *A first order branch point of periodic orbits of* $(\bar{\mathcal{P}}_\epsilon)$ *is a simple zero of the function*

$$B_1^n(r,\lambda^n) := \int_0^{2\pi} \left(\bar{p}(r\cos t, r\sin t)\cos t + \bar{q}(r\cos t, r\sin t)\sin t\right) dt, \tag{2-6}$$

*where* $r$ *is taken in an interval of* $(0,\infty)$.

At this point we must note that the resulting perturbation one-form $\bar{\omega}$ is not necessarily polynomial. Actually formula (2-6) is equivalent to a classic Poincaré formula (See [3])

$$B_1^n(r,\lambda^n) = d_1(r,0,\lambda^n) = -\int_{\gamma(r)} \bar{\omega} \tag{2-7}$$

also called the first Melnikov function $M_1(r)$, along the level line $\gamma(r): \quad H = r$.

**2.1 Higher order Perturbations of the isochrone.**

In general such as in [10], under the linearizing transformation, the resulting perturbation $\bar{\omega}(u,v) = \bar{q}(u,v)du - \bar{p}(u,v)dv$ is not necessarily a polynomial. We consider here the particuliar case where $\bar{\omega}(u,v)$ is a polynomial of $u$ and $v$ of degree $n$, and make use of the relative cohomology decomposition of polynomial one-forms in the plane to analyze the higher order perturbations. It goes as follows.

$\lambda^n$ denotes the set of $n^2+3n$ coefficients of the polynomial one-form $\bar{\omega}$. Assume the first order bifurcation function $B_1^n(r,\lambda^n)$ vanishes identically as a function of $r$, for a value $\lambda_1^n$ of $\lambda^n$. We then need to compute $B_2^n(r,\lambda_1^n) := d_2(r,\lambda_1^n)$, whose positive roots give the branch points at second order. The relative cohomology decomposition (See for instance [3,9] ) states that for such a function $H(u,v)$ in (2-5), and if $\int_{\gamma(r)} \bar{\omega} \equiv 0$, then there are polynomials $g_1^n(u,v)$ and $R_1^n(u,v)$ such that

$$\bar{\omega}(u,v) = g_1^n(u,v)dH + dR_1^n(u,v). \tag{2-8}$$

This leads to

$$B_2^n(r,\lambda_1^n) = \int_{\gamma(r)} (g_1^n\bar{\omega}) \quad (\text{modulo } B_1^n(r,\lambda^n)\equiv 0). \tag{2-9}$$

Thus similarly to formula (2-6), it entails

$$B_2^n(r,\lambda_1^n) = \int_0^{2\pi} \left[(g_1^n\cdot\bar{p})(r\cos t, r\sin t)\cos t + (g_1^n\cdot\bar{q})(r\cos t, r\sin t)\sin t\right] dt. \tag{2-10}$$

We recall briefly the construction [3]. Let $\gamma_\epsilon(r)$ be solution of $0 = \bar{\omega}_\epsilon = dH + \epsilon\bar{\omega}$. From the definitions of $B_1^n$ and the displacement function, integrating over $\gamma_\epsilon$ yields

$$\epsilon B_1^n(r,\lambda^n) + \epsilon\int_{\gamma(r)} \bar{\omega} = 0 \quad (\text{modulo } \epsilon^2). \tag{2-11}$$

That is

$$B_1^n(r,\lambda^n) = -\int_{\gamma(r)} \bar{\omega}. \tag{2-12}$$

Assume $B_1^n(r,\lambda^n) \equiv 0$ and (2-8). Integrating over $\gamma_\epsilon$ the equality

$$(1-\epsilon g_1^n)(dH+\epsilon\bar{\omega}) = d(H+\epsilon R_1^n) - \epsilon^2 g_1^n \bar{\omega} \tag{2-13}$$

gives

$$\epsilon^2 B_2^n(r,\lambda^n) + \epsilon^2 \int_{\gamma(r)} g_1^n \bar{\omega} = 0 \quad (\text{modulo } \epsilon^3). \tag{2-14}$$

Hence formula (2-9). Inductively, given

$$\begin{aligned} B_k^n(r,\lambda_{k-1}^n) &= (-1)^k \int_{\gamma(r)} (g_{k-1}^n \bar{\omega}) \\ &= (-1)^k \int_0^{2\pi} \big(g_{k-1}^n \cdot [\bar{p}(r\cos t, r\sin t)\cos t + \bar{q}(r\cos t, r\sin t)\sin t]\big)\, dt \end{aligned} \tag{2-15}$$

if $B_k^n(r,\lambda_{k-1}^n) \equiv 0$ (as a function of $r$), there exist polynomials $g_k^n$ and $R_k^n$ such that

$$g_{k-1}^n \bar{\omega}(u,v) = g_k^n(u,v)dH + dR_k^n(u,v), \tag{2-16}$$

and therefore the $(k+1)th$ bifurcation function is given by

$$B_{k+1}^n(r,\lambda^n) = (-1)^{k+1} \int_{\gamma(r)} (g_k^n \bar{\omega}). \tag{2-17}$$

Consequently constructing the sequence of polynomials $g_i^n \in \mathbb{R}[u,v], i = 1,\cdots,k$ yields the computation of the first nonzero identically bifurcation function $B_k^n(r,\lambda^n)$ whose positive roots give the branch points of order $k$.

*2.1.1 Computing the relative cohomology decomposition.*

The polynomial $g_1^n(u,v)$ in (2-8) is determined by the following.

**Proposition 2.2.** *Assume that $\bar{p}(u,v)$ and $\bar{q}(u,v)$ in (2-5) are polynomials in $u$ and $v$. If $B_1^n(r,\lambda^n)$ vanishes identically then the polynomial $g_1^n(u,v)$ such that*

$$\bar{\omega}(u,v) = g_1^n(u,v)dH + dR_1^n(u,v)$$

*is given by the partial differential equation*

$$u\frac{\partial g_1^n(u,v)}{\partial v} - v\frac{\partial g_1^n(u,v)}{\partial u} = Div(\bar{p},\bar{q})(u,v), \tag{2-18}$$

*where $Div(\bar{p},\bar{q})$ is the divergence of $\bar{p}$ and $\bar{q}$.*

*Proof.* The cohomology decomposition

$$\bar{\omega}(u,v) = g_1^n(u,v)dH + dR_1^n(u,v) = \bar{q}(u,v)du - \bar{p}(u,v)dv \tag{2-19}$$

yields

$$d\bar{\omega} = dg_1^n \wedge dH = d\bar{q} \wedge du - d\bar{p} \wedge dv. \tag{2-20}$$

This entails

$$\left(v\frac{\partial g_1^n}{\partial u} - u\frac{\partial g_1^n}{\partial v}\right) du \wedge dv = -\left(\frac{\partial \bar{p}}{\partial u} + \frac{\partial \bar{q}}{\partial v}\right) du \wedge dv. \tag{2-21}$$

Hence the claim. The $kth, k \geq 1$ cohomology decomposition factor $g_k^n$ is computed the same way using the polynomials $g_{k-1}^n \bar{p}$ and $g_{k-1}^n \bar{q}$. $\square$

Consequently $g_1^n(u, v)$ is a polynomial of maximum degree $d = n-1$. The second order bifurcation function is then

$$B_2^n(r, \lambda_1^n) = \int_0^{2\pi} [g_1^n(r\cos t, r\sin t)(\bar{p}(r\cos t, r\sin t)\cos t + \bar{q}(r\cos t, r\sin t)\sin t)]\, dt. \tag{2-22}$$

This yields an algorithmic construction of the bifurcation function $B_k^n(r, \lambda^n)$ modulo $B_j^n(r, \lambda^n) \equiv 0$ for $j < k$.

*Remarks 2.3.*

(1) As a consequence of the explicit decomposition (2-19) and the algorithmic construction, we have $B_k^n(r, \lambda^n) \in \mathbb{R}[\lambda^n]$, i.e., the bifurcation function $B_k^n(r, \lambda^n)$ depends polynomially on the system coefficients $\lambda^n$.

(2) This construction yields an increasing sequence of ideals generated by the polynomials $B_k^n$ in the Noetherian ring $\mathbb{R}[\lambda^n]$ of polynomials in $\lambda^n$.

(3) By Hilbert's basis theorem the ideal $I_{\bar{\omega}} =< B_1^n, B_2^n, \cdots, B_k^n, \cdots >$ of all the bifurcation polynomials is finitely generated, i.e., there exists a positive integer $\tau = \tau(n)$ such that $I_{\bar{\omega}} = I_{\tau(n)} =< B_1^n, \cdots, B_{\tau(n)}^n >$. We call $I_{\tau(n)}$ the Bautin-like ideal associated to the polynomial perturbation $\bar{\omega}$.

(4) Therefore whenever the resulting perturbation $\bar{\omega}$ is polynomial under the linearizing transformation, the relative cohomology decomposition allows to compute explicitly the Bautin-like ideal [1] which contains all the informations for finding the bound $\mathcal{M}^{\tau(n)}(n)$ to the number of limit cycles to be born to the origin in a perturbation of the isochrone.

For the sake of illustration, first we address the case of the linear isochrone. Next as an example of a nonlinear isochrone we discuss the cubic Hamiltonian isochrone. This isochronous system admits a linearization that preserves the polynomial perturbation allowing the use of the relative cohomology decomposition-based approach.

## 3. Higher order Perturbations of the linear isochrone

Consider a perturbation of degree $n$ of the linear isochrone in the form

$$\mathcal{I}_\epsilon := (-y + \epsilon p(x, y))\frac{\partial}{\partial x} + (x + \epsilon q(x, y))\frac{\partial}{\partial y}, \tag{3-1}$$

with $p(x, y)$ and $q(x, y)$ given in (1-1), and the set of system coefficients $\lambda^n = (p_{ij}, q_{ij}, 1 \leq i + j \leq n)$. Computing the first order bifurcation function from (2-6) yields

$$B_1^n(r, \lambda^n) = \sum_{i=1}^{n} r^i C_i(\lambda^n), \tag{3-2}$$

where (terms of negative subindex assumed zero)

$$C_i(\lambda^n) = \sum_{k=0}^{i+1} (p_{i-k,k} + q_{i-k+1,k-1}) \int_0^{2\pi} \cos t^{i-k+1} \sin t^k dt. \tag{3-3}$$

Simplifying through the well-known rules $\int_0^{2\pi} \cos t^m \sin t^l dt = 0$ for $m$ or $l$ odd we get

$$C_i(\lambda^n) \equiv 0 \quad (\text{resp. } C_i(\lambda^n) \not\equiv 0) \text{ for } i \text{ even (resp. odd).} \tag{3-4}$$

Note that the coefficients $C_i(\lambda^n)$ are of degree one in the component of $\lambda^n$. They are also linearly independent. For instance

$$C_1(\lambda^n) = \pi(p_{10} + q_{01}); \quad C_3(\lambda^n) = \frac{\pi}{4}(3p_{30} + p_{12} + q_{21} + 3q_{03}). \tag{3-5}$$

From (3-2) the branch points are the real positive roots $\rho = r^2$ of

$$\bar{B}_1^n(\rho, \lambda^n) = C_1(\lambda^n) + C_3(\lambda^n)\rho + \cdots + C_{2N+1}(\lambda^n)\rho^N, \tag{3-6}$$

where $N = \frac{n-2}{2}$ (resp. $\frac{n-1}{2}$) for $n$ even (resp. $n$ odd). Hence the following theorems we proved in [10].

**Theorem 3.1.** *To first order, no more than $\mathcal{M}^1(n) = (n-1)/2$, (resp. $(n-2)/2$) continuous families of limit cycles can bifurcate from the linear isochrone in the direction of any autonomous polynomial perturbation of degree $n$, for $n$ odd (resp. even). We can construct small perturbations with the maximum number of limit cycles. Moreover the limit cycles are asymptotic to the circles whose radii are simple positive roots of the bifurcation function.*

For $n = 2$, (resp. $n = 3$) we have

**Corollary 3.2.** *No (resp. at most one) continuous family of limit cycles bifurcates from the linear isochrone in the direction of the quadratic (resp. cubic ) autonomous perturbation $(p, q)$. In the cubic case the maximum number one is attained if and only if the coefficients satisfy the condition $C_1(\lambda^3) \cdot C_3(\lambda^3) < 0$, where $C_1(\lambda^3)$ and $C_3(\lambda^3)$ are given in* (3-5)*. In this instance, this family emerges from the real positive simple roots of the function*

$$\Delta(\rho, \lambda^3) := C_1(\lambda^3) + C_3(\lambda^3)\rho. \tag{3-7}$$

We now proceed to the higher orders and prove the following.

**Theorem 3.3.** *From the linear isochrone, to second order, no more than $\mathcal{M}^2(n) = n - 2$ continuous families of limit cycles can bifurcate in the direction of any autonomous polynomial perturbation of degree $n$ independently of the parity of $n$. These families emerge from the real positive simple roots of the $(n-2)$th degree polynomial equation*

$$\overline{B}_2^n(\rho, \lambda_1^n) := C_3(\lambda_1^n) + C_5(\lambda_1^n)\rho + \cdots + C_{2n-1}(\lambda_1^n)\rho^{n-2}. \tag{3-8}$$

*Moreover we can construct small perturbations with the maximum number of limit cycles as below.*

*Proof.* First note that in (3-8) there are $\frac{n+1}{2}$ (resp. $\frac{n}{2}$) $C_i(\lambda^n)$ for $n$ odd (resp. $n$ even.) Let $\lambda_1^n = \lambda^n|_{C_i(\lambda^n)=0}$ the set of system coefficients $(p_{ij}, q_{ij})$ such that, from (3-6)

$$C_1(\lambda_1^n) = C_3(\lambda_1^n) = \cdots = C_i(\lambda_1^n) = \cdots = C_{2N+1}(\lambda_1^n) = 0. \tag{3-9}$$

That is $B_1^n(r,\lambda_1^n)\equiv 0$. Important to our analysis is the fact that every equation $C_i(\lambda_1^n)=0$ allows to derive one system coefficient in terms of the remaining in its expression. Therefore we have

$$\operatorname{card}(\lambda_1^n)=\begin{cases} n^2+3n-\frac{n+1}{2}=\frac{2n^2+5n-1}{2}, & \text{for } n \text{ odd}\\ n^2+3n-\frac{n}{2}=\frac{2n^2+5n}{2}, & \text{for } n \text{ even,}\end{cases} \tag{3-10}$$

where $\operatorname{card}(\lambda_1^n)$ is the number of components $p_{ij},q_{ij}$ in $\lambda_1^n$. Using the relative cohomology decomposition we compute the $(n-1)th$ degree polynomial $g_1^n(x,y)$ by solving equation (2-18). Take $g_1^n(x,y)$ as

$$g_1^n(x,y)=\sum_{i=1}^{n-1}\sum_{k=0}^{i} g^1_{i-k,k}x^{i-k}y^k. \tag{3-11}$$

The coefficients $g^1_{i-k,k}=g^1_{i-k,k}(\lambda_1^n)$ are determined by the relation

$$(k{+}1)g^1_{i-k-1,k+1}-(i{-}k{+}1)g^1_{i-k+1,k-1}=(i{-}k{+}1)p_{i-k+1,k}+(k{+}1)q_{i-k,k+1}. \tag{3-12}$$

Set

$$\begin{aligned} G_i(\lambda_1^n,t)&=\sum_{k=0}^{i} g^1_{i-k,k}\cos^{i-k}t\sin^k t\\ F_{i+1}(\lambda_1^n,t)&=\sum_{k=0}^{i+1}(p_{i-k,k}+q_{i-k+1,k-1})\cos t^{i-k+1}\sin t^k, \end{aligned} \tag{3-13}$$

and compute the second order bifurcation function using (2-10). It entails

$$B_2^n(r,\lambda_1^n)=\sum_{i=2}^{2n-1} r^iC_i(\lambda_1^n), \tag{3-14}$$

with

$$C_i(\lambda_1^n)=\sum_{k=1}^{i-1}\int_0^{2\pi} G_{i-k}(\lambda_1^n,t)F_{k+1}(\lambda_1^n,t)dt, \tag{3-15}$$

terms of negative subindex are assumed zero, $G_j(\lambda_1^n,t)=0$ for $j>n-1$, and $F_j(\lambda_1^n,t)=0$ for $j>n+1$. Through the rules $\int_0^{2\pi}\cos t^m\sin t^l dt=0$ for $m$ or $l$ odd it results

$$C_i(\lambda_1^n)\equiv 0 \quad (\text{resp. } C_i(\lambda_1^n)\not\equiv 0), \text{ for } i \text{ even (resp. } i \text{ odd).} \tag{3-16}$$

In particular $C_2(\lambda_1^n)=0$, and $C_{2n-1}(\lambda_1^n)\not\equiv 0$, independently of the parity of $n$. Hence the claim. $\square$

We repeat the above outlined process in the following $S_j, j=1,\cdots,M_n$ steps after which we obtain the first non identically zero $B_\tau^n$ and derive the overall upper bound $\mathcal{M}^\tau(n)$. This procedure is called the *Step Reduction Process.* We prove

**Theorem 3.4.**

(1) *For $n$ odd (resp. $n$ even), the first odd (resp. even) integer $\tau = \tau(n) = M_n$ determined by* (3-20) *(resp.* (3-22)*) yields $B^n_{\tau-1} \not\equiv 0$ (resp. $B^n_\tau \not\equiv 0$).*

(2) *At most*

$$\mathcal{M}^\tau(n) = \begin{cases} \frac{\tau n-(\tau+2)}{2}, & \text{for } n \text{ odd} \\ \frac{\tau n-(\tau+3)}{2}, & \text{for } n \text{ even} \end{cases}$$

*branch points of limit cycles bifurcate from the linear isochrone in a $n-$degree polynomial perturbation.*

(3) *At any arbitrary order $1 \le k \le \tau$ the $kth$ order upper bound of limit cycles is given by* (3-18)*.*

*Proof.* At every step $S_j$ we compute the relative cohomology decomposition factor $g^n_k$ which is a polynomial of degree $k(n-1)th$ for $k = j+1$. At the corresponding coefficients $\lambda^n_k|_{C_i(\lambda^n_{k-1})=0}$, the number of bifurcation coefficients $C_i(\lambda^n_{k-1})$ is

$$\operatorname{card}(C_i(\lambda^n_{k-1})) = \begin{cases} \frac{kn-k}{2}, & \text{for } k \text{ odd, } n \text{ odd} \\ \frac{kn-(k+1)}{2}, & \text{for } k \text{ odd, } n \text{ even.} \end{cases} \tag{3-17}$$

we determine the $kth$ order bifurcation function $B^n_k(r, \lambda^n_{k-1})$ that yields a $kth$ order upper bound of branch points

$$\mathcal{M}^k(n) = \begin{cases} \frac{kn-(k+2)}{2}, & \text{for } k \text{ even and every } n;\ k \text{ odd and } n \text{ odd.} \\ \frac{kn-(k+3)}{2}, & \text{for } k \text{ odd and } n \text{ even.} \end{cases} \tag{3-18}$$

As above we derive some system coefficients in function of others in solving $C_i(\lambda^n_{k-1}) = 0$. Finally, we know from remark (2.3) that the process must stop giving the overall upper bound. Recall that the coefficients $C_i(\lambda^n_{k-1})$ are linearly independent and polynomials of degree $k$ in the components of $\lambda^n_{k-1}$. After the last $M_n$ step the number of remaining system coefficients is less or equal to the number of bifurcation coefficients $C_i(\lambda^n_{M_n})$. Thus at least the last $C_i$ is necessarily nonzero yielding $B^n_{M_n} \not\equiv 0$, as illustrated in the quadratic and cubic cases below. We next determine $M_n$.

(1) For $n$ odd, after $M_n$ steps, from (3-10), we have

$$\frac{2n^2+5n-1}{2} \le \sum_{k=2}^{M_n} \frac{kn-k}{2} \tag{3-19}$$

This leads to $M_n$ satisfying

$$M_n(M_n+1) \ge 4\frac{n^2+3n-1}{n-1} \tag{3-20}$$

(2) For $n$ even, it amounts to determining $\overline{M}_n = M_n/2$ such that

$$\frac{2n^2+5n}{2} \le \sum_{k=2}^{\overline{M}_n} \Big(\frac{kn-k}{2} + \frac{(k+1)n-(k+2)}{2}\Big). \tag{3-21}$$

We get

$$\overline{M}_n(\overline{M}_n+1) \ge \frac{2n^2+9n-6}{n-1} \tag{3-22}$$

Hence the result. $\square$

For example, for $n = 2$ we have

**Corollary 3.5.** *In a quadratic perturbation of the linear isochrone*

(1) *The maximum number of continuous families of limit cycles which can bifurcate is three.*
(2) *To first order, second order, and third order no limit cycles can bifurcate.*
(3) *The number of continuous families of limit cycles which can bifurcate is at most one to fourth order and fifth order, at most two to sixth order and seventh order, at most three to eighth order.*

*Proof.* The result is straightforward by taking $n = 2$ in formulas (3-18) and (3-22). We obtain $M_2 \geq 8$. Thus $B_8^2 \not\equiv 0$. $\square$

Item one in the above corollary confirms results in [2, section 3.1, and Theorem 4.8] whereas items $2, 3$ correct and improve concluding remarks in [4]. The case $n = 3$ yields

**Corollary 3.6.** *In a cubic perturbation of the linear isochrone*

(1) *The maximum number of continuous families of limit cycles which can bifurcate is five.*
(2) *The number of continuous families of limit cycles which can bifurcate is at most one to first order and second order, at most two to third order, at most three to fourth order, at most four to fifth order, at most five to sixth order.*

*Proof.* The result follows from $n = 3$ in formulas (3-18) and (3-20). We get $M_3 \geq 5.3$. Thus $B_6^3 \not\equiv 0$. $\square$

Similar corollaries can be formulated for fourth, fifth, $\cdots$, nth order perturbation of the linear isochrone. We now discuss the nonlinear isochrone case of the cubic polynomial Hamiltonian isochrones. Unlike the Kukles isochrone [10], it admits a polynomial linearizing transformation that preserves the polynomial nature of the perturbation one-form, allowing the use of the relative cohomology decomposition.

## 4. Cubic Hamiltonian Isochrones

Assuming the degenerate singularity on the $y-$axis without loss of generality, a cubic Hamiltonian system may be written as

$$\begin{aligned} \dot{x} &= -y - a_1x^2 - 2a_2xy - 3a_3y^2 - a_4x^3 - 2a_5x^2y \\ \dot{y} &= x + 3a_6x^2 + 2a_1xy + a_2y^2 + 4a_7x^3 + 3a_4x^2y + 2b_5xy^2, \end{aligned} \tag{$\mathcal{H}_3$}$$

with Hamiltonian function

$$H(x,y) = \frac{x^2+y^2}{2} + a_6x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + a_7x^4 + a_4x^3y + a_5x^2y^2. \tag{4-1}$$

Mardešić et al have established the following characterization in [8].

**Theorem 4.1.** *The Hamiltonian cubic system* $(\mathcal{H}_3)$ *is Darboux linearizable if and only if it is of the form*

$$\begin{aligned} \dot{x} &= -y - Cx^2 \\ \dot{y} &= x + 2Cxy + 2C^2x^3. \end{aligned} \tag{$\mathcal{H}_i$}$$

*This system is linearizable through the canonical change of coordinates*

$$(u(x,y), v(x,y)) = (x, y + Cx^2). \tag{$\mathcal{T}_l$}$$

**4.1 First Order Perturbation.**

Consider a cubic autonomous perturbation $(\mathcal{H}_\epsilon)$ of system $(\mathcal{H}_i)$

$$\begin{aligned}\dot{x} =& -y - Cx^2 + \epsilon p(x,y)\\ \dot{y} =& x + 2Cxy + 2C^2x^3 + \epsilon q(x,y),\end{aligned} \qquad (\mathcal{H}_\epsilon)$$

where, along with small values of the parameter $\epsilon \in \mathbb{R}$, and $C \neq 0$ we take

$$p(x,y) = \sum_{i=1}^{3}\sum_{k=0}^{i} p_{i-k,k}x^{i-k}y^k, \quad q(x,y) = \sum_{i=1}^{3}\sum_{k=0}^{i} q_{i-k,k}x^{i-k}y^k. \qquad (4\text{-}2)$$

The system coefficients set is $\lambda^3 = (C, p_{ij}, q_{ij}, 1 \le i+j \le 3)$ with $\mathrm{card}(\lambda^3) = 19$. The linearizing change of coordinates $(\mathcal{T}_l)$ transforms $(\mathcal{H}_\epsilon)$ into system

$$\begin{aligned}\dot{u} =& -v + \epsilon\bar{p}(u,v)\\ \dot{v} =& u + \epsilon\bar{q}(u,v),\end{aligned} \qquad (\bar{\mathcal{H}}_\epsilon)$$

with

$$\begin{aligned}\bar{p}(u,v) =& \sum_{i=1}^{3}\sum_{k=0}^{i} p_{i-k,k}u^{i-k}(v-Cu^2)^k = \sum_{i=1}^{6}\sum_{k=0}^{i} \bar{p}_{i-k,k}u^{i-k}v^k\\
=& p_{10}u + p_{01}v + (p_{20} - Cp_{01})u^2 + p_{11}uv + p_{02}v^2 + (p_{30} - Cp_{11})u^3 +\\
& (p_{21} - 2Cp_{02})u^2v + p_{12}uv^2 + p_{03}v^3 + (c^2p_{02} - Cp_{21})u^4 - 2Cp_{12}u^3v -\\
& 3Cp_{03}u^2v^2 + C^2p_{12}u^5 + 3C^2p_{03}u^4v - C^3p_{03}u^6,\\
\bar{q}(u,v) =& 2Cu\bar{p}(u,v) + \sum_{i=1}^{3}\sum_{k=0}^{i} q_{i-k,k}u^{i-k}(v-Cu^2)^k = \sum_{i=1}^{7}\sum_{k=0}^{i} \bar{q}_{i-k,k}u^{i-k}v^k\\
=& q_{10}u + q_{01}v + (2Cp_{01} + q_{20} - Cq_{01})u^2 + (2Cp_{01} + q_{11})uv + q_{02}v^2 +\\
& (2C(p_{20} - Cp_{01}) + q_{30} - Cq_{11})u^3 + (2Cp_{11} + q_{21} - 2Cq_{02})u^2v + (2Cp_{02} +\\
& q_{12})uv^2 + q_{03}v^3 + (2C(p_{30} - Cp_{11}) + C^2q_{02} - Cq_{21})u^4 + (2Cp_{21} -\\
& 4C^2p_{02} - 2Cq_{12})u^3v + (2Cp_{12} - 3Cq_{03})u^2v^2 + 2Cp_{03}uv^3 + C^2(2Cp_{02} - 2q_{21} +\\
& q_{12})u^5 + C^2(-4p_{12} + 3q_{03})u^4v - 6C^2p_{03}u^3v^2 + C^3(2p_{12} - q_{03})u^6 +\\
& 6C^3p_{03}u^5v - 2C^4p_{03}u^7.\end{aligned} \qquad (4\text{-}3)$$

Therefore the resulting one-form $\bar{\omega} = \bar{q}du - \bar{p}dv$ is polynomial of degree $deg(\bar{\omega}) := max(deg(\bar{p}), deg(\bar{q})) = 7$. Denoting $\overline{\lambda}^7$ the system coefficients set after linearization $\mathrm{card}(\bar{\lambda}^7) = \mathrm{card}(\lambda^3) = 19$. We then prove the following.

**Theorem 4.2.**

*From a periodic trajectory in the period annulus $\mathbb{A}$ of the nonlinear isochrone $(\mathcal{H}_i)$, at most two local families of limit cycles bifurcate to first order in the direction of the cubic perturbation $(p,q)$. Moreover there are autonomous perturbations with exactly $0 \le N_+ \le 2$ families of limit cycles. These families emerge from the real positive simple roots of the quadratic function*

$$\Delta(\rho, \lambda^3) := C_1(\lambda^3) + C_3(\lambda^3)\rho + C_5(\lambda^3)\rho^2, \qquad (4\text{-}4)$$

*with the coefficients* $C_i(\lambda^3), i=1,3,5$ *given below.*

*Proof.* Computation of the first order bifurcation function

$$B_1^n(r,\lambda^3)=\int_0^{2\pi}(\bar{p}(r\cos t,r\sin t)\cos t+\bar{q}(r\cos t,r\sin t)\sin t)dt \tag{4-5}$$

gives

$$B_1^n(r,\lambda^3)=r\left(C_1(\lambda^3)+C_3(\lambda^3)\rho+C_5(\lambda^3)\rho^2\right), \tag{4-6}$$

with $\rho=r^2$,

$$\begin{aligned}&C_1(\lambda^3)=\pi(p_{10}+q_{01});\quad C_3(\lambda^3)=\frac{\pi}{4}\left(3(p_{30}+q_{03})+p_{12}+q_{21}-C(p_{11}+2q_{02})\right);\\&C_5(\lambda^3)=\frac{\pi}{8}(p_{12}+3q_{03})C^2.\end{aligned} \tag{4-7}$$

The upper bound $\mathcal{M}^1(3)$ is clearly two, more accurate than $\mathcal{M}^1(n)=\frac{n-1}{2}=3$ for $n=7$ one might predict from the previous section.

A construction of small perturbations with an indicated number $N_+$ of families of limit cycles may be done using for instance Descartes rule of signs. We outline the technique, not really necessary for this quadratic case but effective for higher orders. Indeed denoting $\nu$ the number of sign changes in the sequence of coefficients of $\Delta(\rho)=C_1(\lambda^3)+C_3(\lambda^3)\rho+C_5(\lambda^3)\rho^2$, the number $N_+$ of positive zeros is such that $N_+-\nu=2k,\quad k\in\mathbb{N}$. Therefore

$$\begin{gathered}C_1(\lambda^3)\cdot C_3(\lambda^3)<0 \text{ and } C_3(\lambda^3)\cdot C_5(\lambda^3)<0, \text{ we get } N_+=2 \text{ or } 0,\\ C_1(\lambda^3)\cdot C_3(\lambda^3)<0 \text{ and } C_3(\lambda^3)\cdot C_5(\lambda^3)>0, \text{ gives } N_+=1 \text{ or } 0.\\ C_1(\lambda^3),\quad C_3(\lambda^3),\quad C_5(\lambda^3) \text{ of same sign, there is no positive zeros.}\end{gathered} \tag{4-8}$$

The analysis is completed by the following lemma.

**Lemma 4.3.**
*Let $s(x)$ be a real polynomial, $s\neq 0$, and let $s_0(x),s_1(x),\dots,s_m(x)$ be the sequence of polynomials generated by the Euclidean algorithm started with $s_0:=s(x)$; $s_1:=s'(x)$. Then for any real interval $[\alpha,\beta]$ such that $s(\alpha)\cdots s(\beta)\neq 0$, $s(x)$ has exactly $\nu(\alpha)-\nu(\beta)$ distinct zeros in $[\alpha,\beta]$ where $\nu(x)$ denotes the number of changes of sign in the numerical sequence $(s_0(x),s_1(x),\dots,s_m(x))$. Moreover all zeros of $s(x)$ in $[\alpha,\beta]$ are simple if and only if $s_m$ has no zeros in $[\alpha,\beta]$.*

For a detailed proof, see [5, Theorem 6.3d]. Assume $C_5(\lambda^3)\neq 0$ for a more general treatment, and set

$$\Delta(\rho)=\rho^2+\alpha_2\rho+\alpha_0,\quad \text{with } \alpha_0:=\frac{C_1(\lambda^3)}{C_5(\lambda^3)};\quad \alpha_2:=\frac{C_3(\lambda^3)}{C_5(\lambda^3)}. \tag{4-9}$$

We derive the following Euclidean sequence (up to constant factors):

$$\begin{gathered}s_0(x)=\Delta(r), \text{ and } s_1(x)=\Delta'(r)\\ s_2(x)=-\frac{\alpha_2}{2}r^2-\alpha_0, \text{ and } s_3(x)=\beta r\\ s_4(x)=\alpha_0,\end{gathered} \tag{4-10}$$

with $\beta = \frac{-2\alpha_2^2+8\alpha_0}{\alpha_2}$. We further assume $\alpha_0 \neq 0$ and $\alpha_2 \neq 0$, i.e., $C_1(\lambda^3)$ and $C_3(\lambda^3)$ nonzero. At $x = 0$ we obtain the sequence $(\alpha_0, 0, -\alpha_0, 0, \alpha_0)$; hence $\nu(0) = 2$. At $\infty$, where the leading terms dominate, we get $(1, 4, -\frac{\alpha_2}{2}, \beta, \alpha_0)$. As a result, to make $N_+ = 2$, (resp. 1) we must have $\nu(\infty) = 0$ (resp. 1). It amounts to taking all the terms $-\frac{\alpha_2}{2}$, $\beta$, and $\alpha_0$ positive. Then it suffices to realize $C_1(\lambda^3) \cdot C_5(\lambda^3) > 0$, $C_3(\lambda^3) \cdot C_5(\lambda^3) < 0$ and $4C_1(\lambda^3) \cdot C_5(\lambda^3) < C_3^2(\lambda^3)$. And respectively $C_1(\lambda^3) \cdot C_3(\lambda^3) < 0$ and $C_3(\lambda^3) \cdot C_5(\lambda^3) < 0$. Moreover for $\alpha_0 \neq 0$, $s_4(x)$ is constant; therefore all zeros made to appear by the previous construction are simple. $\square$

*Remarks 4.4.* One may see the resulting system $(\bar{\mathcal{H}}_\epsilon)$ as a $7th$ degree perturbation of the linear isochrone and use the formulas in the previous section to predict the successive upper bound $\mathcal{M}^k(7), k = 1, 2, 3....$ Although the results are not incorrect, the bound obtained is not the best one. To obtain the most accurate upper bound one must consider the explicit expression of each perturbation polynomial in the building up of the combined cohomology decomposition-step reduction process.

Indeed $(\bar{\mathcal{H}}_\epsilon)$ is not a typical $7th$ degree polynomial perturbation of the linear isochrone so as to literally apply the previous section. For such a perturbation $\mathrm{card}(\lambda^7) = 70$, which yields a more complicated step-reduction procedure than do the actual 19 coefficients.

### 4.2 Higher Order Perturbations.

Set $\lambda_1^3 = \lambda^3|_{C_i(\lambda^3)=0, i=1,3,5}$ that is

$$p_{10} + q_{01} = p_{12} + 3q_{03} = 3p_{30} + q_{21} - C(p_{11} + 2q_{02}) = 0. \tag{4-11}$$

Thus $B_1^3(r, \lambda_1^3) \equiv 0$. We then analyze the second order perturbation and obtain the following result.

**Theorem 4.5.**

*At second order there is a choice of the relative cohomology decomposition first factor leading to a maximum of three, and four continuous families of limit cycles bifurcating in the direction of the cubic perturbation $(p, q)$ of the nonlinear isochrone $(\mathcal{H}_i)$.*

*Proof.*

The particular expression of the resulting polynomial perturbation $\bar{\omega}$ impose the search of a $5th$ degree first relative cohomology decomposition polynomial $g_1^3(u, v)$. From formula (3-12) we obtain

$$\begin{aligned} g_1^3(u, v) = g_{10}^1 u + g_{01}^1 v + g_{20}^1 u^2 + g_{02}^1 v^2 + g_{21}^1 u^2 v + g_{03}^1 v^3 + g_{40}^1 u^4 \\ + g_{22}^1 u^2 v^2 + g_{04}^1 v^4 + g_{50}^1 u^5 + g_{05}^1 v^5, \end{aligned} \tag{4-12}$$

with

$$\begin{aligned} &g_{10}^1 = -(p_{11} + 2q_{02}); \quad g_{01}^1 = 2p_{20} + q_{11}; \quad g_{02}^1 - g_{20}^1 = p_{21} + q_{12} \\ &g_{21}^1 = -2C(p_{21} + q_{12}); \quad g_{03}^1 = -4C(p_{21} + q_{12}) = 2g_{21}^1 \\ &g_{22}^1 = 2g_{40}^1 = 2g_{04}^1; \quad g_{50}^1 = g_{05}^1. \end{aligned} \tag{4-13}$$

This expression of $g_1^3(u, v)$ is particularly interesting. It shows the non-uniqueness of the cohomology decomposition in this case. Indeed, whereas in (4-13) the coefficients $g_{10}^1$, $g_{01}^1$, $g_{21}^1$, $g_{03}^1$ are fixed in terms of the components of $\lambda^3$ we have multiple choices for $g^1_{02}$ and $g^1_{20}$. Moreover $g^1_{22}$, $g^1_{40}$, $g^1_{04}$, $g^1_{50}$, $g^1_{05}$ are arbitrary. Consequently we may consider the following possibilities for $g^3_1$.

(1) A cubic polynomial $\bar{g}^3_1$ by making $g^1_{20} = g^1_{04} = g^1_{05} = 0$.
(2) A $4th$ degree $\tilde{g}^3_1$ with $g^1_{04} \neq 0$; $\quad g^1_{05} = 0$.
(3) A $5th$ degree $\hat{g}^3_1$ for $g^1_{05} \neq 0$.

Of course the upper bounds $\mathcal{M}^k(3), k \geq 2$ vary accordingly. Indeed following the process outlined previously the second bifurcation function $B^3_2(r, \lambda^3_1)$ reduces to

$$B^3_2(r, \lambda^3_1) = \sum_{i=3, iodd}^{N} r^i C_i(\lambda^3_1), \tag{4-14}$$

where the bifurcation coefficients $C_i(\lambda^3_1)$ are computed as in (3-15). We get respectively $N = 11$, for the $5th$ and $4th$ degree polynomial $\bar{g}^3_1$ and $\widetilde{g}^3_1$ yielding a $2nd$ order upper bound $\mathcal{M}^2(3) = (N-3)/2 = 4$. Whereas for the cubic polynomial $\hat{g}^3_1$ we get $N = 9$ leading to $\mathcal{M}^2(3) = (N-3)/2 = 3$. $\square$

In the sequel we choose the "best" relative cohomology decomposition first factor $\hat{g}^3_1$ which we denote again $g^3_1$ for convenience, by assuming zero the arbitrary coefficients in (4-13). We follow the step procedure of the previous section to analyze the higher orders. We obtain

**Theorem 4.5.** *In a cubic perturbation of the nonlinear cubic Hamiltonian isochrone*

(1) *To third order (resp. fourth order) at most six (resp. nine) continuous families of limit cycles can bifurcate.*
(2) *The maximum number of branch points of limit cycles is nine.*

*Proof.* For $\lambda^3_2 = \lambda^3_1|_{C_i(\lambda^3_1)=0, i=3,5,7,9}$, $\text{card}(\lambda^3_2) = 12$, and $B^3_2(r, \lambda^3_2) \equiv 0$. It yields the determination of a $8th$ degree relative cohomology decomposition second factor $g^3_2$. We then compute the third order bifurcation function $B^3_3(r, \lambda^3_2)$ and the bifurcation coefficients $C_i(\lambda^3_2), i = 3, 5, 7, 8, 9, 11, 13, 15$ as in (3-15). This entails the third order upper bound $\mathcal{M}^3(3) = 6$.

The equations $C_i(\lambda^3_2) = 0, i = 3, 5, 7, 8, 9, 11, 13, 15$ yield a coefficient set $\lambda^3_3 = \lambda^3_2|_{C_i(r, \lambda^3_2)=0, i=3,5,7,8,9,11,13,15}$ such that $B^3_3(r, \lambda^3_3) \equiv 0$, and $\text{card}(\lambda^3_3) = 6$. This leads to compute a $14th$ degree cohomology decomposition factor $g^3_3$, and ten bifurcation coefficients $C_i(\lambda^3_3), i = 3, \cdots, 21; odd$. It entails a $4th$ order bifurcation function non identically zero. We obtain the $4th$ order upper bound $\mathcal{M}^4(3) = 9$ as claimed. $\square$

## 5. Concluding Remarks

The relative cohomology decomposition of polynomial one-forms complemented with the step reduction procedure described above provides a useful technique for the investigation of higher order branching of periodic orbits of polynomial isochrones when the linearization preserves the polynomial characteristic of the perturbation. It yields a complete analysis of an arbitrary $n-$degree polynomial perturbation of the linear isochrone, and the nonlinear cubic Hamiltonian isochrone, by providing an explicit formula for any order bifurcation function, as well as for the overall upper bound $\mathcal{M}(n)$ of the branch points of limit cycles, i.e, the finite number of the generators of the corresponding Bautin-like ideals.

A similar technique might be obtained when the resulting perturbation after linearization is rational.

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