Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 87, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF INFINITELY MANY SOLUTIONS OF p-LAPLACIAN EQUATIONS IN $\mathbb{R}^N_+$

JUNFANG ZHAO, XIANGQING LIU, JIAQUAN LIU

Abstract. In this article, we study the p-Laplacian equation

$$
\begin{gathered}
-\Delta_p u = 0, \quad \text{in } \mathbb{R}^N_+,\\
|\nabla u|^{p-2}\frac{\partial u}{\partial n} + a(y)|u|^{p-2}u = |u|^{q-2}u, \quad \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1},
\end{gathered}
$$

where $1 < p < N$, $p < q < \bar{p} = \frac{(N-1)p}{N-p}$, $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ the p-Laplacian operator, and the positive, finite function $a(y)$ satisfies suitable decay assumptions at infinity. By using the truncation method, we prove the existence of infinitely many solutions.

1. Introduction

In this article, we study the existence of infinitely many solutions of the p-Laplacian equation in $\mathbb{R}^N_+$,

$$
\begin{gathered}
-\Delta_p u = 0, \quad \text{in } \mathbb{R}^N_+,\\
|\nabla u|^{p-2}\frac{\partial u}{\partial n} + a(y)|u|^{p-2}u = |u|^{q-2}u, \quad \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1},
\end{gathered} \tag{1.1}
$$

where $1 < p < N$, $p < q < \bar{p} = \frac{(N-1)p}{N-p}$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ the p-Laplacian operator, and the positive, finite function $a(y)$ satisfies suitable decay assumptions at infinity.

For $\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+})$, define a norm

$$
\|\varphi\| = \Big(\int_{\mathbb{R}^N_+} |\nabla\varphi|^p \,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} |\varphi|^p \,\mathrm{d}y\Big)^{1/p}. \tag{1.2}
$$

Let W be the completion of $C_0^\infty(\overline{\mathbb{R}^N_+})$ with respect to the above norm. Problem (1.1) has a variational structure, given by the functional

$$
I(u) = \frac{1}{p}\int_{\mathbb{R}^N_+} |\nabla u|^p \,\mathrm{d}x + \frac{1}{p}\int_{\partial\mathbb{R}^N_+} a(y)|u|^p \,\mathrm{d}y - \frac{1}{q}\int_{\partial\mathbb{R}^N_+} |u|^q \,\mathrm{d}y, \quad u \in W. \tag{1.3}
$$

The embedding $W \hookrightarrow L^s(\partial\mathbb{R}^N_+), p \le s < \bar{p}$ is continuous, but not compact. Consequently, the functional I does not satisfy the Palais-Smale condition. Note

2010 *Mathematics Subject Classification.* 35B05, 35B45.

Key words and phrases. p-Lalacian equation; half space; boundary value problem; multiple solutions; truncation method.

Submitted September 22, 2018. Published July 16, 2019.

that the Sobolev space $W^{1,p}(\mathbb{R}^N_+)$ is continuously embedded into W, but the two spaces $W^{1,p}(\mathbb{R}^N_+)$ and W are different.

The weak form of problem (1.1) is as follows. Look for $u \in W$ satisfying

$$\int_{\mathbb{R}^N_+} |\nabla u|^{p-2}\nabla u \nabla\varphi \,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} a(y)|u|^{p-2}u\varphi \,\mathrm{d}y = \int_{\partial\mathbb{R}^N_+} |u|^{q-2}u\varphi \,\mathrm{d}y, \quad \forall\varphi \in W. \tag{1.4}$$

A function $u \in W$ is a weak solution if and only if u is a critical point of I.

Since the celebrated paper by Brezis and Nirenberg [3], there have been many results for nonlinear problems, involving the lack of compactness. In particular, Devillanova and Solimini [6] considered the problem

$$\begin{gathered} -\Delta u = u^{2^*-2} + \mu u, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.5}$$

where $2^* = \frac{2N}{N-2}, \mu > 0$ and Ω is an open regular domain of $\mathbb{R}^N, N \geq 3$. On the other hand, Cerami, Devillanova and Solimini [4] considered the subcritical equation in $\mathbb{R}^N$,

$$\begin{gathered} -\Delta u + a(x)u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty. \end{gathered} \tag{1.6}$$

Both problems (1.5) and (1.6) have a variational structure, but the Palais-Smale condition is not satisfied by the corresponding functionals. In the case of (1.5), the lack of compactness is due to the scalings, and in the case (1.6) due to the translation. The authors of [6, 4] found the solutions as limits of solutions of suitable approximated problems in bounded domains with subcritical growth. The fact that one solves the approximated problems under suitable assumptions and with the use of a local Pohožaev identity provides some extra information, which lead to a proof of desired convergence. Finally, to obtain infinitely many solutions, one has to distinguish the limits of the multiple approximated solutions. The estimate on the Morse index plays a role in this last step.

As to problems involving p-Laplacian operator, we have no information on the Morse index, therefore the approach of [6, 4] to distinguish the limit of solutions cannot be extended in a straightforward way to problems involving p-Laplacian operator with $p \neq 2$.

In this article, we use the truncation method. Following the idea in [10, 9], we first consider the truncated problems depending on a parameter λ, to which the functionals corresponding satisfy the Palais-Smale condition. Then by a concentration compactness analysis, similar to that in [6, 4], in particular with the use of a local Pohožaev identity, convergence theorem is proved. Our method is different from [6, 4] in the last step, the original problem and the approximated problems share some common solutions, and more and more solutions of the original problem are obtained as the parameter λ tends to zero. In this way, we obtain infinitely many solutions of the original problem. Up to our knowledge, there are few results concerning the existence of infinitely many solutions of the boundary value problems in $\mathbb{R}^N_+$ involving the p-Laplacian operator.

To describe the approximated problem, we need to introduce some auxilary functions. Let $\psi \in C_0^\infty(\mathbb{R}, [0,1])$ be such that $\psi(t) = 1$ for $|t| \leq 1$ and $\psi(t) = 0$ for

$|t| \geq 2$, is even and decreasing in $[1,2]$. For $\lambda > 0, y \in \partial\mathbb{R}^N_+ \sim \mathbb{R}^{N-1}, s \in \mathbb{R}$, define

$$\begin{aligned} b_\lambda(y,s) &= \psi(\lambda(1+|y|^2)^{\alpha/2}s), \\ m_\lambda(y,s) &= \int_0^s b_\lambda(y,\tau)\,\mathrm{d}\tau, \\ F_\lambda(y,s) &= \frac{1}{q}|s|^r|m_\lambda(y,s)|^{q-r}, \\ f_\lambda(y,s) &= \frac{\partial}{\partial s}F_\lambda(y,s), \end{aligned} \tag{1.7}$$

where $\alpha = \frac{N-p}{p-1}, r \in (p,q)$ is a fixed number. For $\lambda = 0$, we understand $m_0(y,s) \equiv s, F_0(y,s) \equiv \frac{1}{q}|s|^q$ and $f_0(y,s) \equiv |s|^{q-2}s$. The approximated equation is

$$\begin{gathered} -\Delta_p u = 0, \quad \text{in } \mathbb{R}^N_+, \\ |\nabla u|^{p-2}\nabla u\frac{\partial u}{\partial n} + a(y)|u|^{p-2}u = f_\lambda(y,u), \quad \text{on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}. \end{gathered} \tag{1.8}$$

Problem (1.8) has a variational structure, given by the functional

$$I_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N_+}|\nabla u|^p\,\mathrm{d}x + \frac{1}{p}\int_{\partial\mathbb{R}^N_+}a(y)|u|^p\,\mathrm{d}y - \frac{1}{q}\int_{\partial\mathbb{R}^N_+}F_\lambda(y,u)\,\mathrm{d}y. \tag{1.9}$$

The critical points of I_λ are weak solutions of (1.8) satisfying

$$\int_{\mathbb{R}^N_+}|\nabla u|^{p-2}\nabla u\nabla\varphi\,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+}a(y)|u|^{p-2}u\varphi\,\mathrm{d}y = \int_{\partial\mathbb{R}^N_+}f_\lambda(y,u)\varphi\,\mathrm{d}y. \tag{1.10}$$

Notice that the function $f_\lambda(y,u)$ decays polynomially as $|y| \to +\infty$ (see Lemma 2.1), therefore the functional I_λ satisfies the Palais-Smale condition. On the other hand, if we have a good estimate, namely

$$|u(y)| \leq \lambda^{-1}(1+|y|^2)^{-\alpha/2}, \quad y \in \partial\mathbb{R}^N_+,$$

then, $f_\lambda(y,u(y)) = |u|^{q-2}u(y), y \in \partial\mathbb{R}^N_+$, and u will be a solution of the original problem.

Now we state the assumptions on the potential function a.

- (A1) $a \in C(\mathbb{R}^{N-1},\mathbb{R})$.
- (A2) There exist $a_0, a_1 > 0$ such that $a_0 \leq a(y) \leq a_1, y \in \mathbb{R}^{N-1}$.
- (A3) There exists $\bar{c} > 1$ such that $\frac{\partial a}{\partial r}a(y) = (\frac{y}{|y|},\nabla a) \geq 0$ and $|\nabla a(y)| \leq \bar{c}\frac{\partial a}{\partial r}a(y)$, for $y \in \mathbb{R}^{N-1}, |y| \geq \bar{c}$.
- (A4) $\lim_{|y|\to+\infty}|\frac{\partial}{\partial r}a(y)|(1+|y|^2)^{\alpha/2} = +\infty$, $\alpha = \frac{N-p}{p-1}$.

Remark 1.1. By (A4), we have $a(y) \geq c(1+|y|)^{-\alpha+1}$. For asumptions (A2) and (A4) to be consistent, we need to assume $\alpha = \frac{N-p}{p-1} > 1$; whence $\alpha > 1$, we choose $\beta \in (1,\alpha)$. Then the function $a(y) = 2 - (1+|y|^2)^{-\alpha/2}$ satisfies (A1)–(A4).

Here are our main results.

Theorem 1.2. *Assume* $1 < p < q < N$, $\alpha = \frac{N-p}{p-1} > 1$. *Assume (A1)–(A4). Given* $M > 0$, *there exists* $\mu = \mu(M)$ *such that if* $u \in W$ *is a solution of* (1.8), $\lambda > 0$ *and* $\|u\| \leq M$, *then*

$$u(y) \leq \frac{1}{\mu}(1+|y|^2)^{-\alpha/2}, \quad \forall y \in \mathbb{R}^{N-1}.$$

Theorem 1.3. *Assume* $1 < p < q < N$, $\alpha = \frac{N-p}{p-1} > 1$. *Assume (A1)–(A4). Then* (1.1) *has infinitely many solutions.*

Throughout this article, we use the following notation: $|\cdot|_p$ for the norm in $L^p(\mathbb{R}^{N-1})$, $\|\cdot\|$ for the norm in W, $\to$ for the strong convergence, $\rightharpoonup$ for the weak convergence, $B_R^+ = \{x | x \in \mathbb{R}_+^N, |x| < R\}, D_R = \{y | y \in \partial\mathbb{R}_+^N = \mathbb{R}^{N-1}, |y| < R\}$.

2. Uniform bounds

As mentioned in the introduction, we use solutions of the truncated problems as approximate solutions of the original problem. In this section, we prove uniform bounds for the approximate solutions by making a concentration compactness analysis and with the help of local Pohožaev identity.

Let $u_n \in W$ be a solution of (1.8) with $\lambda = \lambda_n \geq 0$, $n = 1, 2, \dots$. Assume $\|u\| \leq M$. By [13, Theorem 2.1], $\{u_n\}$ has a profile decomposition

$$u_n = u + \sum_{k\in\Lambda} U_k(\cdot - y_{n,k}) + r_n, \tag{2.1}$$

where $u, U_k, r_n \in W, \{y_{n,k}\} \subset \partial\mathbb{R}_+^N = \mathbb{R}^{N-1}$, $k \in \Lambda$ and Λ is an index set. It holds that

(1) $u_n \rightharpoonup u, u_n(\cdot + y_{n,k}) \rightharpoonup U_k$ in W as $n \to \infty, k \in \Lambda$.
(2) $|y_{n,k}| \to +\infty, |y_{n,k} - y_{n,l}| \to \infty$ as $n \to \infty, k, l \in \Lambda, k \neq l$.
(3) $|u|_q^q + \sum_{k\in\Lambda} |U_k|_q^q \leq \lim_{n\to\infty} |u_n|_q^q$, $p < q < \bar{p}$.
(4) $|r_n|_q \to 0$ in $L^q(\mathbb{R}^{N-1})$ as $n \to \infty$, $p < q < \bar{p}$.

In the following lemma, we list some elementary properties of the auxiliary functions.

Lemma 2.1. *For* $(y, s) \in \mathbb{R}^{N-1} \times \mathbb{R}$ *and* $\lambda > 0$ *the following holds:*

(1) $sm_\lambda(y, s) \geq 0$, $|s|b_\lambda(y, s) \leq |m_\lambda(y, s)|$.
(2) $\min\{|s|, \frac{1}{\lambda}(1 + |y|^2)^{-\alpha/2}\} \leq |m_\lambda(y, s)| \leq \min\{|s|, \frac{2}{\lambda}(1 + |y|^2)^{-\alpha/2}\}$ *and* $m_\lambda(y, s) = s$, *if* $|y| \leq \frac{1}{\lambda}(1 + |y|^2)^{-\alpha/2}$.
(3) $|f_\lambda(y, s)| \leq |s|^{r-1}|m_\lambda(y, s)|^{q-r} \leq |s|^{q-1}$.
(4) $\frac{1}{r} sf_\lambda(y, s) - F_\lambda(y, s) = \frac{q-r}{qr} |s|^{r-1}|m_\lambda(y, s)|^{q-r-1}b_\lambda(y, s) \geq 0$.
(5) $\nabla_y m_\lambda(y, s) = -\alpha \frac{y}{1+|y|^2} (m_\lambda(y, s) - sb_\lambda(y, s))$

$$\nabla_y F_\lambda(y, s) = -\big(1 - \frac{r}{q}\big)\alpha \frac{y}{1 + |y|^2} |s|^r |m_\lambda(y, s)|^{q-r-1}|m_\lambda(y, s) - sb_\lambda(y, s)|.$$

Proof. The proof is elementary and straightforward. We prove only (3)–(5). For (3) and (4), we have

$$\begin{aligned} f_\lambda(y, s) &= \frac{\partial F_\lambda(y, s)}{\partial s} \\ &= \frac{r}{q} |s|^{r-2} s|m_\lambda(y, s)|^{q-r} + \frac{q - r}{q} |s|^r |m_\lambda(y, s)|^{q-r-2} m_\lambda(y, s)b_\lambda(y, s), \end{aligned}$$

since $0 \leq \frac{sb_\lambda(y,s)}{m_\lambda(y,s)} \leq 1$, we have

$$\begin{aligned} |f_\lambda(y, s)| &\leq \frac{r}{q} |s|^{r-1}|m_\lambda(y, s)|^{q-r} + \frac{q - r}{q} |s|^{r-1}|m_\lambda(y, s)|^{q-r} \\ &= |s|^{r-1}|m_\lambda(y, s)|^{q-r} \leq |s|^{q-1}, \end{aligned}$$

and

$$\begin{aligned}\frac{1}{r}sf_\lambda(y,s)-F_\lambda(y,s)&=\big(\frac{1}{r}-\frac{1}{q}\big)|s|^r s|m_\lambda(y,s)|^{q-r-2}m_\lambda(y,s)b_\lambda(y,s)\\&=\big(\frac{1}{r}-\frac{1}{q}\big)|s|^{r+1}|m_\lambda(y,s)|^{q-r-1}b_\lambda(y,s)\ge 0.\end{aligned}$$

For (5), by the definition of m_λ, we have

$$\begin{aligned}\nabla_y m_\lambda(y,s)&=\int_0^s\nabla_y b_\lambda(y,\tau)d\tau\\&=\int_0^s\psi'(\lambda(1+|y|^2)^{\alpha/2}\tau)\cdot\lambda(1+|y|^2)^{\frac{\alpha}{2}-1}\tau\cdot\alpha y d\tau\\&=\int_0^s\alpha\frac{y}{1+|y|^2}\cdot\tau\,\mathrm{d}\psi(\lambda(1+|y|^2)^{\alpha/2}\tau)\\&=\alpha\frac{y}{1+|y|^2}s\psi(\lambda(1+|y|^2)^{\alpha/2}s)-\int_0^s\psi(\lambda(1+|y|^2)^{\alpha/2}\tau)\,\mathrm{d}\tau\\&=-\alpha\frac{y}{1+|y|^2}\big(m_\lambda(y,s)-sb_\lambda(y,s)\big).\end{aligned}$$

Thus

$$\begin{aligned}\nabla_y F_\lambda(y,s)&=\frac{q-r}{q}|s|^r|m_\lambda(y,s)|^{q-r-2}m_\lambda(y,s)\nabla_y m_\lambda(y,s)\\&=-\big(1-\frac{r}{q}\big)\alpha\frac{y}{1+|y|^2}|s|^r|m_\lambda(y,s)|^{q-r-1}|m_\lambda(y,s)-sb_\lambda(y,s)|.\end{aligned}$$

The proof is complete. $\square$

In the following few lemmas we study the profile decomposition 2.1. In particular in Lemmas 2.4 and 2.5, we prove that the weak limit functions u, U_k satisfy differential inequality and decay polynomially at the infinity.

Lemma 2.2. *Let $u\in W$ be a solution of* (1.8), $\lambda\ge 0$. *Then $v=|u|$ satisfies the differential inequality*

$$\int_{\mathbb{R}^N_+}|\nabla v|^{p-2}\nabla v\nabla\varphi\,\mathrm{d}x+\int_{\partial\mathbb{R}^N_+}a(y)v^{p-1}\varphi\,\mathrm{d}y\le\int_{\partial\mathbb{R}^N_+}v^{q-1}\varphi\,\mathrm{d}y,\tag{2.2}$$

for $\varphi\in W$ and $\varphi\ge 0$.

Proof. This lemma is somewhat similar to Kato's inequality setting that if $u\in H^1(\mathbb{R}^N)$ (for instance), then $\Delta|u|\ge \operatorname{sign}u\cdot\Delta u$.

To prove Lemma 2.2, we set $v_\varepsilon=(u^2+\varepsilon^2)^{1/2}-\varepsilon,\varepsilon>0$. Then $v_\varepsilon\to v$ in W as $\varepsilon\to 0$. For $\varphi\in C_0^\infty(\overline{\mathbb{R}^N_+})$, $\varphi\ge 0$ we have

$$\begin{aligned}&\int_{\mathbb{R}^N_+}|\nabla v|^{p-2}\nabla v_\varepsilon\nabla\varphi\,\mathrm{d}x\\&=\int_{\mathbb{R}^N_+}|\nabla u|^{p-2}\frac{u\nabla u}{(u^2+\varepsilon^2)^{1/2}}\nabla\varphi\,\mathrm{d}x\\&=\int_{\mathbb{R}^N_+}|\nabla u|^{p-2}\nabla u\nabla\Big(\frac{u}{(u^2+\varepsilon^2)^{1/2}}\varphi\Big)\,\mathrm{d}x-\int_{\mathbb{R}^N_+}|\nabla u|^p\frac{\varepsilon^2}{(u^2+\varepsilon^2)^{1/2}}\varphi\,\mathrm{d}x\\&\le\int_{\mathbb{R}^N_+}|\nabla u|^{p-2}\nabla u\nabla\Big(\frac{u}{(u^2+\varepsilon^2)^{1/2}}\varphi\Big)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\partial\mathbb{R}^N_+} a(y)|u|^{p-2}u\frac{u}{(u^2+\varepsilon^2)^{1/2}}\varphi\,\mathrm{d}y + \int_{\partial\mathbb{R}^N_+} f_\lambda(y,u)\frac{u}{(u^2+\varepsilon^2)^{1/2}}\varphi\,\mathrm{d}y \\
&\le -\int_{\partial\mathbb{R}^N_+} a(y)v^{p-1}\frac{v}{(v^2+\varepsilon^2)^{1/2}}\varphi\,\mathrm{d}y + \int_{\partial\mathbb{R}^N_+} v^{q-1}\frac{v}{(v^2+\varepsilon^2)^{1/2}}\varphi\,\mathrm{d}y\,.
\end{aligned}
$$

Here we used that $|f_\lambda(y,s)| \le |s|^{q-1}$. Let $\varepsilon \to 0$ in in the above inequality, by Lebesgue's dominated convergence theorem, we obtain (2.2) for $\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+})$, $\varphi \ge 0$. By a density argument, (2.2) holds for $\varphi \in W, \varphi \ge 0$. □

Lemma 2.3. *Let $u_n \in W$ be a solution of (1.8) with $\lambda = \lambda_n \ge 0$, $n = 1,2,\dots$, $\{y_n\} \subset \partial\mathbb{R}^N_+ \sim \mathbb{R}^{N-1}$. Suppose $\widetilde{u}_n = u_n(\cdot + y_n) \rightharpoonup U$ in W. Then $\widetilde{u}_n \to U$ in W locally (equivalently $\widetilde{u}_n \to U$ in $W^{1,p}_{loc}(\mathbb{R}^N_+)$).*

Proof. $\widetilde{u}_n$ satisfies the equation

$$
\begin{aligned}
&\int_{\mathbb{R}^N_+} |\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n\nabla\varphi\,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} a(y+y_n)|\widetilde{u}_n|^{p-2}\widetilde{u}_n\varphi\,\mathrm{d}y \\
&= \int_{\partial\mathbb{R}^N_+} f_{\lambda_n}(y+y_n,\widetilde{u}_n)\varphi\,\mathrm{d}y
\end{aligned} \tag{2.3}
$$

for $\varphi \in W$. Let $R > 0$, $\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+}, [0,1])$ such that $\varphi(x) = 1$ for $|x| \le R$, $\varphi(x) = 0$ for $|x| \ge 2R$. Since $\widetilde{u}_n$ converges in $L^q_{\rm loc}(\partial\mathbb{R}^N_+), 1 \le q < \bar{p}$ and in $L^q_{\rm loc}(\mathbb{R}^N_+), 1 \le q < \bar{p}$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N_+} \big(|\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n - |\nabla\widetilde{u}_m|^{p-2}\nabla\widetilde{u}_m, \nabla\widetilde{u}_n - \nabla\widetilde{u}_m\big)\varphi\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^N_+} \big(|\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n - |\nabla\widetilde{u}_m|^{p-2}\nabla\widetilde{u}_m, \nabla\varphi\big)(\widetilde{u}_n - \widetilde{u}_m)\,\mathrm{d}x \\
&\quad + \int_{\partial\mathbb{R}^N_+} \big(a(y+y_n)|\widetilde{u}_n|^{p-2}\widetilde{u}_n - a(y+y_m)|\widetilde{u}_m|^{p-2}\widetilde{u}_m\big)(\widetilde{u}_n - \widetilde{u}_m)\varphi\,\mathrm{d}y \\
&\quad + \int_{\partial\mathbb{R}^N_+} (f_{\lambda_n}(y+y_n,\widetilde{u}_n) - f_{\lambda_m}(y+y_m,\widetilde{u}_m))(\widetilde{u}_n - \widetilde{u}_m)\varphi\,\mathrm{d}y \\
&\le c\Big(\int_{B^+_{2R}} |\widetilde{u}_n - \widetilde{u}_m|^p dx\Big)^{1/p} + c\Big(\int_{D_{2R}} |\widetilde{u}_n - \widetilde{u}_m|^p dy\Big)^{1/p} \\
&\quad + c\Big(\int_{D_{2R}} |\widetilde{u}_n - \widetilde{u}_m|^q dy\Big)^{1/q} \to 0, \quad \text{as } n,m \to \infty.
\end{aligned} \tag{2.4}
$$

The following elementary inequalities are very useful (see [5]). There exists a constant $c_{p,N}$ such that for $\xi,\eta \in \mathbb{R}^N$,

$$
\begin{gathered}
\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\big) \ge c_{p,N}|\xi - \eta|^p, \text{ if } p \ge 2, \\
\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\big) \ge c_{p,N}|\xi - \eta|^2(|\xi|^p + |\eta|^p)^{-\frac{2-p}{p}}, \quad \text{if } 1 < p < 2\,.
\end{gathered} \tag{2.5}
$$

For $p \ge 2$, by (2.4) and (2.5), we have

$$
\begin{aligned}
\int_{B^+_R} |\nabla\widetilde{u}_n - \nabla\widetilde{u}_m|^p\,\mathrm{d}x &\le c\int_{\mathbb{R}^N_+} (|\nabla\widetilde{u}_n|^{p-2}\nabla\widetilde{u}_n - |\nabla\widetilde{u}_m|^{p-2}\nabla\widetilde{u}_m, \nabla\widetilde{u}_n - \nabla\widetilde{u}_m)\varphi\,\mathrm{d}x \\
&\to 0
\end{aligned}
$$

as $n, m \to \infty$. For $1 < p \le 2$, by (2.4) and (2.5), we have

$$
\begin{aligned}
&\int_{B_R^+} |\nabla \widetilde{u}_n - \nabla \widetilde{u}_m|^p \,\mathrm{d}x \\
&\le c \int_{B_R^+} |(|\nabla \widetilde{u}_n|^{p-2}\nabla \widetilde{u}_n - |\nabla \widetilde{u}_m|^{p-2}\nabla \widetilde{u}_m, \nabla \widetilde{u}_n - \nabla \widetilde{u}_m)|^{p/2} \\
&\quad \times (|\nabla \widetilde{u}_n|^p + |\nabla \widetilde{u}_m|^p)^{\frac{2-p}{2}} \,\mathrm{d}x \\
&\le c\Big(\int_{B_R^+} (|\nabla \widetilde{u}_n|^{p-2}\nabla \widetilde{u}_n - |\nabla \widetilde{u}_m|^{p-2}\nabla \widetilde{u}_m, \nabla \widetilde{u}_n - \nabla \widetilde{u}_m) \,\mathrm{d}x\Big)^{p/2} \\
&\quad \times \Big(\int_{B_R^+} (|\nabla \widetilde{u}_n|^p + |\nabla \widetilde{u}_m|^p) \,\mathrm{d}x\Big)^{\frac{2-p}{2}} \\
&\le c\Big(\int_{B_{2R}^+} (|\nabla \widetilde{u}_n|^{p-2}\nabla \widetilde{u}_n - |\nabla \widetilde{u}_m|^{p-2}\nabla \widetilde{u}_m, \nabla \widetilde{u}_n - \nabla \widetilde{u}_m)\varphi \,\mathrm{d}x\Big)^{p/2} \\
&\to 0, \quad \text{as } n, m \to \infty.
\end{aligned}
$$

Hence $\{\widetilde{u}_n\}$ converges locally in W (and in $W^{1,p}(\mathbb{R}^N_+)$). □

Lemma 2.4. *Let the profile decomposition* (2.1) *hold for* $\{u_n\}$*. Then*

(1) $v = |u|, V_k = |U_k|$ *satisfy the differential inequalities*

$$
\int_{\mathbb{R}^N_+} |\nabla v|^{p-2}\nabla v \nabla\varphi \,\mathrm{d}x + a_0 \int_{\partial\mathbb{R}^N_+} v^{p-1}\varphi \,\mathrm{d}y \le \int_{\partial\mathbb{R}^N_+} v^{q-1}\varphi \,\mathrm{d}y, \tag{2.6}
$$

for $\varphi \in W$ *and* $\varphi \ge 0$.

$$
\int_{\mathbb{R}^N_+} |\nabla V_k|^{p-2}\nabla V_k \nabla\varphi \,\mathrm{d}x + a_0 \int_{\partial\mathbb{R}^N_+} V_k^{p-1}\varphi \,\mathrm{d}y \le \int_{\partial\mathbb{R}^N_+} V_k^{q-1}\varphi \,\mathrm{d}y, \tag{2.7}
$$

for $\varphi \in W$ *and* $\varphi \ge 0$.

(2) *The index set* Λ *is finite.*

Proof. (1) Denote $v_n = |u_n|$. By Lemma 2.2, v_n satisfies the differential inequality

$$
\int_{\mathbb{R}^N_+} |\nabla v_n|^{p-2}\nabla v_n \nabla\varphi \,\mathrm{d}x + a_0 \int_{\partial\mathbb{R}^N_+} v_n^{p-1}\varphi \,\mathrm{d}y \le \int_{\partial\mathbb{R}^N_+} v_n^{q-1}\varphi \,\mathrm{d}y \tag{2.8}
$$

for $\varphi \in W$ and $\varphi \ge 0$. By Lemma 2.3, $u_n \to u$ in W locally, consequently $v_n \to v$ in W locally. Take the limit $n \to \infty$ in (2.8), we obtain (2.6) for $\varphi \in C_0^\infty(\mathbb{R}^N_+), \varphi \ge 0$. By a density argument, this inequality holds for $\varphi \in W, \varphi \ge 0$. Similarly, we can prove that V_k satisfies the inequality (2.7).

(2) By (2.7) and the Sobolev embedding theorem,

$$
\Big(\int_{\partial\mathbb{R}^N_+} V_k^q \,\mathrm{d}y\Big)^{p/q} \le S_{p,q}^{-1}\Big(\int_{\mathbb{R}^N_+} |\nabla V_k|^p \,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} V_k^p \,\mathrm{d}y\Big) \le c \int_{\partial\mathbb{R}^N_+} V_k^q \,\mathrm{d}y\,,
$$

where $S_{p,q}$ is the Sobolev constant for the embedding from W to $L^q(\partial\mathbb{R}^N_+)$:

$$
S_{p,q} = \inf_{u \in W\setminus\{0\}} \frac{\int_{\mathbb{R}^N_+} |\nabla u|^p \,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} |u|^p \,\mathrm{d}y}{(\int_{\partial\mathbb{R}^N_+} |u|^q \,\mathrm{d}y)^{p/q}}.
$$

Hence $\int_{\partial\mathbb{R}^N_+} |U_k|^q \,\mathrm{d}y = \int_{\partial\mathbb{R}^N_+} V_k^q \,\mathrm{d}y \ge m$ for some $m > 0$. By the property (3) of the decomposition (2.1), Λ is finite. □

Lemma 2.5. *Let $v \in W$, $v \geq 0$ satisfy the differential inequality* (2.6),

$$\int_{\mathbb{R}^N_+} |\nabla v|^{p-2}\nabla v \nabla\varphi \,\mathrm{d}x + a_0 \int_{\partial\mathbb{R}^N_+} v^{p-1}\varphi \,\mathrm{d}y \leq \int_{\partial\mathbb{R}^N_+} v^{q-1}\varphi \,\mathrm{d}y$$

for $\varphi \in W$ and $\varphi \geq 0$. Then there exists a positive constant c such that

$$\begin{gathered} v(x) \leq c(1+|x|)^{-\frac{N-p}{p-1}}, \\ \int_{\mathbb{R}^N_+ \setminus B^+_R} |\nabla v|^p \,\mathrm{d}x \leq cR^{-\frac{N-p}{p-1}}, \\ \int_{\partial\mathbb{R}^N_+ \setminus D_R} v^p dy \leq cR^{-\frac{N-p}{p-1}}. \end{gathered}$$

Proof. The proof is divided into three steps, by using Moser's iteration and the Wolff potential for the p-Laplacian equation.

Step 1. Use Moser's iteration to prove that, given $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$v(y) \leq \varepsilon, \quad \text{if } y \in \partial\mathbb{R}^N_+,\ |y| \geq R_0. \tag{2.9}$$

In particular, $v^{q-p}(y) \leq \frac{1}{2}a_0$, if $y \in \partial\mathbb{R}^N_+, |y| \geq R_0$. We prove that

$$|v|_{L^\infty(D_{\frac{1}{2}}(y))} \leq c\Big(|v|_{L^p(B^+_1(y))} + |v|_{L^{\frac{\bar p}{d}}(D_1(y))}\Big), \quad y \in \partial\mathbb{R}^N_+, \tag{2.10}$$

where $d = \frac{\bar p - q + p}{p} > 1$. Since $v \in L^{p^*}(\mathbb{R}^N_+) \cap L^{\frac{\bar p}{d}}(\partial\mathbb{R}^N_+)$, we have

$$|v|_{L^p(B^+_{\frac{1}{2}}(y))} + |v|_{L^{\frac{\bar p}{d}}(D_{\frac{1}{2}}(y))} \to 0, \quad \text{as } |y| \to +\infty, \quad y \in \partial\mathbb{R}^N_+.$$

Hence, the estimate (2.9) follows from (2.10).

Now, we prove (2.10) by using Moser's iteration. Let $\varphi \in C^\infty_0(\overline{R^N_+})$, $r \geq 1$. Take $v^{p(r-1)+1}\varphi^p$ as the test function in (2.6),

$$\int_{\mathbb{R}^N_+} |\nabla v|^{p-2}\nabla v \nabla(v^{p(r-1)+1}\varphi^p) \,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} v^{pr}\varphi^p \,\mathrm{d}y \leq c\int_{\partial\mathbb{R}^N_+} v^{q-p}v^{pr}\varphi^p \,\mathrm{d}y.$$

By Hölder inequality we have

$$\begin{aligned} &\frac{1}{r^p}\int_{R^N_+} |\nabla(v^r\varphi)|^p \,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} (v^r\varphi)^p \,\mathrm{d}y \\ &\leq \int_{R^N_+} v^{pr}|\nabla\varphi|^p \,\mathrm{d}x + c\int_{\partial\mathbb{R}^N_+} v^{q-p}v^{pr}\varphi^p \,\mathrm{d}y \\ &\leq c\int_{\mathbb{R}^N_+} v^{pr}|\nabla\varphi|^p \,\mathrm{d}x + c\Big(\int_{\partial\mathbb{R}^N_+} v^{\bar p} \,\mathrm{d}y\Big)^{\frac{q-p}{\bar p}} \Big(\int_{\partial\mathbb{R}^N_+} (v^r\varphi)^{\frac{p\bar p}{\bar p - q + p}} \,\mathrm{d}y\Big)^{\frac{\bar p - q + p}{\bar p}} \\ &\leq c\int_{\mathbb{R}^N_+} v^{pr}|\nabla\varphi|^p \,\mathrm{d}x + c\Big(\int_{\partial\mathbb{R}^N_+} (v^r\varphi)^{\frac{\bar p}{d}} \,\mathrm{d}y\Big)^{pd/\bar p}. \end{aligned} \tag{2.11}$$

By the Sobolev embedding theorem,

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^N_+}(v^r\varphi)^{p^*}\,\mathrm{d}x\Big)^{p/p^*}+\Big(\int_{\partial\mathbb{R}^N_+}(v^r\varphi)^{\bar p}\,\mathrm{d}y\Big)^{p/\bar p}\\
&\le c\int_{R^N_+}\big|\nabla\big(v^r\varphi\big)\big|^p\,\mathrm{d}x \qquad (2.12)\\
&\le cr^p\Big(\int_{\mathbb{R}^N_+}v^{pr}|\nabla\varphi|^p\,\mathrm{d}x+\int_{\partial\mathbb{R}^N_+}(v^r\varphi)^{\frac{\bar p}{d}}\,\mathrm{d}y\Big)^{d/\bar p}.
\end{aligned}$$

Now choose $y_0\in\partial\mathbb{R}^N_+$, assume that the support of the function $\varphi\in C_0^\infty(\overline{\mathbb{R}^N_+})$ is contained in $B_2(y_0)=\{x\mid x\in\mathbb{R}^N,\ |x-y_0|<2\}$. Then

$$\Big(\int_{\mathbb{R}^N_+}(v^r\varphi)^{pd}\,\mathrm{d}x\Big)^{\frac{1}{pdr}}\le\Big(c\int_{\mathbb{R}^N_+}(v^r\varphi)^{p^*}\,\mathrm{d}x\Big)^{\frac{1}{p^*r}}. \qquad (2.13)$$

Since $pd=\bar p-q+p<p^*$. By (2.12) and (2.13), we obtain

$$\begin{aligned}
&\max\Big\{\Big(\int_{\mathbb{R}^N_+}(v^r\varphi)^{pd}\,\mathrm{d}x\Big)^{\frac{1}{pdr}},\ \Big(\int_{\partial\mathbb{R}^N_+}(v^r\varphi)^{\bar p}\,\mathrm{d}y\Big)^{\frac{1}{\bar pr}}\Big\}\\
&\le(cr)^{1/r}\max\Big\{\Big(\int_{\mathbb{R}^N_+}v^{pr}|\nabla\varphi|^p\,\mathrm{d}x\Big)^{\frac{1}{pr}},\ \Big(\int_{\partial\mathbb{R}^N_+}(v^r\varphi)^{\frac{\bar p}{d}}\,\mathrm{d}y\Big)^{d/\bar p}\Big\}.
\end{aligned} \qquad (2.14)$$

Denote

$$\begin{gathered}
s_n=\frac12+\frac{1}{2^{n+1}},\quad n=0,1,2,\dots,\\
B^+_{s_n}=\{x\in\mathbb{R}^N_+:|x-y_0|<s_n\},\\
D_{s_n}=\{y\in\partial\mathbb{R}^N_+:|y-y_0|<s_n\}.
\end{gathered}$$

Let $\varphi=\varphi_n$ be such that $\varphi_n=1$ for $x\in B^+_{s_{n+1}}$; $\varphi_n=0$ for $x\notin B^+_{s_n}$ and $|\nabla\varphi_n|\le\frac{1}{2^n}$, $r=r_n=d^n$. Then by (2.14),

$$\begin{aligned}
&\max\Big\{\Big(\int_{B^+_{s_{n+1}}}v^{r_{n+1}p}\,\mathrm{d}x\Big)^{\frac{1}{r_{n+1}p}},\ \Big(\int_{D_{s_{n+1}}}v^{r_{n+1}\frac{\bar p}{d}}\,\mathrm{d}y\Big)^{\frac{d}{r_{n+1}\bar p}}\Big\}\\
&\le\big(c2^nd^n\big)^{\frac{1}{d_n}}\max\Big\{\Big(\int_{B^+_{s_n}}v^{r_np}\,\mathrm{d}x\Big)^{\frac{1}{r_np}},\ \Big(\int_{D_{s_n}}v^{r_n\frac{\bar p}{d}}\,\mathrm{d}y\Big)^{\frac{d}{r_n\bar p}}\Big\}\\
&\le\prod_{n=0}^{\infty}(c2^nd_n)^{\frac{1}{d_n}}\max\Big\{\Big(\int_{B^+_1}v^p\,\mathrm{d}x\Big)^{1/p},\ \Big(\int_{D_1}v^{\frac{\bar p}{d}}\,\mathrm{d}y\Big)^{d/\bar p}\Big\}\\
&=c\max\Big\{\Big(\int_{B^+_1}v^p\,\mathrm{d}x\Big)^{1/p},\ \Big(\int_{D_1}v^{\frac{\bar p}{d}}\,\mathrm{d}y\Big)^{d/\bar p}\Big\}.
\end{aligned} \qquad (2.15)$$

Taking the limit $n\to\infty$ in (2.15), we obtain the desired estimate.

Step 2. Using the Wolff potential for the p-Laplacian operator in $\mathbb{R}^N_+$ we prove that there exists $c>0$ such that

$$V(x)\le c(1+|x|)^{-\frac{N-p}{p-1}},\quad x\in\overline{\mathbb{R}^N_+}.$$

Let R_0 be as defined in Step 1,

$$v^{q-p}(y)\le\frac12a_0,\quad\text{for } y\in\partial\mathbb{R}^N_+,\ |y|\ge R_0. \qquad (2.16)$$

Choose $K > 0$ large enough such that

$$K \ge \big|a(y)v^{p-1} - v^{q-1}\big| + v^{p-1} \quad \text{for } y \in \partial\mathbb{R}^N_+,\ |y| \le R_0. \tag{2.17}$$

Let $w \in W$ be the solution of the p-Laplacian equation

$$\begin{gathered} -\Delta_p w = 0, \quad \text{in } \mathbb{R}^N_+, \\ |\nabla w|^{p-2}\frac{\partial w}{\partial n} = g, \quad \text{on } \partial\mathbb{R}^N_+, \end{gathered} \tag{2.18}$$

where $g \ge 0, g(y) = 0$ if $|y| \ge R_0, g(y) = K$ if $|y| < R_0$. For $y \in \partial\mathbb{R}^N_+, |y| \ge R_0$, by the choice of K,

$$\begin{aligned} \Big(\big(a(y)v^{p-1} - v^{q-1}\big) + g\Big)(v-w)_+ &= \Big(\big(a(y)v^{p-1} - v^{q-1}\big) + K\Big)(v-w)_+ \\ &\ge v^{p-1}(v-w)_+ \ge (v-w)^p_+. \end{aligned}$$

For $y \in \partial\mathbb{R}^N_+, |y| \ge R_0$, by the choice of R_0,

$$\begin{aligned} \big(\big(a(y)v^{p-1} - v^{q-1}\big) + g\big)(v-w)_+ &= (a(y)v^{p-1} - v^{q-1})(v-w)_+ \\ &\ge \frac{1}{2}a_0 v^{p-1}(v-w)_+ \ge \frac{1}{2}a_0(v-w)^p_+. \end{aligned}$$

We have

$$\begin{aligned} 0 &\ge \int_{\mathbb{R}^N_+} \Big(|\nabla v|^{p-2}\nabla v - |\nabla w|^{p-2}\nabla w, \nabla(v-w)_+\Big)\,\mathrm{d}x \\ &\quad + \int_{\partial\mathbb{R}^N_+} \Big(\big(a(y)v^{p-1} - v^{q-1}\big) + g\Big)(v-w)_+\,\mathrm{d}y \\ &\ge \int_{\mathbb{R}^N_+} \Big(|\nabla v|^{p-2}\nabla v - |\nabla w|^{p-2}\nabla w, \nabla(v-w)_+\Big)\,\mathrm{d}x + c\int_{\partial\mathbb{R}^N_+} (v-w)^p_+\,\mathrm{d}y, \end{aligned}$$

hence

$$v(x) \le w(x), \quad \text{for } x \in \overline{\mathbb{R}^N_+}. \tag{2.19}$$

We claim that

$$w(x) \le c(1+|x|)^{-\frac{N-p}{p-1}}, \quad \text{for } x \in \overline{\mathbb{R}^N_+}. \tag{2.20}$$

Since W is bounded, we need only to prove (2.20) for $|x| \ge 2R_0$. By the Wolff potential for the p-Laplacian operator in $\mathbb{R}^N_+$ [12, 8, Corollary 4.13],

$$\begin{aligned} w(x) &\le c\int_0^\infty \Big(\frac{1}{t^{N-p}}\int_{B_t(x)\cap\partial\mathbb{R}^N_+} g\,\mathrm{d}y\Big)^{1/p}\frac{1}{t}\,\mathrm{d}t \\ &= c\int_0^\infty \Big(\frac{1}{t^{N-p}}\int_{B_t(x)\cap\operatorname{supp} g} g\,\mathrm{d}y\Big)^{\frac{1}{p-1}}\frac{1}{t}\,\mathrm{d}t, \end{aligned}$$

where $\operatorname{supp} g = D_{R_0} = \{y | y \in \partial\mathbb{R}^N_+, |y| \le R_0\}$. If $|x| \ge 2R_0$ and $t < \frac{1}{2}|x|$, then $B_t(x) \cap D_{R_0} = \emptyset$, hence

$$\begin{aligned} w(x) &\le c\int_{\frac{1}{2}|x|}^\infty \Big(\frac{1}{t^{N-p}}\int_{B_t(x)\cap D_{R_0}} g\,\mathrm{d}y\Big)^{\frac{1}{p-1}}\frac{1}{t}\,\mathrm{d}t \\ &\le c\int_{\frac{1}{2}|x|}^\infty \Big(\frac{1}{t^{N-p}}\int_{D_{R_0}} g\,\mathrm{d}y\Big)^{\frac{1}{p-1}}\frac{1}{t}\,\mathrm{d}t \\ &= c|x|^{-\frac{N-p}{p-1}}, \quad \text{for } |x| \ge 2R. \end{aligned}$$

Consequently, we obtain (2.20) for some $c > 0$.

Step 3. We prove that there exists $c>0$ such that

$$\int_{\mathbb{R}^N_+\setminus B^+_R}|\nabla v|^p\,\mathrm{d}x\le cR^{-\frac{N-p}{p-1}},\quad \int_{\mathbb{R}^N_+\setminus D_R}v^p\,\mathrm{d}y\le cR^{-\frac{N-p}{p-1}}.$$

Choose $\varphi\in C^\infty(\mathbb{R}^N_+,[0,1])$ such that $\varphi(x)=0,|x|\le\frac12R$, $\varphi(x)=1$, $|x|\ge R$, $|\nabla\varphi|\le 4/R$. Take $v\varphi^p$ as test function in (2.6).

$$\int_{\mathbb{R}^N_+}|\nabla v|^{p-2}\nabla v\nabla(v\varphi^p)\,\mathrm{d}x+a_0\int_{\partial\mathbb{R}^N_+}v^p\varphi^p\,\mathrm{d}y\le\int_{\partial\mathbb{R}^N_+}v^q\varphi^p\,\mathrm{d}y.$$

Assume $R\ge 2R_0$, then

$$\begin{aligned}
&\int_{\mathbb{R}^N_+\setminus B^+_R}|\nabla v|^p\,\mathrm{d}x+a_0\int_{\partial\mathbb{R}^N_+\setminus D_R}v^p\,\mathrm{d}y\\
&\le\int_{\mathbb{R}^N_+}|\nabla v|^p\varphi^p\,\mathrm{d}x+a_0\int_{\partial\mathbb{R}^N_+}v^p\varphi^p\,\mathrm{d}y\\
&\quad -p\int_{\mathbb{R}^N_+}|\nabla v|^{p-2}\nabla v v\cdot\varphi^{p-1}\nabla\varphi\,\mathrm{d}x+\int_{\partial\mathbb{R}^N_+}v^q\varphi^p\,\mathrm{d}y\\
&\le\varepsilon\int_{\mathbb{R}^N_+}|\nabla v|^p\varphi^p\,\mathrm{d}x+c\int_{\mathbb{R}^N_+}v^p|\nabla\varphi|^p\,\mathrm{d}x+\frac12a_0\int_{\partial\mathbb{R}^N_+}v^p\varphi^p\,\mathrm{d}y,
\end{aligned}$$

hence

$$\begin{aligned}
\int_{\mathbb{R}^N_+\setminus B^+_R}|\nabla v|^p\,\mathrm{d}x+\int_{\partial\mathbb{R}^N_+\setminus D_R}v^p\,\mathrm{d}y&\le c\int_{\mathbb{R}^N_+}v^p|\nabla\varphi|^p\,\mathrm{d}x\\
&\le cR^{-p}\int_{\mathbb{R}^N_+\setminus B^+_{\frac12R}}v^p\,\mathrm{d}x\\
&\le cR^{-p}\big(R^{-\frac{N-p}{p-1}}\big)^pR^N=cR^{-\frac{N-p}{p-1}}.
\end{aligned}$$

The proof is complete. □

Remark 2.6. Let $v_n\in W,v_n\ge0$ and satisfy the differential inequality (2.6), $n=1,2,\dots$. Suppose $v_n\to v$ in $L^q(\partial\mathbb{R}^N_+)$ and $L^r(R^{N-1}\times(0,2))$ for some $r\in(p,p^*)$, then by checking the proof of Lemma 2.5, v_n is uniformly bounded.

Lemma 2.7. *Let $u_n\in W$ be a solution of the Problem* (1.8) *with $\lambda=\lambda_n>0,n=1,2,\dots$. Assume $\{u_n\}$ is bounded in W and the profile decomposition* (2.1) *holds. Then there exists a positive constant c, independent of n, such that*

$$|u_n(x)|\le c\big(1+d_n(x)\big)^{-\frac{N-p}{p-1}}$$
$$\int_{\Omega^{(n)}_R}|\nabla u_n|^p\,\mathrm{d}x\le c\bar R,$$
$$\int_{\sum^{(n)}_R}|u_n|^p\,\mathrm{d}x\le cR^{-\frac{N-p}{p-1}},$$

where

$$\begin{gathered}
d_n(x)=\min\big\{|x|,|x-y_{n,k}|,\ k\in\Lambda\big\},\\
\Omega^{(n)}_R=\big\{x\in\mathbb{R}^N_+:d_n(x)>R\big\}=\mathbb{R}^N_+\setminus\big(\overline{B^+_R}\cup\cup_{k\in\Lambda}\overline{B^+_R}(y_{n,k})\big),\\
\Sigma^{(n)}_R=\big\{y\in\partial\mathbb{R}^N_+:d_n(y)>R\big\}=\partial\mathbb{R}^N_+\setminus\big(\overline{D_R}\cup\cup_{k\in\Lambda}\overline{D_R}(y_{n,k})\big).
\end{gathered}\tag{2.21}$$

Proof. The proof is similar to that of Lemma 2.5, and is divided into three steps.

Step 1. Given $\varepsilon > 0$, there exists $R_0 > 0$, independent of n, such that

$$|u_n(y)| \le \varepsilon, \quad \text{if } y \in \partial\mathbb{R}^N_+,\ d_n(x) \ge R_0. \tag{2.22}$$

In particular, $|u_n(y)|^{q-p} \le \frac{1}{2}a_0$ for $y \in \partial\mathbb{R}^N_+, d_n(x) \ge R_0$.

As in Step 1 of the proof of Lemma 2.5, we have

$$|u_n|_{L^\infty(D_{\frac{1}{2}}(y))} \le c\big(|u_n|_{L^p(B_1^+(y))} + |u_n|_{L^{\frac{\bar p}{d}}(D_1(y))}\big), \quad y \in \partial\mathbb{R}^N_+.$$

By Lemma 2.5 and the property (4) of the profile decomposition (2.1), it holds for $p < r < \bar p$,

$$\begin{aligned}
&\int_{\Sigma_R^{(n)}} |u_n|^r \,\mathrm{d}y \\
&\le c\int_{\Sigma_R^{(n)}} |u|^r \,\mathrm{d}y + c\sum_{k\in\Lambda}\int_{\Sigma_R^{(n)}} |U_k(\cdot - y_{n,k})|^r \,\mathrm{d}y + c\int_{\Sigma_R^{(n)}} |r_n|^r \,\mathrm{d}y \\
&\le c\int_{\mathbb{R}^N_+\setminus B_R^+} |u|^r \,\mathrm{d}y + c\sum_{k\in\Lambda}\int_{\partial\mathbb{R}^N_+\setminus D_R} |U_k|^r \,\mathrm{d}y + c\int_{\partial\mathbb{R}^N_+} |r_n|^r \,\mathrm{d}y \\
&\le cR^{-\frac{N-p}{p-1}} + o_n(1) = o_k(1) + o_n(1).
\end{aligned} \tag{2.23}$$

Note that the space W is continuously embedded into $W^{1,p}\big(R^{N-1}\times(0,2)\big)$. Let D be the translation group

$$D = \big\{g :|\ gu(\cdot) = u(\cdot - y),\ y \in \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}\times\{0\}\big\}. \tag{2.24}$$

The embedding from $W^{1,p}\big(\mathbb{R}^{N-1}\times(0,2)\big)$ into $L^r(\mathbb{R}^{N-1}\times(0,2))$, $p < r < p^*$, is cocompact with respect to the group D. So we may assume $r_n \to 0$ in $L^r(\mathbb{R}^{N-1}\times(0,2))$, $p < r < p^*$. In parallel to (2.23), we have

$$\begin{aligned}
&\int_{\Sigma_R^{(n)}\times(0,2)} |u_n|^r \,\mathrm{d}x \\
&\le c\int_{\Sigma_R^{(n)}\times(0,2)} |u|^r \,\mathrm{d}x + c\sum_{k\in\Lambda}\int_{\Sigma_R^{(n)}\times(0,2)} |u_n(\cdot - y_{n,k})|^p \,\mathrm{d}x \\
&\quad + c\int_{\Sigma_R^{(n)}\times(0,2)} |r_n|^r \,\mathrm{d}y \\
&= o_R(1) + o_n(1)\,.
\end{aligned} \tag{2.25}$$

For $y \in \Sigma_R^{(n)}$, $B_1^+(y) \subset \Sigma_{R-1}^{(n)}\times(0,2), D_1(y) \subset \Sigma_{R-1}^{(n)}$. The estimate (2.22) follows from (2.23),(2.21) and (2.25).

Step 2. There exists $C > 0$, independent of n, such that

$$|u_n(x)| \le C(1 + d_n(x))^{-\frac{N-p}{p-1}}, \quad x \in \overline{R^N_+}. \tag{2.26}$$

Let R_0 be as defined in Step 1,

$$|u_n(y)|^{q-p} \le \frac{1}{2}a_0, \quad \text{for } y \in \partial\mathbb{R}^N_+,\ \ d_n(y) \ge R_0.$$

Choose $K > 0$ large enough such that

$$k \ge \big(a(y)|u_n|^{p-2}u_n\big) - |f_{\lambda_n}(y,u_n)| + |u_n|^{p-1}, \quad \text{for } y \in \partial\mathbb{R}^N_+,\ d_n(y) \le R_0.$$

Let $w_n \in W$ be the solution of the p-Laplacian equation

$$\begin{gathered}
-\Delta_p w_n = 0, \quad \text{in } \mathbb{R}^N_+, \\
|\nabla w_n|^{p-2}\frac{\partial w_n}{\partial n} = g_n, \quad \text{on } \partial\mathbb{R}^N_+,
\end{gathered} \tag{2.27}$$

where $g_n \geq 0$, $g_n(y) = 0$ if $d_n(y) \geq R_0, g_n(y) = R$ if $d_n(y) < R_0$. For $y \in \partial\mathbb{R}^N_+$, $d_n(y) \geq R_0$, by the choice of K,

$$\begin{aligned}
&\Big(a(y)|u_n|^{p-2}u_n - f_{\lambda_n}(y, u_n) + g_n\Big)(u_n - w_n)_+ \\
&\geq \Big(a(y)|u_n|^{p-2}u_n - f_{\lambda_n}(y, u_n) + K\Big)(u_n - w_n)_+ \\
&\geq |u_n|^{p-1}(u_n - w_n)_+ \geq (u_n - w_n)_+^p.
\end{aligned}$$

For $y \in \partial\mathbb{R}^N_+$, $d_n(y) \geq R_0$, by the choice of R_0,

$$\begin{aligned}
&\Big(a(y)|u_n|^{p-2}u_n - f_{\lambda_n}(y, u_n) + g\Big)(u_n - w_n)_+ \\
&= \Big(a(y)|u_n|^{p-2}u_n - f_{\lambda_n}(y, u_n)\Big)(u_n - w_n)_+ \\
&\geq \Big(a(y)|u_n|^{p-1} - |u_n|^{q-1}\Big)(u_n - w_n)_+ \\
&\geq \frac{1}{2}a_0|u_n|^{p-1}(u_n - w_n)_+ \geq \frac{1}{2}a_0(u_n - w_n)_+.
\end{aligned}$$

We have

$$\begin{aligned}
0 &= -\int_{\mathbb{R}^N_+}(\Delta_p u_n - \Delta_p u_m)(u_n - w_n)_+ \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N_+}\Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla w_n|^{p-2}\nabla w_n, \nabla(u_n - w_n)_+\Big)\,\mathrm{d}x \\
&\quad + \int_{\partial\mathbb{R}^N_+}\Big(a(y)|u_n|^{p-2}u_n - f_{\lambda_n}(y, u_n)\Big)(u_n - w_n)_+ \,\mathrm{d}y \\
&\geq \int_{\mathbb{R}^N_+}\Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla w_n|^{p-2}\nabla w_n, \nabla(u_n - w_n)_+\Big)\,\mathrm{d}x \\
&\quad + c\int_{\partial\mathbb{R}^N_+}(u_n - w_n)^p \,\mathrm{d}y;
\end{aligned}$$

hence

$$u_n(x) \leq w_n(x), \quad \text{for } x \in \overline{\mathbb{R}^N_+}. \tag{2.28}$$

Similarly we have $-u_n(x) \leq w_n(x)$ for $x \in \overline{\mathbb{R}^N_+}$.

We claim that

$$w_n(x) \leq c(1 + d_n(x))^{-\frac{N-p}{p-1}}, \quad \text{for } x \in \overline{\mathbb{R}^N_+}. \tag{2.29}$$

Since w_n is uniformly bounded, we need only to prove (2.29) for $x \in \overline{\mathbb{R}^N_+}$, $d_n(x) \geq 2R_0$. Again by the Wolff potential for the p-Laplacian in $\mathbb{R}^N_+$, we have

$$w_n(x) \leq c\int_0^\infty \Big(\frac{1}{t^{N-p}}\int_{B_t(x)\cap \operatorname{supp} g_n} g_n \,\mathrm{d}y\Big)^{\frac{1}{p-1}}\frac{1}{t}\,\mathrm{d}t,$$

where $\operatorname{supp} g = \overline{D}_{R_0} \cup_{k\in\Lambda} \overline{D}_{R_0}(y_{n,k}) = \Big\{y | y \in \partial\mathbb{R}^N_+,\ d_n(y) \leq R_0\Big\}$.

If $x \in \overline{\mathbb{R}^N_+}$, $d_n(x) \geq 2R_0$ and $t \leq \frac{1}{2}d_n(x)$, then $B_t(x) \cap \operatorname{supp} g = \emptyset$, hence

$$\begin{aligned} w_n(x) &\leq c \int_{\frac{1}{2}d_n(x)}^{\infty} \Big(\frac{1}{t^{N-p}} \int_{B_t(x)\cap \operatorname{supp} g_n} g_n \, \mathrm{d}y \Big)^{\frac{1}{p-1}} \frac{1}{t} \, \mathrm{d}t, \\ &\leq c \int_{\frac{1}{2}d_n(x)}^{\infty} \Big(\frac{1}{t^{N-p}} \int_{\operatorname{supp} g_n} g_n \, \mathrm{d}y \Big)^{\frac{1}{p-1}} \frac{1}{t} \, \mathrm{d}t = c d_n^{-\frac{N-p}{p-1}}(x), \end{aligned}$$

for $d_n(x) \geq 2R_0$. Consequently, we obtain (2.29) for some $c > 0$.

Step 3. There exists $c > 0$, independent of n, such that

$$\int_{\Omega_R^{(n)}} |\nabla u_n|^p \, \mathrm{d}x \leq c R^{-\frac{N-p}{p-1}}, \quad \int_{B_R^+ \setminus B_{\frac{1}{2}R}^+} |u_n|^p \, \mathrm{d}y \leq c R^{-\frac{N-p}{p-1}}.$$

The proof is similar to that of Lemma 2.5. Choose $\varphi_n \in C_0^\infty\big(\mathbb{R}^N_+, [0,1]\big)$ such that $\varphi_n(x) = 0$, if $d_n(x) \leq \frac{1}{2}R$, $\varphi_n(x) = 1$, if $d_n(x) \geq R$, $|\nabla \varphi_n| \leq \frac{4}{R}$. Testing equation (1.8) by $u_n \varphi_n^p$ with $\lambda = \lambda_n$, and assuming $R \geq 2R_0$, we have

$$\begin{aligned} \int_{\Omega_R^{(n)}} |\nabla u_n|^p \, \mathrm{d}x + \int_{\Sigma_R^{(n)}} |u_n|^p \, \mathrm{d}x &\leq c \int_{\mathbb{R}^N_+} |\nabla u_n|^p |\nabla \varphi_n|^p \, \mathrm{d}x \\ &\leq c R^{-p} \int_{\Omega_{\frac{1}{2}R}^{(n)} \setminus \Omega_R^{(n)}} |u_n|^p \, \mathrm{d}x \\ &\leq c R^{-p} (R^{-\frac{N-p}{p-1}})^p R^N = c R^{-\frac{N-p}{p-1}}. \end{aligned}$$

We follow the idea in [6] to derive a local Pohožaev type identity with a form as in [4], which is much closer to our case. $\square$

Lemma 2.8. *Let $u \in W$ be a solution of Problem* (1.8), *$t \in \partial\mathbb{R}^N_+$ and $\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+})$. Then the following Pohožaev type identity holds*

$$\begin{aligned} &\frac{1}{p} \int_{\partial\mathbb{R}^N_+} (t, \nabla a) |u|^p \varphi dy - \int_{\partial\mathbb{R}^N_+} a(y) \big(t, \nabla_y F_\lambda(y,u)\varphi\big) \, \mathrm{d}y \\ &= \frac{1}{p} \int_{\mathbb{R}^N_+} |\nabla u|^p (t, \nabla\varphi) \, \mathrm{d}x - \int_{\mathbb{R}^N_+} |\nabla u|^{p-2} (t, \nabla u)(\nabla u, \nabla\varphi) \, \mathrm{d}x \\ &\quad - \frac{1}{p} \int_{\partial\mathbb{R}^N_+} a(y) |u|^p (t, \nabla\varphi) \, \mathrm{d}y + \int_{\partial\mathbb{R}^N_+} F_\lambda(y,u)(t, \nabla\varphi) dy. \end{aligned} \tag{2.30}$$

Proof. Taking $(t, \nabla u)\varphi$ as the test function in equation (1.10) and integrating by parts, we obtain the identity. $\square$

Assume that $u_n \in W$ is a solution of the problem (1.8) with $\lambda = \lambda_n \geq 0$, $\|u_n\| \leq M$, $n = 1, 2, \dots$. Assume that the profile decomposition (2.1) for the sequence $\{u_n\}$ holds.

$$u_n = u + \sum_{k \in \Lambda} U_k(\cdot - y_{n,k}) + r_n.$$

Without loss of generality, we assume $|y_{n,1}| = \min\{|y_{n,k}|, k \in \Lambda\}$. Denote $y_n = y_{n,1}$. According to [4], we can construct a sequence of cones C_n, having vertex $\frac{1}{2}y_n$ and generated by the semiball $B^+_{R_n}(y_n)$ as follows:

$$C_n^+ = \big\{ w \in \mathbb{R}^N_+ : w = \frac{1}{2}y_n + \lambda(x - \frac{1}{2}y_n), \, x \in B^+_{R_n}(y_n), \lambda \geq 0 \big\},$$

where R_n satisfies

$$\frac{\hat{r}}{k_0}\cdot\frac{|y_n|}{2}=r_n\le R_n\le R_0r_n=\hat{r}\frac{|y_n|}{2},\quad \hat{r}=\frac{1}{5(\bar{c}+1)},$$

and $\bar{c}$ is the constant in the definition (A4), $\Lambda=\{1,2,\dots,k_0\}$.

The cone C_n^+ has the following property, let ∂C_n^+ be the boundary of C_n^+ in $\overline{\mathbb{R}^N_+}$, then

$$\partial C_n^+\cap\big\{B^+_{\frac12 r_n}\cup\cup_{k\in\Lambda}B^+_{\frac12 r_n}(y_{n,k})\big\}=\emptyset\,. \tag{2.31}$$

Now we apply the Pohožaev type identity (2.31). Take $u=u_n$, $t=t_n=\frac{y_n}{|y_n|}$ and $\varphi=\chi\varphi_R$, where $\chi,\varphi_R\in C_0^\infty(\mathbb{R}^N)$ such that $\chi(x)=0$ for $x\notin C_n^+$, $\chi(x)=1$ for $x\in C_n^+$ and $\operatorname{dist}(x,\partial C_n^+)\ge1$, $\varphi_R(x)=1$ for $|x|\le R,\varphi_R(x)=0$ for $|x|\ge 2R$. Let $R\to\infty$, we obtain

$$\begin{aligned}
&\frac1p\int_{\partial\mathbb{R}^N_+}(t_n,\nabla u)|u_n|^p\chi dy-\int_{\partial\mathbb{R}^N_+}(t_n,\nabla_yF_{\lambda_n}(y,u_n)\chi)\,\mathrm{d}y\\
&=-\frac1p\int_{\mathbb{R}^N_+}|\nabla u_n|^p(t_n,\nabla\chi)\,\mathrm{d}x+\int_{\mathbb{R}^N_+}|\nabla u_n|^{p-2}(t_n,\nabla u_n)(\nabla u_n,\nabla\chi)\,\mathrm{d}x\\
&\quad-\frac1p\int_{\partial\mathbb{R}^N_+}a(y)|u_n|^p(t_n,\nabla\chi)\,\mathrm{d}y+\int_{\partial\mathbb{R}^N_+}F_{\lambda_n}(y,u_n)(t_n,\nabla\chi)\,\mathrm{d}y.
\end{aligned}\tag{2.32}$$

By (2.31) and the definition of χ, the support of $\nabla\chi$ is contained in the set $\Omega_R^{(n)}\cup\Sigma_R^{(n)}$ with $R=\frac12r_n-1$. By Lemma 2.7, the right-hand side of (2.32) decays polynomially. More precisely,

$$\begin{aligned}
&-\frac1p\int_{\mathbb{R}^N_+}|\nabla u_n|^p(t_n,\nabla\chi)\,\mathrm{d}x+\int_{\mathbb{R}^N_+}|\nabla u_n|^{p-2}(t_n,\nabla u_n)(\nabla u_n,\nabla\chi)\,\mathrm{d}x\\
&-\frac1p\int_{\partial\mathbb{R}^N_+}a(y)|u_n|^p(t_n,\nabla\chi)\,\mathrm{d}y+\int_{\partial\mathbb{R}^N_+}F_{\lambda_n}(y,u_n)(t,\nabla\chi)dy\\
&\le c\Big(\int_{\Omega_R^{(n)}}|\nabla u_n|^p\,\mathrm{d}x+\int_{\sum_R^{(n)}}(|u_n|^p+|u_n|^q)\,\mathrm{d}y\Big)\\
&\le cR^{-\frac{N-p}{p-1}}\le cr_n^{-\frac{N-p}{p-1}}\le c|y_n|^{-\frac{N-p}{p-1}}.
\end{aligned}\tag{2.33}$$

To estimate the left-hand side of (2.32), we use some estimates from [4]. By [4, Lemma 4.2],

$$(t_n,y)\ge0,\quad \langle t_n,\nabla a(y)\rangle\ge\frac12\frac{\partial}{\partial r}a(y)\quad\text{for } y\in\overline{C_n^+}\cap\partial\mathbb{R}^N_+.$$

Moreover, by Lemma 2.1(5),

$$\big(t_n,\nabla_yF_\lambda(y,u_n)\big)=-\big|\nabla_yF_\lambda(y,u_n)(t_n,\frac{y}{|y|})\big|\le0,\quad\text{for } y\in\overline{C_n^+}\cap\partial\mathbb{R}^N_+.$$

Hence, the left-hand side of (2.33) can be as estimated as

$$\begin{aligned}
&\frac{1}{p}\int_{\partial\mathbb{R}^N_+}(t_n,\nabla a)|u_n|^p\,\mathrm{d}y-\int_{\partial\mathbb{R}^N_+}(t_n,\nabla_y F_{\lambda_n}(y,u_n))\chi\,\mathrm{d}y\\
&\geq \frac{1}{2p}\int_{\partial\mathbb{R}^N_+}\frac{\partial}{\partial r}a(y)|u_n|^p\chi dy\\
&\geq \frac{1}{2p}\inf_{D_L(y_n)}\frac{\partial a}{\partial r}\int_{D_L(y_n)}|u_n|^p\,\mathrm{d}y,
\end{aligned}\tag{2.34}$$

where $D_L(y_n)\subset\Sigma_R^{(n)}\subset\overline{C_n^+}$, L is a large number such that

$$\int_{D_L}|U_1|^p\,\mathrm{d}y=m>0.$$

Since $\tilde{u}_n=u_n(\cdot-y_n)\rightharpoonup U_1$ in W, we have

$$\int_{D_L(y_n)}|u_n|^p\,\mathrm{d}y=\int_{D_L}|\tilde{u}_n|^p\,\mathrm{d}y\to\int_{D_L}|U_1|^p\,\mathrm{d}y=m.\tag{2.35}$$

By (2.34), (2.35), the left-hand side of (2.33),

$$\begin{aligned}
&\frac{1}{p}\int_{\partial\mathbb{R}^N_+}\Big(t_n,\nabla a\Big)|u_n|^p\chi\,\mathrm{d}y-\int_{\partial\mathbb{R}^N_+}(t_n,\nabla_y F_{\lambda_n}(y,u_n))\chi\,\mathrm{d}y\\
&\geq \frac{m}{4p}\inf_{D_L(y_n)}\frac{\partial a}{\partial r}.
\end{aligned}\tag{2.36}$$

Finally by (2.33), (2.36),

$$\frac{1}{4p}\inf_{D_L(y_n)}\frac{\partial a}{\partial r}\leq c|y_n|^{\frac{N-p}{p-1}},$$

which contradicts (A4). Thus $\Lambda=\emptyset$, and by the profile decomposition (2.1) $u_n=u+r_n\to u$ in $L^q(\partial\mathbb{R}^N_+)$. As mentioned before, the space W is continuously embedded into $W^{1,p}\big(\mathbb{R}^{N-1}\times(0,2)\big)$, and in turn $W^{1,p}\big(\mathbb{R}^{N-1}\times[0,2]\big)$ is embedded into $L^s\big(\mathbb{R}^{N-1}\times[0,2]\big), p<s<p^*$, compactly with respect to the translation group D. We also have $u_n\to u$ in $L^s\big(\mathbb{R}^{N-1}\times[0,2]\big), p<s<p^*$. Namely we have the following proposition.

Proposition 2.9. *Let $u_n\in W$ be a solution of* (1.8) *with $\lambda=\lambda_n, n=1,2,\dots$. Assume $\|u_n\|\leq M, u_n\rightharpoonup u$ in W. Then $u_n\to u$ in $L^s\big(\partial\mathbb{R}^N_+\big), p<s\leq\bar{p}$ and in $L^s(\mathbb{R}^{N-1}\times(0,2))$, $p<s<p^*$.*

Proof of Theorem 1.2. We use an indirect argument. Let $u_n\in W$ be a solution of Problem (1.8) with $\lambda=\lambda_n\geq 0$, $\|u_n\|\leq M$, $n=1,2,\dots$, but it holds that

$$\sup_{y\in\partial\mathbb{R}^N_+}\frac{1}{n}(1+|y|)^{\frac{N-p}{p-1}}|u_n(y)|>1.\tag{2.37}$$

By Proposition 2.9, $u_n\to u$ in $L^s\big(\mathbb{R}^N_+\big)$, $p<s<\bar{p}$ and in $L^s(\mathbb{R}^{N-1}\times(0,2))$, $p<s<p^*$, the index set Λ in the profile decomposition for the sequence $\{u_n\}$ is empty. Hence $d_n(x)=\min\{|x|,|x-y_{n,k}|,k\in\Lambda\}=|x|$, and by Lemma 2.7, there exists $c>0$, independent of n, such that

$$|u_n(y)|\leq c(1+|y|)^{-\frac{N-p}{p-1}},\quad y\in\partial\mathbb{R}^N_+,$$

we arrive at a contradiction. $\square$

Corollary 2.10. *Let $u_n \in W$ be a solution of (1.8) with $\lambda = \lambda_n \geq 0$, $n = 1, 2, \dots$. Assume $I_{\lambda_n}(u_n) \leq M$, then there exists a constant $c > 0$ independent of n, such that*
$$|u_n(y)| \leq c(1 + |y|)^{-\frac{N-p}{p-1}} \quad \text{for } y \in \partial\mathbb{R}^N_+. \tag{2.38}$$
Moreover, up to a subsequence, $\{u_n\}$ converges in W.

Proof. By Lemma 2.1(4), we have
$$\begin{aligned}
M \geq I_{\lambda_n}(u_n) &= I_{\lambda_n}(u_n) - \frac{1}{r}\langle DI_{\lambda_n}(u_n), u_n\rangle \\
&= \big(\frac{1}{p} - \frac{1}{r}\big)\Big(\int_{\mathbb{R}^N_+} |\nabla u_n|^p \,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} a(y)|u_n|^p \,\mathrm{d}y\Big) \\
&\quad + \big(\frac{1}{r} - \frac{1}{q}\big)\int_{\partial\mathbb{R}^N_+} |u_n|^{r+1}|m_{\lambda_n}(y, u_n)|^{q-r-1} b_{\lambda_n}(y, u_n) \,\mathrm{d}y \\
&\geq \big(\frac{1}{p} - \frac{1}{r}\big)\Big(\int_{\mathbb{R}^N_+} |\nabla u_n|^p \,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} |u_n|^p \,\mathrm{d}y\Big).
\end{aligned}$$
The sequence $\{u_n\}$ is bounded in W. By Theorem 1.2, (2.38) holds. Moreover, by Proposition 2.9, up to a subsequence $\{u_n\}$ converges in $L^q(\partial\mathbb{R}^N_+)$, and
$$\begin{aligned}
&\int_{\mathbb{R}^N_+} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m, \nabla u_n - \nabla u_m) \,\mathrm{d}x \\
&+ \int_{\partial\mathbb{R}^N_+} a(y)(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m) \,\mathrm{d}y \\
&= \int_{\partial\mathbb{R}^N_+} (f_{\lambda_n}(y, u_n) - f_{\lambda_m}(y, u_m))(u_n - u_m) \,\mathrm{d}y \\
&\leq c\int_{\partial\mathbb{R}^N_+} (|u_n|^{q-1} + |u_m|^{q-1})|u_n - u_m| \,\mathrm{d}y \\
&\leq c|u_n - u_m|_{L^q(\partial\mathbb{R}^N_+)} \to 0 \quad \text{as } n, m \to \infty\,.
\end{aligned}$$
The sequence $\{u_n\}$ converges in W. ☐

3. Existence of infinitely many solutions

In this section, we prove the existence of infinitely many solutions of the original problem (1.1). First we construct a sequence of critical values of the truncated functionals $I_\lambda, \lambda > 0$, by the symmetric mountain pass lemma due to Ambosetti and Rabinowitz [1].

Lemma 3.1. *The functional I_λ, $\lambda > 0$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset W$ be a Palais-Smale sequence of I_λ. By Lemma 2.1 (4), hence we have

$$\begin{aligned} &I_\lambda(u_n) - \frac{1}{r}\langle DI_\lambda(u_n), u_n\rangle \\ =&(\frac{1}{p}-\frac{1}{r})\Big(\int_{\mathbb{R}^N_+} |\nabla u_n|^p\,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} |u_n|^p\,\mathrm{d}y\Big) \\ &+ (\frac{1}{r}-\frac{1}{q})\int_{\partial\mathbb{R}^N_+} |u_n|^{r+1}|m_\lambda(y,u_n)|^{q-r-1} b_\lambda(y,u_n)\,\mathrm{d}y \\ &\geq (\frac{1}{p}-\frac{1}{r})\Big(\int_{\mathbb{R}^N_+} |\nabla u_n|^p\,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} a(y)|u_n|^p\,\mathrm{d}y\Big). \end{aligned} \tag{3.1}$$

Hence $\{u_n\}$ is bounded in W. Assume $u_n \rightharpoonup u$ in W, $u_n \to u$ in $L^s_{\rm loc}(\partial\mathbb{R}^N_+)$, $p \leq s < \bar{p}$. By Lemma 2.1, we have

$$\begin{aligned} &\int_{\mathbb{R}^N_+} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m, \nabla u_n - \nabla u_m)\,\mathrm{d}x \\ &+ \int_{\partial\mathbb{R}^N_+} a(y)(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n-u_m)\,\mathrm{d}y \\ &= \int_{\partial\mathbb{R}^N_+} \big(f_\lambda(y,u_n) - f_\lambda(y,u_m)\big)(u_n-u_m)\,\mathrm{d}y \\ &\leq \int_{\partial\mathbb{R}^N_+} \Big(\frac{2}{\lambda}(1+|y|^{-\frac{N-p}{p-1}})\Big)^{q-r}(|u_n|^{r-1}+|u_m|^{r-1})|u_n-u_m|\,\mathrm{d}y \\ &\leq cR^{-\frac{N-p}{p-1}(p-r)}\int_{\partial\mathbb{R}^N_+\setminus D_R} \big(|u_n|^r + |u_m|^r\big)\,\mathrm{d}y \\ &\quad + C_R\Big(\int_{D_R}(|u_n|^r+|u_m|^r)\,\mathrm{d}y\Big)^{\frac{r-1}{r}}\Big(\int_{D_R}|u_n-u_m|^r\,\mathrm{d}x\Big)^{1/r} \\ &\leq cR^{-\frac{N-p}{p-1}(p-r)} + C_R|u_n-u_m|_{L^r(D_r)} \\ &\to 0 \quad \text{as } n,m\to\infty\,. \end{aligned} \tag{3.2}$$

By (3.2) and the elementary inequalities (2.5), $\{u_n\}$ is a Cauchy sequence, hence a convergent sequence in W. ☐

Now we define a sequence of critical values of I_λ as follows.

$$c_k(\lambda) = \inf_{A\in\Gamma_k}\sup_{u\in A} I_\lambda(u), \quad \lambda>0,\ k=1,2,\dots, \tag{3.3}$$

where

$$\begin{gathered} \Gamma_k = \{A\subset W : A \text{ is compact },\ -A=A,\ \gamma(A\cap\sigma^{-1}(S_\rho))\geq k,\ \forall\sigma\in G\}, \\ G=\{\sigma\in C(W,W) : \sigma(-u)=-\sigma(u),\ \forall u\in W; \sigma(u)=u,\ \text{if } I_1(u)<0\}, \\ S_\rho = \{u\in W : \|u\|=\rho\}, \end{gathered}$$

where $\rho>0$ is a fixed number to be chosen as follows. For $u\in S_\rho$ we have

$$\begin{aligned} I_1(u) &= \frac{1}{p}\Big(\int_{\mathbb{R}^N_+}|\nabla u|^p\,\mathrm{d}x + \frac{1}{p}\int_{\partial\mathbb{R}^N_+} a(y)|u|^p\,\mathrm{d}y\Big) - \frac{1}{q}\int_{\partial\mathbb{R}^N_+}|u|^q\,\mathrm{d}y \\ &\geq c_0\rho^p - c_1\rho^q \geq \frac{1}{2}c_0\rho^p, \end{aligned}$$

provided $c_1\rho^{q-p} \le \frac{1}{2}c_0$ and $I(u) = \frac{1}{2}c_0\rho^p$ for $u \in S_\rho$. The following proposition is known, see [1, 2, 11].

Proposition 3.2. *Assume* $0 < \lambda \le 1$. *Then*

(1) $c_k(\lambda) > 0$, $k = 1, 2, \dots$ *are critical values of* I_λ.
(2) *If* $c_k(\lambda) = c_{k+1}(\lambda) = \dots = c_{k+m-1}(\lambda) = c$, *then* $\gamma\big(K_c(I_\lambda)\big) \ge m$, *where* $K_c(I_\lambda) = \{u | u \in W, DI_\lambda(u) = 0, I_\lambda(u) = c\}$.
(3) *Assume* $p = 2$. *Then there exists* $u \in W$ *such that* $I_\lambda(u) = c_k(\lambda), DI_\lambda(u) = 0$ *and* $m^*(u) \ge k$, *where* $m^*(\cdot)$ *is the augmented Morse index.*

Given $k \in N$, by Corollary 2.10, there exists $\mu_k > 0$ such that if $0 < \lambda \le 1$, $u \in W$, $DI_\lambda(u) = 0$, $I_\lambda(u) = c_k(\lambda) \le \alpha_k := c_k(1)$, then

$$|u(y)| \le \frac{1}{\mu_k}(1 + |y|^2)^{-\frac{N-p}{2(p-1)}}, \quad y \in \partial\mathbb{R}^N_+ . \tag{3.4}$$

Choose $0 < \lambda_k < \min\{1, \mu_k\}$. Let $u_1(\lambda), \dots, u_k(\lambda)$ be the solutions of (1.8) with $\lambda = \lambda_k$, corresponding to the critical values $c_1(\lambda_k) \le \dots \le c_k(\lambda_k)$. Since I_λ is increasing in λ, we have $c_1(\lambda_k) \le \dots \le c_k(\lambda_k) \le \alpha_k$, $u_1(\lambda_k), \dots, u_k(\lambda_k)$ satisfy the estimate (3.4), hence they are solutions of the original problem (1.1). Now k is arbitrary, we obtain infinitely many solutions of Problem (1.1).

Remark 3.3. We have proved that Problem (1.1) has infinite many solutions. We can prove a little more, namely claim the functional I has an infinitely sequence of critical values.

We use an indirect argument. Assume I has only a finite number of critical values $c_1, \dots, c_k$. Denote $K = \{u | u \in W, DI(u) = 0\}$. Then by Corollary 2.10, K is compact. Assume $\gamma(K) = m < +\infty$. For $0 < \lambda < 1$, the functional I_λ has critical values $c_1(\lambda) \le c_2(\lambda) \le \dots \le c_{km+1}(\lambda)$. If λ is sufficiently small, they will be critical values of I. We claim $c_1(\lambda) < c_{m+1}(\lambda) < \dots < c_{km+1}(\lambda)$. Otherwise suppose, say $c = c_1(\lambda) = c_{m+1}(\lambda)$. By Proposition 3.2, $\gamma(K_c) \ge m + 1$, where $K_c = \{u | u \in W, DI_\lambda(u) = 0, I_\lambda(u) = c\} \subset K$, which is a contradiction. We obtain $k + 1$ different critical values of I. Since k is arbitrary, I has a infinite sequence of critical values.

For $p = 2$, by the information on the Morse index, one can prove that I has an unbounded sequence of critical values(see [6, 4]).

Acknowledgments. J. Zhao was supported by the NSFC 11601493 and by the Fundamental Research Funds for the Central Universities 2652018058. X. Liu was supported by the NSFC 11361077 and by the Yunnan Province, Young Academic and Technical Leaders Program (2015HB028). J. Liu was supported by the NSFC 11671364.

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Junfang Zhao (corresponding author)
School of Science, China University of Geosciences, Beijing 100083, China
Email address: zhao_junfang@163.com

Xiangqing Liu
Department of Mathematics, Yunnan Normal University, Kunming 650500, China
Email address: lxq8u8@163.com

Jiaquan Liu
LMAM, School of Mathematics, Peking University, Beijing 100871, China
Email address: jiaquan@math.pku.edu.cn