Electronic Journal of Differential Equations, Vol. 2007(2007), No. 67, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

RADIAL SELFSIMILAR SOLUTIONS OF A NONLINEAR ORNSTEIN-UHLENBECK EQUATION

ARIJ BOUZELMATE, ABDELILAH GMIRA, GUILLERMO REYES

Abstract. This paper concerns the existence, uniqueness and asymptotic properties (as $r = |x| \to \infty$) of radial self-similar solutions to the nonlinear Ornstein-Uhlenbeck equation

$$v_t = \Delta_p v + x \cdot \nabla(|v|^{q-1} v)$$

in $\mathbb{R}^N \times (0, +\infty)$. Here $q > p - 1 > 1$, $N \geq 1$, and Δ_p denotes the p-Laplacian operator. These solutions are of the form

$$v(x, t) = t^{-\gamma} U(cxt^{-\sigma}),$$

where γ and σ are fixed powers given by the invariance properties of differential equation, while U is a radial function, $U(y) = u(r)$, $r = |y|$. With the choice $c = (q-1)^{-1/p}$, the radial profile u satisfies the nonlinear ordinary differential equation

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + \frac{q+1-p}{p} ru' + (q-1)r(|u|^{q-1}u)' + u = 0$$

in $\mathbb{R}_+$. We carry out a careful analysis of this equation and deduce the corresponding consequences for the Ornstein-Uhlenbeck equation.

1. Introduction and Main Results

We are interested in radial, selfsimilar solutions of the nonlinear degenerate parabolic equation

$$v_t = \Delta_p v + x \cdot \nabla(|v|^{q-1} v), \tag{1.1}$$

posed in $\mathbb{R}^N \times (0, +\infty)$, where $q > p - 1 > 1$, $N \geq 1$. As usual, ∇ denotes the spatial gradient, while $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ stands for the p-Laplacian operator. Equation (1.1) can be considered as a nonlinear version of the Ornstein-Uhlenbeck equation (see for example [7] and [4]), which is an important model of diffusion.

The study of radial self-similar solutions is motivated by the role that they have played in the general theory for related equations. Thus, it is well known that for the purely p-laplacian equation, the so called Barenblatt solutions having the same (invariant) norm $\|U(t)\|_{L^1}$ describe the asymptotics of general solutions with integrable data, see [5]. In the same spirit, in the papers [11, 6] it is proved that

2000 *Mathematics Subject Classification.* 34L30, 35K55, 35K65.
Key words and phrases. p-laplacian; Ornstein-Uhlenbeck diffusion equations; self-similar solutions; shooting technique.

Submitted January 11, 2007. Published May 9, 2007.

the long time behaviour of solutions to the diffusion-absorption equation

$$v_t = \Delta_p v - v^q$$

is also given by a family of radial self-similar solutions of the equation itself or of some reduced equation. The particular member of the family depends on the behaviour of initial data at infinity. The same questions for related equations can be found in [2, 3, 8, 9, 10]. The radial self-similar solutions to (1.1) constructed in the present paper are also related to the long time behaviour of solutions to the initial value problem. The authors intend to report on this in a forthcoming paper.

We show that, under certain assumptions on p, q and N, the equation (1.1) admits a family of radial self-similar solutions of the form

$$v(x,t) = t^{-\gamma} U(cxt^{-\sigma}), \tag{1.2}$$

defined for $x \in \mathbb{R}^N$ and $t > 0$. Here $U : \mathbb{R}^N \to \mathbb{R}$, is a radial function. The scaling powers γ, σ are determined by the equation in the usual manner (dimensional analysis):

$$\gamma = \frac{1}{q-1}, \quad \sigma = \frac{q+1-p}{p(q-1)}. \tag{1.3}$$

With the choice of the scaling constant $c = (q-1)^{-1/p}$, it can be easily checked that U satisfies the degenerate elliptic equation

$$\Delta_p U + \frac{q+1-p}{p} x \cdot \nabla U + (q-1) x \cdot \nabla(|U|^{q-1}U) + U = 0$$

in $\mathbb{R}^N$. If we put $U(x) = u(|x|)$, it is easy to see that $u : \mathbb{R}^+ \to \mathbb{R}$ is a solution of the ODE

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + \frac{q+1-p}{p} r u' + (q-1) r (|u|^{q-1}u)' + u = 0. \tag{1.4}$$

We study this equation by classical methods, suitably modified in order to deal with its degenerate character at $r = 0$ as well as at points where $u' = 0$. This is particularly important for local existence, since we are interested in radial solutions and it is natural to impose $u'(0) = 0$. Actually, we will deal with a more general equation, containing (1.4) as a particular case. Thus, consider the following initial value problem.

Problem (P). Find a function u, defined on $[0, +\infty[$ such that $|u'|^{p-2}u'$ is in $C^1([0, +\infty[)$ and

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha r u' + \beta r (|u|^{q-1}u)' + u = 0 \quad \text{in }]0, +\infty[, \tag{1.5}$$

$$u(0) = A, \quad u'(0) = 0, \tag{1.6}$$

where $q > p - 1 > 1$, $N \geq 1$, $\alpha \geq 0$, $\beta > 0$, $A \neq 0$. Note that in the application to the nonlinear Ornstein-Uhlenbeck equation (1.1) the choice of parameters is

$$\alpha = \frac{q+1-p}{p}, \quad \beta = (q-1). \tag{1.7}$$

Our results concerning problem (P) can be summarized as follows. For this brief account we assume that $A > 0$, for the sake of clarity.

By reducing the initial value problem (P) to a fixed point problem for a suitable integral operator, we prove that for each $A \neq 0$ there exists a unique function

$u(\cdot, A)$ defined in $[0, +\infty[$ satisfying (1.5)and (1.6). This is the content of Theorem 2.1.

Once these basic facts are established, we perform a careful analysis of the qualitative properties of the solutions to (P). First, we prove that the solutions are ordered. Moreover, we show that they are strictly ordered while the smaller is positive. See Theorem 2.6.

Next, we consider the behavior of solutions as $r \to +\infty$. It turns out that this behaviour strongly depends on the size of α.

More precisely, we show that $\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} u'(r) = 0$. If $\alpha > 0$, we prove that there exists the finite limit $L := \lim_{r\to+\infty} r^{1/\alpha} u(r) \geq 0$. Moreover, $L > 0$ if $\alpha N > 1$. See Theorems 3.1, 3.3 and 3.6 for the precise statements.

Concerning the sign of u, we prove that (i) if $\alpha N \geq 1$, then solutions are strictly positive, whereas (ii) if $0 < \alpha N < 1$ solutions with small data change sign, while those with large data are strictly positive. As a direct consequence, we obtain the existence and uniqueness of a compactly supported solution in this range. See Theorems 3.7, 4.2, 4.3 and 4.9.

This paper is organized as follows: Section 2 is devoted to basic theory: we prove local existence, uniqueness and extendability of solutions for problem (P), as well as monotonicity of solutions with respect to the datum A. In Section 3 we describe the asymptotic behavior of positive solutions as $r \to \infty$. In Section 4 we give a fairly complete classification of solutions according to their behaviour at infinity (strictly positive, compactly supported or oscillating), depending on the parameters α, and β. Finally, in Section 5, we apply the obtained results to the original equation, taking into account the relations (1.7).

2. Basic Theory

Unless otherwise specified, we assume throughout that

$$q > p - 1 > 1, \quad N \geq 1, \quad \alpha \geq 0, \quad \beta > 0.$$

Moreover, we restrict ourselves to the case $A > 0$ in (1.6), since equation (1.5) is invariant under the change of unknown $u \to v = -u$; i.e., if u solves (P) with $u(0) = A$, then $v = -u$ solves the same problem with $v(0) = -A$. We start with existence and uniqueness result.

Theorem 2.1. *Problem* (P) *has a unique solution* $u(\cdot, A, \alpha, \beta)$. *Moreover,*

$$(|u'|^{p-2}u')'(0) = -A/N. \tag{2.1}$$

Some ideas for the proof are inspired by [2] and [3].

Proof. The proof will be done in three steps.
Step 1: Existence of a local solution. Integrating (1.5)twice from 0 to r and taking into account (1.6), we see that problem (P) is equivalent to the integral equation

$$u(r) = A - \int_0^r G(F[u](s))\, ds, \tag{2.2}$$

where

$$G(s) = |s|^{(2-p)/(p-1)} s, \quad s \in \mathbb{R} \tag{2.3}$$

and the nonlinear mapping F is given by the formula

$$F[\varphi](s) = \alpha s\varphi(s)+\beta s|\varphi|^{q-1}\varphi(s)+s^{1-N}\int_0^s \sigma^{N-1}[-\beta N|\varphi|^{q-1}(\sigma)+(1-N\alpha)]\varphi(\sigma)d\sigma. \tag{2.4}$$

Let us introduce the functional setting. For $R > 0$ we denote by $C([0,R])$, the Banach space of real continuous functions on $[0,R]$ with the uniform norm, denoted by $\|\cdot\|_0$. Then $F[\varphi]$ is well defined as an operator from $C([0,R])$ into itself. Given $A, M > 0$ we consider the complete metric space

$$E_{A,M;R} = \{\varphi \in C([0,R]) : \|\varphi - A\|_0 \le M\}. \tag{2.5}$$

Next we define the mapping $\mathcal{T}$ on $E_{A,M;R}$ by

$$\mathcal{T}[\varphi](r) = A - \int_0^r G(F[\phi](s))ds. \tag{2.6}$$

Claim 1: $\mathcal{T}$ maps $E_{A,M;R}$ into itself for some small M and $R > 0$.

Proof. Obviously $\mathcal{T}[\varphi] \in C([0,R])$. Let us first choose M. From the definition of the space $E_{A,M;R}$, $\varphi(r) \in [A-M, A+M]$, for any $r \in [0,R]$. Simple calculations show the existence of M_1 with $0 < M_1 < A$, such that for any $M \in [0, M_1]$, $F[\varphi]$ has a constant sign in $[0,R]$ for every $\varphi \in E_{A,M;R}$. Fix one such M. Moreover, there exists a constant $K > 0$, depending on p, q, N, A, R, M, α and β, such that

$$|F[\varphi](s)| \ge Ks \quad \text{for all } s \in [0,R]. \tag{2.7}$$

Taking into account that the function $r \to G(r)/r$ is decreasing on $(0,+\infty)$, we have

$$|\mathcal{T}[\varphi](r) - A| \le \int_0^r \frac{G(F[\varphi](s))}{F[\varphi](s)}|F[\varphi](s)|ds \le \int_0^r \frac{G(Ks)}{Ks}|F[\varphi](s)|ds$$

for $r \in (0,R)$. On the other hand,

$$|F[\varphi](s)| \le Cs, \quad C = [|\alpha| + |1/N - \alpha| + 2\beta(M+A)^{q-1}](M+A).$$

We thus get

$$|\mathcal{T}[\varphi](r) - A| \le \frac{p-1}{p} C K^{\frac{2-p}{p-1}} r^{\frac{p}{p-1}}$$

for every $r \in (0,R)$. Choose R small enough such that

$$|\mathcal{T}[\varphi](r) - A| \le M, \quad \text{for } \varphi \in E_{A,M;R}, \tag{2.8}$$

and thereby $\mathcal{T}[\varphi] \in E_{A,M,R}$ (observe that the value of K may be kept fixed). The claim is thus proved. □

Claim 2: $\mathcal{T}$ is a contraction in some interval $[0, r_A]$.

Proof. According to Claim 1, if r_A is small enough, the space $E_{A,M;r_A}$ applies into itself. For such r_A and any $\varphi, \psi \in E_{A,M_A;r_A}$ we have

$$|\mathcal{T}[\varphi](r) - T[\psi](r)| \le \int_0^r |G(F[\varphi](s)) - G(F[\psi](s))|ds \tag{2.9}$$

where $F[\varphi]$ is given by (2.4). Next, let

$$\Phi(s) = \min(|F[\varphi](s)|, |F[\psi](s)|).$$

As a consequence of estimate (2.7) (which is also valid for $F[\psi](s)$), we have

$$\Phi(s) \ge Ks \quad \text{for } 0 \le s \le r < r_A$$

and then

$$\begin{aligned} |G(F[\varphi](s)) - G(F[\psi](s))| &\le \frac{G(\Phi(s))}{\Phi(s)} |F[\varphi](s) - F[\psi](s)| \\ &\le \frac{G(Ks)}{Ks} |F[\varphi](s) - F[\psi](s)|. \end{aligned} \tag{2.10}$$

Moreover,

$$|F[\varphi](s) - F[\psi](s)| \le C' \|\varphi - \psi\|_0 s; \quad C' = [|\alpha| + |1/N - \alpha| + 6\beta(M+A)^{q-1}]. \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we have

$$|\mathcal{T}[\varphi](r) - \mathcal{T}[\psi](r)| \le \frac{p-1}{p} C' K^{\frac{2-p}{p-1}} r^{\frac{p}{p-1}} \|\varphi - \psi\|_0$$

for any $r \in [0, r_A]$. Choosing r_A small enough, $\mathcal{T}$, is a contraction. This proves the claim. $\square$

The Banach Fixed Point Theorem then implies the existence of a unique fixed point of $\mathcal{T}$ in $E_{A,M;r_A}$, which is a solution of (2.2) and, consequently, of problem (P). As usual, this solution can be extended to a maximal interval $[0, r_{\max}[$, $0 < r_{\max} \le +\infty$.

Step 2: Existence of a global solution. We define the energy function

$$E(r) = \frac{p-1}{p} |u'|^p(r) + \frac{1}{2} u^2(r). \tag{2.12}$$

According to equation (1.5), the energy satisfies

$$E'(r) = -ru'^2 \{ \frac{N-1}{r^2} |u'|^{p-2}(r) + \alpha + \beta q |u|^{q-1} \}. \tag{2.13}$$

From our hypothesis it follows that E is decreasing, hence it is bounded. Consequently, u and u' are also bounded and the local solution constructed above can be extended to $\mathbb{R}_+$.

Step 3: $(|u'|^{p-2}u')'(0) = -A/N$. Taking the first derivative in (2.2), inverting the function G , dividing both members by r and passing to the limit as $r \to 0$ gives, after a standard application of L'Hopital's rule,

$$\lim_{r \to 0} \frac{|u'|^{p-2} u'(r)}{r} = -\frac{A}{N}, \tag{2.14}$$

as desired. The proof of Theorem 2.1 is complete. $\square$

Remark 2.2. (i) It is not difficult to see that the solutions of (P) are C^∞ functions in the set $\{r > 0 : u'(r) \ne 0\}$. However, we only have $u \in C^{1+1/(p-1)}$ as global regularity, see (2.15) and (4.17) below. This is exactly the regularity of the Barenblatt solutions to the pure p-laplacian equation.

(ii) It is easy to see from (1.5) that local minima (resp. maxima) of the function u can take place only at points where $u \le 0$ (resp. $u \ge 0$).

Remark 2.3. In the general case, near the origin we have $E'(r) \sim -\frac{N-1}{r} |u'|^p(r)$.

Remark 2.4. Since the vector field $\mathbf{F}(r, X, Y)$, is locally Lipschitz continuous in the set

$$\{(r, X, Y) \in \mathbb{R}_+^* \times \mathbb{R}^* \times \mathbb{R}^*\},$$

there exists a unique solution of (2.11) in a neighborhood of (r_0, A, B) if $r_0 > 0$, $A, B \neq 0$. The same method above can be used to extend this result to the cases $r_0 = 0$ or $B = 0$.

Remark 2.5. Local existence holds with $\beta \leq 0$, $\alpha < 0$ and the same proof applies.

The following result concerns monotonicity of solutions with respect to the initial data.

Theorem 2.6. *Let $u(\cdot, A)$ and $u(\cdot, B)$ be two solutions of problem* (P) *with $0 < A < B$. Then $u(\cdot, A)$ and $u(\cdot, B)$ can not intersect each other before their first zero.*

Before giving the proof of theorem 2.6, we need an asymptotic expansion near $r = 0$, which is given by the following lemma.

Lemma 2.7. *Let u be the solution to (P) with $A > 0$. There holds*
$$u(r) = A - C_1 r^{p/(p-1)} + C_2 r^{2p/(p-1)} + o(r^{2p/(p-1)}) \quad as \quad r \to 0 \tag{2.15}$$
with
$$C_1 = \frac{p-1}{p}\Big(\frac{A}{N}\Big)^{1/(p-1)}; \quad C_2 = \frac{C_1^2 \kappa N}{2A(N+p\kappa)}[\beta p q \kappa A^{q-1} + \alpha p \kappa + 1]$$
and $\kappa := 1/(p-1)$.

Proof. It follows from (2.14) that, as $r \to 0$,
$$-u'(r) = \big(\frac{A}{N}\big)^{\kappa} r^{\kappa} + o(r^{\kappa}).$$
Integrating on $[0, r]$ we obtain
$$u(r) = A - C_1 r^{p\kappa} + \dots \tag{2.16}$$
with C_1 as above (in the sequel we omit the $o's$ for simplicity). L'Hopital's rule and the integral equation (2.1) imply
$$C_2 := \lim_{r\to 0} \frac{u(r) - A + C_1 r^{p\kappa}}{r^{2p\kappa}} = \lim_{r \to 0} \frac{C_1 p\kappa r^{\kappa} - G(F_u(r))}{2p\kappa r^{2p\kappa - 1}}.$$
Write
$$F_u(r) = \alpha r u + \beta r u^q + H(r); \quad H(r) := r^{1-N} \int_0^r \sigma^{N-1}[-\beta N u^q(\sigma) + (1 - N\alpha)u(\sigma)] d\sigma.$$
Again, l'Hopital's rule and the fact that $u(r) \to A$ as $r \to 0$ imply
$$\begin{aligned} \lim_{r\to 0}\frac{H(r)}{r} &= \lim_{r\to 0} \frac{\int_0^r \sigma^{N-1}[-\beta N u^q(\sigma) + (1-N\alpha)u(\sigma)]d\sigma}{r^N} \\ &= \frac{-\beta N A^q + (1-N\alpha)A}{N} =: C'. \end{aligned}$$
This information does not allow to find the value of C_2, since the leading order of the numerator is still unknown. We need the next term. Toward this end, we compute
$$H(r) - C'r = [\beta N C_1 q A^{q-1} - C_1(1-N\alpha)] r^{1-N} \int_0^r \sigma^{N+\kappa} d\sigma + \dots = C'' r^{(2p-1)\kappa} + \dots,$$
where
$$C'' := \frac{[\beta N C_1 q A^{q-1} - C_1(1-N\alpha)]}{N + p\kappa}.$$

(here we have used (2.16)). Consequently, $H(r) = C'r + C''r^{(2p-1)\kappa} + \dots$. Plugging this expansion into F_u and using again (2.16) we obtain

$$F_u(r) = \frac{A}{N}r + C'''r^{(2p-1)\kappa} + \dots; \quad C''' = C'' - \alpha C_1 - \beta C_1 q A^{q-1}.$$

Therefore, since $F_u(r) > 0$ for $r \sim 0$,

$$G(F_u(r)) = C_1 p\kappa r^{\kappa} + \frac{C_1 C''' p\kappa^2 N}{A} r^{2p\kappa - 1} + \dots$$

where G is given by (2.3) and then

$$C_2 = -\frac{C_1 C''' \kappa N}{2A} = \frac{C_1^2 \kappa}{2A(N + p\kappa)}[\beta pq\kappa A^{q-1} + \alpha p\kappa + 1].$$

This concludes the proof of the Lemma. ☐

Proof of Theorem 2.6. Let R_1 (resp. R_2) denote the first zero of $u(\cdot, A)$ (resp. $u(\cdot, B)$) or $R_1 = +\infty$ (resp $R_2 = +\infty$) if $u(\cdot, A) > 0 \in \mathbb{R}_+$(resp. $u(\cdot, B) > 0 \in \mathbb{R}_+$). For ease of notation, we write $u(r) = u(r, A)$ and $v(r) = u(r, B)$.

We argue by contradiction. Suppose there exists $R_0 \in]0, \min(R_1, R_2)[$ such that $u(r) < v(r)$ for $r \in [0, R_0[$and $u(R_0) = v(R_0)$. For $k > 0$ we define

$$u_k(r) = k^{-\frac{p}{p-2}} u(kr). \tag{2.17}$$

Since u is decreasing, we can choose $k \in]0, 1[$ such that $u_k(r) > v(r)$ for $r \in [0, R_0]$. Set

$$K = \sup\big\{0 < k < 1 : u_k(r) > v(r) \quad \text{for } r \in [0, R_0]\big\}.$$

We claim that $u_K(r) \geq v(r)$ for $r \in [0, R_0]$. Indeed, assume there exists $r_1 \in [0, R_0]$ such that $u_K(r_1) < v(r_1)$. Since the function $k \to u_k(r_1)$ is continuous, there exists $\varepsilon > 0$ such that $u_k(r_1) < v(r_1)$ for $K - \varepsilon < k < K$, contrary to the definition of K. Hence $u_K(r) \geq v(r)$ for $r \in [0, R_0]$.

Moreover $u_K(R_0) > v(R_0)$, and $u_K(0) > v(0)$. In fact, observe first that if $u_K(R_0) = v(R_0) = u(R_0)$, then $K = 1$, this contradicts $u(r) < v(r)$ for $0 < r < R_0$. Secondly assume $u_K(0) = v(0) = B$. Then u_K solves the same problem as v but for a different value of β. Put $\beta_K := \beta K^{\frac{p(q-1)}{p-2}} < \beta$ (recall that $K < 1$). Applying the lemma above to u_K and v we have

$$u_K(r) = B - C_1 r^{p/(p-1)} + \widetilde{C}_2 r^{2p/(p-1)} + \dots$$
$$v(r) = B - C_1 r^{p/(p-1)} + C_2 r^{2p/(p-1)} + \dots,$$

where $\widetilde{C}_2 < C_2$, since $\beta_K < \beta$. Thus $u_K < v$ in a right neighborhood of $r = 0$, a contradiction. Then necessarily $u_K(0) > v(0)$.

Next, we claim that there exists $R \in]0, R_0[$ such that $u_K(R) = v(R)$. Assume the opposite. Then there would exist $\varepsilon > 0$ such that $u_K - v > \varepsilon$ on $[0, R_0]$ and therefore, by continuity of $k \to u_k$ as a map from $]0, 1[$ to $C[0, R_0]$, the same would hold with K replaced by $k, K < k < K + \delta$, for some $\delta(\varepsilon) > 0$, contradicting the definition of K. This proves the claim.

Clearly, $u_K'(R) = v'(R)$, since otherwise one would have $u_K < v$ on some one-sided neighborhood of R, which is impossible. Thus the function $\varphi = u_K - v$ has a local minimum at R. On the other hand, it is easy to see that u_K satisfies the equation

$$(|u_K'|^{p-2}u_K')' + \frac{N-1}{r}|u_K'|^{p-2}u_K' + \alpha r u_K' + u_K + \beta K^{\frac{p(q-1)}{p-2}} r(u_K^q)' = 0.$$

Subtracting this equation from the one satisfied by v we obtain at $r = R$:

$$(p-1)|v'|^{p-2}(R)\varphi''(R) + \beta[K^{\frac{p(q-1)}{p-2}} - 1]R(u_K^q)' = 0.$$

Since $K < 1$, $\beta > 0$, $u_K(R) = v(R) > 0$ and $u'_K(R) = v'(R) < 0$, we deduce $\varphi''(R) < 0$, thus contradicting the fact that φ has a local minimum at R. The obtained contradiction proves the assertion. This completes the proof of the theorem. □

3. Behaviour at infinity

This section deals with some qualitative properties of solutions of problem (P).

Theorem 3.1. *Let u be a solution of (P). Then,*

$$\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} u'(r) = 0.$$

Proof. By (2.12), it is enough to show that $\lim_{r\to+\infty} E(r) = 0$. Since $E'(r) \leq 0$ and $E(r) \geq 0$ for all $r > 0$, there exists a constant $l \geq 0$ such that $\lim_{r\to+\infty} E(r) = l$. Suppose $l > 0$. Then, there exists $r_1 > 0$, such that

$$E(r) \geq l/2 \quad \text{for } r \geq r_1. \tag{3.1}$$

Now consider the function

$$\begin{aligned} D(r) &= E(r) + \frac{N-1}{2r}|u'|^{p-2}u'(r)u(r) + \frac{\alpha(N-1)}{4}u^2(r) \\ &\quad + \frac{q\beta(N-1)}{2(q+1)}|u|^{q+1} + \alpha\int_0^r su'^2(s)ds. \end{aligned}$$

Then

$$D'(r) = -q\beta r|u|^{q-1}u'^2 - \frac{(N-1)}{2r}[|u'|^p + \frac{N}{r}|u'|^{p-2}u'u + u^2].$$

Since $\beta > 0$, we have

$$D'(r) \leq -\frac{N-1}{2r}[|u'|^p + u^2 + \frac{N}{r}|u'|^{p-2}u'u].$$

Recalling that u and u' are bounded,

$$\lim_{r\to+\infty} \frac{|u'|^{p-2}u'\ u(r)}{r} = 0.$$

Moreover, by (3.1) we have

$$|u'(r)|^p + u^2(r) \geq \frac{p-1}{p}|u'(r)|^p + \frac{u^2(r)}{2} = E(r) > l/2 \quad \text{for } r \geq r_1.$$

Consequently, there exist two constants $c > 0$ and $r_2 \geq r_1$ such that

$$D'(r) \leq -c/r \quad \text{for } r \geq r_2.$$

Integrating this last inequality between r_2 and r, we get

$$D(r) \leq D(r_2) - c\ \ln(r/r_2) \quad \text{for } r \geq r_2.$$

In particular we obtain $\lim_{r\to+\infty} D(r) = -\infty$. Since

$$E(r) + \frac{N-1}{2r}|u'|^{p-2}u'u(r) \leq D(r),$$

we get $\lim_{r\to+\infty} E(r) = -\infty$. This is impossible, hence the conclusion. □

Theorem 3.2. *Let $P_u := \{r > 0 : u(r) > 0\}$. Then $u' < 0$ in the connected component of P_u containing a right neighborhood of $r = 0$.*

Proof. We argue by contradiction. Let $r_0 > 0$ be the first zero of u' in the connected component of P_u containing a right neighborhood of $r = 0$. Then, it follows from (1.5)that $(|u'|^{p-2}u')'(r_0) = -u(r_0) < 0$. On the other hand, we know from (2.14) that $u' < 0$ for $r \sim 0$. By continuity and the definition of r_0, there exists a left neighborhood $]r_0 - \varepsilon, r_0[$ (for some $\varepsilon > 0$) where u' is strictly increasing and strictly negative, that is $(|u'|^{p-2}u')'(r) > 0$ for any $r \in]r_0 - \varepsilon, r_0[$; hence by letting $r \to r_0$ we get $(|u'|^{p-2}u')'(r_0) \geq 0$, a contradiction. □

Theorem 3.3. *Assume $\alpha > 0$. Let u be a strictly positive solution of (P). Then $\lim_{r\to+\infty} r^{1/\alpha}u(r) = L$ exists and lies in $[0, +\infty[$.*

We postpone the proof of this theorem until establishing some preliminary results.

Lemma 3.4. *Assume $\alpha > 0$. Let u be a strictly positive solution of (P). Suppose there are some $\sigma \geq 0$and $r_0 > 0$ such that*

$$u(r) \leq K(1 + r)^{-\sigma} \quad \text{for } r \geq r_0. \tag{3.2}$$

Then, there exists a constant M depending on K, σ, and r_0 such that

$$|u'(r)| \leq M(1 + r)^{-\sigma-1} \quad \text{for } r \geq r_0. \tag{3.3}$$

Proof. Consider the function ρ defined by

$$\rho(r) = \exp[\frac{\alpha}{p-1}\int_{r_0}^{r} s|u'(s)|^{2-p}ds].$$

Note that the function u' is strictly negative and then $\rho(r)$ is well defined for $r \geq r_0$ and it is an increasing C^∞ function. Set

$$F(r) = (p-1)u'(r)r^{\frac{N-1}{p-1}}\rho(r) \quad \text{for } r \geq r_0.$$

Using the fact that $\rho'(r) = \frac{\alpha}{p-1}r|u'(r)|^{2-p}\rho(r)$ and equation (1.5), we deduce that, for any $r \geq r_0$,

$$F'(r) = -\frac{p-1}{\alpha}r^{\frac{N-p}{p-1}}\rho'(r)[u(r) + \beta qru(r)^{q-1}u'(r)]$$

Integrating this last equation in (r_0, r) with $r > r_0$ and using the expression of $F(r)$, we get

$$u'(r) = \frac{r^{\frac{1-N}{p-1}}}{\rho(r)}u'(r_0)r_0^{\frac{N-1}{p-1}} - \frac{1}{\alpha}\frac{r^{\frac{1-N}{p-1}}}{\rho(r)}\int_{r_0}^{r} s^{\frac{N-p}{p-1}}\rho'(s)[u(s) + \beta qsu(s)^{q-1}u'(s)]ds.$$

Since $u'(r) < 0, \alpha > 0, \beta > 0$ and $\rho'(r) > 0$ it follows that

$$|u'(r)| \leq \frac{r^{\frac{1-N}{p-1}}}{\rho(r)}r_0^{\frac{N-1}{p-1}}|u'(r_0)| + \frac{1}{\alpha}\frac{r^{\frac{1-N}{p-1}}}{\rho(r)}I \quad \text{for } r \geq r_0, \tag{3.4}$$

where

$$I = \int_{r_0}^{r} s^{\frac{N-p}{p-1}}\rho'(s)u(s)ds.$$

Since u' is continuous in $]0, +\infty[$ and $\lim_{r\to+\infty} u'(r) = 0$ (from Theorem 3.1), there exists a constant K_0 depending on r_0 such that

$$|u'(r)|^{2-p} \geq K_0 \quad \text{for } r \geq r_0. \tag{3.5}$$

As a consequence,

$$\rho(r) \geq K_2 \exp(K_1 r^2) \quad \text{for } r \geq r_0, \tag{3.6}$$

where $K_1 = \frac{\alpha}{2(p-1)} K_0$ and $K_2 = \exp[-K_1 r_0^2]$. Therefore, the first term in the right hand side of (3.4) can be estimated as

$$\frac{r^{\frac{1-N}{p-1}}}{\rho(r)} r_0^{\frac{N-1}{p-1}} |u'(r_0)| \leq \frac{1}{K_2} |u'(r_0)| \exp(-K_1 r^2). \tag{3.7}$$

Next we estimate the second term in the right-hand side of (3.4). It follows from (3.2) that, for $r \geq 2r_0$,

$$I \leq C \int_{r_0}^{\frac{r}{2}} s^{\frac{N-p}{p-1}} \rho'(s)(1+s)^{-\sigma} ds + C \int_{r/2}^{r} s^{\frac{N-p}{p-1}} \rho'(s)(1+s)^{-\sigma} ds. \tag{3.8}$$

Plainly,

$$\int_{r_0}^{\frac{r}{2}} s^{\frac{N-p}{p-1}} \rho'(s)(1+s)^{-\sigma} ds \leq (1+r_0)^{-\sigma} \max_{(r_0, r/2)} (s^{\frac{N-p}{p-1}}) \rho(r/2), \tag{3.9}$$

and

$$\int_{r/2}^{r} s^{\frac{N-p}{p-1}} \rho'(s)(1+s)^{-\sigma} ds \leq (1+r/2)^{-\sigma} \max_{(r/2, r)} (s^{\frac{N-p}{p-1}}) \rho(r). \tag{3.10}$$

Now note that

$$\frac{\rho(r/2)}{\rho(r)} = \exp\big[-\frac{\alpha}{p-1} \int_{r/2}^{r} s|u'(s)|^{2-p} ds\big] \leq \exp(-K_3 r^2), \tag{3.11}$$

where $K_3 = \frac{3\alpha}{8(p-1)} K_0$. Combining (3.8)—(3.11), we obtain

$$\frac{1}{\alpha} \frac{r^{\frac{1-N}{p-1}}}{\rho(r)} I \leq C(1+r)^{-\sigma-1} + C r^{\frac{1-N}{p-1}} \max_{(r_0, \frac{r}{2})} (s^{\frac{N-p}{p-1}}) \exp(-K_3 r^2), \tag{3.12}$$

where $C > 0$ is a constant depending on r_0, p, N and σ. Putting together (3.7) and (3.12) the desired estimate (3.3) follows. ∎

Lemma 3.5. *Assume $\alpha > 0$. Let u be a strictly positive solution of (P). Then*

$$u(r) \leq C\, r^{-1/\alpha} \quad \textit{for large } r. \tag{3.13}$$

Proof. Multiplying the equation (1.5) by u/r and rearranging we obtain

$$\frac{u^2(r)}{r} = \frac{|u'|^p}{r} - \alpha u u' - \frac{N}{r^2} u|u'|^{p-2}u' - [\frac{u|u'|^{p-2}u'}{r}]' - \beta q u^q u'(r).$$

Recalling the definition of the energy function given by (2.12) we have the inequality

$$\frac{E(r)}{r} \leq \frac{3p-2}{2p} \frac{|u'|^p}{r} - \frac{\alpha}{2} u u' + \frac{N}{2r^2} u|u'|^{p-1} - \frac{1}{2} [\frac{u|u'|^{p-2}u'}{r}]' - \frac{\beta}{2} q u^q u'(r).$$

By Theorem 3.2, $u' < 0$. Integrating the last inequality on some interval (r, R) we obtain

$$\begin{aligned} \int_r^R \frac{E(s)}{s} ds \leq & \frac{3p-2}{2p} \int_r^R \frac{|u'(s)|^p}{s} ds + \frac{N}{2} \int_r^R \frac{u(s)|u'(s)|^{p-1}}{s^2} ds \\ & + \frac{u(R)|u'(R)|^{p-1}}{2R} + \frac{\alpha}{4} u^2(r) + \frac{q\beta}{2(q+1)} u^{q+1}(r). \end{aligned}$$

Since E is strictly decreasing and converges to zero when $r \to \infty$, we deduce that $E' \in L^1(]r_0, \infty[)$ for any $r_0 > 0$. Thereby $|u'|^p/r \in L^1(]r_0, \infty[)$. Letting $R \to +\infty$ the following inequality holds

$$\begin{aligned}\int_r^\infty \frac{E(s)}{s} ds \leq & \frac{\alpha}{4} u^2(r) + \frac{3p-2}{2p} \int_r^\infty \frac{|u'(s)|^p}{s} ds + \\ & + \frac{N}{2} \int_r^\infty \frac{u(s)|u'(s)|^{p-1}}{s^2} ds + \frac{q\beta}{2(q+1)} u^{q+1}(r).\end{aligned} \tag{3.14}$$

Set

$$H(r) = \int_r^\infty \frac{E(s)}{s} ds. \tag{3.15}$$

From the expression of $E(r)$, we have $u^2(r) \leq 2E(r)$, then inequality (3.14) gives

$$\begin{aligned}H(r) + \frac{\alpha}{2} r H'(r) \leq & \frac{3p-2}{2p} \int_r^\infty \frac{|u'(s)|^p}{s} ds + \frac{N}{2} \int_r^\infty \frac{u(s)|u'(s)|^{p-1}}{s^2} ds \\ & + \frac{q\beta}{2(q+1)} u^{q+1}(r).\end{aligned} \tag{3.16}$$

Assume now that u satisfies

$$u(r) \leq C r^{-\sigma} \quad \text{for } r \geq 1 \tag{3.17}$$

with some fixed $\sigma \geq 0$ and some constant $C > 0$ (this is possible because $u(r) \leq A$ for $r \geq 0$). Then Lemma (3.4) implies $|u'(r)| \leq C r^{-\sigma-1}$, for large r, and then (3.16) and (3.17) imply

$$[r^{2/\alpha} H(r)]' \leq C r^{-p(1+\sigma)+2/\alpha-1} + C r^{-\sigma(q+1)+2/\alpha-1}. \tag{3.18}$$

We claim that

$$u(r) \leq C r^{-m} \quad \text{for large } r, \tag{3.19}$$

with

$$m = \min\Big\{\frac{1}{\alpha}, \frac{\sigma(p+2)+p}{4}, \frac{\sigma(1+\frac{q+1}{2})}{2}\Big\}.$$

In fact, we have to distinguish two cases.
Case (I): $[2/\alpha - p(1+\sigma)][2/\alpha - \sigma(q+1)] \neq 0$. Using (3.18) there holds

$$H(r) \leq C r^{-2/\alpha} + C r^{-p(1+\sigma)} + C r^{-\sigma(q+1)} \quad \text{for large } r. \tag{3.20}$$

Case (II): $[2/\alpha - p(1+\sigma)][2/\alpha - \sigma(q+1)] = 0$. Here we have three subcases:
Case (IIa): $2/\alpha = p(1+\sigma)$ and $2/\alpha \neq \sigma(q+1)$. Then

$$H(r) \leq C r^{\frac{-\sigma(p+2)-p}{2}} + C r^{-\sigma(q+1)} \quad \text{for large } r. \tag{3.21}$$

Case (IIb): $2/\alpha \neq p(1+\sigma)$ and $2/\alpha = \sigma(q+1)$. Then

$$H(r) \leq C r^{-p(\sigma+1)} + C r^{-\sigma(1+\frac{q+1}{2})} \quad \text{for large } r. \tag{3.22}$$

Case (IIc): $2/\alpha = p(1+\sigma)$ and $2/\alpha = \sigma(q+1)$. Then

$$H(r) \leq C r^{\frac{-\sigma(p+2)-p}{2}} \quad \text{for large } r. \tag{3.23}$$

Now using the inequality

$$H(r) \geq \int_r^{2r} \frac{E(s)}{s} ds \geq \frac{E(2r)}{2} \geq \frac{u^2(2r)}{4}, \tag{3.24}$$

and combining (3.20), (3.21), (3.22), (3.23) and (3.24), the estimate (3.19) follows. If $m = 1/\alpha$ we have exactly the estimate (3.13). Otherwise, observe that $m > \sigma$

and (3.13) follows by induction starting with $\sigma = \min\left\{\frac{\sigma(p+2)+p}{4}, \frac{\sigma[1+(q+1)/2]}{2}\right\}$. This finishes the proof of the lemma. □

Now we turn to the proof of Theorem 3.3.

Proof of Theorem 3.3. . Let u be the solution of (P). Set

$$I(r) = r^{1/\alpha}[u + \frac{1}{\alpha r}|u'|^{p-2}u']. \tag{3.25}$$

By a simple computation we get

$$I'(r) = -\frac{1}{\alpha}r^{1/\alpha}[(N-1/\alpha)\frac{|u'|^{p-2}u'(r)}{r^2} + \beta(u^q)'(r)]$$

By lemmas 3.4 and 3.5, the functions $r \to r^{1/\alpha}(u^q)'(r)$ and $r \to r^{1/\alpha-2}|u'(r)|^{p-1}$ belong to $L^1(]r_0,\infty[)$ for any $r_0 > 0$; therefore $I'(r) \in L^1(]r_0,\infty[)$, and consequently, the limit

$$\lim_{r\to+\infty} I(r) = \int_{r_0}^{\infty} I'(r)dr + I(r_0) \tag{3.26}$$

exists and is finite. Again (3.13) and (3.3) imply

$$r^{1/\alpha-1}|u'|^{p-1} \le Cr^{-(p-2)/\alpha-p} \tag{3.27}$$

for large r. As a consequence, $\lim_{r\to+\infty} r^{1/\alpha}u(r) = L$ exists and is finite, thus concluding the proof. □

Theorem 3.6. *Assume $L = 0$ in Theorem 3.6. Then $r^m u(r) \to 0$ and $r^m u'(r) \to 0$ as $r \to +\infty$ for all positive integers m.*

Proof. Since $L = \lim_{r\to+\infty} r^{1/\alpha}u(r) = 0$, $\lim_{r\to+\infty} I(r) = 0$ (where I is given by (3.25)). Hence $I(r) = -\int_r^{+\infty} I'(t)dt$. This yields

$$\begin{aligned} u(r) \le & \frac{1}{\alpha r}|u'|^{p-1} + \frac{q\beta}{\alpha}r^{-1/\alpha}\int_r^{+\infty} s^{1/\alpha}u^{q-1}|u'|(s)ds \\ & + \frac{1}{\alpha}|N - \frac{1}{\alpha}|r^{-1/\alpha}\int_r^{+\infty} s^{1/\alpha-2}|u'(s)|^{p-1}ds \end{aligned} \tag{3.28}$$

In view of Lemma 3.5,

$$u(r) \le C(r^{-p-(p-1)/\alpha} + r^{-q/\alpha}),$$

for some $C > 0$. If we define the sequence $\{m_k\}_{k\in\mathbb{N}}$ by

$$\begin{cases} m_0 = \frac{1}{\alpha} \\ m_k = \min\{p + (p-1)m_{k-1},\ qm_{k-1}\}; \quad k \ge 1, \end{cases}$$

then $\lim_{r\to+\infty} m_k = +\infty$, and the theorem follows by induction starting with $m_0 = 1/\alpha$. This completes the proof. □

Theorem 3.7. *Suppose $\alpha N \ge 1$. Then any solution of (P) is strictly positive.*

Proof. We argue by contradiction. Thus, assume that $u(r_0) = 0$ (where $r_0 > 0$ is the first zero of u). Then $u'(r_0) \le 0$. On the other hand, multiplying the equation (E_1) by r^{N-1} and integrating on $(0, r_0)$ we get

$$r_0^{N-1}|u'(r_0)|^{p-2}u'(r_0) = (\alpha N - 1)\int_0^{r_0} s^{N-1}u(s)ds + \beta N\int_0^{r_0} s^{N-1}u^q(s)ds. \tag{3.29}$$

Under our hypotheses, the right-hand side of (3.29) is strictly positive. The obtained sign contradiction proves our assertion. □

Next, we consider the function

$$J(r) = [u(r) + \frac{1}{\alpha r}|u'(r)|^{p-2}u'(r)]r^N. \tag{3.30}$$

Lemma 3.8. *Let the hypotheses in Theorem 3.7 hold. Then the function J is strictly positive for any $r > 0$.*

Proof. It easy to see that

$$J'(r) = \frac{1}{\alpha}r^{N-1}[\alpha N - 1 - q\beta r u^{q-2}(r)u'(r)]u(r). \tag{3.31}$$

Since u' is strictly negative, $J(r)$ is strictly increasing for $r > 0$. On the other hand, $(|u'|^{p-2}u')'(0)$ is finite $(= -A/N)$, hence $J(0) = 0$. Consequently, $J(r) > 0$ for $r > 0$, concluding the proof. □

Theorem 3.9. *Assume $\alpha N > 1$. Then $L = \lim_{r\to+\infty} r^{1/\alpha}u(r) > 0$.*

Proof. By Theorem 3.7, solutions are strictly positive. Then, by Theorem 3.3, the limit $L \in [0, +\infty[$ exists. Suppose $L = 0$. By Theorem 3.6 we have

$$\lim_{r\to+\infty} r^m u(r) = \lim_{r\to+\infty} r^m u'(r) = 0$$

for any $m > 0$ and thereby $\lim_{r\to+\infty} J(r) = 0$, in contradiction with the fact that J is strictly increasing and strictly positive. The theorem is proved. □

4. Classification of Solutions

In this section we give a classification of solutions of problem (P), according to whether they are strictly positive, change sign or are compactly supported. We define the following sets:

$$\mathcal{S}_+ = \{A > 0 : u(r, A) > 0 \text{ for } r > 0\}\,;$$
$$\mathcal{S}_- = \{A > 0 : \exists r_0 > 0 : u(r_0, A) = 0 \text{ and } u'(r_0; A) < 0\}\,;$$
$$\mathcal{S}_C = \{A > 0 : \exists r_0 > 0 : u(r, A) > 0 \text{ for } r \in [0, r_0[\text{ and } u(r, A) = 0 \text{ for } r \geq r_0\}\,,$$

corresponding respectively to strictly positive, changing sign and compactly supported solutions.

Observe that $\mathbb{R}_+ = \mathcal{S}_- \cup \mathcal{S}_+ \cup \mathcal{S}_C$. Indeed, let $A > 0$ be given. Suppose $A \notin \mathcal{S}_+$. Then there exists r_0 such that $u(r) > 0$, for $0 \leq r < r_0$, and $u(r_0) = 0$. If $u'(r_0) < 0$,, then u changes sign and $A \in \mathcal{S}_-$. Assume now $u'(r_0) = 0$. Since the energy function given by (2.12) is non-negative and decreasing, we get $u(r) = 0$ for any $r \geq r_0$ and $A \in \mathcal{S}_C$.

Remark 4.1. Theorem 3.7 can be reformulated as: If $\alpha N \geq 1$ then $\mathcal{S}_+ = \mathbb{R}_+$.

Next, we investigate the range $0 < \alpha N < 1$. It turns out that in this range non of the sets $\mathcal{S}_+$, $\mathcal{S}_-$, $\mathcal{S}_C$ is empty. To show this, we apply below the well known shooting technique.

Theorem 4.2. *Assume $0 < \alpha N < 1$. Then $\mathcal{S}_-$ is an open nonempty set.*

Proof. **Step 1.** First, we prove that $\mathcal{S}_- \neq \emptyset$. More precisely, we claim that there exists a constant $A_0 > 0$ such that for each $A \in (0, A_0)$, the solution $u(\cdot, A)$ changes sign. To this end, we make the following scaling transformation

$$u(r) = Av(\zeta), \quad \text{where } \ \zeta = A^{-\frac{p-2}{p}} r.$$

Then v solves the problem

$$\begin{gathered} (|v'|^{p-2}v')' + \frac{N-1}{\zeta}|v'|^{p-2}v' + \alpha\zeta v' + v + \beta q A^{q-1}\zeta|v|^{q-1}v' = 0 \quad \zeta > 0; \\ v(0) = 1, \quad v'(0) = 0. \end{gathered} \tag{4.1}$$

Recall that the energy function given by (2.12) is positive and decreasing. Then

$$E(r) \leq E(0) = \frac{A^2}{2},$$

which implies

$$|u(r)| \leq A \quad \text{and} \quad |u'(r)| \leq (\frac{p}{2(p-1)})^{1/p} A^{\frac{2}{p}}.$$

Therefore, v and v' are bounded. More precisely

$$|v(\zeta)| \leq 1 \quad \text{and} \quad |v'(\zeta)| \leq (\frac{p}{2(p-1)})^{1/p}.$$

Consequently, for small A, the problem (4.1) can be seen as a perturbation of the problem

$$\begin{gathered} (|w'|^{p-2}w')' + \frac{N-1}{\zeta}|w'|^{p-2}w'(\zeta) + \alpha\zeta w' + w = 0 \quad \text{for } \zeta > 0; \\ w(0) = 1, \quad w'(0) = 0. \end{gathered} \tag{4.2}$$

The first equation of the last problem can be written as

$$[\zeta^{N-1}|w'|^{p-2}w' + \alpha\zeta^N w]' = (\alpha N - 1)\zeta^{N-1}w(\zeta).$$

We claim that w changes sign. In fact, if w were strictly positive, we would have $w' < 0$. On the other hand, as $\alpha N < 1$, the function

$$\varphi : \zeta \to \zeta^{N-1}|w'|^{p-2}w' + \alpha\zeta^N w$$

is strictly decreasing; hence $\varphi(\zeta) \leq \varphi(0) = 0$ for $\zeta > 0$. That is,

$$|w'|^{p-2}w'(\zeta) \leq -\alpha\zeta w(\zeta),$$

which gives

$$\frac{p-1}{p-2}(w^{\frac{p-2}{p-1}})'(\zeta) \leq -\frac{p-1}{p}\alpha^{\frac{1}{p-1}}\big(\zeta^{\frac{p}{p-1}}\big)',$$

and after integration,

$$w^{\frac{p-2}{p-1}}(\zeta) \leq 1 - \frac{p-2}{p}\alpha^{\frac{1}{p-1}}\zeta^{\frac{p}{p-1}}.$$

By letting $\zeta \to +\infty$, we get a contradiction. Thereby w and also u change sign. That is, $u \in \mathcal{S}_-$.
Step 2. $\mathcal{S}_-$ is open. This follows easily from local continuous dependence of solutions on the initial value. The proof is concluded. □

Theorem 4.3. *Assume* $0 < \alpha N < 1$. *Then* $\mathcal{S}_+$ *is an open nonempty set.*

The proof of the theorem will be done in several lemmas.

Lemma 4.4. *Assume $\alpha > 0$. Then, for large initial data A, the solution $u(\cdot, A)$ is strictly positive.*

Proof. As in the previous theorem, we introduce a new function v defined by
$$u(r) = Av(\zeta), \quad \text{where } \zeta = A^{\frac{q+1-p}{p}} r.$$
It is easy to see that v solves the problem
$$\begin{gathered}(|v'|^{p-2}v')' + \frac{N-1}{\zeta}|v'|^{p-2}v' + \beta\zeta\left(|v|^{q-1}v\right)' + A^{1-q}(\alpha\zeta v' + v) = 0, \quad \zeta > 0;\\ v(0) = 1, \quad v'(0) = 0.\end{gathered} \tag{4.3}$$
Similarly, we have
$$|v(\zeta)| \le 1 \quad \text{and} \quad |v'(\zeta)| \le (\frac{p}{2(p-1)})^{1/p}A^{\frac{1-q}{p}}.$$
Then, for large $A > 0$, (4.3) is a perturbation of the problem
$$\begin{gathered}(|w'|^{p-2}w')' + \frac{N-1}{\zeta}|w'|^{p-2}w' + \beta\zeta\left(|w|^{q-1}w\right)' = 0 \quad \text{for } \zeta > 0;\\ w(0) = 1, \quad w'(0) = 0.\end{gathered} \tag{4.4}$$
We claim that w is strictly positive. Otherwise, let r_0 the first zero of w. Then $w'(r_0) \le 0$. On the other hand, multiplying the equation in (4.4) by ζ^{N-1} and integrating on $(0, r_0)$ we obtain
$$r_0^{N-1}|w'|^{p-2}w'(r_0) = \beta N \int_0^{r_0} \zeta^{N-1}w^q(\zeta)d\zeta,$$
which is impossible. Consequently, $u(\cdot, A) \in \mathcal{S}_+$. □

Proving that $\mathcal{S}_+$ is open requires much more effort. For $c > 0$, define the function
$$E_c(r) = cu(r) + ru'(r), \quad r > 0. \tag{4.5}$$
Note that if $E_c(r_0) = 0$ for some $r_0 > 0$, equation (1.5) gives
$$\begin{aligned}&(p-1)|u'|^{p-2}(r_0)E_c'(r_0)\\ &= r_0u(r_0)\Big[\alpha c - 1 + c\beta q|u|^{q-1}(r_0) + c^{p-1}(p-1)(\frac{N-p}{p-1} - c)\frac{|u|^{p-2}(r_0)}{r_0^p}\Big].\end{aligned} \tag{4.6}$$
from which the sign of $E_c(r)$ for large r can be obtained.

Theorem 4.5. *Let u be a strictly positive solution of (P) and $\alpha > 0$. Then, for large r, $E_c(r)$ has a constant sign in the following cases.*

(i) $c \ne \frac{1}{\alpha}$;
(ii) $c = \frac{1}{\alpha} = \frac{N-p}{p-1}$;
(iii) $c = \frac{1}{\alpha} \ne \frac{N-p}{p-1}$, *and* $\lim_{r\to+\infty} r^{1/\alpha}u(r) = 0$.

Proof. Assume there is a sequence $\{r_n\}$ with $r_n \to +\infty$ and such that $E_c(r) > 0$ for $r_{2k} < r < r_{2k+1}$ and $E_c(r) < 0$ for $r_{2k+1} < r < r_{2k+2}$, $(k = 0, 1, \dots)$. (i) Since $\lim_{r\to+\infty} u(r) = 0$ and according to (4.6), we have $E_c'(r_n) > 0$ (respectively $E_c'(r_n) < 0$) for $c > 1/\alpha$ (respectively $c < 1/\alpha$) and large n. On the other hand, it is clear that $E_c'(r_{2k+1}) \le 0$ (respectively $E_c'(r_{2k}) \ge 0$). The obtained contradiction proves our assertion for $c \ne 1/\alpha$.

When $c = 1/\alpha$, equation (4.6) at point $r = r_n$ becomes

$$(p-1)|u'|^{p-2}(r_n)E'_{1/\alpha}(r_n)$$
$$= r_n u^q(r_n)\Big[\frac{\beta q}{\alpha} + (\frac{1}{\alpha})^{p-1}(p-1)(\frac{N-p}{p-1} - \frac{1}{\alpha})\frac{u^{p-1-q}(r_n)}{r_n^p}\Big].$$

Then, if $\frac{1}{\alpha} = \frac{N-p}{p-1}$, the leading term of the right hand of the above equality is $\frac{\beta q}{\alpha} r_n u^q(r_n)$. Hence (ii) follows.

Finally, for (iii), recalling Theorem 3.6, we get that $E'_{1/\alpha}(r_n)$ has the same sign as

$$(p-1)\left(\frac{1}{\alpha}\right)^{p-1}\left(\frac{N-p}{p-1} - \frac{1}{\alpha}\right)\frac{u^{p-1}(r_n)}{r_n^{p-1}}.$$

This completes the proof. □

Next we introduce the following auxiliary function:

$$g(r) = u(r) + |u'|^{p-2}u'(r). \tag{4.7}$$

Lemma 4.6. *Assume $\alpha > 0$. Let u be a strictly positive solution of (P). Then the function $g(r)$ is strictly positive for large r.*

Proof. From Theorem 3.3 we know that $\lim_{r\to+\infty} r^{1/\alpha}u(r) = L \in [0, +\infty[$.
Case A. $L > 0$. Then $u(r) \sim Lr^{-1/\alpha}$ as $r \to +\infty$ and, by Lemma 3.4, $\lim_{r\to+\infty} r^{1/\alpha}|u'|^{p-2}u' = 0$. Hence, for large r, $g(r) \sim u(r) \sim Lr^{-1/\alpha}$ and thereby g is strictly positive.
Case B. $L = 0$. The proof will be done in two steps.
Step 1. g is not negative for large r. Suppose the opposite holds; i.e., that there exists a large R_1 such that $g(r) \le 0$ for $r \ge R_1$. Integrating this inequality on (R_1, r), we get

$$u^{\frac{p-2}{p-1}}(r) \le u^{\frac{p-2}{p-1}}(R_1) - \frac{p-2}{p-1}r + \frac{p-2}{p-1}R_1.$$

By letting $r \to +\infty$, we obtain a contradiction.
Step 2. $g(r)$ is monotone for large r. First note that for any $r > 0$,

$$g'(r) = u'(r) - \frac{N-1}{r}|u'|^{p-2}u' - \beta q r u^{q-1}u' - \alpha r u' - u.$$

By (ii) of Theorem 4.5, $E_{1/\alpha}(r)$ has a constant sign for large r, while u and u', go to zero as $r \to +\infty$. This implies $g'(r) \sim -\alpha E_{1/\alpha}(r)$ as $r \to +\infty$. Consequently, the function g is monotone for large r. Combining steps 1 and 2 we get the desired result. □

Now we use the phase plane arguments introduced by [1]. For this purpose we consider the equivalent non-autonomous first order system in the plane (X, Y):

$$\begin{aligned} X' &= |Y|^{-\frac{p-2}{p-1}}Y, \\ Y' &= -\frac{N-1}{r}Y - \alpha r|Y|^{-\frac{p-2}{p-1}}Y - \beta q r|X|^{q-1}|Y|^{-\frac{p-2}{p-1}}Y - X, \end{aligned} \tag{4.8}$$

where $X = u$, $Y = |u'|^{p-2}u'$, and $'$ is the derivative d/dr.

For $\lambda > 0$, we consider the following triangular subset of the (X, Y)- plane:

$$\mathcal{L}_\lambda = \{(X, Y) : 0 < X < 1,\ -\lambda X < Y < 0\} \tag{4.9}$$

Lemma 4.7. *For any $\lambda > 0$ let $r_\lambda = \frac{\lambda}{\alpha}(1 + \lambda^{-\frac{p}{p-1}})$. If $(X(r_\lambda), Y(r_\lambda)) \in \mathcal{L}_\lambda$ then the semi-orbit $(X(r), Y(r))_{r \geq r_\lambda}$ of (4.8) can leave $\mathcal{L}_\lambda$ only through $(1, 0)$.*

Proof. We shall show that if $r \geq r_\lambda$, then the vector field determined by (4.8) points into $\mathcal{L}_\lambda$. Indeed, on the line $Y = -\lambda X$,

$$\begin{aligned}\frac{Y'}{X'} = \frac{Y'}{|Y|^{-\frac{p-2}{p-1}} Y} &= -\frac{N-1}{r} |Y|^{\frac{p-2}{p-1}} - \alpha r - \frac{X}{Y} |Y|^{\frac{p-2}{p-1}} - q\beta r |X|^{q-1} \\ &= -\frac{N-1}{r} \lambda^{\frac{p-2}{p-1}} |X|^{\frac{p-2}{p-1}} - \alpha r + \lambda^{-\frac{1}{p-1}} |X|^{\frac{p-2}{p-1}} - q\beta r |X|^{q-1}.\end{aligned}$$

To have $Y'/X' < -\lambda$ it suffices that $-\alpha r + \lambda^{-\frac{1}{p-1}} < -\lambda$ or, equivalently, $r > r_\lambda$.

On the top $(Y = 0)$,

$$Y' = -\frac{N-1}{r} Y - \alpha r |Y|^{-\frac{p-2}{p-1}} Y - X - \beta q r |Y|^{-\frac{p-2}{p-1}} Y |X|^{q-1} = -X < 0$$

for all $r > 0$. Consequently, if the orbit leaves $\mathcal{L}_\lambda$, it must be through the point $(1, 0)$. The proof is complete. □

Remark 4.8. As a consequence of the previous Lemma, the orbits $(X(r), Y(r))$ corresponding to strictly positive solutions (hence strictly decreasing), can not leave $\mathcal{L}_\lambda$.

Proof of Theorem 4.3. . First, note that from Lemma 4.4, the set $\mathcal{S}_+$ is non empty. To prove that $\mathcal{S}_+$ is open, take $A_0 \in \mathcal{S}_+$ and fix $r_0 > 0$ large, such that $u(r_0, A_0) < 1$ and $g(r_0) = u(r_0, A_0) + |u'|^{p-2} u'(r_0, A_0) > 0$ (this is possible by virtue of Lemma 4.6). Then, by continuous dependence of solutions on the initial data, there is a neighborhood $\mathcal{O}(A_0)$ of A_0 such that

$$0 < u(r_0, A) < 1; \quad g(r_0) = u(r_0, A) + |u'|^{p-2} u'(r_0, A) > 0, \tag{4.10}$$

for any $A \in \mathcal{O}(A_0)$. In terms of the first order system (4.8), (4.10) reads $0 < X(r_0) < 1$ and $(X + Y)(r_0) > 0$; i.e., $(X, Y)(r_0) \in \mathcal{L}_1$. By Remark 4.8, $(X, Y)(r) \in \mathcal{L}_1$ for $r \geq r_0$. Thus in particular $X(r) = u(r, A) > 0$ for any $r \geq r_0$ and $A \in O(A_0)$. Consequently, $O(A_0) \subset \mathcal{S}_+$. The proof is complete. □

Theorem 4.9. *Assume $0 < \alpha N < 1$. Then there exists a unique $A > 0$ such that $u(\cdot, A)$ has compact support; i.e., $\mathcal{S}_c \neq \emptyset$.*

Existence follows easily from Theorem 4.2 and Theorem 4.3. For the proof of uniqueness the keystone is to compute series development of a generic compactly supported solution around the point where it vanishes. which is the content of the following lemma.

Lemma 4.10. *Assume $0 < \alpha N < 1$ and $\beta > 0$. Let u be a solution with compact support $[0, R]$.*

(i) *If $\frac{kp-(2k-1)}{p-1} < q < \frac{(k+1)p-(2k+1)}{p-1}$, $(k = 1, 2, 3, \dots)$, then*

$$\frac{|u'|^{p-1}}{u}(r) = \sum_{i=0}^{k-1} C_i (R - r)^i + \tilde{C} (R - r)^{(q-1)(p-1)/(p-2)} + \dots \tag{4.11}$$

(ii) *If* $q = \frac{kp-(2k-1)}{p-1}$ $(k = 2, 3, \dots)$, *then*

$$\frac{|u'|^{p-1}}{u}(r) = \sum_{i=0}^{k-1} D_i (R-r)^i + \dots \tag{4.12}$$

where

(a) $C_i = D_i$ *for* $i = 0, 1, \dots k-2$ *and depend on* p, N, α, R;
(b) C_{k-1} *depends on* p, N, α, R *for* $k \geq 2$;
(c) $\tilde{C} = \beta R C^{q-1}$, $C = \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)}$;
(d) $D_{k-1} = C_{k-1} + \tilde{C}$ *for* $k \geq 2$,

and the dots denote higher order infinitesimals as $r \to R$.

Proof. For the sake of simplicity and clarity, we give the proof for $k = 1$ and $k = 2$. From this it will be clear how to proceed by induction. Let $\varepsilon > 0$ be small. By integrating equation (1.5) on $(r, R) \subset (R-\varepsilon, R)$, we get

$$r^{N-1}|u'|^{p-1}(r) = \alpha r^N u(r) + \beta r^N u^q(r) - \int_r^R [1 - \alpha N - \beta N u^{q-1}(s)] s^{N-1} u(s) ds. \tag{4.13}$$

Dividing both sides by $r^{N-1}u(r)$,

$$\frac{|u'|^{p-1}(r)}{u(r)} = \alpha r + \beta r u^{q-1}(r) - \frac{1}{r^{N-1}u(r)} \int_r^R [1 - \alpha N - \beta N u^{q-1}(s)] s^{N-1} u(s)\, ds. \tag{4.14}$$

Note that, as $0 < \alpha N < 1$ and $u(R) = 0$, then $1 - \alpha N - \beta N u^{q-1}(s) > 0$ in (r, R) if ε is sufficiently small. Thereby,

$$\frac{|u'|^{p-1}(r)}{u(r)} < \alpha r + \beta r u^{q-1}(r) \tag{4.15}$$

and

$$\frac{|u'|^{p-1}(r)}{u(r)} \geq \alpha r + \beta r u^{q-1}(r) - \frac{1}{r^{N-1}u(r)} \int_r^R [1 - \alpha N + |\beta| N u^{q-1}(s)] s^{N-1} u(s) ds \tag{4.16}$$

Since $0 < u(s) < u(r)$ for $s \in (r, R)$, by letting $r \to R$ in (4.15) and (4.16), it follows that

$$\frac{|u'|^{p-1}}{u}(r) = C_0 + o(1); \quad C_0 = \alpha R,$$

as $r \to R$. Integrating this equation, we get

$$u(r) = C(R-r)^{(p-1)/(p-2)} + o((R-r)^{(p-1)/(p-2)}), \tag{4.17}$$

with C as in (c) above. Plugging this expression in (4.14) and taking into account that $\alpha r = \alpha R - \alpha(R-r)$ and

$$\left|\frac{1}{r^{N-1}u(r)} \int_r^R [1 - \alpha N - \beta N u^{q-1}(s)] s^{N-1} u(s) ds\right| \leq C'(R-r)$$

for $r \sim R$, part (i) of the assertion follows for $k = 1$. For part (ii) we need to refine the last estimate. An application of L'Hopital's rule gives

$$\lim_{r \to R} \frac{\int_r^R [1 - \alpha N - \beta N u^{q-1}(s)] s^{N-1} u(s) ds}{u R^{N-1}(R-r)} = C'' := \frac{(1-\alpha N)(p-2)}{2p-3},$$

and therefore in case (ii) with $k=1$ we get the desired result

$$\frac{|u'|^{p-1}(r)}{u(r)} = \alpha R + [\tilde{C} - (\alpha + C'')](R-r) + \dots = D_0 + D_1(R-r) + \dots,$$

since $(q-1)(p-1)/(p-2) = 1$.

Assume now $k=2$. In case (i), the previous calculations give

$$\frac{|u'|^{p-1}(r)}{u(r)} = \alpha R - (\alpha + C'')(R-r) + \dots = C_0 + C_1(R-r) + \dots,$$

which is not enough for our purposes, since the dependence on β is still unknown. Integrating the last equality, we get the more precise development

$$u(r) = C(R-r)^{(p-1)/(p-2)} + D(R-r)^{(p-1)/(p-2)+1} + \dots; \quad D = \frac{CC_1}{2C_0(p-2)}.$$

Using this last formula and the assumption $q > (2p-3)/(p-1)$, we compute

$$\begin{aligned}
&\lim_{r\to R} \frac{\int_r^R [1-\alpha N - \beta N u^{q-1}(s)] s^{N-1} u(s) ds - C'' R^{N-1}(R-r)u}{uR^{N-1}(R-r)^2} = \\
&= C''' := -\frac{(3p-5)C''DR - (p-2)(1-\alpha N)DR + (p-2)(N-1)(1-\alpha N)C}{(3p-5)CR}.
\end{aligned}$$

Since $q < (3p-5)/(p-1)$, we have $(q-1)(p-1)/(p-2) < 2$ and therefore

$$\begin{aligned}
\frac{|u'|^{p-1}(r)}{u(r)} &= \alpha R - (\alpha + C'')(R-r) + \tilde{C}(R-r)^{(q-1)(p-1)/(p-2)} + \dots \\
&= C_0 + C_1(R-r) + \tilde{C}(R-r)^{(q-1)(p-1)/(p-2)} + \dots
\end{aligned}$$

If (ii) holds, $(q-1)(p-1)/(p-2) = 2$ and the last formula is replaced by

$$\begin{aligned}
\frac{|u'|^{p-1}(r)}{u(r)} &= \alpha R - (\alpha + C'')(R-r) + (\tilde{C} + C''')(R-r)^2 + \dots \\
&= D_0 + D_1(R-r) + D_2(R-r)^2 + \dots
\end{aligned}$$

□

Now we are able to establish Theorem 4.9.

Poof of Theorem 4.9. Since $\mathcal{S}_+$ and $\mathcal{S}_-$ are nonempty, open and disjoint, the connectedness of $\mathbb{R}^+$ implies that there exists $A \in \mathbb{R}^+ \setminus (\mathcal{S}_+ \cup \mathcal{S}_-) = \mathcal{S}_c$. This settles the existence question.

For uniqueness we use the same ideas of the proof of Theorem 2.6. For this purpose let $u = u(\cdot, A)$ and $v = u(\cdot, B)$ be two solutions of problem (P) with $0 < A < B$, $\operatorname{supp} u = [0, R]$, $\operatorname{supp} v = [0, R_1]$. Much as in the aforementioned proof, consider the rescaled versions of u given by (2.17). By Theorem 2.6, we know that $u < v$ in $[0, R[$. Hence $R \le R_1$.

Define K as in that proof. The same arguments allow to conclude that $u_K \ge v$ on $[0, R_1]$ and that there exists $R_0 \in]0, R_1]$ such that $u_K(R_0) = v(R_0)$.

If $R = R_1$, then $R_0 = R$ is easily discarded, thus necessarily $R_0 \in]0, R[$ and the proof concludes exactly as that of Theorem 2.6.

If, on the contrary, $R < R_1$, then $R_0 = R_1$. Hence, both u_K and v are supported on $[0, R_1]$. Applying to them the lemma above, and taking into account the equation satisfied by u_K

$$(|u_K'|^{p-2}u_K')' + \frac{N-1}{r}|u_K'|^{p-2}u_K' + \alpha r u_K' + u_K + \tilde{\beta} r (u_K^q)' = 0, \tag{4.18}$$

where $\tilde{\beta} = \beta K^{\frac{p(q-1)}{p-2}} < \beta$, we conclude that, in some left neighborhood of $r = R_1$,

$$\frac{(-u_K')^{p-1}}{u_K} < \frac{(-v')^{p-1}}{v}.$$

Equivalently, $(u_K^{(p-2)/(p-1)})' > (v^{(p-2)/(p-1)})'$. Integrating on $[r, R_1]$ with r sufficiently close to R_1, we obtain $u_K(r) < v(r)$. This is impossible, hence $R_0 \neq R_1$ and we conclude as in the proof of Theorem 2.6. □

5. Results for the Ornstein-Uhlenbeck equation

In this section we apply the results obtained in the previous sections with the particular choice of the constants (1.7), related to the Ornstein-Uhlenbeck equation (1.1).

Theorem 5.1. *Let $q \geq p(1 + 1/N) - 1$. Then, for every $A > 0$ equation* (1.1) *admits a radial, strictly positive self-similar solution $U_A(x, t)$, of the form (1.2)–(1.3), with $A = U_A(0, 1)$. Moreover, $|x|^{p/(q+1-p)}U_A(x, t)$ is bounded for each $t > 0$ and there exists $L(A) \geq 0$ such that*

$$\lim_{t \to 0^+} U_A(x, t) = L(A)|x|^{-p/(q+1-p)} \quad \textit{for each } x \neq 0. \tag{5.1}$$

If $q > p(1 + 1/N) - 1$, then $L(A) > 0$. If $L(A) = 0$, $|x|^m U(x, t)$ is bounded for every $t > 0$ and $m > 0$.

Theorem 5.2. *Let $p - 1 < q < p(1 + 1/N) - 1$. Then, for every $A > 0$ equation* (1.1) *admits a radial, self-similar solution $U_A(x, t)$, of the form (1.2)–(1.3), with $A = U_A(0, 1)$. These solutions change sign for small A and are strictly positive for large A. In the later case, $|x|^{p/(q+1-p)}U(x, t)$ is bounded for each $t > 0$ and there exists $L(A) \geq 0$ such that* (5.1) *holds. Moreover, there exists a unique non-negative and compactly supported element U_{A_0} in the family with support*

$$\operatorname{supp} U_{A_0}(\cdot, t) = \{x \in \mathbb{R}^N : |x| < Ct^{(q+1-p)/p(q-1)}\}, \quad C > 0. \tag{5.2}$$

Proof of Theorem 5.1. Put $U_A(x, t) = t^{-1/(q-1)}u(|x|t^{-(q+1-p)/p(q-1)})$, where $u(r)$ is the solution of (P). This range of p, q corresponds to $\alpha \geq 1/N,$, $\beta > 0$. Existence and uniqueness follow from Theorems 2.1. Positivity follows from Theorem 3.7. By Lemma 3.5,

$$|x|^{p/(q+1-p)}U_A(x, t) = t^{-1/(q-1)}r^{p/(q+1-p)}u(rt^{-(q+1-p)/p(q-1)}) \leq C$$

for each $t > 0$. Moreover, by Theorem 3.3 we have

$$\lim_{t \to 0^+} U_A(x, t) = \lim_{y \to +\infty} y^{-p/(q+1-p)}u(y) = L(A) \geq 0.$$

The remaining assertions follow at once from Theorems 3.9 and 3.6. □

The proof of Theorem 5.2 is completely analogous to the proof above, and we omit it.

Remark 5.3. It is worth mentioning that the compactly supported solution U_{A_0} from Theorem 5.2 is *very singular* in the sense of [1]. Indeed, since $q > p-1$, the support (5.2) shrinks to $\{0\}$ as $t \to 0^+$, while an easy calculation shows that

$$\|U_{A_0}(t)\|_{L^1(\mathbb{R}^N)} = C_1 t^{\frac{N(q+1-p)-p}{p(q-1)}} \to +\infty$$

as $t \to 0^+$.

Acknowledgment. The authors would like to express their deep gratitude to Prof. J. L. Vázquez, who kindly suggested the idea of the paper and supported them through many fruitful discussions.

References

[1] H. Brezis, L. A. Peletier and D. Terman; *A very singular solution of the heat equation with absorption*, Arch. Rat. Mech. Anal, **96** (1986), pp. 185-209.

[2] A. Gmira and B. Bettioui; *On the radial solutions of a degenerate quasilinear elliptic equation in* $\mathbb{R}^N$, Ann. Fac. Sci. Toulouse **8** (3) (1999) pp.411-438.

[3] A. Gmira and B. Bettioui; *On the selfsimilar solutions of a diffusion-convection equation,* NoDEA, Nonlinear Differ. Equ. Appl. **9** (2002), pp. 277-294.

[4] W. G. Faris; *Ornstein-Uhlenbeck and renormalisation semigroups*, Moscow Mathematical Journal **1** (3), (2001), pp. 389-405.

[5] S. Kamin, J. L. Vázquez; *Fundamental solutions and asymptotic behaviour for the p-Laplacian equation*, Rev. Mat. Iberoamericana **4** (2) (1988), pp. 339-354.

[6] M. Kwak, K. Yu; *Asymptotic behaviour of solutions of a degenerate parabolic equation*, Nonlinear Anal. **45** (2001), pp. 109-121.

[7] A. Lunardi; *On the Ornstein-Uhlenbeck operator in* L^2 *spaces with respect to invariant measures*, Trans. Amer. Math. Soc. **349** (1) (1997), pp. 155-169.

[8] Yuan-Wei Qi and M. Wang; *The self-similar profiles of generalized KPZ equation*, Pacific Journal of Mathematics **201** (1) (2001), pp. 223-240.

[9] Yuan-Wei Qi and M. Wang; *The global existence and finite time extinction of quasilinear prabolic equations*, Advances Diff. Equ. **4** (5) (1999), pp. 731-753.

[10] Yuan-Wei Qi; *The global existence and nonuniqueness of a nonlinear degenerate equation*, Nonlinear Anal. **31** (1/2) (1998), pp. 117-136.

[11] J. Zhao; *The asymptotic behaviour of solutions of a quasilinear degenerate parabolic equation*, J. Diff. Equ. **102** (1993), pp. 33-52.

Arij Bouzelmate
Département de Mathématiques et Informatique, Faculté des Sciences, BP 2121, Tétouan, Maroc
E-mail address: `bouzelmatearij@yahoo.fr`

Abdelilah Gmira
Département de Mathématiques et Informatique, Faculté des Sciences, BP 2121, Tétouan, Maroc
E-mail address: `gmira@fst.ac.ma` or `gmira.i@menara.ma`

Guillermo Reyes
Departamento de Matemáticas, Universidad Carlos III de Madrid, Leganés, Madrid 28911, Spain
E-mail address: `greyes@math.uc3m.es`