*Electronic Journal of Differential Equations*, Vol. 2018 (2018), No. 178, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# POSITIVE SOLUTION CURVES OF AN INFINITE SEMIPOSITONE PROBLEM

RAJENDRAN DHANYA

*Communicated by Ratnasingham Shivaji*

Abstract. In this article we consider the infinite semipositone problem $-\Delta u = \lambda f(u)$ in $\Omega$, a smooth bounded domain in $\mathbb{R}^N$, and $u = 0$ on $\partial\Omega$, where $f(t) = t^q - t^{-\beta}$ and $0 < q$, $\beta < 1$. Using stability analysis we prove the existence of a connected branch of maximal solutions emanating from infinity. Under certain additional hypothesis on the extremal solution at $\lambda = \Lambda$ we prove a version of Crandall-Rabinowitz bifurcation theorem which provides a multiplicity result for $\lambda \in (\Lambda, \Lambda + \epsilon)$.

## 1. Introduction

Consider the infinite semi-positone problem

$$\begin{gathered} -\Delta u = \lambda f(u) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

where $f(t) = t^q - t^{-\beta}$, $0 < q < 1$ and $\beta \in (0,1)$ and $\lambda$ a positive parameter. Here $\Omega$ is assumed to be a bounded domain with smooth boundary in $\mathbb{R}^N$. Note that $f(0) = -\infty$ (hence the name infinite semipositone problem) and $f$ is an increasing concave function in $\mathbb{R}^+$. Finding a positive solution for semipositone problems are always challenging and in fact proving the existence of multiple positive solutions are even more difficult. The existence of a positive solution for (1.1) when $\lambda$ large is studied using sub-super solutions technique in [18]. Later in [10], it was additionally shown that when $\lambda$ is large there exists a maximal positive solution for (1.1) which is in fact bounded below by the distance function $d(x, \partial\Omega) = \inf\{|x-y| : y \in \partial\Omega\}$. The aim of this work is to further understand this maximal branch of solution of (1.1) which emanates from $\infty$.

**Definition 1.1.** We say $u$ is a solution of (1.1), if $u \in C^2(\Omega) \cap C^1_0(\overline{\Omega})$ and $u(x) \geq c\, d(x, \partial\Omega)$ for some positive constant $c = c(\lambda)$.

Suppose that $\partial\Omega$ is smooth and $u$ is a solution of $(P_\lambda)$, then the outward normal derivative $\frac{\partial u}{\partial \nu}(x_0) < 0$ for all $x_0 \in \partial\Omega$. Conversely if we assume that $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} < 0$ then by the tubular neighbourhood lemma $u(x) \geq c$, $d(x, \partial\Omega)$ for some $c > 0$.

---

2010 *Mathematics Subject Classification.* 35J25, 35J61, 35J75.
*Key words and phrases.* Semipositone problems; topological methods; bifurcation theory.

Submitted May 3, 2018. Published November 1, 2018.

**Definition 1.2.** Let $\mathcal{S} = \{(\lambda, u_\lambda) : u_\lambda$ is a solution to (1.1), as in Definition 1.1$\}$ and let $\Lambda = \inf\{\lambda > 0 : (1.1)$ admits at least one solution$\}$.

**Definition 1.3.** We say $\lambda_\infty = \infty$ is a bifurcation point at infinity for (1.1) if there exists a sequence $(\lambda_n, u_{\lambda_n}) \in \mathcal{S}$ such that $\lambda_n \to \lambda_\infty$ and $\|u_{\lambda_n}\| \to \infty$.

The principal eigenvalue of the linearized operator associated to (1.1) is denoted by $\Lambda_1(\lambda)$ and defined as

$$\Lambda_1(\lambda) = \inf_{\varphi \in H_0^1(\Omega), \|\varphi\|_2 = 1} \Big( \int_\Omega |\nabla \varphi|^2 - \lambda \int_\Omega f'(u)\varphi^2 \Big). \tag{1.2}$$

where $u$ solves (1.1) as in definition 1.1. Since the solution $u(x)$ behaves like $d(x)$ near $\partial\Omega$, by Hardy's inequality the term $\int_\Omega f'(u)\varphi^2$ make sense. The functional $\int_\Omega |\nabla\varphi|^2 - \lambda \int_\Omega f'(u)\varphi^2$ is bounded below and coercive on the set $\{\varphi \in H_0^1(\Omega) : \|\varphi\|_2 = 1\}$ and hence a minimizer exists. Also one can show that $\Lambda_1(\lambda)$ satisfies the differential equation $-\Delta\psi - \lambda f'(u)\psi = \Lambda_1(\lambda)\psi$ for some non-negative $\psi \in H_0^1(\Omega)$. We say that a solution $u$ of (1.1) is stable if $\Lambda_1(\lambda)$ is strictly positive. Our main result is the following theorem.

**Theorem 1.4.** *Assume that $\Omega$ is a bounded open set in $\mathbb{R}^N$ with smooth boundary and consider the infinite semipositone problem* (1.1) $-\Delta u = \lambda(u^q - u^{-\beta})$ *in* $\Omega$ *for* $0 < q, \beta < 1$ *and* $u = 0$ *on* $\partial\Omega$.

(a) *There exists a $\Lambda \in (0, \infty)$ and for all $\lambda > \Lambda$, there exists a maximal positive solution $u_\lambda$ solving* (1.1). *And $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$, i.e. $\lambda_\infty$ is a bifurcation point at infinity. Also if $\lambda \in (0, \Lambda)$, the problem* (1.1) *does not admit any positive solution.*
(b) *The maximal solution $u_\lambda$ is stable for all $\lambda > \Lambda$.*
(c) *There exists an unbounded connected branch $\mathcal{C}$ of solutions of* (1.1) *emanating from $(\infty, \infty)$ consisting of the maximal solution $u_\lambda$. The map $(\Lambda, \infty) \ni \lambda \to u_\lambda$ is of class $C^2$ in $\mathbb{R} \times C_e(\overline{\Omega})$.*

We prove results (a) and (b) in Section 2 (see Theorems 2.1 and 2.5). We introduce the operator $\mathcal{A}$ and the space $C_e(\overline{\Omega})$ in section 3 and prove the differentiability of the map $\mathcal{A}$ (in fact we prove $\mathcal{A}$ is a $C^2$ map) in the Appendix. Using the stability analysis and smoothness of the map $\mathcal{A}$ we prove $(c)$ in Theorem 3.3. Existence of a positive solution for large $\lambda$ for similar problems are well studied in literature. For example Shi-Yao[21] and Hernández et al. [16] consider the semipositone problem of the type $-\Delta u = \lambda u^q - u^{-\beta}$ with Dirichlet boundary condition in an arbitrary smooth domain $\Omega$ and establish the existence of positive solution bounded below by the distance function using sub-super solution techniques. We also use similar techniques to prove the existence of solution for large $\lambda$, but here in this work we additionally show that the maximal solution curve $\lambda \to u_\lambda$ is in fact smooth. Also see [19, 9, 14] for related problems where they prove stability results for infinite semipositone problems. In [2] the authors discuss a bifurcation phenomenon for semipositone problems ($f(0) \in (-\infty, 0)$) depending on the behaviour of $f(t)$ at infinity, i.e. depending on if $f$ is sublinear, superlinear or asymptotically linear at infinity. Positive solutions curves of concave semipositone problems are also studied in [8] and [7].

In Section 4, existence of a non-negative weak solution at $\lambda = \Lambda$ is proved using a limiting argument (see Proposition 4.1). We conclude our paper by proving the following result.

**Theorem 1.5.** *Either of the following two alternatives hold:*

(a) *The extremal solution $u_\Lambda(x)$ does not belong to the interior of $C_e(\overline{\Omega})$, or*
(b) *The point $\lambda = \Lambda$ is a bifurcation point, i.e. there exists a $C^2$ curve $(\lambda(s), u(s)) \in \mathcal{S}$ where $s \in (-\epsilon, \epsilon)$ with $\lambda(0) = \Lambda$, $\lambda'(0) = 0$, $\lambda''(0) < 0$ and $u(0) = u_\Lambda$.*

To the best of our knowledge a complete bifurcation diagram for semipositone problem is understood in either of the following two situations: (a) in case of $f(0) = -\infty$ and dimension $N = 1$ (see [17]) or (b) in case of strictly semipositone problems, i.e. $-\infty < f(0) < 0$ in a ball (see [6]). In the latter work the results were obtained by using shooting methods for ODE as any positive solution for a semipositone problem in a ball is known to be radially symmetric. In Theorem 1.5 we make an attempt to understand the bifurcation curve in arbitrary domain $\Omega$ under certain additional hypothesis on extremal function $u_\Lambda$. The second alternative gives a precise description of the bifurcation branch at $\lambda = \Lambda$. At least in dimension $N = 1$ and $\beta \in (0, \frac{1}{2})$, it is clear from [17, Theorem 2] that the first case does not arise. The second alternative also suggests the existence of multiple positive solutions for (1.1) when $\lambda \in (\Lambda, \Lambda + \epsilon)$ for some $\epsilon > 0$. In fact the solution in the lower branch (the non-maximal solution) is also bounded below by $\tilde{c}(\lambda)d(x, \partial\Omega)$. It is expected that the solutions exhibit a "free boundary" condition(i.e. a non negative solution becomes zero in a set of positive measure) beyond $\Lambda + \epsilon$.

## 2. Stability analysis

**Theorem 2.1.** *There exists a $\Lambda \in (0, \infty)$ and for all $\lambda > \Lambda$, there exists a positive function $u_\lambda$ solving (1.1) as defined in 1.1. In fact, the function $u_\lambda$ is the maximal solution for (1.1).*

*Proof.* For $\lambda$ large enough the existence of a positive solution bounded below by $d(x, \partial\Omega)$ is obtained in Section 5 of [10] for more general nonlinear function $f$. Here we briefly explain the sub and supersolution to be chosen for our particular nonlinearity $f(t) = t^q - t^{-\beta}$. Following the lines of proof of [10, Example 5.6] we define $\psi = \lambda^r(\phi_1 + \phi_1^{\frac{2}{1+\beta}})$, where $\phi_1$ is the first eigenfunction of $-\Delta$, and $1 < r < \frac{1}{1-q+\epsilon}$ is chosen so that $-\Delta\psi \leq \lambda(\psi^q - \psi^{-\beta})$. We define a super-solution $\phi = v_\lambda$ where

$$-\Delta v_\lambda = \lambda v_\lambda^q \text{ in } \Omega, \quad v_\lambda = 0 \text{ on } \partial\Omega. \tag{2.1}$$

Then we know that $v_\lambda = \lambda^{\frac{1}{1-q}} v_1$ and hence for large $\lambda$ we have $\psi \leq \phi$. Now by [10, Theorem 5.5] there exists a maximal solution $u_\lambda$ in the ordered interval $[\psi, \phi]$. Thus the solution is bounded below by $\psi$ and hence

$$u_\lambda(x) \geq \psi = \lambda^r(\phi_1 + \phi_1^{\frac{1}{1+\beta}}), \quad \text{i.e. } \|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to \infty. \tag{2.2}$$

Suppose $u$ is a solution of (1.1). Then, $-\Delta u \leq \lambda u^q$ and by comparison [20, Lemma 2.2] $u \leq v_\lambda$. Thus the $u_\lambda$ that we constructed via sub-super solution is in fact the maximal positive solution of (1.1). Now define $\Lambda = \inf\{\lambda > 0 : (P_\lambda)$ admits at least one solution$\}$. Next we claim that

$$0 < \Lambda < \infty. \tag{2.3}$$

Clearly from our previous discussion $\Lambda < \infty$. We shall now prove that $\Lambda > 0$. Suppose on the contrary that $\Lambda = 0$, then there exists a sequence $(\lambda_m, u_{\lambda_m}) \in \mathcal{S}$ and $\lambda_m \to 0$. By comparison Lemma we have $0 < u_{\lambda_m} \leq v_{\lambda_m}$. Therefore for large

$m$, since $v_{\lambda_m} = \lambda_m^{\frac{1}{1-q}} v_1$ we have $0 < u_{\lambda_m} < 1$ and $-\Delta u_{\lambda_m} = \lambda_m(u_{\lambda_m}^q - u_{\lambda_m}^{-\beta}) < 0$. This leads to a contradiction, since by maximum principle any such solution $u_{\lambda_m}$ has to be necessarily negative and hence $\Lambda > 0$.

Next we claim that for any $\lambda > \Lambda$ there exists at least one solution for (1.1). Fix $\lambda > \Lambda$, then by definition there exists a $\lambda' \in (\Lambda, \lambda)$ such that (1.1) with $\lambda = \lambda'$ admits at least one solution which we call $\psi$. Note that we do not claim $\psi$ is a sub-solution for (1.1), but still we prove that there exists a $u_\lambda > \psi$ solving (1.1). Clearly, $\psi < v_{\lambda'} < v_\lambda =: u_0$. Let

$$
\begin{gathered}
-\Delta u_1 = \lambda(u_0^q - u_0^{-\beta}) \quad \text{in } \Omega \\
u_1 = 0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

By the standard weak comparison principle for the functions in $W^{2,p}(\Omega)$ we obtain $u_1 < u_0$. We claim that $\psi < u_1 < u_0$. In fact,

$$
\begin{aligned}
-\Delta(u_1 - \psi) &= \lambda f(u_0) - \lambda' f(\psi) \geq \lambda f(\psi) - \lambda' f(\psi) \\
&= \Big(\frac{\lambda - \lambda'}{\lambda'}\Big)\lambda' f(\psi) = -\Delta(\delta\psi)
\end{aligned}
$$

where $\delta = (\lambda - \lambda')/\lambda' > 0$. Thus once again by comparison method we prove the claim. Iteratively if we define the sequence

$$
\begin{gathered}
-\Delta u_{n+1} = \lambda(u_n^q - u_n^{-\beta}) \quad \text{in } \Omega \\
u_{n+1} = 0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

by mathematical induction we can easily prove that

$$
\psi < \dots \leq u_{n+1} \leq u_n \leq \dots u_1 < u_0.
$$

Thanks to the lower and upper bound of the sequence $\{u_n\}$, we have have $u_n \in C_0^{1,\gamma}(\overline{\Omega}) \cap C^2(\Omega)$ (see [10, Theorem 5.2] and [13]). Hence the sequence $\{u_n\}$ is bounded say in $H_0^1(\Omega)$ and if we define $u_\lambda = \lim_{n\to\infty} u_n$, then $u_\lambda$ is the maximal solution of (1.1). $\square$

Our next aim is to prove that the principal eigenvalue of the linearized operator about the maximal solution $u_\lambda$ is positive. As a first step towards it we prove the following proposition.

**Proposition 2.2.** *The maximal solution $u_\lambda$ is semi-stable or the principal eigenvalue of the linearized operator*

$$
\Lambda_1(\lambda) = \inf_{\varphi \in H_0^1(\Omega), \|\varphi\|_2 = 1} \Big( \int_\Omega |\nabla\varphi|^2 - \lambda \int_\Omega f'(u_\lambda)\varphi^2 \Big) \geq 0\,.
$$

*Proof.* For a fixed $\lambda > \Lambda$ we consider the $\epsilon$-approximate regular problem

$$
\begin{gathered}
-\Delta w = \lambda\left((w+\epsilon)^q - (w+\epsilon)^{-\beta}\right) \quad \text{in } \Omega, \\
w > 0 \quad \text{in } \Omega, \\
w = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{2.4}
$$

Let

$$
-\Delta v_\lambda^\epsilon = \lambda(v_\lambda^\epsilon + \epsilon)^q \text{ in } \Omega \quad v_\lambda^\epsilon > 0 \text{ in } \Omega; \quad v_\lambda^\epsilon = 0 \text{ on } \partial\Omega.
$$

It is easy to check that $v_\lambda^\epsilon$ exists and $v_\lambda^\epsilon < v_\lambda$, Note that $u_\lambda$ and $v_\lambda^\epsilon$ are respectively sub and super solutions of (2.4) and by standard monotone iteration there exists a $w_\epsilon \in [u_\lambda, v_\lambda]$ solving (2.4). In fact $w_\epsilon$ is the maximal solution of (2.4). By Hopf's maximum principle for some $\theta_1 > 0$ we have $w_\epsilon(x) + \theta_1 d(x, \partial\Omega) \leq v_\lambda^\epsilon$. Next we observe that the sequence $\{w_\epsilon\}$ is bounded independent of $\epsilon$ since

$$\int_\Omega |\nabla w_\epsilon|^2 \leq \lambda \int_\Omega (w_\epsilon + 1)^{q+1} \leq \lambda \int_\Omega (v_\lambda + 1)^{q+1} < \infty.$$

Clearly $w_\epsilon$ converges to some function $\tilde{w}$ which is a weak solution of (1.1) and $u_\lambda \leq \tilde{w} \leq v_\lambda^\epsilon$. Since $u_\lambda$ is the maximal solution of (1.1) we must have

$$\lim_{\epsilon \to 0} w_\epsilon = u_\lambda. \tag{2.5}$$

Let us write $f_\epsilon(t) = (t + \epsilon)^q - (t + \epsilon)^{-\beta}$.

**Claim:** $\Lambda_1^\epsilon(\lambda) = \inf_{\varphi \in H_0^1(\Omega), \|\varphi\|_2 = 1} \left(\int_\Omega |\nabla \varphi|^2 - \lambda \int_\Omega f_\epsilon'(w_\epsilon)\varphi^2\right) \geq 0$. On the contrary suppose that $\Lambda_1^\epsilon(\lambda) < 0$ and $\varphi_\epsilon \in H_0^1(\Omega)$ be the associated non-negative eigenfunction of

$$-\Delta\varphi_\epsilon - \lambda f_\epsilon'(w_\epsilon)\varphi_\epsilon = \Lambda_1^\epsilon(\lambda)\varphi_\epsilon.$$

We will show that $(w_\epsilon + \theta\varphi_\epsilon)$ is a sub solution of (2.4). For a non-negative $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned}
&\int_\Omega \nabla(w_\epsilon + \theta\varphi_\epsilon)\nabla\varphi - \lambda \int_\Omega f_\epsilon(w_\epsilon + \theta\varphi_\epsilon)\varphi \\
&= \lambda \int_\Omega f_\epsilon(w_\epsilon)\varphi - f_\epsilon(w_\epsilon + \theta\varphi_e)\varphi + \theta f_\epsilon'(w_\epsilon)\varphi_\epsilon\varphi + \theta\Lambda_1^\epsilon(\lambda) \int_\Omega \varphi\varphi_e \\
&= o(\theta) + \theta\Lambda_1^\epsilon(\Lambda) \int_\Omega \varphi\varphi_\epsilon
\end{aligned}$$

Choosing $\theta > 0$ small enough we have $(w_\epsilon + \theta\varphi_\epsilon)$ is a sub-solution of (2.4). If required we may choose $\theta$ smaller so that $w_\epsilon(x) + \theta\varphi_\epsilon \leq v_\lambda^\epsilon$. Thus $w_\epsilon + \theta\varphi_\epsilon$ and $v_\lambda^\epsilon$ forms an ordered pair of sub and super solution of (2.4) and we obtain a solution $\tilde{w}_\epsilon \in [w_\epsilon + \theta\varphi_\epsilon, v_\lambda]$ of (2.4). This contradicts the fact that $w_\epsilon$ is the maximal solution of (2.4) and hence the claim is verified. Thus for every $\varphi \in H_0^1(\Omega)$ such that $\|\varphi\|_2 = 1$,

$$\int_\Omega |\nabla\varphi|^2 - \lambda \int_\Omega f_\epsilon'(w_\epsilon)\varphi^2 \geq 0.$$

Now passing through the limit using (2.5) and Hardy's inequality we obtain that $\Lambda_1(\lambda) \geq 0$. □

**Proposition 2.3.** *The semi-stable solution of* (1.1) *is unique.*

*Proof.* Let $u_\lambda$ be the maximal solution of (1.1) and $v_\lambda$ be any other solution of (1.1). We know that $u_\lambda$ is semi-stable by Proposition 2.2 and assume that $v_\lambda$ is also semi-stable. Then

$$\int_\Omega |\nabla w|^2 \geq \lambda \int_\Omega f'(v_\lambda)w^2$$

for all $w \in H_0^1(\Omega)$. In particular,

$$\int_\Omega |\nabla(u_\lambda - v_\lambda)|^2 \geq \lambda \int_\Omega f'(v_\lambda)(u_\lambda - v_\lambda)^2. \tag{2.6}$$

Since $v_\lambda$ and $u_\lambda$ are both the solutions of (1.1)

$$\int_\Omega |\nabla(u_\lambda - v_\lambda)|^2 = \lambda \int_\Omega (f(u_\lambda) - f(v_\lambda))(u_\lambda - v_\lambda). \tag{2.7}$$

Combining the above two equations we have

$$\int_\Omega \{f(u_\lambda)-f(v_\lambda)-f'(v_\lambda)(u_\lambda-v_\lambda)\}(u_\lambda-v_\lambda)\geq 0.$$

Since $u_\lambda$ is the maximal solution this implies

$$\int_{\{u_\lambda>v_\lambda\}} \{f(u_\lambda)-f(v_\lambda)-f'(v_\lambda)(u_\lambda-v_\lambda)\}(u_\lambda-v_\lambda)\geq 0.$$

Since $f$ is strictly concave the above integral is strictly negative if the Lebesgue measure of the set $\{x : u_\lambda(x) > v_\lambda(x)\}$ is non-zero. Thus $u_\lambda \equiv v_\lambda$, or the semi-stable solution is unique. $\square$

Next we shall prove our main result of this section, the maximal $u_\lambda$ is stable. We consider here a different approximate problem (2.8) for a parameter $\theta < 0$.

$$\begin{gathered} -\Delta z = \lambda\big(z^q - z^{-\beta} + \theta)\big) \quad \text{in } \Omega,\\ z > 0 \quad \text{in } \Omega,\\ z = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.8}$$

**Lemma 2.4.** *For each $\theta \in (\theta_0, 0)$ there exists a function $z_\theta$ which is a maximal solution of* (2.8)*. If $\theta < \theta'$ then $z_\theta \leq z_{\theta'}$ and $z_\theta \neq z_{\theta'}$.*

*Proof.* Fix a $\lambda \in (\Lambda, \infty)$ and choose $\lambda' \in (\Lambda, \lambda)$. Let

$$-\Delta V_\lambda = \lambda \text{ in } \Omega; \quad V_\lambda = 0 \text{ in } \partial\Omega$$

and $u_\lambda$ be the maximal solution of (1.1). Define $\underline{z}_\epsilon = \frac{\lambda}{\lambda'} u_{\lambda'} - \epsilon V_\lambda$. Then for some positive constants $C_1, C_2$

$$\underline{z}_\epsilon - u_{\lambda'} = \Big(\frac{\lambda-\lambda'}{\lambda'}\Big) u_{\lambda'} - \epsilon V_\lambda \geq (C_1 - \epsilon C_2)\, d(x, \partial\Omega).$$

If we choose $0 < \epsilon < |\theta_0|$ for some small $\theta_0 < 0$, we have $\underline{z}_\epsilon > u_{\lambda'}$. For all $\theta \in (\theta_0, 0)$ define

$$\underline{z}_\theta = \frac{\lambda}{\lambda'} u_{\lambda'} + \theta V_\lambda. \tag{2.9}$$

Then $-\Delta \underline{z}_\theta = \lambda(u_{\lambda'}^q - u_{\lambda'}^{-\beta} + \theta) \leq \lambda(\underline{z}_\theta^q - \underline{z}_\theta^{-\beta} + \theta)$ and hence a sub solution of (2.8). It is easy to check that $\overline{z}_\theta = v_\lambda$ is a super solution of (2.8) for all $\theta < 0$. Since $u_{\lambda'} < v_{\lambda'}$

$$\underline{z}_\theta - \overline{z}_\theta = \frac{\lambda}{\lambda'} u_{\lambda'} + \theta V_\lambda - v_\lambda < \frac{\lambda}{\lambda'} v_{\lambda'} - v_\lambda = \Big(\frac{\lambda}{\lambda'}(\lambda')^{\frac{1}{1-q}} - \lambda^{\frac{1}{1-q}}\Big) v_1 < 0.$$

Thus there exists a solution $z_\theta$ of (2.8) in between the ordered pair $[\underline{z}_\theta, \overline{z}_\theta]$. As before using comparison lemma one can easily observe that $z_\theta$ is the maximal solution of (2.8). Now let $\theta < \theta'$ and $z_\theta, z_{\theta'}$ be the maximal solutions of (2.8) and (2.8) with $\theta = \theta'$ respectively. Then

$$-\Delta z_\theta \leq \lambda(z_\theta^q - z_\theta^{-\beta} + \theta') \quad \text{and} \quad z_\theta \leq \overline{z}_{\theta'}.$$

Since $z_{\theta'}$ is the maximal solution of (2.8) with $\theta = \theta'$ we conclude that $z_\theta \leq z_{\theta'}$. $\square$

**Theorem 2.5.** *The maximal solution $u_\lambda$ of* (1.1) *is stable.*

*Proof.* Let $\Lambda_1^\theta(\lambda)$ denote the principal eigenvalue of (2.8). Repeating the calculations of Proposition 2.2 we can show that $\Lambda_1^\theta(\lambda) \geq 0$. If $\theta_1 < \theta_2$ using the strict concavity of $f$ and Lemma 2.4 we have for all $\varphi \in H_0^1(\Omega)$, $\|\varphi\|_2 = 1$,

$$\int_\Omega |\nabla\varphi|^2 - \lambda \int_\Omega f'(z_{\theta_1})\varphi^2 < \int_\Omega |\nabla\varphi|^2 - \lambda \int_\Omega f'(z_{\theta_2})\varphi^2.$$

Since $\inf_{\varphi\in H_0^1(\Omega)} \int_\Omega |\nabla\varphi|^2 - \lambda \int_\Omega f'(z_\theta)\varphi^2$ is attained, we have $\Lambda_1^{\theta_1}(\lambda) < \Lambda_1^{\theta_2}(\lambda)$. Observe that $z_\theta \to u_\lambda$ as $\theta \to 0^-$ and $\lim_{\theta\to 0^-} \Lambda_1^\theta(\lambda) = \Lambda_1(\lambda)$. Thus

$$\Lambda_1(\lambda) > \Lambda_1^\theta(\lambda) \geq 0$$

which is the main result. □

## 3. Bifurcation analysis

In the previous section we have shown that for each $\lambda > \Lambda$ there exists a maximal solution for (1.1). In this section we try to understand this maximal branch of solution using bifurcation theory. For $\lambda > \Lambda$, consider the function $u_{\lambda'}$ which is a solution of (1.1) with $\lambda = \lambda'$ for some $\lambda' \in [\Lambda, \lambda)$ and $v_\lambda$ as in (2.1). To ease notation we omit the subscript $\lambda$ and denote $\psi = \psi_\lambda = u_{\lambda'}$ and $\phi = \phi_\lambda = v_\lambda$, then clearly $\psi < \phi$. Let

$$\mathcal{C}_\lambda = \{u \in C_0(\overline{\Omega}) : \psi \leq u \leq \phi\}. \tag{3.1}$$

For each $u \in \mathcal{C}_\lambda$ there exists $w \in C_0^1(\overline{\Omega}) \cap C^2(\Omega)$ which is a solution of

$$-\Delta w = \lambda f(u) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \tag{3.2}$$

The existence of $w \in W^{2,p}(\Omega)$ easily follows from the lower estimate on $u$ and the regularity of $w$ by [13] (see Section 5 of [10] for the details). Since we would repeatedly use the regularity result of Gui-Lin [13], for the sake of completeness we quote the result below.

**Theorem 3.1** (Gui-Lin [13, Prop. 3.4])**.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$|\Delta u(x)| \leq M d(x)^{-\beta} \quad \text{and} \quad |u(x)| \leq M d(x)^\alpha$$

*for some positive constants $M, \alpha$. Then there exists some $\gamma \in (0,1)$ depending upon $\beta$ and $\alpha$ such that $\|u\|_{C^{1,\gamma}(\overline{\Omega})} \leq C(M,\alpha,\beta)$.*

We can in fact prove that the solution $w$ of (3.2) belongs to $\mathcal{C}_\lambda$. One can observe that $w \leq \phi$ since $\phi$ is a supersolution of (3.2). It is not clear if $\psi$ is a sub solution of (3.2) or not. But still by the specific choice of $\psi$ we can show that

$$-\Delta(w - \psi) = \lambda f(u) - \lambda' g(\psi) \geq \frac{\lambda - \lambda'}{\lambda'}(-\Delta\psi) \tag{3.3}$$

Since $\lambda' < \lambda$ it follows that $w > \psi$ and hence $w \in \mathcal{C}_\lambda$. For a fixed $\lambda \in (\Lambda, \infty)$ we define the map

$$\mathcal{A}{:}\mathcal{C}_\lambda \to \mathcal{C}_\lambda \text{ is defined as } \mathcal{A}(u) = w \text{ if } w \text{ is a solution of (3.2).} \tag{3.4}$$

We aim to employ the well known abstract setting of bifurcation theory to prove the existence of a connected branch of solutions. If we consider the map $\mathcal{A}{:}\mathcal{C}_\lambda \to \mathcal{C}_\lambda$ it is not possible to use the implicit function theorem since the set $\mathcal{C}_\lambda \subset C_0(\overline{\Omega})$ has empty interior. Hence we introduce the space $C_e(\overline{\Omega})$ as in [1] and consider the set $\mathcal{C}_\lambda$ with the topology induced from $C_e(\overline{\Omega})$ in which $\mathcal{C}_\lambda$ has nonempty interior.

Let $e \in C^2(\overline{\Omega})$ denote the unique positive solution of

$$-\Delta e = 1 \quad \text{in } \Omega$$
$$e = 0 \quad \text{on } \partial\Omega.$$

Then $e(x) > 0$ in $\Omega$, $\frac{\partial e}{\partial \nu} < 0$ on $\partial\Omega$ and thus $e(x) \geq kd(x, \partial\Omega)$ for some constant $k > 0$. $C_e(\overline{\Omega})$ is the set of functions in $u \in C_0(\overline{\Omega})$ such that $-te \leq u \leq te$ for some $t \geq 0$. $C_e(\overline{\Omega})$ equipped with $\|u\|_e = \inf\{t > 0 : -te \leq u \leq te\}$ is a Banach space. Also the following continuous embedding holds:

$$C_0^1(\overline{\Omega}) \hookrightarrow C_e(\overline{\Omega}) \hookrightarrow C_0(\overline{\Omega}).$$

Further $C_e(\overline{\Omega})$ is an ordered Banach space(OBS) whose positive cone $P_e = \{u \in C_e(\overline{\Omega}) : u(x) \geq 0\}$ is normal and has non empty interior. In particular the interior of $P_e$ consists of all those functions $u \in C(\overline{\Omega})$ with $t_1 e \leq u \leq t_2 e$ for some $t_1, t_2 > 0$. Define

$$\mathcal{M}_\lambda = \{u \in C_e(\overline{\Omega}) : \psi \leq u \leq \phi\} \tag{3.5}$$

Using the lower and upper bounds for $\psi$ and $\phi$ in terms of $d(x, \Omega)$ we find that set theoretically $\mathcal{C}_\lambda$ is same as $\mathcal{M}_\lambda$. But topologically they are different and in fact $\mathcal{M}_\lambda$ has non empty interior which we denote by $\mathcal{U}_\lambda$ where

$$\mathcal{U}_\lambda = \{u \in \mathcal{M}_\lambda : \psi + t_1 e \leq u \leq \phi - t_2 e \text{ for some } t_1, t_2 > 0\}. \tag{3.6}$$

By definition the set $\mathcal{U}_\lambda$ is open and we denote the restriction of the map $\mathcal{A}$ to $\mathcal{U}_\lambda$ as $\mathcal{A}$ itself. From (3.4) $\mathcal{A}$ maps $\mathcal{U}_\lambda$ to $\mathcal{C}_\lambda$. In the next proposition we prove that $\mathcal{A}$ maps $\mathcal{U}_\lambda$ to itself and it is a $C^2$ map.

**Proposition 3.2.** *The map $\mathcal{A}$:$\mathcal{U}_\lambda \to \mathcal{U}_\lambda$ is twice continuously differentiable. The map $\mathcal{A}'(u)$:$C_e(\overline{\Omega}) \to C_e(\overline{\Omega})$ is continuous linear and compact.*

*Proof.* Let $u \in \mathcal{U}_\lambda$, i.e there exists some $t_1, t_2 > 0$ such that $\psi + t_1 e \leq u \leq \phi - t_2 e$ and let $\mathcal{A}(u) = w$. Then $-\Delta(w - \phi) < 0$ in $\Omega$ and $w - \phi = 0$ on $\partial\Omega$, and by Hopf Maximum principle there exists a $\tilde{t}_2 > 0$ for which $w \leq \phi - \tilde{t}_2 e$. From our previous discussion (3.3) if we take $\tilde{t}_1 = \frac{\lambda - \lambda'}{\lambda}$ we find $w \geq \psi + \tilde{t}_1 e$. Thus $\mathcal{A}$ maps $\mathcal{U}_\lambda$ into itself. Proof of the smoothness of the map $\mathcal{A}$ and the compactness of $\mathcal{A}'(u)$ is much technical and we shall give the details in the Appendix. $\square$

Next we shall treat $\lambda$ as a variable and define the map $A$:$(\Lambda, \infty) \times \mathcal{U}_\lambda \to \mathcal{U}_\lambda$ as $A(\lambda, u) = w$ if $w$ is a solution of

$$-\Delta w = \lambda f(u) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \tag{3.7}$$

Fix $\lambda_1, \lambda_2$ such that $\Lambda < \lambda_1 < \lambda_2 < \infty$. Then for all $\lambda \in [\lambda_1, \lambda_2]$ we can in fact fix the indexed set $\mathcal{U}_\lambda$ independent of $\lambda$ in the following way. By the definition of $\Lambda$ there exists a $\lambda' \in [\Lambda, \lambda_1)$ and (1.1) with $\lambda = \lambda'$ is solvable. Let $\psi = u_{\lambda'}$ and $\phi = v_{\lambda_2}$ and let $\mathcal{M}_\lambda$ and $\mathcal{U}_\lambda$ defined as before in (3.5) and (3.6) for this choice of $\psi$ and $\phi$. Now $\mathcal{U}_\lambda$ is independent of $\lambda$ for all $\lambda \in [\lambda_1, \lambda_2]$. For this particular choice of $\mathcal{U} = \mathcal{U}_\lambda$ we can prove that the map $A$ is $C^2$ in $\lambda$ and $u$ variable in $(\lambda_1, \lambda_2) \times \mathcal{U}$.

**Theorem 3.3.** *There exists a connected branch of positive maximal solutions of* (1.1) *bifurcating from $\lambda_\infty = \infty$.*

*Proof.* Fix an open interval $I \subset (\Lambda, \infty)$ and $\bar{I}$ compactly contained in $(\Lambda, \infty)$. Let $I = (\lambda_1, \lambda_2)$ and $\psi = u_{\lambda'}$ and $\phi = v_{\lambda_2}$ as before. Thus for all $\lambda \in I$ we define

$\mathcal{M} = \{u \in C_e(\overline{\Omega}) : \psi \leq u \leq \phi\}$ and $\mathcal{U}$ to be the interior of $\mathcal{M}$. Consider the map $F{:}I \times \mathcal{U} \to \mathcal{U}$ defined as

$$F(\lambda, u) = u - A(\lambda, u). \tag{3.8}$$

Clearly the zeroes of $F$ are the solutions of (1.1) and $F(\lambda, u_\lambda) = 0$ where $u_\lambda$ is the maximal solution of (1.1). Note that $F{:}I \times \mathcal{U} \to \mathcal{U}$ is a $C^2$ map and $\partial_u F(\lambda, u) = I - \partial_u A(\lambda, u)$ is a compact perturbation of identity. Fix $\lambda_0 \in I$ and let $u_0 = u_{\lambda_0}$ be the maximal solution of (1.1) with $\lambda = \lambda_0$, then $F(\lambda_0, u_0) = 0$. From Theorem 2.5 we know that $u_0$ is a stable solution and hence $\partial_u F(\lambda_0, u_0)$ is one-one. Now by Fredholm alternative it is onto as well. Thus the linear map $\partial_u F(\lambda_0, u_0)$ is bijective and continuous, hence by open mapping theorem $\partial_u F(\lambda_0, u_0)$ has a continuous inverse. Now we can apply implicit function theorem around $(\lambda_0, u_0)$ and deduce that there exists a $C^2$ curve $(\lambda, u(\lambda)) \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \times \mathcal{U}$ such that the set of all solutions of $F(\lambda, u) = 0$ in a neighbourhood of $(\lambda_0, u_0)$ is given by $(\lambda, u(\lambda))$. Note that this $u(\lambda)$ may be different from the maximal solution $u_\lambda$.

If we can show that $\lambda \longmapsto u_\lambda$ (where $u_\lambda$ is the maximal solution) is continuous then by the uniqueness of the solution near $(\lambda_0, u_0)$ we have a $u(\lambda) = u_\lambda$. On the contrary suppose $\lambda \longmapsto u_\lambda$ is not continuous at $\lambda_0$. i.e. there exists a sequence $\lambda_n \to \lambda_0$ such that $u_{\lambda_n} \not\to u_0$. One can use Hardy's inequality to prove that $\{u_{\lambda_n}\}$ is bounded in $H_0^1(\Omega)$ and hence up to a sub sequence $u_{\lambda_n} \rightharpoonup \tilde{u}$ in $H_0^1(\Omega)$. It is also easy to check that $\tilde{u}$ is a solution of $(P_{\lambda_0})$. Since $u_0$ is the maximal solution of $(P_{\lambda_0})$ we have

$$\tilde{u} \leq u_0 \quad \text{and} \quad \tilde{u} \neq u_0. \tag{3.9}$$

On the other hand we have $u(\lambda_n) \to u_0$ and $u(\lambda_n) \leq u_{\lambda_n}$. Taking limit as $n \to \infty$ we find $u_0 \leq \tilde{u}$ which contradicts (3.9). We have now $u(\lambda) = u_\lambda$ and hence by implicit function theorem $\lambda \to u_\lambda$ is a $C^2$ map which completes the proof of theorem. □

**Remark 3.4.** The smoothness of the map $\lambda \to u_\lambda$ for $\lambda \in (\Lambda, \infty)$ is completely determined by the smoothness of the operator $\mathcal{A}$. We can in fact prove that the map is infinitely many times differentiable, hence $\lambda \to u_\lambda$ is a $C^\infty$ map.

The proof of our main result now follows from Theorem 2.1, equations (2.2), (2.3), Theorems 2.5, 3.3 and Remark 3.4.

## 4. Bifurcation analysis at $\lambda = \Lambda$

**Proposition 4.1.** *There exists a non-negative solution $u_\Lambda$ solving* (1.1) *with $\lambda = \Lambda$ in the weak sense. The Lebesgue measure of the set $\{x : u_\Lambda(x) = 0\}$ is zero.*

*Proof.* Let $\{u_n\}$ denote the sequence of maximal solutions of $(P_{\lambda_n})$ where $\lambda_n \downarrow \Lambda$ and $\lambda_n < \overline{\lambda}$. If $\overline{v}$ denote the solution of (2.1) for $\lambda = \overline{\lambda}$, we have $0 < u_{n+1} \leq u_n \leq \overline{v}$ and

$$\int_\Omega |\nabla u_n|^2 = \lambda_n \int_\Omega (u_n^{q+1} - u_n^{1-\beta}) \leq \lambda_n \int_\Omega u_n^{q+1} \leq \overline{\lambda} \int_\Omega \overline{v}^{q+1}. \tag{4.1}$$

Thus the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and denote the weak limit of $u_n$ as

$$u_\Lambda := \lim_{n\to\infty} u_n. \tag{4.2}$$

We will show that $u_\Lambda$ is in fact a solution of (1.1) with $\lambda = \Lambda$ in the weak sense. As a first step we shall prove that $\{x \in \Omega : u_\Lambda(x) = 0\}$ has Lebesgue measure zero. Let $\phi_1$ be the first eigenfunction of $-\Delta$ and $\gamma \in (0, 1), \epsilon > 0$. Consider the function

$\psi = (\phi_1 + \epsilon)^\gamma - \epsilon^\gamma \in H_0^1(\Omega)$. Then from a direct computation we find $-\Delta\psi \geq 0$ and hence $< -\Delta u_n, \psi >_{H_0^1(\Omega)\times H^{-1}(\Omega)} \geq 0$ which implies

$$\lambda_n \int_\Omega (u_n^q - u_n^{-\beta})\psi \geq 0. \tag{4.3}$$

Thus

$$\int_\Omega u_n^{-\beta}((\phi_1 + \epsilon)^\gamma - \epsilon^\gamma) \leq \int_\Omega u_n^q((\phi_1 + \epsilon)^\gamma - \epsilon^\gamma).$$

Now letting $\epsilon \to 0$ and $\gamma \to 0$ we have $\int_\Omega u_n^{-\beta} \leq \int_\Omega u_n^q \leq \int_\Omega \overline{v}^q < \infty$. Once again using Fatou's lemma,

$$\int_\Omega u_\Lambda^{-\beta} < \infty \tag{4.4}$$

which in turn implies $\{x \in \Omega : u_\Lambda(x) = 0\}$ is of Lebesgue measure zero. Now we will prove that $u_\Lambda$ is a weak solution of (1.1) with $\lambda = \Lambda$. We have

$$\int_\Omega \nabla u_n \nabla\varphi = \lambda_n \int_\Omega (u_n^q - u_n^{-\beta})\varphi \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

The only difficulty arises while passing through the limit in the term involving $u_n^{-\beta}$. But note that $u_n^{-\beta}|\varphi| \leq u_\Lambda^{-\beta}\|\varphi\|_\infty \in L^1(\Omega)$ and by dominated convergence theorem $u_\Lambda$ is a weak solution of (1.1) with $\lambda = \Lambda$. □

Next we shall discuss a sufficient condition that ensures the existence of multiple solutions for (1.1). We make a crucial assumption that the non-negative solution $u_\Lambda$ belongs to $C_e(\overline{\Omega})$ and is bounded below by $cd(x, \partial\Omega)$ for some $c > 0$. By the above assumption $u_\Lambda$ is positive and it can be shown that the (1.1) with $\lambda = \Lambda$ admits a unique positive solution. Indeed, if $\tilde{u}_\Lambda$ is another positive solution of (1.1) with $\lambda = \Lambda$ then we can show that a convex combination of $u_\Lambda$ and $\tilde{u}_\Lambda$ is a positive solution of (1.1) with $\lambda = \lambda'$ for some $\lambda' < \Lambda$ which is impossible (see [19, Proposition 5] for details). Now the uniqueness in the class of positive solutions imply that $u_\Lambda$ is maximal and by Proposition 2.2, $\Lambda_1(\Lambda) = 0$. Indeed, since $u_\Lambda$ is maximal it is clear that $\Lambda_1(\Lambda) \geq 0$. Suppose $\Lambda_1(\Lambda) > 0$, then implicit function theorem would guarantee the existence of a positive solution for some $\lambda < \Lambda$ which would contradict the definition of $\Lambda$. Next we shall prove a local bifurcation result of Crandall-Rabinowitz [3] for an infinite semipositone problem. Similar ideas of the proof were used in [4, 11] when the authors studied a positone convex non-linearity.

**Lemma 4.2.** *The solutions of $F(\lambda, u) = 0$ near $(\Lambda, u_\Lambda)$ are described by a curve $(\lambda(s), u(s)) = (\Lambda + \tau(s), u_\Lambda + s\phi_\Lambda + x(s))$ where $s \to (\tau(s), x(s)) \in \mathbb{R} \times C_e(\overline{\Omega})$ is a continuously differentiable function near $s = 0$ with $\tau(0) = \tau'(0) = 0$, $\tau''(0) > 0$ and $x(0) = x'(0) = 0$. Moreover $\tau$ is of class $C^2$ near $0$.*

*Proof.* Consider the map $F(\lambda, u)$ and the Gateaux derivative of $F$ at $(\Lambda, u_\Lambda)$. Clearly $\partial_\lambda F(\Lambda, u_\Lambda) = -\partial_\lambda A(\Lambda, u_\Lambda) = -\frac{u_\Lambda}{\Lambda}$. Now consider the null space of the linear operator $\partial_u F(\Lambda, u_\Lambda)$. Since $\Lambda_1(\Lambda) = 0$, there exists a $\phi_\Lambda \in H_0^1(\Omega)$ such that

$$\begin{gathered} -\Delta\phi_\Lambda = \Lambda f'(u_\Lambda)\phi_\Lambda \quad \text{in } \Omega, \\ \phi_\Lambda = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

By the interior regularity results the eigenfunction $\phi_\Lambda \in C^2(\Omega) \cap H_0^1(\Omega)$ itself. Now by [12, Theorem 8.16], the principal eigenvalue $\Lambda_1(\Lambda)$ is simple and the corresponding eigenfunction $\phi_\Lambda$ is positive. Hence $\ker(\partial_u F(\Lambda, u_\Lambda))$ is one dimensional and is spanned by $\phi_\Lambda$. We claim that $\partial_\lambda F(\Lambda, u_\Lambda) \notin \ker \partial_u F(\Lambda, u_\Lambda)$. If so, then for some constant $k$ we have $u_\Lambda = k\phi_\Lambda$. This implies $f(u_\Lambda) = kf'(u_\Lambda)\phi_\Lambda$ which is impossible since RHS is has a constant sign and LHS changes its sign inside $\Omega$ and hence that $\partial_\lambda F(\Lambda, u_\Lambda) \notin \ker \partial_u F(\Lambda, u_\Lambda)$.

Let $X$ be any complement of the span of $\{\phi_\Lambda\}$ in $C_e(\overline{\Omega})$ and the map $\theta:\mathbb{R} \times \mathbb{R} \times X \to C_e(\overline{\Omega})$ be defined as

$$\theta(s, \tau, x) = F(\Lambda + \tau, u_\Lambda + s\phi_\Lambda + x)$$

Then, we claim that $\partial_{\tau,x}\theta(0, 0, 0) = (\partial_\lambda F(\Lambda, u_\Lambda), \partial_u F(\Lambda, u_\Lambda))$ is an isomorphism from $\mathbb{R}\times X$ on to $X$. Since $\partial_\lambda F(\Lambda, u_\Lambda) \notin \operatorname{Range} \partial_u F(\Lambda, u_\Lambda)$ the map $\partial_{\tau,x}\theta(0, 0, 0)$ is one-one in $\mathbb{R} \times X$. Now by Fredholm alternative $\partial_{\tau,x}\theta(0, 0, 0)$ is also onto. Now by implicit function theorem there exists an $\epsilon > 0$ and a $C^2$ function $p:(-\epsilon, \epsilon) \to \mathbb{R}\times X$ such that $p(s) = (\tau(s), x(s))$ and $\theta(s, p(s)) = 0$, $\tau(0) = 0$ and $x(0) = 0$. i.e., $F(\Lambda + \tau(s), u_\Lambda + s\phi_\Lambda + x(s)) = 0$. Now differentiating with respect to $s$ variable and evaluating at $s = 0$, we obtain

$$\partial_\lambda F(\Lambda, u_\Lambda)\tau'(0) + \partial_u F(\Lambda, u_\Lambda)x'(0) = 0.$$

Since $\partial_\lambda F(\Lambda, u_\Lambda) \notin \mathit{Range}(\partial_u F(\Lambda, u_\Lambda))$ we have $\tau'(0) = x'(0) = 0$. Once again differentiating $F(\Lambda + \tau(s), u_\Lambda + s\phi_\Lambda + x(s))$ we obtain

$$\partial_\lambda F(\Lambda, u_\Lambda)\tau''(0) + \partial_{uu} F(\Lambda, u_\Lambda)\phi_\Lambda^2 + \partial_u F(\Lambda, u_\Lambda)x''(0) = 0. \tag{4.5}$$

Let us write the middle term in the above expression as $W = \partial_{uu} F(\Lambda, u_\Lambda)\phi_\Lambda^2$. Then one can easily check that

$$\begin{gathered} \Delta W = \Lambda f''(u_\Lambda)\phi_\Lambda^2 \quad \text{in } \Omega, \\ W = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Since $f$ is concave, by maximum principle $W \geq 0$. Now call $w = \partial_u F(\Lambda, u_\Lambda)x''(0)$ which by definition is equal to $x''(0) - \partial_u A(\Lambda, u_\Lambda)x''(0)$. If $w_1 = \partial_u A(\Lambda, u_\Lambda)x''(0)$ then $w_1$ solves

$$\begin{gathered} -\Delta w_1 = \Lambda f'(u_\Lambda)x''(0) \quad \text{in } \Omega, \\ w_1 = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Thus $\int_\Omega \nabla w_1 \nabla\phi_\Lambda = \int_\Omega \Lambda f'(u_\Lambda)x''(0)\phi_\Lambda$. From the definition of $\phi_\Lambda$, we also have $\int_\Omega \nabla w_1 \nabla\phi_\Lambda = \int_\Omega \Lambda f'(u_\Lambda)w_1\phi_\Lambda$. Thus

$$\int_\Omega \Lambda f'(u_\Lambda)\phi_\Lambda w = 0.$$

Now multiplying (4.5) by $\Lambda f'(u_\Lambda)\phi_\Lambda$ and integrating over $\Omega$,

$$-\tau''(0)\int_\Omega u_\Lambda f'(u_\Lambda)\phi_\Lambda + \int_\Omega W\Lambda f'(u_\Lambda)\phi_\Lambda = 0.$$

We know $f$ is monotonically increasing and $\phi_\Lambda$ is a non-negative function and $W \geq 0$. Thus $\tau'' \geq 0$ which completes the proof. $\square$

*Proof of Theorem 1.5.* Suppose that alternative (a) does not hold. Then from the properties of $C_e(\overline{\Omega})$ (see Section 3) there exists a constant $c_\Lambda > 0$ such that $u_\Lambda(x) \geq c_\Lambda d(x, \partial\Omega)$. Thus $\Lambda_1(\Lambda)$ is well defined and is non-negative. Now by the definition of $\Lambda$ the principal eigenvalue $\Lambda_1(\Lambda)$ cannot be positive and hence the proof of Lemma 4.2 is applicable and which completes the Theorem 1.5. $\square$

## 5. Appendix

**Proposition 5.1.** *The map $\mathcal{A}:\mathcal{U}_\lambda \to \mathcal{U}_\lambda$ is a $C^2$ map.*

*Proof.* Let $u \in \mathcal{U}_\lambda$, i.e. there exists some $t_1, t_2 > 0$ such that $\psi + t_1 e \le u \le \phi - t_2 e$ and let $\mathcal{A}(u) = w$. Then $-\Delta(w - \phi) < 0$ in $\Omega$ and $w - \phi = 0$ on $\partial\Omega$, and by Hopf Maximum principle there exists a $\tilde{t}_2 > 0$ for which $w \le \phi - \tilde{t}_2 e$. From our previous discussion (3.3) if we take $\tilde{t}_1 = \frac{\lambda - \lambda'}{\lambda}$ we find $w \ge \psi + \tilde{t}_1 e$. Thus $\mathcal{A}$ maps $\mathcal{U}_\lambda$ into itself.

**Step I.** $\mathcal{A}:\mathcal{U}_\lambda \to \mathcal{U}_\lambda$ is continuous. Let $h \in C_e(\overline{\Omega})$ with $\|h\|_{C_e(\overline{\Omega})}$ small so that $u + h \in \mathcal{U}_\lambda$ and $\mathcal{A}(u + h) = w_h$. Then $(w_h - w)$ satisfies $-\Delta(w_h - w) = \lambda\left(f(u + h) - f(u)\right)$ in $\Omega$ and $w_h - w = 0$ on $\partial\Omega$. For $p \in (1, \frac{1}{\beta})$ using $L^p$ estimate and dominated convergence theorem we find

$$\|w_h - w\|_{W^{2,p}(\Omega)} \le C\|f(u + h) - f(u)\|_{L^p(\Omega)} \to 0 \quad \text{as } \|h\|_{C_e(\overline{\Omega})} \to 0. \tag{5.1}$$

Now since $w_h$ and $w$ belongs to $\mathcal{U}_\lambda$ we have $|w_h - w| \le Cd(x, \partial\Omega)$. Now we can apply Theorem 3.1 and obtain $\|w_h - w\|_{C^{1,\gamma}(\Omega)}$ is bounded. Thanks to Ascoli-Arzela theorem and (5.1) we have $w_h \to w$ in $C_0^1(\overline{\Omega})$. Finally using the continuity of the embedding $C_0^1(\overline{\Omega}) \hookrightarrow C_e(\overline{\Omega})$ we conclude that $\mathcal{A}:\mathcal{U}_\lambda \to \mathcal{U}_\lambda$ is continuous.

**Step II.** The map $\mathcal{A}:\mathcal{U}_\lambda \to \mathcal{U}_\lambda$ is $C^1$. For a given $u \in \mathcal{U}_\lambda$ and $h \in C_e(\overline{\Omega})$ consider the solution operator $z$ defined as

$$-\Delta z = \lambda f'(u)h \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{5.2}$$

Let us denote $\xi_\lambda \in C_0^1(\overline{\Omega}) \cap C^2(\Omega)$ be the unique solution of

$$-\Delta\xi_\lambda = \lambda\xi_\lambda^{-\beta} \text{ in } \Omega \quad \text{and} \quad \xi_\lambda = 0 \text{ on } \partial\Omega.$$

The existence and behaviour of the solution $\xi_\lambda$ near $\partial\Omega$ is studied in [5]. It is well known that $\xi_\lambda \sim d(x, \partial\Omega)$ and $d(x, \partial\Omega) \sim e(x)$ and thus $\xi_\lambda \sim e(x)$. We can estimate $f'(u)h$ in terms of $\xi_\lambda$ as

$$|f'(u)h| \le C_0.e(x)^{-(\beta+1)}|h(x)| \le \frac{C_1\|h\|_{C_e(\overline{\Omega})}}{\xi_\lambda^\beta}$$

for some positive constant $C_1$. Thus,

$$C_1\|h\|\Delta\xi_\lambda \le -\Delta z = \lambda f'(u)h \le \lambda C_1\|h\|\xi_\lambda^{-\beta} = -C_1\|h\|\Delta\xi_\lambda,$$

By the comparison principle and since $\xi_\lambda(x) \sim e(x)$ we have for some $C > 0$,

$$|z(x)| \le C\|h\|_{C_e(\overline{\Omega})}e(x) \tag{5.3}$$

Now as in Step I, let $w_h = A(u + h)$ and $w = A(u)$, then using Taylor's theorem

$$-\Delta(w_h - w - z) = \lambda f''(u + \theta h)\frac{h^2}{2} \quad \text{for some } \theta(x) \in (0, 1).$$

Since $|f''(u + \theta h)h^2| \le C\|h\|^2_{C_e(\overline{\Omega})}e(x)^{-\beta}$ we have

$$\Big\|\frac{w_h - w - z}{\|h\|_{C_e(\overline{\Omega})}}\Big\|_{W^{2,p}(\Omega)} \le C\|h\|_{C_e(\overline{\Omega})}.$$

Up to a sub sequence $(w_h - w - z)/\|h\|_{C_e(\overline{\Omega})}$ converges to 0 as $\|h\|_{C_e(\overline{\Omega})} \to 0$. It can be shown that $|w_h - w - z|/\|h\|_{C_e(\overline{\Omega})} \le Cd(x, \partial\Omega)$ and thus $(w_h - w - z)/\|h\|_{C_e(\overline{\Omega})}$

satisfies the assumptions of theorem 3.1. Hence,

$$\frac{w_h - w - z}{\|h\|_{C_e(\overline{\Omega})}} \text{ is bounded in } C^{1,\gamma}(\overline{\Omega}) \tag{5.4}$$

Now by using Ascoli-Arzela theorem and continuity of the embedding $C_0^1(\Omega) \hookrightarrow C_e(\overline{\Omega})$ we deduce that $\frac{w_h - w - z}{\|h\|_{C_e(\overline{\Omega})}} \to 0$ in $C_e(\overline{\Omega})$. If we call $\mathcal{A}'(u)h = z$ then

$$\|\mathcal{A}(u+h) - \mathcal{A}(u) - \mathcal{A}'(u)h\|_{C_e(\overline{\Omega})} = o(\|h\|).$$

Now from (5.3) we note that $\mathcal{A}'(u){:}C_e(\overline{\Omega}) \to C_e(\overline{\Omega})$ is a bounded linear map and hence the map $\mathcal{A}{:}\mathcal{U}_\lambda \to \mathcal{U}_\lambda$ is differentiable. It remains to show that $\mathcal{A}$ is continuously differentiable, i.e. $u \to \mathcal{A}'(u)$ is continuous. Let $\tilde{u} \in C_e(\overline{\Omega})$ such that $\|\tilde{u} - u\| < \delta$ and $\mathcal{A}'(\tilde{u})h = \tilde{z}$ for some $h \in C_e(\overline{\Omega})$. Using Taylor's theorem there exists some $\theta(x) \in [u, \tilde{u}]$ and

$$|f'(\tilde{u}) - f'(u)| = \lambda |f''(\theta)|\,|(\tilde{u} - u)h| \le \frac{C_0 e(x)^2}{d(x)^{\beta+2}} \delta \|h\|_{C_e(\overline{\Omega})} \le \frac{C_1 \delta}{\xi_1^\beta} \|h\|_{C_e(\overline{\Omega})}$$

where the constant $C_1$ is independent of $u$ and $\tilde{u}$. As before estimating $-\Delta(\tilde{z} - z)$ from above and below and using maximum principle we have $|\tilde{z}(x) - z(x)| \le C\delta\|h\|e(x)$. Now taking supremum over $\|h\|_{C_e(\overline{\Omega})} \le 1$ we have

$$\|\mathcal{A}'(\tilde{u}) - \mathcal{A}'(u)\| \le C\|\tilde{u} - u\|_{C_e(\overline{\Omega})}$$

and thus $\mathcal{A}$ is continuously differentiable.

**Step III.** The map $\mathcal{A}$ is $C^2$. Now that we have proved $\mathcal{A}{:}\mathcal{U}_\lambda \to \mathcal{U}_\lambda$ is $C^1$, using the same idea we can prove that $\mathcal{A}$ is twice continuously differentiable.In order to avoid the repetition of the same arguments we skip the details of the proof of step III. □

From (5.4) of above proposition we know that $\|\frac{w_h - w - z}{\|h\|}\|_{C^{1,\gamma}(\overline{\Omega})}$ is bounded and similarly $\|\frac{w_h - w}{\|h\|}\|_{C^{1,\gamma}}$ is also bounded. So

$$\begin{aligned}
\|\mathcal{A}'(u)h\|_{C^{1,\gamma}(\overline{\Omega})} &= \|z\|_{C^{1,\gamma}(\overline{\Omega})} \le \|w_h - w - z\|_{C^{1,\gamma}(\overline{\Omega})} + \|w_h - w\|_{C^{1,\gamma}(\overline{\Omega})} \\
&= \|h\| \|\frac{w_h - w - z}{\|h\|}\|_{C^{1,\gamma}(\overline{\Omega})} + \|h\| \|\frac{w_h - w}{\|h\|}\|_{C^{1,\gamma}(\overline{\Omega})} \\
&\le M\|h\|_{C_e(\overline{\Omega})}
\end{aligned}$$

which implies $\mathcal{A}'(u) \in BL(C_e(\overline{\Omega}), C^{1,\gamma}(\overline{\Omega}))$ and hence $\mathcal{A}'(u){:}C_e(\overline{\Omega}) \to C_e(\overline{\Omega})$ is compact.

**Corollary 5.2.** $\mathcal{A}'(u){:}C_e(\overline{\Omega}) \to C_e(\overline{\Omega})$ *is continuous linear and compact.*

**Acknowledgments.** This research was supported by INSPIRE faculty fellowship (DST/INSPIRE/04/2015/003221) at the Indian Statistical Institute, Bangalore Centre.

## References


[1] H. Amann; *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces,* SIAM Rev. 18 (1976), no. 4, 620–709.

[2] A. Ambrosetti, D. Arcoya, B. Buffoni; *Positive solutions for some semi-positone problems via bifurcation theory.* Differential Integral Equations 7 (1994), no. 3-4, 655–663.

[3] M. G. Crandall, P. H. Rabinowitz; *Bifurcation from simple eigenvalues,* J. Funct. Anal., 8 (1971), 321–340.

[4] M. G. Crandall, P. H. Rabinowitz, *Bifurcation, perturbation of simple eigenvalues and linearized stability.* Arch. Rational Mech. Anal. 52 (1973), 161–180.
[5] M. G. Crandall, P. H. Rabinowitz, L. Tartar; *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations, 2 (1977), 193–222.
[6] Alfonso Castro, Gadam Sudhasree; *Uniqueness of stable and unstable positive solutions for semipositone problems.* Nonlinear Anal. 22 (1994), no. 4, 425–429.
[7] Alfonso Castro, Gadam, Sudhasree; R. Shivaji; *Positive solution curves of semipositone problems with concave nonlinearities.* Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), no. 5, 921–934.
[8] Alfonso Castro, R. Shivaji; *Positive solutions for a concave semipositone Dirichlet problem.* Nonlinear Anal. 31 (1998), no. 1-2, 91–98.
[9] Juan Davila, Marcelo Montenegro; *Positive versus free boundary solutions to a singular elliptic equation.* J. Anal. Math. 90 (2003), 303–335.
[10] R. Dhanya, E. Ko, R. Shivaji; *A three solution theorem for singular nonlinear elliptic boundary value problems.* J. Math. Anal. Appl. 424 (2015), no. 1, 598–612.
[11] R. Dhanya, J. Giacomoni, K. Saoudi, S. Prashanth; *Global bifurcation and local multiplicity results for elliptic equation with singular nonlinearity of super exponential growth in* $\mathbb{R}^2$. Adv. Differential Equations 17 (2012), No 3-4, 369-400.
[12] M. Ghergu, V. D. Radulescu; *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, Oxford University Press, 2008.
[13] C. Gui, F.-H. Lin; *Regularity of an elliptic problem with a singular nonlinearity.* Proc. Roy. Soc. Edinburgh Sect. A, 123 (1993), 1021–1029.
[14] Habib Maagli, Jacques Giacomoni, Paul Sauvy; *Existence of compact support solutions for a quasilinear and singular problem.* Differential Integral Equations 25 (2012), no. 7-8, 629–656.
[15] J. Hernández, F. Mancebo, J. M. Vega; *On the linearization of some singular nonlinear elliptic problems and applications.* Ann. Inst. H. Poincare Anal. Non Lineaire, 19 (2002), 777–813.
[16] J. Hernández, F. Mancebo, J. M. Vega; *Positive solutions for singular nonlinear elliptic equations.* Proc. Roy. Soc. Edinburgh Sect. A 137 (2007), no. 1, 41—62.
[17] J. I. Díaz, J. Hernández; F. J. Mancebo; *Branches of positive and free boundary solutions for some singular quasilinear elliptic problems.* J. Math. Anal. Appl. 352 (2009), no. 1, 449–474.
[18] E. K. Lee, R. Shivaji, J. Ye; *Subsolution: A journey from positone to infinite semipositone problems.* Elec. J. Diff. Eqns., Conf. 17 (2009), 123–131.
[19] Paul Sauvy; *Stability of the solutions for a singular and sublinear elliptic problem.* Twelfth International Conference Zaragoza-Pau on Mathematics, 195-206, Monogr. Mat. Garca Galdeano, 39, Prensas Univ. Zaragoza, Zaragoza, 2014.
[20] J. Shi, R. Shivaji; *Global bifurcations of concave semipositone problems.* Evolution equations, pp. 385-398, Lecture Notes in Pure and Appl. Math., Vol. 234, Dekker, New York, 2003.
[21] Junping Shi, Miaoxin Yao; *On a singular nonlinear semilinear elliptic problem.* Proc. Roy. Soc. Edinburgh Sect. A 128 (1998), no. 6, 1389–1401.

Rajendran Dhanya
School of Mathematics and Computer Science, Indian Institute of Technology, Goa 403401, India
*E-mail address*: dhanya.tr@gmail.com