Electronic Journal of Differential Equations, Vol. 2005(2005), No. 36, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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AN LP-APPROACH FOR THE STUDY OF DEGENERATE PARABOLIC EQUATIONS

RABAH LABBAS, AHMED MEDEGHRI, BOUBAKER-KHALED SADALLAH

Abstract. We give regularity results for solutions of a parabolic equation in non-rectangular domains $U = \cup_{t\in]0,1[}\{t\} \times I_t$ with $I_t = \{x : 0 < x < \varphi(t)\}$. The optimal regularity is obtained in the framework of the space L^p with $p > 3/2$ by considering the following cases: (1) When $\varphi(t) = t^\alpha$, $\alpha > 1/2$ with a regular right-hand side belonging to a subspace of $L^p(U)$ and under assumption $p > 1 + \alpha$. We use Labbas-Terreni results [11]. (2) When $\varphi(t) = t^{1/2}$ with a right-hand side taken only in $L^p(U)$. Our approach make use of the celebrated Dore-Venni results [2].

1. Introduction

Statement of the Problem. We study the autonomous parabolic equation

$$\begin{gathered} D_t u(t,x) - D_x^2 u(t,x) + \lambda m(t,x) u(t,x) = f(t,x), \quad (t,x) \in U \\ u\big|_{\partial U - \Gamma_1} = 0, \end{gathered} \tag{1.1}$$

with a positive spectral parameter λ and some positive weight functions $m(\cdot)$ which will be specified below. The right-hand term f is taken in $L^p(U)$ with

$$p > \frac{3}{2}. \tag{1.2}$$

Here, $\Gamma_1 = \{1\}\times]0,1[$ and U is a non-cylindrical domain of the form

$$U = \cup_{t\in]0,1[}\{t\} \times I_t,$$

where

$$I_t = \{x : 0 < x < \varphi(t)\},$$

and $\varphi : [0,1] \to \mathbb{R}$ is a given continuous function such that $\varphi(0) = 0$. The non-cylindrical space-time of the boundary

$$\partial U_\varphi = \{(t,\varphi(t)) : 0 < t < 1\}$$

can be considered as an approximation of a flame front; so that, equation (1.1) is of interest in combustion theory.

On the other hand, the analysis of (1.1) with a spectral parameter λ and with a weight multiplier $m(\cdot)$ plays an important rôle since it arises naturally from

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2000 *Mathematics Subject Classification.* 35K05, 35K65, 35K90.
Key words and phrases. Sum of linear operators, diffusion equation, non rectangular domain, bounded imaginary powers of operators.

Submitted July 16, 2004. Published March 29, 2005.

nonlinear diffusion equations when they are linearized [7, 8]. The function $m(\cdot)$ can also model the exchange coefficient with the exterior environment.

In Savaré [17], parabolic problems in non cylindrical domains are considered in the Hilbertian case. The author obtains some regularity results under assumption on the geometrical behavior of the boundary which cannot include our triangular domain.

Hofmann and Lewis [9] have also considered some boundary value problems for the heat equation in non cylindrical domains satisfying some conditions of Lipschitz's type. They showed that the optimal L^p regularity holds for $p = 2$ and the situation gets progressively worse as p approaches 1.

In the work [13], the Authors have studied existence, uniqueness and regularity of the solutions in general mixed semilinear parabolic problems set in a non-cylindrical domain U. They proved the existence of a weak solution u in some anisotropic Hölder continuous functions space $C^{\gamma/2,\gamma}(\overline{U})$ with continuity of ∇u in U and in ∂U_φ under the assumption that the second member f is hölderian.

We will see below that, after some change of variables, Problem (1.1) is transformed into a degenerate parabolic equation set in a cylindrical domain.

In this paper we have considered the unidimensional case in x as a model to exemplify parabolic problems set in non rectangular domains. Our study can be naturally extended to an upper dimension in x, such as, for example, the following problem

$$D_t u(t,x) - \Delta_x u(t,x) + \lambda m(t,x)u(t,x) = f(t,x)$$

in the domain

$$\{(t,x_1,x_2) : t \in (0,1),\ x_1, x_2 > 0 \text{ and } (x_1/\varphi_1(t), x_2/\varphi_2(t)) \in D\},$$

where D is some given cylindrical domain in $\mathbb{R}^2_+$ and φ_i, $i = 1,2$ are similar to φ.

The change of variables $(t,x) \mapsto (t,y) = (t, x/\varphi(t))$ transforms U into the rectangle $\Omega =]0,1[\times]0,1[$. Putting $u(t,x) = w(t,y)$ and $f(t,x) = g(t,y)$, Problem (1.1) is transformed, in Ω into the degenerate evolution problem

$$\begin{gathered}\varphi(t)^2 D_t w(t,y) - D_y^2 w(t,y) - \varphi'(t)\varphi(t) y D_y w(t,y) + \lambda M(t,y)\varphi(t)^2 w(t,y) \\ = \varphi(t)^2 g(t,y) \\ w\big|_{\partial\Omega - \Gamma_1} = 0.\end{gathered} \tag{1.3}$$

with $\Gamma_1 = \{1\}\times]0,1[$ and $M(t,y) = m(t,x)$.

It is easy to see that $f \in L^p(U)$ if and only if $\varphi^{-2+1/p} h \in L^p(\Omega)$ which implies that $h \in L^p(\Omega)$, since

$$h = (\varphi^{-2+1/p} h)\varphi^{2-1/p}$$

and $2 - 1/p > 0$. Then the function $h = \varphi^2 g$ lies in the closed subspace of $L^p(\Omega)$ defined by

$$E_\varphi = \{h \in L^p(0,1; L^p(0,1)) :\ \varphi^{-2+1/p} h \in L^p(0,1; L^p(0,1))\}.$$

This space is equipped with the norm

$$\|h\|_{E_\varphi} = \|\varphi^{-2+1/p} h\|_{L^p(0,1;L^p(0,1))}.$$

We can find in the Favini-Yagi book [5] an important study of abstract problems of parabolic type with degenerate terms in the time derivative. In particular, these authors have given an alternative approach to the study of Equation (1.3), when it takes a particular following form

$$t^l D_t w(t,y) - D_y^2 w(t,y) = t^l g(t,y)$$
$$w\big|_{\partial\Omega-\Gamma_1} = 0.$$

The authors of this article have considered the three cases $l > 1$, $l = 1$ and $l < 1$, see [5, p. 111.]. They used the notion of multivalued linear operators and constructed fundamental solutions when the right-hand side has a Hölder regularity with respect to the time.

In this work, we are especially interested in the question: What conditions the functions φ and m must verify in order that Problem (1.1) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_p^{1,2}(U) = \{u \in L^p(U) : D_t u, D_x^j u \in L^p(U),\ j = 1,2\}?$$

Our approach is different from that used in the previous methods : it is based on the direct use of the sum theory of operators in Banach spaces. This is naturally suggested by Equation (1.3).

We will prove that the answer to the previous question is positive in the following two cases.

1. When f is regular, the function $t \mapsto \varphi'(t)\varphi(t)$ is Hölderian and the weight multiplier coefficient m is function of the parabolic boundary in the sense that

$$m(t,x) = m(t) = (\varphi(t))^{-2}.$$

It corresponds for instance to the model case $\varphi(t) = t^\alpha$, with $\alpha > 1/2$ and $m(t) = t^{-2\alpha}$. The approach uses Labbas-Terreni results [11].

2. When f is taken only in $L^p(U)$, $\varphi(t) = \sqrt{t}$ and $m(t,x) = t^{-1}$. Here, we use a celebrated Dore-Venni Theorem given in [2].

This work is also an extension of the Hilbertian case ($p = 2$) studied in Sadallah [16]. The author has considered the cases $\lambda = 0$, $m = 0$.

Assumptions and main results.

First case. Let $\varphi(t) = t^\alpha$, with $\alpha > 1/2$, $m(t) = t^{-2\alpha}$ and assume that

$$p - 1 > \alpha. \tag{1.4}$$

For $\sigma \in]0,1[$, we introduce the following subspace of $L^p(U)$ (with a slight abuse)

$$\begin{aligned} &L^p_{\varphi^{2\sigma}}(0,1;W_\varphi^{2\sigma,p}) \\ &= \{f \in L^p(U) : \int_0^1 \varphi(t)^{2\sigma p} \int_0^{\varphi(t)} \int_0^{\varphi(t)} \frac{|f(t,x) - f(t,x')|^p}{|x-x'|^{2\sigma p+1}}\, dx\, dx'\, dt < \infty\}. \end{aligned}$$

The main result regarding this case is the following.

Theorem 1.1. *Assume* (1.4). *Let* $\sigma \in]0,1[$ *such that* $0 < \sigma < 1/2p$. *Then, there exists* λ^* *such that* $\forall \lambda \geqslant \lambda^*$ *and for all* $f \in L^p_{\varphi^{2\sigma}}(0,1;W_\varphi^{2\sigma,p})$*, Problem* (1.1) *has a unique solution* $u \in H_p^{1,2}(U)$ *fulfilling the regularity properties:* u, $D_t u$, $D_x u$ *and* $D_x^2 u$ *belong to* $L^p_{\varphi^{2\sigma}}(0,1;W_\varphi^{2\sigma,p})$.

Second case. For $\varphi(t) = \sqrt{t}$ and $m(t) = t^{-1}$, we will prove the following statement.

Theorem 1.2. *For $f \in L^p(U)$ and $\lambda \geqslant 1/2p$, Problem* (1.1) *has a unique solution $u \in H_p^{1,2}(U)$.*

Note that in the second case, we obtain maximal results for any $\lambda \geqslant 1/2p$.

2. On the sum of linear operators

Definitions. Let Λ be a closed linear operator in a complex Banach space E. Then, Λ is said sectorial if

(i) $D(\Lambda)$ and $\mathrm{Im}(\Lambda)$ are dense in E,
(ii) $\ker(\Lambda) = \{0\}$,
(iii) $]-\infty, 0[\subset \rho(\Lambda)$ and there exists a constant $M \geqslant 1$ such that
$$\|t(\Lambda + tI)^{-1}\| \leqslant M$$
for all $t > 0$.

We recall that for a sectorial operator Λ, the complex powers Λ^z, $z \in \mathbb{C}$, are well defined but not necessarily bounded, see Komatsu [10]. When $\Lambda^{is} \in L(E)$ for any $s \in \mathbb{R}$ and
$$\sup_{s\in[-1,1]} \|\Lambda^{is}\| < \infty,$$
we say that $\Lambda \in Bip(E)$, see Prüss-Sohr [15].

In general, the knowledge of Λ^{is} is important for the determination of the domain of Λ^z, see Triebel [18].

Consider two closed linear operators A and B with domains $D(A)$ and $D(B)$ in E. Their sum is defined by
$$Sv = Av + Bv, \quad v \in D(S) = D(A) \cap D(B).$$

First approach. Let us assume that A and B satisfy the following assumptions:

(LT1) There exists r, C_A, $C_B > 0$, ϵ_A, $\epsilon_B > 0$ such that
(i) $\rho(-A) \supset \sum_{\epsilon_A} = \{z : |z| \geq r,\ |Arg(z)| < \epsilon_A\}$
and for all $z \in \sum_{\epsilon_A}$, $\|(A + zI)^{-1}\|_{L(E)} \leqslant C_A/|z|$,
(ii) $\rho(-B) \supset \sum_{\epsilon_B} = \{z : |z| \geq r,\ |Arg(z)| < \epsilon_B\}$
and for all $z \in \sum_{\epsilon_B}$, $\|(B + zI)^{-1}\|_{L(E)} \leqslant C_B/|z|$,
(iii) $\epsilon_A + \epsilon_B > \pi$,

where $\rho(-A)$ and $\rho(-B)$ denote the resolvent sets of $(-A)$ and $(-B)$ respectively.

We suppose that there exist $C > 0$, $\lambda_0 > 0$ (with $\lambda_0 \in \rho(-A)$), τ and ρ such that

(LT2) (i) For all $\lambda \in \rho(-A)$ and all $\mu \in \rho(-B)$,
$$\|(A + \lambda_0)(A + \lambda I)^{-1}[(A + \lambda_0)^{-1}; (B + \mu I)^{-1}]\|_{L(E)} \leqslant \frac{C}{|\lambda|^{1-\tau}.|\mu|^{1+\rho}},$$
(ii) $0 \leqslant \tau < \rho \leqslant 2$.

For any $\sigma \in]0, 1[$ and $1 \leqslant p \leqslant +\infty$, let us introduce the real Banach interpolation spaces $D_A(\sigma, p)$ between $D(A)$ and E (or $D_B(\sigma, p)$ between $D(B)$ and E) which are characterized by
$$D_A(\sigma, p) = \{\xi \in E : t \mapsto \|t^\sigma A(A + tI)^{-1}\xi\|_E \in L^p_*\},$$

where L^p_* denotes the space of p-integrable functions on $(0,+\infty)$ with the measure dt/t. For $p=+\infty$,

$$D_A(\sigma,\infty)=\big\{\xi\in E:\ \sup_{t>0}\|t^\sigma A(A+tI)^{-1}\xi\|_E<\infty\big\}.$$

Then the main result proved in Labbas-Terreni [11] is

Theorem 2.1. *Under Assumptions (LT1) and (LT2), there exists λ^* such that $\forall\lambda \geqslant \lambda^*$ and $\forall h\in D_A(\sigma,p)$, Equation $Aw+Bw+\lambda w=h$, has a unique solution $w\in D(A)\cap D(B)$ with the regularities $Aw, Bw\in D_A(\theta,p)$ and $Aw\in D_B(\theta,p)$ for any θ verifying $\theta\leqslant\min(\sigma,\rho-\tau)$.*

Second approach. Here, we assume that E is an U.M.D complex Banach space (Unconditional Martingales Differences Property). It means that for all $p\in]1,\infty[$, the Hilbert transform is bounded on $L^p(\mathbb{R},E)$, (see Burckholder [1]). In concrete terms, any space built on L^p is U.M.D if p belongs to $(1,\infty)$.

We suppose that A and B satisfy

(DV1) (i) $D(A)$ and $D(B)$ are dense in E
(ii) $\rho(A)\supset]-\infty,0]$ and there exists $M_A\geqslant 1$ so that for all $\lambda\geqslant 0$, $\|(A+\lambda I\ )^{-1}\|_{L(E)}\leqslant M_A/(1+\lambda)$
(iii) $\rho(B)\supset]-\infty,0]$ and there exists $M_B\geqslant 1$ such that for all $\lambda\geqslant 0$, $\|(B+\lambda I)^{-1}\|_{L(E)}\leqslant M_B/(1+\lambda)$.

(DV2) There exist $K>0$, θ_A, $\theta_B\in[0,\pi[$ such that
(i) $A^{is}\in L(E)$ and for all $s\in\mathbb{R}$, $\|A^{is}\|\le Ke^{|s|\theta_A}$,
(ii) $B^{is}\in L(E)$ and for all $s\in\mathbb{R}$, $\|B^{is}\|\le Ke^{|s|\theta_B}$,
(iii) $\theta_A+\theta_B<\pi$.

(DV3) For all $\xi\in\rho(-A)$ and all $\eta\in\rho(-B)$,

$$(A+\xi I)^{-1}(B+\eta I)^{-1}=(B+\eta I)^{-1}(A+\xi I)^{-1}.$$

In general, Hypothesis (DV2) is hard to check in concrete cases.

We shall use the following Dore-Venni Theorem (see [2]).

Theorem 2.2. *Assume (DV1), (DV2), (DV3). Then $A+B$ is closed, invertible and $(A+B)^{-1}\in L(E)$.*

3. Proof of Theorem 1.1

Let $\varphi(t)=t^\alpha$ with $\alpha>1/2$, $m(t)=t^{-2\alpha}$ and assume

$$p>1+\alpha. \tag{3.1}$$

Set $X=L^p(0,1)$ and $w(t)=w(t,.)$, then (1.3) is equivalent to the abstract degenerate Cauchy problem

$$\begin{gathered}t^{2\alpha}w'(t)+L(t)w(t)+\lambda w(t)=t^{2\alpha}g(t),\quad t\in(0,1),\\ w(0)=0,\end{gathered}\tag{3.2}$$

where the family $(L(t))_{t\in[0,1]}$ is defined by

$$\begin{gathered}D(L(t))=\{\psi\in W^{2,p}(0,1):\psi(0)=\psi(1)=0\}\\ ([L(t)]\psi)(y)=-\psi''(y)-\alpha t^{2\alpha-1}y\psi'(y)\text{ for a.e. } t\in(0,1).\end{gathered}\tag{3.3}$$

Observe that

$$\overline{D(L(t))}=X. \tag{3.4}$$

Then we must solve

$$\begin{gathered} t^{2\alpha}w'(t)+L(t)w(t)+\lambda w(t)=h(t), \quad t\in(0,1) \\ w(0)=0, \end{gathered} \tag{3.5}$$

where h belongs to the space

$$E_1=\big\{h\in L^p(0,1;X): t^{-2\alpha+\alpha/p}h\in L^p(0,1;X)\big\}.$$

Equation (3.5) can be written in the form

$$Bw+Aw+\lambda w=h,$$

where

$$\begin{gathered} D(A)=\{w\in E_1: t^{-2\alpha+\alpha/p}w\in L^p(0,1;W^{2,p}(0,1)\cap W_0^{1,p}(0,1))\} \\ (Aw)(t)=L(t)w(t), \quad t\in(0,1), \end{gathered} \tag{3.6}$$

and

$$\begin{gathered} D(B)=\{w\in E_1: t^{\alpha/p}w'\in L^p(0,1;X) \text{ and } w(0)=0\} \\ (Bw)(t)=t^{2\alpha}w'(t), \quad t\in(0,1). \end{gathered} \tag{3.7}$$

Note that the trace $w(0)$ is well defined in $D(B)$. In fact, we have

$$t^{\alpha/p}w\in L^p(0,1;X), \quad t^{\alpha/p}w'\in L^p(0,1;X),$$

and $\alpha/p+1/p<1$ in virtue of (3.1). Then w is continuous on $[0,1]$, (see [18, Lemma, p. 42]).

Now, to apply Theorem (2.1), we will verify Assumptions $(LT1)$ and $(LT2)$.

Proposition 3.1. *The operators A and B are linear closed with dense domains in E_1. Moreover, they satisfy Assumption $(LT1)$.*

The properties of operator B are based on the solvability of the spectral equation $Bw+zw=h$. Fix some positive μ_0 and let z such that $\operatorname{Re}(z)\geqslant\mu_0$. Then the problem

$$\begin{gathered} t^{2\alpha}w'(t)+zw(t)=h(t) \\ w(0)=0, \end{gathered}$$

admits the solution

$$\begin{aligned} w(t)&=\big((B+zI)^{-1}h\big)(t) \\ &=\exp\big(\frac{z}{(2\alpha-1)t^{2\alpha-1}}\big)\int_0^t \sigma^{-2\alpha}h(\sigma)\exp\big(\frac{-z}{(2\alpha-1)\sigma^{2\alpha-1}}\big)d\sigma \end{aligned}$$

Let us check that this formula is well defined on $[0,1]$ and gives $w(0)=0$. Set $\mu=z/(2\alpha-1)$, then

$$\begin{aligned} \|w(t)\|&\leqslant\exp\big(\frac{\operatorname{Re}\mu}{t^{2\alpha-1}}\big)\int_0^t\|\sigma^{-2\alpha+\alpha/p}h(\sigma)\|\sigma^{-\alpha/p}\exp\big(\frac{-\operatorname{Re}\mu}{\sigma^{2\alpha-1}}\big)d\sigma \\ &\leqslant\Big(\int_0^t\|\sigma^{-2\alpha+\alpha/p}h(\sigma)\|^pd\sigma\Big)^{1/p}\Big(\int_0^t\sigma^{-\alpha q/p}d\sigma\Big)^{1/q} \\ &\leqslant\big(\frac{1}{1-\alpha q/p}\big)^{1/q}t^{1/q-\alpha/p}\|h\|_E \end{aligned}$$

where $1/p+1/q=1$. Hence $w(t)$ is defined and $w(0)=0$ since

$$1/q-\alpha/p=1-1/p-\alpha/p>0$$

means $p > 1+\alpha$. On the other hand we can write

$$\begin{aligned} t^{-2\alpha+\alpha/p}((B+zI)^{-1}h)(t) &= t^{-2\alpha+\alpha/p}w(t) \\ &= \int_0^1 \sigma^{-2\alpha+\alpha/p}h(\sigma)K_\mu(t,\sigma)d\sigma, \end{aligned} \tag{3.8}$$

where

$$K_\mu(t,\sigma) = \begin{cases} \frac{1}{t^{2\alpha-\alpha/p}\sigma^{\alpha/p}} \exp \mu(t^{-2\alpha+1} - \sigma^{-2\alpha+1}) & \text{if } t > \sigma \\ 0 & \text{if } t < \sigma. \end{cases} \tag{3.9}$$

Therefore,

$$\begin{aligned} \int_0^1 |K_\mu(t,\sigma)|d\sigma &= \frac{1}{t^{2\alpha-\alpha/p}} \exp(\operatorname{Re}(\mu).t^{-2\alpha+1}) \int_0^t \frac{\exp(-\operatorname{Re}(\mu).\sigma^{-2\alpha+1})}{\sigma^{\alpha/p}} d\sigma \\ &\leqslant \frac{1}{2\alpha-1} \exp(\operatorname{Re}\mu.t^{-2\alpha+1}) \int_{t^{-2\alpha+1}}^{+\infty} \exp(-\operatorname{Re}(\mu).s)ds \\ &\leqslant \frac{1}{\operatorname{Re}(z)}, \end{aligned}$$

and

$$\max_{t\in[0,1]} \int_0^1 |K_\mu(t,\sigma)|d\sigma \leqslant \frac{1}{\operatorname{Re}(z)}. \tag{3.10}$$

Furthermore, one has

$$\begin{aligned} &\int_0^1 |K_\mu(t,\sigma)|dt \\ &= \frac{\exp(-\operatorname{Re}\mu.\sigma^{-2\alpha+1})}{\sigma^{\alpha/p}} \int_\sigma^1 \frac{1}{t^{2\alpha-\alpha/p}} \exp(\operatorname{Re}\mu.t^{-2\alpha+1})dt \\ &= \frac{1}{2\alpha-1} \frac{\exp(-\operatorname{Re}(\mu).\sigma^{-2\alpha+1})}{\sigma^{\alpha/p}} \int_1^{\sigma^{-2\alpha+1}} \frac{1}{s^{\frac{\alpha}{p(2\alpha-1)}}} \exp(\operatorname{Re}\mu.s)ds, \end{aligned}$$

and

$$\begin{aligned} &\int_1^{\sigma^{-2\alpha+1}} \frac{1}{s^{\frac{\alpha}{p(2\alpha-1)}}} \exp(\operatorname{Re}\mu s)ds \\ &= \int_1^{\frac{1+\sigma^{-2\alpha+1}}{2}} \frac{1}{s^{\frac{\alpha}{p(2\alpha-1)}}} \exp(\operatorname{Re}\mu s)ds + \int_{\frac{1+\sigma^{-2\alpha+1}}{2}}^{\sigma^{-2\alpha+1}} \frac{1}{s^{\frac{\alpha}{p(2\alpha-1)}}} \exp(\operatorname{Re}\mu s)ds \\ &\leqslant \int_1^{\frac{1+\sigma^{-2\alpha+1}}{2}} \exp(\operatorname{Re}\mu s)ds + \frac{1}{(\frac{1+\sigma^{-2\alpha+1}}{2})^{\frac{\alpha}{p(2\alpha-1)}}} \int_{\frac{1+\sigma^{-2\alpha+1}}{2}}^{\sigma^{-2\alpha+1}} \exp(\operatorname{Re}\mu s)ds \\ &= I_1 + I_2. \end{aligned}$$

Then

$$\begin{aligned} I_1 &\leqslant \frac{1}{\operatorname{Re}\mu}\Big[\exp(\operatorname{Re}\mu\frac{(1+\sigma^{-2\alpha+1})}{2}) - \exp(\operatorname{Re}\mu)\Big] \\ &\leqslant \frac{1}{\operatorname{Re}\mu}\exp(\operatorname{Re}\mu\frac{(1+\sigma^{-2\alpha+1})}{2}), \end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\alpha-1}\frac{\exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1})}{\sigma^{\alpha/p}}I_1\\
&\leqslant \frac{1}{\operatorname{Re}(z)}\frac{\exp(-\operatorname{Re}\mu(\sigma^{-2\alpha+1}-1)/2)}{\sigma^{\alpha/p}}\\
&\leqslant \frac{1}{\operatorname{Re}(z)}\frac{\exp(-\mu_0(\sigma^{-2\alpha+1}-1)/2)}{\sigma^{\alpha/p}}\\
&\leqslant \frac{C_1(\alpha,p)}{\operatorname{Re}(z)},
\end{aligned}$$

since the function

$$\sigma \mapsto \frac{\exp(-\mu_0(\sigma^{-2\alpha+1}-1)/2)}{\sigma^{\alpha/p}}$$

is continuous on $[0,1]$. Moreover

$$\begin{aligned}
&\frac{1}{2\alpha-1}\frac{\exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1})}{\sigma^{\alpha/p}}I_2\\
&\leqslant \frac{1}{2\alpha-1}\frac{\exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1})}{\sigma^{\alpha/p}\big(\frac{1+\sigma^{-2\alpha+1}}{2}\big)^{\frac{\alpha}{p(2\alpha-1)}}}\int_{\frac{1+\sigma^{-2\alpha+1}}{2}}^{\sigma^{-2\alpha+1}}\exp(\operatorname{Re}\mu s)ds\\
&\leqslant \frac{C_2(\alpha,p)}{\operatorname{Re}(z).\sigma^{\alpha/p}\big(\frac{1+\sigma^{-2\alpha+1}}{2}\big)^{\frac{\alpha}{p(2\alpha-1)}}}\\
&\leqslant \frac{C_3(\alpha,p)}{\operatorname{Re}(z)}
\end{aligned}$$

in virtue of the fact that

$$\lim_{\sigma\to 0}\frac{1}{\sigma^{\alpha/p}(1+\sigma^{-2\alpha+1})^{\frac{\alpha}{p(2\alpha-1)}}}=1.$$

Consequently, there exists some constant $C(\alpha,p)>0$ such that

$$\max_{\sigma\in[0,1]}\int_0^1|K_\mu(t,\sigma)|dt\leqslant\frac{C(\alpha,p)}{\operatorname{Re}(z)}. \tag{3.11}$$

Now, using Schur interpolation Lemma together with (3.10) and (3.11), we obtain

$$\|t^{-2\alpha+\alpha/p}w\|_{L^p(0,1;X)}\leqslant\frac{C(\alpha,p)}{\operatorname{Re}(z)}\|t^{-2\alpha+\alpha/p}h\|_{L^p(0,1;X)},$$

from which it follows

$$\|(B+zI)^{-1}\|_{L(E_1)}\leqslant\frac{C(\alpha,p)}{\operatorname{Re}(z)}.$$

Thus, we can take $\epsilon_B=\pi/2-\epsilon_0$ (for each $\epsilon_0\in]0,\pi/2[$).

The study of the spectral properties of the operator A are based on those of operators $L(t)$. For each t, we write

$$(L(t)\Psi)=L_0\Psi+P(t)\Psi$$

with

$$\begin{gathered}
D(L_0)=\{\Psi\in W_p^2(0,1):\Psi(0)=\Psi(1)=0\}\\
L_0\Psi=-\Psi'',
\end{gathered}\tag{3.12}$$

and

$$\begin{gathered} D(P(t)) = W^{1,p}(0,1) \\ P(t)\Psi = -\alpha t^{2\alpha-1} y\Psi' = -b(t)y\Psi'. \end{gathered} \tag{3.13}$$

It is easy to see that the operator L_0 is sectorial. Moreover, thanks to Hölder inequality, for $\psi \in D(L_0) \subset D(P(t))$ we have

$$\begin{aligned}
\|P(t)\Psi\|_{L^p(0,1)} &= \Big(\int_0^1 \Big| -b(t)y\Psi'(y)|^p dy\Big)^{1/p} \\
&= \Big(\int_0^1 |-b(t)y\Big(\int_0^y (s\Psi''(s))ds + \int_y^1 (1-s)\Psi''(s)ds\Big)\Big|^p dy\Big)^{1/p} \\
&\leqslant \Big(\int_0^1 \Big| -b(t)y\int_0^y (s\Psi''(s))ds\Big|^p dy\Big)^{1/p} \\
&\quad + \Big(\int_0^1 \Big| -b(t)y\int_y^1 (1-s)\Psi''(s)ds\Big|^p dy\Big)^{1/p} \\
&\leqslant b(t)\big(C_1(p)\|\Psi''\|_{L^p(0,1)} + C_2(p)\|\Psi''\|_{L^p(0,1)}\big) \\
&\leqslant C(p)\|\Psi\|_{D(L_0)}.
\end{aligned}$$

On the other hand, let us set

$$\begin{aligned}
m(t) : L^p(0,1) &\to L^p(0,1) \\
\psi &\mapsto [m(t)\psi](y) = -b(t)y\psi(y), \\
i : W^{1,p}(0,1) &\to L^p(0,1) \\
\psi &\mapsto \psi, \\
d : W^{2,p}(0,1) &\to W^{1,p}(0,1) \\
\psi &\mapsto d(\psi) = \psi'.
\end{aligned}$$

Then $P(t) = m(t) \circ i \circ d$. Thus, $P(t)$ is compact from $D(L_0)$ into E_1 since i is compact and d, $m(t)$ are continuous. So for any $t \in [0,1]$, the operator $L(t)$ is sectorial (see Lunardi [14, Proposition 2.4.3, p.65]) and consequently there exists some $r_0 > 0$ such that

$$\rho(-L(t)) \supset \Sigma_{\pi-\epsilon_1} = \{z : |z| \geqslant r_0,\ |\arg z| < \pi - \epsilon_1\}$$

where $\epsilon_1 \in]0,\pi/2[$. Now, for $h \in E_1$ and $\lambda \in \Sigma_{\pi-\epsilon_1}$ the spectral equation

$$Aw + zw = h$$

is equivalent to

$$L(t)w(t) + zw(t) = h(t), \quad t \in (0,1)$$

which admits a unique solution,

$$w(t) = (L(t)+z)^{-1}h(t).$$

Hence

$$\|w(t)\|_{L^p(0,1)} \leqslant \frac{K}{|z|}\|h(t)\|_{L^p(0,1)}$$

and

$$\|w\|_{E_1} = \Big(\int_0^1 \|t^{-2\alpha+\alpha/p}w(t)\|^p dt\Big)^{1/p} \leqslant \frac{K}{|z|}\|h\|_{E_1}.$$

Finally A and B satisfy $(LT1)$.

Proposition 3.2. *Operators A and B satisfy Hypothesis (LT2).*

In our case, the domains $D(L(t)) = D(L(0))$ are constant. Let us verify the Sobolevskii's estimate: There exists $M > 0$ such that for all $t, \sigma \in [0,1]$,

$$\|[(L(t)L(\sigma)^{-1} - I)]\|_{L(X)} \leqslant M|t-\sigma|^{\rho}\,, \tag{3.14}$$

where $\rho = \min(1, 2\alpha - 1)$.

For $g \in X = L^p(0,1)$, the equation $\psi = L(\sigma)^{-1}g$ is equivalent to

$$\begin{gathered}([L(\sigma)]\psi)(y) = -D_y^2\psi(y) - \varphi(\sigma)\varphi'(\sigma)yD_y\psi(y) = g(y),\\ \psi(0) = \psi(1) = 0,\end{gathered}$$

and

$$[(L(t) - L(\sigma))L(\sigma)^{-1}g](y) = (\varphi(\sigma)\varphi'(\sigma) - \varphi(t)\varphi'(t))yD_y\psi(y). \tag{3.15}$$

Then, we get

$$\begin{aligned}\|[(L(t) - L(\sigma))L(\sigma)^{-1}g]\|_X &\leqslant \alpha|t^{2\alpha-1} - \sigma^{2\alpha-1}|\|y\psi'\|_X\\ &\leqslant M_1|t-\sigma|^{\min(1,2\alpha-1)}\|\psi''\|_X\\ &\leqslant M_2|t-\sigma|^{\min(1,2\alpha-1)}\|\psi\|_{W^{2,p}(0,1)}\\ &\leqslant M|t-\sigma|^{\min(1,2\alpha-1)}\|g\|_{L^p(0,1)}.\end{aligned}$$

To prove (LT2), it is sufficient to estimate

$$\|A(A+\lambda I)^{-1}[A^{-1}; (B+zI)^{-1}]\|_{L(E_1)}$$

where $\lambda \in \rho(-A)$,$z \in \rho(-B)$. Let $h \in E_1$. Then

$$\begin{aligned}\Delta &= (t^{-2\alpha+\alpha/p}A(A+\lambda I)^{-1}[A^{-1}; (B+zI)^{-1}]h)(t)\\ &= t^{-2\alpha+\alpha/p}(A(A+\lambda I)^{-1}(A^{-1}(B+zI)^{-1} - (B+zI)^{-1}A^{-1})h)(t)\\ &= t^{-2\alpha+\alpha/p}L(t)(L(t)+\lambda I)^{-1}\{L(t)^{-1}\big((B+zI)^{-1}h\big)(t)\\ &\quad - \big((B+zI)^{-1}A^{-1}h\big)(t)\}.\end{aligned}$$

Using the representation given in (3.8) and the kernel defined in (3.9) where we have put $\mu = z/(2\alpha-1)$, we obtain

$$\begin{aligned}\Delta &= L(t)(L(t)+\lambda I)^{-1}\int_0^1 \sigma^{-2\alpha+\alpha/p}K_\mu(t,\sigma)(L(t)^{-1} - L(\sigma)^{-1})h(\sigma)d\sigma\\ &= \int_0^1 \sigma^{-2\alpha+\alpha/p}K_\mu(t,\sigma)L(t)(L(t)+\lambda I)^{-1}(L(t)^{-1} - L(\sigma)^{-1})h(\sigma)d\sigma\\ &= \int_0^1 \sigma^{-2\alpha+\alpha/p}K_\mu(t,\sigma)(L(t)+\lambda I)^{-1}(I - L(t)L(\sigma)^{-1})h(\sigma)d\sigma\end{aligned}$$

since the domains $D(A(t))$ are constant. Also

$$\|\Delta\|_X = \frac{K}{|\lambda|}\int_0^1 |K_\mu(t,\sigma)||t-\sigma|^{\rho}\sigma^{-2\alpha+\alpha/p}\|h(\sigma)\|_X d\sigma,$$

with $\rho = \min(1, 2\alpha - 1)$. Recall that

$$
\begin{aligned}
&\int_0^1 K_\mu(t,\sigma)|t-\sigma|^\rho d\sigma \\
&= \frac{1}{t^{2\alpha-\alpha/p}} \exp(\operatorname{Re}\mu t^{-2\alpha+1}) \int_0^t \sigma^{-\alpha/p}(t-\sigma)^\rho \exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1}) d\sigma\,.
\end{aligned}
$$

Then by Hölder inequality, one has

$$
\begin{aligned}
&\int_0^t \sigma^{-\alpha/p}(t-\sigma)^\rho \exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1}) d\sigma \\
&\leqslant \Big(\int_0^t \sigma^{-\alpha/p} \exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1}) d\sigma\Big)^{1-\rho} \\
&\quad \times \Big(\int_0^t \sigma^{-\alpha/p}(t-\sigma) \exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1}) d\sigma\Big)^{\rho}
\end{aligned}
$$

and

$$
\begin{aligned}
J_1 &= \Big(\int_0^t \sigma^{2\alpha-\alpha/p}\sigma^{-2\alpha} \exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1}) d\sigma\Big)^{1-\rho} \\
&\leqslant \frac{(t^{2\alpha-\alpha/p})^{1-\rho}}{(2\alpha-1)^{1-\rho}} \frac{1}{(\operatorname{Re}\mu)^{1-\rho}} (\exp(-\operatorname{Re}\mu t^{-2\alpha+1}))^{1-\rho}
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \Big(\int_0^t \sigma^{-\alpha/p}(t-\sigma) \exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1}) d\sigma\Big)^{\rho} \\
&= \Big(\int_0^t \sigma^{2\alpha-\alpha/p}(t-\sigma)\sigma^{-2\alpha} \exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1}) d\sigma\Big)^{\rho} \\
&\leqslant \frac{(t^{2\alpha-\alpha/p})^{\rho}}{(2\alpha-1)^{\rho}} \frac{1}{(\operatorname{Re}\mu)^{\rho}} \Big(\int_0^t (t-\sigma)\varkappa'(\sigma) d\sigma\Big)^{\rho}
\end{aligned}
$$

where $\varkappa(\sigma) = \exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1})$. Using an integration by parts, we obtain

$$
\begin{aligned}
\int_0^t (t-\sigma)\varkappa'(\sigma) d\sigma &= \int_0^t \exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1}) d\sigma \\
&= \int_0^t \sigma^{2\alpha}\sigma^{-2\alpha} \exp(-\operatorname{Re}\mu\sigma^{-2\alpha+1}) d\sigma \\
&\leqslant \frac{t^{2\alpha}}{(2\alpha-1)} \frac{1}{\operatorname{Re}\mu} \exp(-\operatorname{Re}\mu t^{-2\alpha+1})
\end{aligned}
$$

from which we deduce that

$$
J_2 \leqslant \frac{(t^{2\alpha-\alpha/p})^{\rho}}{(2\alpha-1)^{\rho}} \frac{1}{(\operatorname{Re}\mu)^{\rho}} \frac{(t^{2\alpha})^{\rho}}{(2\alpha-1)^{\rho}} \frac{1}{(\operatorname{Re}\mu)^{\rho}} (\exp(-\operatorname{Re}\mu t^{-2\alpha+1}))^{\rho}.
$$

Finally we have

$$\begin{aligned}
&\int_0^1 K_\mu(t,\sigma)|t-\sigma|^\rho d\sigma \\
&\leqslant \frac{\exp(\operatorname{Re}\mu t^{-2\alpha+1})}{t^{2\alpha-\alpha/p}} \frac{(t^{2\alpha-\alpha/p})^{1-\rho}}{(2\alpha-1)^{1-\rho}} \frac{(\exp(-\operatorname{Re}\mu t^{-2\alpha+1}))^{1-\rho}}{(\operatorname{Re}\mu)^{1-\rho}} \\
&\quad \times \frac{(t^{2\alpha-\alpha/p})^{\rho}}{(2\alpha-1)^{\rho}} \frac{1}{(\operatorname{Re}\mu)^{\rho}} \frac{(t^{2\alpha})^{\rho}}{(2\alpha-1)^{\rho}} \frac{(\exp(-\operatorname{Re}\mu t^{-2\alpha+1}))^{\rho}}{(\operatorname{Re}\mu)^{\rho}} \\
&\leqslant \frac{(t^{2\alpha})^{\rho}}{(2\alpha-1)^{1+\rho}} \frac{1}{(\operatorname{Re}\mu)^{1+\rho}},
\end{aligned}$$

and

$$\max_t \int_0^1 K_\mu(t,\sigma)|t-\sigma|^\rho d\sigma \leqslant \frac{C}{(\operatorname{Re}\mu)^{1+\rho}}.$$

Similarly one has

$$\max_{\sigma\in[0,1]} \int_0^1 K_\mu(t,\sigma)|t-\sigma|^\rho dt \leqslant \frac{C}{(\operatorname{Re}\mu)^{1+\rho}}.$$

In virtue of Schur's lemma, we conclude that

$$\|A(A+\lambda I)^{-1}[A^{-1};(B+zI)^{-1}]\|_{L(E_1)} \leqslant \frac{C}{|\lambda|(\operatorname{Re}\mu)^{1+\rho}} = \frac{C}{|\lambda|(\operatorname{Re}z)^{1+\rho}}$$

which implies

$$\|A(A+\lambda I)^{-1}[A^{-1};(B+zI)^{-1}]\|_{L(E_1)} \leqslant \frac{C}{|\lambda||z|^{1+\rho}}$$

for any $\lambda\in\rho(-A)$ and any μ belonging to a suitable sectorial curve. Then (LT2) is verified with $\tau=0$ and $\rho=\min(1,2\alpha-1)$. For more details concerning the commutator used in (LT2), see Labbas-Terreni [11].

Applying Theorem 2.1, we deduce the following statement.

Proposition 3.3. *There exists λ^* such that for all $\lambda\geqslant\lambda^*$ and $h\in D_A(\sigma,p)$ Problem* (3.5) *has a unique solution $w\in D(A)\cap D(B)$ such that for all $\theta\leqslant 1/2p$*

- (i) $L(.)w\in D_A(\theta,p)$
- (ii) $\varphi(t)w'\in D_A(\theta,p)$
- (iii) $L(.)w\in D_B(\theta,p)$.

Observe that we have a similar result when $h\in D_B(\sigma,p))$. To make precise the time and the space regularity of w we need to specify the space $D_A(\sigma,p)$. One has

$$D_A(\sigma,p)=\begin{cases}\big\{w\in E_1: t^{-2\alpha+\alpha/p}w\in L^p(0,1;W^{2\sigma,p}(0,1)), & \\ w(t,0)=w(t,1)=0\big\} & \text{if } 2\sigma>1/p,\\ \{w\in E: t^{-2\alpha+\alpha/p}w\in L^p(0,1;W^{2\sigma,p}(0,1))\} & \text{if } 2\sigma<1/p.\end{cases}$$

Indeed we know that

$$D_A(\sigma,p)=\{w\in E_1:\|\xi^{1-\sigma}Ae^{-\xi A}w\|_{E_1}\in L^p_*\},$$

because $(-A)$ is a generator of the analytic semigroup $\{e^{-\xi A}\}$, $\xi\geqslant 0$. Now, $w\in D_A(\sigma,p)$ implies

$$\|\xi^{1-\sigma}Ae^{-\xi A}w\|_{E_1}\in L^p_*$$

or equivalently

$$
\begin{aligned}
&\int_0^\infty \|\xi^{1-\sigma} A e^{-\xi A} w\|_{E_1}^p \frac{d\xi}{\xi}\\
&= \int_0^\infty \|t^{-2\alpha+\alpha/p}\xi^{1-\sigma} A e^{-\xi A} w\|_{L^p(0,1;L^p(0,1))}^p \frac{d\xi}{\xi}\\
&= \int_0^\infty \Big(\int_0^1 \|t^{-2\alpha+\alpha/p}\xi^{1-\sigma} (A e^{-\xi A} w)(t)\|_{L^p(0,1)}^p dt \Big) \frac{d\xi}{\xi} < +\infty.
\end{aligned}
$$

Since

$$
(Ae^{-\xi A} w)(t) = L(t)\ e^{\xi L(t)}(w(t)),
$$

by Fubini's Theorem, we obtain

$$
\begin{aligned}
&\int_0^\infty \|\xi^{1-\sigma} A e^{-\xi A} w\|_{E_1}^p \frac{d\xi}{\xi}\\
&= \int_0^\infty \Big(\int_0^1 \|t^{-2\alpha+\alpha/p}\xi^{1-\sigma} L(t)\ e^{\xi L(t)}(w(t))\|_{L^p(0,1)}^p dt \Big) \frac{d\xi}{\xi}\\
&= \int_0^1 \|t^{-2\alpha+\alpha/p}\|^p \Big(\int_0^\infty \|\xi^{1-\sigma} L(t) e^{\xi L(t)}(w(t))\|_{L^p(0,1)}^p \frac{d\xi}{\xi} \Big) dt < +\infty,
\end{aligned}
$$

which means that, for almost every t, the function $y \mapsto t^{-2\alpha+\alpha/p} w(t)(y)$ is in $D_{L(t)}(\sigma,p)$. It is well known that this last constant space is

$$
\begin{aligned}
D_{L(t)}(\sigma,p) = D_{L(0)}(\sigma,p) &= (W^{2,p}(0,1) \cap W_0^{1,p}(0,1); L^p(0,1))_{1-\sigma,p}\\
&= \begin{cases} \{w \in W^{2\sigma,p}(0,1),\ w(0)=w(1)=0\} & \text{if } 2\sigma > 1/p\\ W^{2\sigma,p}(0,1) & \text{if } 2\sigma < 1/p. \end{cases}
\end{aligned}
$$

Let σ be a fixed positive number satisfying $\sigma < 1/2p$. From the above proposition, we have the following statement.

Proposition 3.4. *For all h such that $t^{-2\alpha+\alpha/p} h \in L^p(0,1;W^{2\sigma,p}(0,1))$, Problem* (3.5) *admits a unique solution w fulfilling the regularity properties:*

- (i) $w \in L^p(\Omega)$, $t^{-2\alpha+\alpha/p} w \in L^p(\Omega)$, $w(0)=0$
- (ii) $t^{-2\alpha+\alpha/p} D_y^2 w \in L^p(\Omega)$
- (iii) $t^{\alpha/p} D_t w \in L^p(\Omega)$,
- (iv) $t^{-2\alpha+\alpha/p} D_y^2 w \in L^p(0,1;W^{2\sigma,p}(0,1))$
- (v) $t^{\alpha/p} D_t w \in L^p(0,1;W^{2\sigma,p}(0,1))$.

Let us recall that $h(t,y) = t^{2\alpha} g(t,y)$, $g(t,y) = f(t,x)$ and $w(t,y) = u(t,x)$ where $(t,y) = (t, x/t^\alpha)$. Then

$$
\begin{aligned}
&\int_0^1 \|t^{-2\alpha+\alpha/p} h(t,.)\|_{W^{2\sigma}(0,1)}^p dt\\
&= \int_0^1 t^{\alpha-2\alpha p} \int_0^1 \int_0^1 \frac{|h(t,y)-h(t,y')|^p}{|y-y'|^{2\sigma p+1}} dy\, dy'\, dt\\
&= \int_0^1 t^{2\sigma\alpha p} \int_0^{t^\alpha} \int_0^{t^\alpha} \frac{|f(t,x)-f(t,x')|^p}{|x-x'|^{2\sigma p+1}} dx\, dx'\, dt,
\end{aligned}
$$

from which we deduce that $\varphi^{-2+1/p}h \in L^p(0,1;W^{2\sigma,p}(0,1))$ signifies that

$$\int_0^1 \varphi(t)^{2\sigma p} \int_0^{\varphi(t)} \int_0^{\varphi(t)} \frac{|f(t,x)-f(t,x')|^p}{|x-x'|^{2\sigma p+1}} dx dx' dt < \infty.$$

We denote this condition by

$$f \in L^p_{\varphi^{2\sigma}}(0,1;W^{2\sigma,p}_{\varphi}).$$

Similarly, we can prove the following equivalences.

Proposition 3.5.

(i) $t^{-2\alpha+\alpha/p}w \in L^p(0,1;L^p(0,1))$ *if and only if* $u \in L^p(U)$

(ii) $t^{-2\alpha+\alpha/p}D_y^2 w \in L^p(0,1;W^{2\sigma,p}(0,1))$ *if and only if* $D_x^2 u \in L^p_{\varphi^{2\sigma}}(0,1;W^{2\sigma,p}_{\varphi})$

(iii) $t^{\alpha/p}D_t w \in L^p(0,1;W^{2\sigma,p}(0,1))$ *if and only if* $D_t u \in L^p_{\varphi^{2\sigma}}(0,1;W^{2\sigma,p}_{\varphi})$.

Then Theorem 1.1 is a consequence of the previous propositions. Note that the regularity of the solution in this result is maximal, when $f \in L^p_{\varphi^{2\sigma}}(0,1;W^{2\sigma,p}_{\varphi})$.

4. Proof of Theorem 1.2

Assume that $\varphi(t) = \sqrt{t}$, $m = t^{-1}$ and $\lambda \geqslant 1/2p$. Recall that $u(t,x) = w(t,y)$. Then problem (1.1) is equivalent to the following problem in $X = L^p(0,1)$

$$\begin{gathered} tw'(t) + Lw(t) + \lambda w(t) = tg(t) \quad t \in (0,1), \\ w(0) = 0, \end{gathered} \tag{4.1}$$

where

$$\begin{gathered} D(L) = \{\psi \in W^{2,p}(0,1) \ : \psi(0) = \psi(1) = 0\} \\ (L\psi)(y) = -\psi''(y) - \frac{1}{2} y\psi'(y). \end{gathered} \tag{4.2}$$

Observe that $\overline{D(L)} = X$. It is sufficient to consider the equation

$$\begin{gathered} tw'(t) + Lw(t) + \frac{1}{2p} w(t) = h_1(t), \quad t \in (0,1), \\ w(0) = 0, \end{gathered} \tag{4.3}$$

where h_1 is in the space

$$E_2 = \{h \in L^p(0,1;L^p(0,1)) : t^{-1+1/2p}h \in L^p(0,1;L^p(0,1))\}.$$

Then, we can write $Bw + Aw = h_1$, where

$$\begin{gathered} D(A) = \{w \in E_2 : t^{-1+\frac{1}{2p}} w \in L^p(0,1;W^{2,p}(0,1) \cap W_0^{1,p}(0,1))\} \\ (Aw)(t) = Lw(t), \quad t \in (0,1), \end{gathered}$$

and

$$\begin{gathered} D(B) = \{w \in E_2 : t^{\frac{1}{2p}} w' \in L^p(0,1;X)\} \\ (Bw)(t) = t\, w'(t) + \frac{1}{2p} w(t), \quad t \in (0,1). \end{gathered} \tag{4.4}$$

Since $p > 3/2$ then $1/2p + 1/p < 1$ and $w(0)$ is well defined in $D(B)$. We will see below that necessarily $w(0) = 0$.

Observe that the condition (DV3) is satisfied. In order to apply Theorem 2.2 we need to verify (DV1), (DV2).

Proposition 4.1. *Operators A and B are linear and closed with dense domains in E_2. Moreover, they satisfy (DV1) and (DV2).*

Regarding the operator B, the proof is based on the solvability of the spectral equation
$$Bw + zw = h_1,$$
where $h_1 \in E_2$ and $z \geqslant 0$. Then
$$t\,w'(t) + \frac{1}{2p}w(t) + zw(t) = h_1(t)$$
admits the following solution
$$w(t) = t^{-z-1/2p}\int_0^t s^{z-1+1/2p}h_1(s)ds.$$
Observe that $w(0) = 0$ since
$$\begin{aligned}\|t^{-z-1/2p}\int_0^t s^{z+1/2p-1}h_1(s)ds\|_X &\leqslant \int_0^t s^{-1/2p}s^{-1+1/2p}\|h_1(s)\|ds\\ &\leqslant \|h_1\|_E\Big(\int_0^t s^{-q/2p}ds\Big)^{1/q}\\ &\leqslant \frac{t^{\frac{2p-3}{2p-2}}}{(1-q/2p)^{1/q}}\|h_1\|_{E_2},\end{aligned}$$
and $p > 3/2$ (recall that $1/p + 1/q = 1$). On the other hand, we have
$$\|(zI\ + B)^{-1}h_1\|_{E_2}^p = \|w\|_{E_2}^p = \int_0^1 \|t^{-1+1/2p}w(t)\|_X^p dt,$$
and
$$t^{-1+1/2p}w(t) = \int_0^t t^{-z-1}s^{z-1+1/2p}h(s)ds = \int_0^1 K_z(t,s)s^{-1+1/2p}h(s)ds,$$
where
$$K_z(t,s) = \begin{cases} t^{-z-1}s^z & \text{if } s < t\\ 0 & \text{if } s > t.\end{cases}$$
So
$$\sup_{t\in[0,1]}\int_0^1 |K_z(t,s)|ds \leqslant \sup_{t\in[0,1]}\int_0^t t^{-z-1}s^z ds \leqslant \frac{1}{z+1},$$
and for all $z > 0$,
$$\sup_{s\in[0,1]}\int_0^1 |K_z(t,s)|dt \leqslant \sup_{s\in[0,1]}\int_s^1 t^{-z-1}s^z dt \leqslant \frac{1}{z}.$$
Then, there exists a constant $C(p)$ such that
$$\|(B + zI)^{-1}\|_{L(E_2)} \leqslant \frac{C(p)}{z}$$
for any large z. Moreover, we can see that
$$(B^{-1}h_1)(t) = t^{-1/2p}\int_0^t s^{-1+1/2p}h(s)ds,$$
and
$$\|(B^{-1}h_1)(t)\|_X \leqslant t^{1-1/2p}\|h_1\|_{E_2},$$

which implies

$$\|B^{-1}h_1\|_{E_2} \leqslant C(p)\|h_1\|_{E_2}.$$

So $B^{-1} \in L(E_2)$ and $ii)$ of $(DV1)$ is verified.

Now, to prove that $B \in Bip(E_2)$, we set

$$Bw = B_0 w + \frac{1}{2p} w,$$

with

$$D(B_0) = \{w \in E : t^{\frac{1}{2p}} w' \in L^p(0,1;X),\, w(0) = 0\} = D(B)$$
$$(B_0 w)(t) = t\, w'(t), \quad t \in (0,1).$$

Consequently, it is sufficient to show that $B_0 \in Bip(E_2)$ (see [15, Theorem 3]). Due to the Dore-Venni similar techniques used in [3], we have, for $t \in (0,1)$ and $\mathrm{Re}(z) \in]0,1[$

$$\begin{aligned}
(B_0^{-z} w)(t) &= \frac{1}{\Gamma(z)\Gamma(1-z)} \int_0^\infty \lambda^{-z} \big((\lambda + B_0)^{-1} w\big)(t) d\lambda \\
&= \frac{1}{\Gamma(z)\Gamma(1-z)} \int_0^\infty \lambda^{-z} t^{-\lambda} \int_0^t s^{\lambda-1} w(s) ds d\lambda \\
&= \frac{1}{\Gamma(z)\Gamma(1-z)} \int_0^\infty \lambda^{-z} \int_0^t (\frac{s}{t})^\lambda w(s) \frac{ds}{s} d\lambda \\
&= \frac{1}{\Gamma(z)\Gamma(1-z)} \int_0^\infty \lambda^{-z} \int_0^t \exp\big(\lambda \ln(\frac{s}{t})\big) w(s) \frac{ds}{s} d\lambda \\
&= \frac{1}{\Gamma(z)\Gamma(1-z)} \int_0^t w(s) \big(\ln(\frac{t}{s})\big)^{z-1} \Big(\int_0^\infty \mu^{-z} e^{-\mu} d\mu\Big) \frac{ds}{s} \\
&= \frac{1}{\Gamma(z)} \int_0^t (\ln \frac{t}{s})^{z-1} w(s) \frac{ds}{s}.
\end{aligned}$$

which implies that, for $\tau < 0$,

$$(B_0^{-z} w)(e^\tau) = \frac{1}{\Gamma(z)} \int_{-\infty}^\tau (\tau - \sigma)^{z-1} w(e^\sigma) d\sigma.$$

Let us set $\omega = -1 + \frac{1}{2p}$, and

$$\psi(\sigma) = \frac{1}{\Gamma(z)} (\max\{\sigma, 0\})^{z-1} e^{\sigma(\omega + 1/p)} \quad \text{for } \sigma \in \mathbb{R},$$
$$\phi(\sigma) = \begin{cases} e^{\sigma(\omega+1/p)} w(e^\sigma) & \text{for } \sigma \in (-\infty, 0), \\ 0 & \text{for } \sigma \in (0, \infty). \end{cases}$$

Then it is easy to verify that

$$(\psi * \phi)(\tau) = e^{\tau(\omega+1/p)} (B_0^{-z} w)(e^\tau),$$

for any $\tau \in \mathbb{R}_-$. Then we have

$$\begin{aligned}
\|\phi\|_{L^p((-\infty,+\infty);X)} &= \Big(\int_{-\infty}^0 \|e^{\sigma(\omega+1/p)} w(e^\sigma)\|_X^p d\sigma\Big)^{1/p} \\
&= \Big(\int_{-\infty}^0 e^{\sigma(\omega p+1)} \|w(e^\sigma)\ \|_X^p d\sigma\Big)^{1/p} \\
&= \Big(\int_0^1 \|t^\omega w(t)\|_X^p dt\Big)^{1/p} \\
&= \|w\|_{E_2}.
\end{aligned}$$

Since $\psi \in L^1(\mathbb{R})$, its Fourier transform $F(\psi)$ is

$$\begin{aligned}
F(\psi)(\xi) &= \frac{1}{\sqrt{2\pi}\Gamma(z)} \int_0^\infty e^{-i\sigma\xi} \sigma^{z-1} e^{\sigma(\omega+1/p)} d\sigma \\
&= \frac{1}{\sqrt{2\pi}\Gamma(z)} \int_0^\infty e^{-\sigma(i\xi-\omega-1/p)} \sigma^{z-1} d\sigma \\
&= \frac{1}{\sqrt{2\pi}} \big(-\frac{1}{p} - \omega + i\xi\big)^{-z}.
\end{aligned}$$

This last equality is obtained by using the curve

$$\gamma_{\epsilon,R} = \gamma^1_{\epsilon,R} \cup \gamma^2_R \cup \gamma^3_{\epsilon,R} \cup \gamma^4_{\epsilon,R},$$

with

$$\begin{gathered}
\gamma^1_{\epsilon,R} : [\epsilon, R] \longrightarrow \mathbb{R}; \sigma \mapsto \sigma, \\
\gamma^2_R : [0,\theta] \longrightarrow \mathbb{R}; \sigma \mapsto \gamma(\sigma) = R\, e^{i\sigma}, \\
\gamma^3_{\epsilon,R} : [R,\epsilon] \longrightarrow \mathbb{R}; \sigma \mapsto \sigma(-\frac{1}{p} - \omega + i\xi), \\
\gamma^4_{\epsilon,R} : [\theta, 0] \longrightarrow \mathbb{R}; \sigma \mapsto \gamma(\sigma) = \epsilon e^{i\sigma},
\end{gathered}$$

where $\theta = \arg(-\frac{1}{p} - \omega + i\xi)$. It follows that

$$\begin{aligned}
0 &= \lim_{\epsilon\to 0, R\to\infty} \int_{\gamma_{\epsilon,R}} e^{-u(i\xi-\omega-1/p)} u^{z-1} du \\
&= \int_0^\infty e^{-\sigma(i\xi-\omega-1/p)} \sigma^{z-1} d\sigma - \Gamma(z)\big(-\frac{1}{p} - \omega + i\xi\big)^{-z}.
\end{aligned}$$

Therefore,

$$\begin{gathered}
\frac{d}{d\xi} F(\psi)(\xi) = -iz(-1/p - \omega + i\xi)^{-1} F(\psi)(\xi) \\
\frac{d^2}{d\xi^2} F(\psi)(\xi) = -z(z+1)(-1/p - \omega + i\xi)^{-2} F(\psi)(\xi)
\end{gathered}$$

Now, let us put $z = \eta - ir$ with $(\eta, r) \in]0,1[\times\mathbb{R}$ and

$$\widetilde{(\psi * \phi)}(\tau) = \begin{cases} (\psi * \phi)(\tau) & \text{for } \tau \in (-\infty, 0) \\ 0 & \text{for } \tau \in (0, +\infty). \end{cases}$$

Then, the extended Mikhlin's multiplier theorem gives

$$\begin{aligned}
&\|(B_0^{-z}w)\|_{E_2}\\
&= \|\widetilde{(\psi * \phi)}\|_{L^p((-\infty,+\infty);X)}\\
&\leqslant C \max_{0\leqslant j\leqslant 2}\Big(\sup_{\xi\in\mathbb{R}}\big|\xi^j \frac{d^j}{d\sigma^j}F(\psi)(\xi)\big|\Big)\|\phi\|_{L^p(X)}\\
&\leqslant C(1+|z|^2)\sup_{\xi\in\mathbb{R}}\Big[\big((\omega+\frac{1}{p})^2+\xi^2\big)^{-\operatorname{Re} z/2}e^{(\operatorname{Im} z\arg(-\omega-\frac{1}{p}+i\xi))}\Big]\|w\|_{E_2}\\
&\leqslant C(1+\eta^2+r^2)e^{\frac{\pi}{2}|r|}\|w\|_{E_2}.
\end{aligned}$$

since $-\omega-1/p = 1-1/2p-1/p > 0$. We then deduce the following proposition.

Proposition 4.2. *For any $r \in \mathbb{R}$, B_0^{ir} is a bounded operator in E_2. Moreover $r \to B_0^{ir}$ is a strongly continuous group, and there exists a constant $C > 0$ such that*

$$\|B_0^{ir}\|_{L(E_2)} \leqslant C(1+r^2)e^{\frac{\pi}{2}|r|},$$

See A.9 in the appendix of Dore-Venni [2]. Consequently, condition (ii) of $(DV2)$ is satisfied with $\theta_B = \pi/2+\varepsilon$ for any $\varepsilon > 0$.

Now, we are concerned with the operator A and its realization defined by

$$(L\Psi) = L_0\Psi + P\Psi$$

where L_0 and P are defined as in (14) and (15) by

$$D(L_0) = \{\Psi \in W_p^2(0,1) : \Psi(0) = \Psi(1) = 0\}, \quad L_0\Psi = -\Psi'';$$

$$D(P) = W^{1,p}(0,1), \quad P\Psi = -\frac{1}{2}y\Psi'.$$

It is easy to see that the operator L_0 is sectorial. Moreover, thanks to Hölder inequality, for $\psi \in D(L_0) \subset D(P)$, we have

$$\|P\Psi\|_{L^p(0,1)} \leqslant C(p)\|\Psi\|_{D(L_0)}.$$

And the operator $P = m \circ i \circ d$ is compact from $D(L_0)$ into E_2 while L is sectorial. Furthermore, there exists some $r_0 > 0$ such that

$$\rho(-L) \supset \Sigma_{\pi-\epsilon_1} = \{\lambda :\ |\lambda| \geqslant r_0\ , |\arg\lambda| < \pi-\epsilon_1\},$$

where $\epsilon_1 \in]0,\pi/2[$. Observe that the classical second order differential equations theory with Dirichlet conditions gives, for all $\delta > 0$,

$$\rho(-(L+\delta I)) \supset \Sigma_{\pi-\epsilon_1} = \{\lambda : |\arg\lambda| < \pi-\epsilon_1\}$$

$$\forall\lambda \in \Sigma_{\pi-\epsilon_1}, \|(L_\delta+\lambda I\ )^{-1}\|_{L(E_2)} \leqslant M_\delta/(1+|\lambda|),$$

where $L_\delta = L+\delta I$. Consequently, L_δ is a positive operator.

According to Labbas-Moussaoui [12], $(L_0+\delta I)^{is}$ forms a strongly continuous group such that for any $\gamma_1 > 0$, there exists $M_0 > 0$ satisfying

$$\|(L_0+\delta I)^{is}\|_{L(E_2)} \leqslant M_0 e^{\gamma_1|s|}, \quad \forall s \in \mathbb{R}\,.$$

On the other hand, for any $\eta \in]0,1[$, $(L_0+\delta I)^\eta$ is defined and its domain $D((L_0+\delta I)^\eta)$ coincides with the complex interpolation space

$$[L^p(0,1), D(L_0+\delta I)]_\eta$$

which is contained in $W^{1,p}(0,1) = D(P)$ when η is near to 1, see Triebel [18]. Then, using Dore-Venni [4], the operator $(L_\delta)^{is}$ is bounded for all $s \in \mathbb{R}$ and

$$\|(L_\delta)^{is}\|_{L(E_2)} \leqslant M'_\delta e^{(\varepsilon+\gamma_1)|s|}, \quad \forall s \in \mathbb{R}\ \forall \varepsilon > 0\,.$$

So, we have a same result for the operator $A + \delta I$, for any $\delta > 0$.

Now, applying the same techniques used in the proofs of Theorem A1 and Lemma A2, pages 89-90 in [6], we obtain, as $\delta \to 0$,

$$\theta_A = \theta_{L_\delta} = \varepsilon + \gamma_1 \in]0, \pi/2[.$$

The condition $\theta_A + \theta_B < \pi$ is verified and, finally, A and B satisfy (DV1), (DV2) and (DV3). Then, Theorem 2.2 leads to

Proposition 4.3. *Given $h_1 \in E_2$, Problem* (4.1) *has a unique solution w satisfying*

$$w \in E_2, w(0) = 0,$$
$$t^{-1+\frac{1}{2p}} w \in L^p(0,1; W^{2,p}(0,1) \cap W_0^{1,p}(0,1)),$$
$$t^{\frac{1}{2p}} w' \in L^p(0,1; L^p(0,1)).$$

In the triangle U, using the changes $h_1(t,y) = t.g(t,y)$, $g(t,y) = f(t,x)$ with $f \in L^p(U)$, we then have $h_1 \in E_2$. Consequently, from this last Proposition, we have

(1) $w \in E_2$, that is $t^{-1}u \in L^p(U)$
(2) $t^{-1+1/2p} D_x^2 w \in L^p(0,1; L^p(0,1))$ that is $D_x^2 u \in L^p(U)$.

Thus, Equation (1.1) implies $D_t u = D_x^2 u - \lambda t^{-1} u + f \in L^p(U)$. This completes the proof of Theorem 1.2.

Note that here, we obtain a maximal regularity of the solution in the triangle when the second member is only in $L^p(U)$.

Remark 4.4. Note that our approach for solving (3.5) and (4.2) can also be used in more general situations of elliptic operators $L(t)$ and L.

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Rabah Labbas
Laboratoire de Mathématiques, Université du Havre, U.F.R. Sciences et Techniques, b.p. 540, 76058 Le Havre, France
E-mail address: labbas@univ-lehavre.fr

Ahmed Medeghri
Université de Mostaganem, b.p. 188, 27000, Mostaganem, Algeria
E-mail address: medeghri@univ-mosta.dz

Boubaker-Khaled Sadallah
Ecole Normale Supérieure, Département, de Maths, 16050 Kouba, Alger, Algeria
E-mail address: sadallah@wissal.dz