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COMPACTNESS OF THE CANONICAL SOLUTION OPERATOR ON LIPSCHITZ q-PSEUDOCONVEX BOUNDARIES

SAYED SABER

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Abstract. Let $\Omega \subset \mathbb{C}^n$ be a bounded Lipschitz q-pseudoconvex domain that admit good weight functions. We shall prove that the canonical solution operator for the $\overline{\partial}$-equation is compact on the boundary of Ω and is bounded in the Sobolev space $W^k_{r,s}(\Omega)$ for some values of k. Moreover, we show that the Bergman projection and the $\overline{\partial}$-Neumann operator are bounded in the Sobolev space $W^k_{r,s}(\Omega)$ for some values of k. If Ω is smooth, we shall give sufficient conditions for compactness of the $\overline{\partial}$-Neumann operator.

1. Introduction

Pseudoconvex domains are central objects in several complex variables analysis as they are natural domains for existence of holomorphic functions. It turns out that boundaries of domains play a leading role in the theory of several complex variables. In this article, we discuss the existence of a compact canonical solution operator $\overline{\partial}^* N$ to the $\overline{\partial}$-equation on the boundary of a Lipschitz q-pseudoconvex domain that admits a good weight function. The connection between finite type and good weight functions was first observed by Catlin [8, 9]. Straube [41] showed that Catlin's result could be used to construct useful weight functions on certain Lipschitz domains. Harrington-Zeytuncu [26] showed that on bounded Lipschitz pseudoconvex domains that admit good weight functions, the $\overline{\partial}$-Neumann operators N, $\overline{\partial}N$ and $\overline{\partial}^* N$ are bounded on L^p spaces, for some values of p greater than 2. Shaw [40] constructed a solution to the tangential Cauchy-Riemann operator $\overline{\partial}_b$ that is regular on L^2 on Lipschitz domains with plurisubharmonic defining functions. In [39], the author extended this result to Lipschitz q-pseudoconvex domains. The first main result in this article proves the compactness of this solution.

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded Lipschitz q-pseudoconvex domain and let $1 \le q \le n$. Let ρ be a defining function of Ω satisfying*

$$i\partial\overline{\partial}\rho \ge i(-\rho)\phi(-\rho)\partial\overline{\partial}|z|^2$$

on Ω, for some positive function $\phi \in C(0,\infty)$ satisfying

$$\lim_{x\to 0^+} \phi(x) = +\infty.$$

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Thus, there exists a compact solution operator $S : L^2_{r,s}(b\Omega) \cap \ker(\overline{\partial}_b) \to L^2_{r,s-1}(b\Omega)$ *such that* $\overline{\partial}_b S = I$*, for every* $s \geq q$.

When Ω has C^1-boundary and has a plurisubharmonic defining function on the boundary $b\Omega$ of Ω, Boas-Straube [5] proved that the Bergman projection maps the Sobolev space $W^k(\Omega)$ into itself for any $k > 0$. On C^2-pseudoconvex domains, Diederich-Fornaess [15] constructed a global defining function ρ so that $-(-\rho)^\alpha$ is a bounded plurisubharmonic function for some $0 < \alpha < 1$. Berndtsson-Charpentier [3] showed that in such cases the Bergman projection and the canonical solution operator $\overline{\partial}^* N$ are regular in any Sobolev space $W^k(\Omega)$, for $0 \leq k < \alpha/2$ (see also [7]). Harrington [25] showed that the result of Diederich-Fornaess and Berndtsson-Charpentier still holds when the boundary is only Lipschitz. However, Diederich-Fornaess [16] used worm domain to show that for any $0 < \alpha < 1$, one can find a smooth pseudoconvex domain where $-(-\rho)^\alpha$ is not plurisubharmonic for any global defining function ρ. Barrett [2] showed that the Bergman projection on a smooth worm domain does not map W^k into W^k for some values of k. On C^2-weakly q-convex domains, Herbig-McNeal [28] constructed a global defining function ρ so that $-(-\rho)^\alpha$ is a bounded strictly plurisubharmonic function for some $0 < \alpha < 1$. In [35], the author showed that in such cases the Bergman projection and the canonical solution operator $\overline{\partial}^* N$ are regular in any Sobolev space $W^k(\Omega)$, for $0 \leq k < \alpha/2$. The second main result in this article extends the result of Berndtsson-Charpentier to all Lipschitz q-pseudoconvex domains.

Theorem 1.2. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded Lipschitz q-pseudoconvex domain and let* $1 \leq q \leq n$. *Suppose that there exists a Lipschitz defining function* ρ *for* Ω *such that there exists some* $0 < \alpha < 1$ *with*

$$i\partial\overline{\partial}(-(-\rho)^\alpha) \geq 0 \quad \text{on } \Omega. \tag{1.1}$$

Thus, for $0 < k < \alpha/2$ *and for* $q + 1 \leq s \leq n - 1$*, the Bergman projection and the canonical solution operator for the* $\overline{\partial}$*-equation are bounded in the Sobolev space* $W^k_{r,s}(\Omega)$.

Cao-Shaw-Wang [7] extend Berndtsson-Charpentier's result to obtain estimates for the $\overline{\partial}$-Neumann operator. In [36] the author proved this result in the case of log δ-pseudoconvexity in a Kähler manifold for forms with values in a holomorphic vector bundle.

Theorem 1.3. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded Lipschitz q-pseudoconvex domain and let* $1 \leq q \leq n$. *Suppose that there exists a Lipschitz defining function* ρ *for* Ω *such that there exists some* $0 < \alpha < 1$ *satisfies* (1.1). *Thus, for* $0 < k < \alpha/2$ *and for* $q+1 \leq s \leq n-1$*, the* $\overline{\partial}$*-Neumann operator is bounded in the Sobolev space* $W^k_{r,s}(\Omega)$.

Also, we provide sufficient conditions for compactness of the $\overline{\partial}$-Neumann problem. Our motivation for studying compactness of the $\overline{\partial}$-Neumann problem comes from its connections to the geometry of the boundaries of q-pseudoconvex domains. There have been two different approaches for compactness of the $\overline{\partial}$-Neumann problem. The first is a potential theory approach. Catlin [8] introduced Property (P) and showed that it implies the compactness of the $\overline{\partial}$-Neumann problem. McNeal [32] introduced Property $(\tilde{P})$ and showed that it still implies compactness of the $\overline{\partial}$-Neumann problem. The second approach is geometric in nature. Straube [42] introduced a geometric condition that implies compactness of the $\overline{\partial}$-Neumann operator on domains in C^2. This problem was considered in [18, 19, 20, 32, 24]. Some recent work on compactness of the $\overline{\partial}$-Neumann operator, for non-pseudoconvex domains, can be found in [37, 38].

Theorem 1.4. *Let Ω be a smooth bounded q-pseudoconvex domain in $\mathbb{C}^n$ and let $1 \le q \le n$. If Ω satisfies a McNeal's Property $(\tilde{P})$, then N is compact (in particular, continuous) as an operator from $W^k_{r,s}(\Omega)$ to itself, for all $k \ge 0$ and for $s \ge q$.*

2. Preliminaries

Let $(z_1, \dots, z_n)$ be the complex coordinates for $\mathbb{C}^n$. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary and ρ be its C^2 defining function. For $0 \le r, s \le n$, an (r, s)-form u on $\overline{\Omega}$, can be expressed as

$$u = {\sum_{I,J}}' u_{I,J}\, dz^I \wedge d\bar{z}^J,$$

where $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$ are multi-indices and $dz^I = dz_{i_1} \wedge \dots \wedge dz_{i_r}$, $d\bar{z}^J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_s}$. The notation $\sum'$ means the summation over strictly increasing multi-indices. Denote by $C^\infty(\mathbb{C}^n)$ the space of complex-valued C^∞ functions on $\mathbb{C}^n$ and $C^\infty_{r,s}(\mathbb{C}^n)$ the space of complex-valued differential (r, s)-forms of class C^∞ on $\mathbb{C}^n$. Let $C^\infty_{r,s}(\overline{\Omega}) = \{u\big|_{\overline{\Omega}} : u \in C^\infty_{r,s}(\mathbb{C}^n)\}$ be the subspace of $C^\infty_{r,s}(\Omega)$ whose elements can be extended smoothly up to the boundary $b\Omega$. Let $\mathcal{D}(\mathbb{C}^n)$ be the space of C^∞-functions with compact support in $\mathbb{C}^n$. A form $u \in C^\infty_{r,s}(\mathbb{C}^n)$ is said to be has compact support in $\mathbb{C}^n$ if its coefficients belongs to $\mathcal{D}(\mathbb{C}^n)$. The subspace of $C^\infty_{r,s}(\mathbb{C}^n)$ which has compact support in $\mathbb{C}^n$ is denoted by $\mathcal{D}_{r,s}(\mathbb{C}^n)$. For $u, v \in C^\infty_{r,s}(\mathbb{C}^n)$, the local inner product (u, v) is denoted by

$$(u, v) = {\sum_{I,J}}' u_{I,J}\, \overline{v}_{I,J}.$$

Let $\phi : \mathbb{C}^n \to \mathbb{R}^+$ be a plurisubharmonic C^2-weight function and define the space

$$L^2(\Omega, \phi) = \{u : \Omega \to \mathbb{C} : \int_\Omega |u|^2 e^{-\phi} dV < \infty\},$$

where dV denotes the Lebesgue measure. Denote the inner product and the norm in $L^2(\Omega, \phi)$ by

$$\langle u, v\rangle_\phi = \int_\Omega u\overline{v}e^{-\phi} dV \quad \text{and} \quad \|u\|_\phi = \int_\Omega |u|^2 e^{-\phi} dV.$$

We also have the inner product and norm defined on the boundary:

$$\langle u, v\rangle_{b\phi} = \langle u, v\rangle_{L^2(b\Omega,\phi)} = \int_{b\Omega} u\overline{v}e^{-\phi} dS,$$
$$\|u\|_{b\phi} = \|u\|_{L^2(b\Omega,\phi)} = \int_{b\Omega} |u|^2 e^{-\phi} dS.$$

We will typically abbreviate $\langle u, v\rangle_0$ as $\langle u, v\rangle$. Recall that $L^2_{r,s}(\Omega, \phi)$ the space of (r, s)-forms with coefficients in $L^2(\Omega, \phi)$. If $u, v \in L^2_{r,s}(\Omega, \phi)$, the L^2-inner product and norms are defined by

$$\langle u,v\rangle_{\phi,\Omega}=\int_\Omega (u,v)e^{-\phi}\,dV=\int_\Omega {}^t u\wedge\star\overline{v}e^{-\phi}\quad\text{and}\quad \|u\|^2_{\phi,\Omega}=\langle u,u\rangle_{\phi,\Omega},$$

where $\star : C^\infty_{r,s}(\mathbb{C}^n)\to C^\infty_{n-s,n-r}(\mathbb{C}^n)$ is the Hodge star operator such that $\overline{\star u}=\star\overline{u}$ (that is $\star$ is a real operator) and $\star\star u=(-1)^{r+s}u$. Set

$$Q(u,u)=\|u\|^2+\|\overline{\partial}u\|^2+\|\overline{\partial}^*u\|^2.$$

For a form u, the vector of all m-th derivatives of all coefficients of u will be denoted $\nabla^m u$ (we treat ∇^0 as the identity). If ρ is the distance function for $b\Omega$, for any real number $-1\le k\le 1$ and integer $m>0$, one defines

$$\langle u,v\rangle_{W^{(k)}(\Omega)}=\int_\Omega (u,v)(\rho(z))^{-2k}\,dV,$$

$$\|u\|^2_{W^{(k)}(\Omega)}=\langle u,u\rangle_{W^{(k)}(\Omega)},$$

$$\langle u,v\rangle_{W^{(m,k)}(\Omega)}=\begin{cases}\langle\nabla^m u,\nabla^m v\rangle_{W^{(k)}(\Omega)}+\langle\nabla^{m-1}u,\nabla^{m-1}v\rangle+\langle u,v\rangle & \text{when } k\le 0,\\ \langle\nabla^m u,\nabla^m v\rangle_{W^{(k)}(\Omega)}+\langle u,v\rangle & \text{when } k>0,\end{cases}$$

$$\|u\|^2_{W^{(m,k)}(\Omega)}=\langle u,u\rangle_{W^{(m,k)}(\Omega)}.$$

The corresponding function spaces are defined by

$$W^{(k)}_{r,s}(\Omega)=\{u\in L^2_{r,s}(\Omega):\|u\|^2_{W^{(k)}(\Omega)}<\infty\},$$

$$W^{(m,k)}_{r,s}(\Omega)=\begin{cases}\{u\in W^{m-1}_{r,s}(\Omega):\|u\|^2_{W^{(m,k)}(\Omega)}<\infty\} & \text{when } k\le 0,\\ \{u\in W^{m}_{r,s}(\Omega):\|u\|^2_{W^{(m,k)}(\Omega)}<\infty\} & \text{when } k>0.\end{cases}$$

Let $a=(a_1,\dots,a_n)$ be a multi-index; that is, $a_1,\dots,a_n$ are nonnegative integers. For $x\in\mathbb{R}^n$, one defines $x^a=x_1^{a_1}\dots x_n^{a_n}$ and D^a is the operator

$$D^a=\Big(\frac{1}{i}\frac{\partial}{\partial x_1}\Big)^{a_1}\dots\Big(\frac{1}{i}\frac{\partial}{\partial x_n}\Big)^{a_n}.$$

Denote by $\mathcal{S}$ the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^n$; that is, $\mathcal{S}$ consists of all functions u which are smooth on $\mathbb{R}^n$ with $\sup_{x\in\mathbb{R}^n}|x^aD^bu(x)|<\infty$ for all multi-indices a, b. The Fourier transform $\hat{u}$ of a function $u\in\mathcal{S}$ is defined by

$$\hat{u}(\xi)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}u(x)e^{-ix\cdot\xi}\,dx,$$

where $x\cdot\xi=\sum_{j=1}^n x_j\xi_j$ and $dx=dx_1\wedge\dots\wedge dx_n$ with $x=(x_1,\dots,x_n)$ and $\xi=(\xi_1,\dots,\xi_n)$. If $u\in\mathcal{S}$, then $\hat{u}\in\mathcal{S}$. The Sobolev space $W^k(\mathbb{R}^n)$, $k\in\mathbb{R}$, is the completion of $\mathcal{S}$ under the Sobolev norm

$$\|u\|^2_{W^k(\mathbb{R}^n)}=\int_{\mathbb{R}^n}(1+|\xi|^2)^k|\hat{u}|^2\,d\xi.$$

Denote by $W^k(\Omega)$, $k\ge 0$, the space of the restriction of all functions $u\in W^k(\mathbb{C}^n)=W^k(\mathbb{R}^{2n})$ to Ω and

$$\|u\|_{W^k(\Omega)}=\inf\{\|f\|_{W^k(\mathbb{C}^n)},f\in W^k(\mathbb{C}^n),f|_\Omega=u\}$$

the $W^k(\Omega)$-norm. Denote by $W^k_0(\Omega)$ the completion of $\mathcal{D}(\Omega)$ under the $W^k(\Omega)$-norm and $W^k_{r,s}(\Omega)$, $k\in\mathbb{R}$, the Hilbert spaces of (r,s)-forms with $W^k(\Omega)$-coefficients and their norms are denoted by $\|u\|_{W^k(\Omega)}$. In addition, for any $(1,1)$-form $\Theta = \Theta_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ we have

$$(u,v)^*_\Theta = u_{iI}\Theta_{i\bar{j}}\overline{v}_{iI}.$$

The $*$ is used to emphasize that these norms are dual to the norms defined by Demailly in [13].

Let $\overline{\partial} : L^2_{r,s}(\Omega) \to L^2_{r,s+1}(\Omega)$ be the maximal closed extensions of the Cauchy-Riemann operator $\overline{\partial} : C^\infty_{r,s}(\Omega) \to C^\infty_{r,s+1}(\Omega)$ and let $\overline{\partial}^*$ be its Hilbert space adjoint. Define

$$\mathcal{H}^2(\Omega) = \{u \in L^2(\Omega) : \triangle u = 0 \text{ on } \Omega\},$$
$$\mathcal{H}^{r,s}(\Omega) = \{u \in L^2_{r,s}(\Omega) : \overline{\partial} u = \overline{\partial}^* u = 0 \text{ on } \Omega\},$$

where $\triangle$ is the real Laplacian operator. The $\overline{\partial}$-Neumann operator $N : L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega)$ is defined as the inverse of the restriction of the complex Laplacian $\square = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ to $(\mathcal{H}^{r,s}(\Omega))^\perp$. Note that N may not always exist. The Bergman projection B is the orthogonal projection from the space of square integrable functions onto the space of square integrable holomorphic functions on a domain. For any $0 \le r \le n$ and $1 \le s \le n$, denote by $B : L^2_{r,s}(\Omega) \to \ker \overline{\partial}$ the Bergman projection operator.

Definition 2.1 ([8])**.** A domain Ω has Property (P), if for every positive number M there exists a smooth plurisubharmonic function λ on $\overline{\Omega}$ such that $0 \le \lambda \le 1$ on $\overline{\Omega}$ and $i\partial\overline{\partial}\lambda \ge iM\partial\overline{\partial}|z|^2$ on the boundary $b\Omega$.

McNeal [32] defined Property $(\tilde{P})$ (a generalization of Catlin's Property (P)) as follows:

Definition 2.2. A domain Ω has the McNeal Property $(\tilde{P})$ if for every positive number M there exists $\lambda = \lambda_M \in C^2(\overline{\Omega})$ such that

(1) $|\partial\lambda|_{i\partial\overline{\partial}\lambda} \le 1$;
(2) the sum of any q eigenvalues of the matrix $\big(\frac{\partial^2\lambda}{\partial z_k \partial \bar{z}_k}\big)(z) \ge M$, for all $z \in b\Omega$.

A bounded domain is called Lipschitz if locally the boundary of the domain is the graph of a Lipschitz function. The defining function associated with a Lipschitz domain is called a Lipschitz defining function.

Definition 2.3. A bounded Lipschitz domain Ω in $\mathbb{C}^n$ is said to have a Lipschitz defining function if there exists a Lipschitz function $\rho : \mathbb{C}^n \to \mathbb{R}$ satisfies $\rho < 0$ in Ω, $\rho > 0$ outside $\overline{\Omega}$ and

$$C_1 < |d\rho| < C_2 \quad \text{a.e. on } b\Omega,$$

where C_1, C_2 are positive constants.

Lemma 2.4 ([23])**.** *Let $\Omega \subset \mathbb{C}^n$ be a bounded Lipschitz domain. For any $0 < k < \frac{1}{2}$, one obtains $W^k(\Omega) \subset W^{(k)}(\Omega)$.*

Lemma 2.5 ([30])**.** *Let $\Omega \subset \mathbb{C}^n$ be a bounded Lipschitz domain. For some constant $0 \le k \le 1$ and integer $m \ge 0$, one obtains*

$$\mathcal{H}^2(\Omega) \cap W^{m+k}(\Omega) = \mathcal{H}^2(\Omega) \cap W^{(m+1,k-1)}(\Omega).$$

Definition 2.6. Let Ω be an open domain. A function $\varphi : \Omega \to \mathbb{R}$ is called an exhaustion function for Ω if the closure of $\{x \in \Omega | \varphi(x) < c\}$ is compact for all real c.

Now, we recall the following definition of q-subharmonic functions which has been introduced by Ahn-Dieu [1] (also see [29]).

Definition 2.7. Let Ω be a bounded domain in $\mathbb{C}^n$ and let q be an integer with $1 \le q \le n$. A semicontinuous function η defined in Ω is called a q-subharmonic function if for every q-dimension space L in $\mathbb{C}^n$, $\eta|_L$ is a subharmonic function on $L \cap \Omega$. This means that for every compact subset $K \subset L \cap \Omega$ and every continuous harmonic function h on K such that $\eta \le h$ on bK, then $\eta \le h$ on K.

The function η is called strictly q-subharmonic if for every $U \subset \Omega$ there exists a constant $C_U > 0$ such that $\eta - C_U|z|^2$ is q-subharmonic.

Proposition 2.8 ([1])**.** *Let Ω be a bounded domain in $\mathbb{C}^n$ and let q be an integer with $1 \le q \le n$. Let $\eta : \Omega \to [-\infty, \infty)$ be a C^2 smooth function. Thus, the following statements are equivalent:*

1. *η is a q-subharmonic function.*
2. *For every smooth (r, s)-form $f = \sum_{I,J} f_{I,J}\, dz^I \wedge d\bar{z}^J$, and for $s \ge q$,*

$$\sum_{I,K}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z^j \partial \bar{z}^k} f_{I,jK} \overline{f}_{I,kK} \ge 0. \tag{2.1}$$

Definition 2.9. A Lipschitz domain $\Omega \subset \mathbb{C}^n$ is said to be (strictly) q-pseudoconvex if there is a (strictly) q-subharmonic exhaustion Lipschitz function on Ω.

Definition 2.10.

- (i) A C^2 smooth function u on $U \subset \mathbb{C}^n$ is called q-plurisubharmonic if its complex Hessian has at least $(n-q)$ non-negative eigenvalues at each point of U.
- (ii) An n-subharmonic function is just subharmonic function in usual sense. An upper semicontinuous function on U is plurisubharmonic exactly when it is 1-subharmonic.

Example 2.11 ([22])**.** Let $\Omega \subset \mathbb{C}^n$ be a bounded domain satisfy the $Z(q)$ condition, that is, the Levi form of a smooth defining function of Ω has, at every boundary point of Ω, at least $n - q$ positive or at least $q + 1$ negative eigenvalues. Thus Ω is strictly q-pseudoconvex.

Remark 2.12. A domain $\Omega \subset \mathbb{C}^n$ is pseudoconvex if and only if it is 1-pseudoconvex, since 1-subharmonic function is just plurisubharmonic.

Remark 2.13 ([22])**.** If $\Omega \subset \mathbb{C}^n$ is a q-pseudoconvex domain, $1 \le q \le n$, then the following hold

1. If $b\Omega$ is of class C^2, thus by (2.1), Ω is weakly q-convex;
2. if $q \le q'$, Thus q-pseudoconvexity implies q'-pseudoconvexity.

Proposition 2.14 ([22])**.** *Let Ω be a domain in $\mathbb{C}^n$ and let $1 \le q \le n$. Thus, one obtains:*

- (i) *If $\{\eta_j\}_{j=1}^{\infty}$ is a decreasing sequence of q-subharmonic functions. Thus $\eta = \lim_{j\to+\infty} \eta_j$ is a q-subharmonic function;*

(ii) *let χ be a nonnegative smooth function in $\mathbb{C}^n$ vanishing outside the unit ball and satisfying $\int_{\mathbb{C}^n} \chi dV = 1$. If f is a q-subharmonic function, one defines*

$$f_\epsilon(z) = (f * \chi_\epsilon)(z) = \int_{\mathbb{B}(0,\,\epsilon)} f(z-w)\chi_\epsilon(w)\, dV_w, \quad \forall z \in \Omega_\epsilon,$$

where $\chi_\epsilon(z) = \chi(z/\epsilon)/|\epsilon|^{2n}$ and $\Omega_\epsilon = \{z \in \Omega : d(z, b\Omega) > \epsilon\}$. Thus f_ϵ is smooth q-subharmonic on Ω_ϵ, and $f_\epsilon \downarrow f$ as $\epsilon \downarrow 0$;

(iii) *if $\eta \in C^2(\Omega)$ such that $\frac{\partial^2 \eta}{\partial z^j \partial \bar{z}^k}(z) = 0$ for all $j \neq k$ and $z \in \Omega$. Thus η is q-subharmonic if and only if $\sum_{j,k\in J} \frac{\partial^2 \eta}{\partial z^j \partial \bar{z}^k}(z) \geq 0$, for all $|J| = s$, for $s \geq q$ and for all $z \in \Omega$.*

If Ω is a bounded Lipschitz domain with distance function ρ. We equip the boundary $b\Omega$ with the induced metric from $\mathbb{C}^n$. Let $C^\infty(b\Omega)$ be the space of the restriction of all smooth functions in $\mathbb{C}^n$ to $b\Omega$. $L^2(b\Omega)$ denote the space of L^2 functions on the boundary of Ω, and $\tilde{L}^2_{r,s}(b\Omega)$ denote the space of (r,s)-forms in Ω such that the restrictions of the coefficients to $b\Omega$ are in $L^2(b\Omega)$. Fix $p \in b\Omega$. Thus for some neighborhood U of p locally choose an orthonormal coordinate patch $\{d\bar{z}_1, \dots, d\bar{z}_n\}$ defined almost everywhere in $U \cap \overline{\Omega}$ such that $d\bar{z}_n = -\overline{\partial}\rho$ a.e. Note that $|\overline{\partial}\rho| = \frac{1}{2}$ because we are using the metric where $|dz_j| = 1$, which is half the size induced by the usual Euclidean metric on $\mathbb{R}^n$. Define $L^2_{r,s}(b\Omega) \subset \tilde{L}^2_{r,s}(b\Omega)$ as the space of all $f \in \tilde{L}^2_{r,s}(b\Omega)$ such that $d\bar{z}_n \vee f = 0$ almost everywhere on $b\Omega$.

Definition 2.15. For $u \in L^2_{r,s}(b\Omega)$ and $f \in L^2_{r,s+1}(b\Omega)$, u is in $\operatorname{dom}\overline{\partial}_b$ and $\overline{\partial}_b u = f$ if

$$\int_{b\Omega} u \wedge \overline{\partial}\phi\, dS = (-1)^{r+s} \int_{b\Omega} f \wedge \phi\; dS, \quad \text{for every } \phi \in C^\infty_{n-r,n-s-1}(\mathbb{C}^n).$$

Thus u is said to be in $\operatorname{dom}\overline{\partial}_b$ and $\overline{\partial}_b u = f$.

Since $\overline{\partial}^2 = 0$, it follows that $\overline{\partial}_b^2 = 0$. Thus $\overline{\partial}_b$ is a complex and one obtains

$$0 \to L^2_{r,0}(b\Omega) \xrightarrow{\overline{\partial}_b} L^2_{r,1}(b\Omega) \xrightarrow{\overline{\partial}_b} L^2_{r,2}(b\Omega) \xrightarrow{\overline{\partial}_b} \dots \xrightarrow{\overline{\partial}_b} L^2_{r,n-1}(b\Omega) \to 0.$$

The $\overline{\partial}_b$ operator is a closed, densely defined, linear operator from $L^2_{r,s-1}(b\Omega)$ to $L^2_{r,s}(b\Omega)$, where $0 \leq r \leq n$, $1 \leq s \leq n-1$.

Definition 2.16. $\operatorname{dom}\overline{\partial}_b^*$ is the subset of $L^2_{r,s}(b\Omega)$ composed of all forms f for which there exists a constant $C > 0$ satisfies

$$|\langle f, \overline{\partial}_b u\rangle_{L^2(b\Omega)}| \leq C\|u\|_{L^2(b\Omega)},$$

for all $u \in \operatorname{dom}\overline{\partial}_b$.

For all $f \in \operatorname{dom}\overline{\partial}_b^*$, let $\overline{\partial}_b^* f$ be the unique form in $L^2_{r,s}(b\Omega)$ satisfying

$$\langle \overline{\partial}_b^* f, u\rangle_{L^2(b\Omega)} = \langle f, \overline{\partial}_b u\rangle_{L^2(b\Omega)},$$

for all $u \in \operatorname{dom}\overline{\partial}_b$. The $\overline{\partial}_b$ Laplacian operator $\Box_b = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b : \operatorname{dom}\Box_b \to L^2_{r,s}(b\Omega)$ is defined on $\operatorname{dom}\Box_b = \{u \in L^2_{r,s}(b\Omega) : u \in \operatorname{dom}\overline{\partial}_b \cap \operatorname{dom}\overline{\partial}_b^* : \overline{\partial}_b u \in \operatorname{dom}\overline{\partial}_b^*$ and $\overline{\partial}_b^* u \in \operatorname{dom}\overline{\partial}_b\}$. The $\overline{\partial}_b$ Laplacian operator is a closed, densely defined self-adjoint operator. The space of harmonic forms $\mathcal{H}_b^{r,s}(b\Omega)$ is denoted by

$$\mathcal{H}_b^{r,s}(b\Omega) = \{u \in \operatorname{dom}\Box_b : \overline{\partial}_b u = \overline{\partial}_b^* u = 0\}.$$

The space $\mathcal{H}_b^{r,s}(b\Omega)$ is a closed subspace of $\operatorname{dom}\Box_b$ since $\Box_b$ is a closed operator. The $\overline{\partial}_b$-Neumann operator $N_b : L^2_{r,s}(b\Omega) \to L^2_{r,s}(b\Omega)$ is defined as the inverse of the restriction of $\Box_b$ to $(\mathcal{H}_b^{r,s}(b\Omega))^{\perp}$.

The Bochner-Martinelli-Koppelman kernel on Lipschitz domains is defined in [27] for (r,s)-forms as follows. Define

$$(\overline{\zeta} - \overline{z}, d\zeta) = \sum_{j=1}^{n} (\overline{\zeta}_j - \overline{z}_i)\, d\zeta_j,$$
$$(d\overline{\zeta} - d\overline{z}, d\zeta) = \sum_{j=1}^{n} (d\overline{\zeta}_j - d\overline{z}_j)\, d\zeta_j,$$

where $(\zeta - z) = (\zeta_1 - z_1, \dots, \zeta_n - z_n)$, $d\zeta = (d\zeta_1, \dots, d\zeta_n)$. Thus, the Bochner-Martinelli-Koppelman kernel $K(\zeta, z)$ is defined by

$$K(\zeta, z) = \frac{1}{(2\pi i)^n} \frac{(\overline{\zeta} - \overline{z}, d\zeta)}{|\zeta - z|^2} \wedge \Big(\frac{(d\overline{\zeta} - d\overline{z}, d\zeta)}{|\zeta - z|^2} \Big)^{n-1} = \sum_{s=0}^{n-1} K_s(\zeta, z),$$

where $K_s(\zeta, z)$is the the component of $K(\zeta, z)$; that is, an (r,s) in z and of degree $(n-r, n-s)$ in ζ. When $n = 1$, $K(\zeta, z) = (2\pi i)^{-1} d\zeta/(\zeta - z)$ is the Cauchy kernel. As in the Cauchy integral case, for any $f \in L^2_{r,s}(b\Omega)$ the Cauchy principal value integral $K_b f$ is defined as

$$K_b f(z) = \lim_{\epsilon \to 0^+} \int\limits_{\substack{b\Omega \\ |\zeta - z| > \epsilon}} K_s(\zeta, z) \wedge f(\zeta),$$

whenever the limit exists. Denote by ν_z the outward unit normal to $b\Omega$ at z. Since $b\Omega$ is Lipschitz, ν_z exists almost everywhere on $b\Omega$. Thus, for $z \in b\Omega$, one defines

$$K_b^- f(z) = \lim_{\epsilon \to 0^+} \int_{b\Omega} K_s(\cdot, z - \epsilon \upsilon_z) \wedge f,$$
$$K_b^+ f(z) = \lim_{\epsilon \to 0^+} \int_{b\Omega} K_s(\cdot, z + \epsilon \upsilon_z) \wedge f.$$

The properties of the Bochner-Martinelli-Koppelman kernel and the related transforms are developed on smooth domains in [11], and on Lipschitz domains in [40]. In [23, Lemma 4.1.1] we find the following result.

Lemma 2.17. *Let Ω be a bounded domain in $\mathbb{C}^n$. Thus, for any $f \in L^2_{r,s}(b\Omega)$, one obtains*

$$\begin{aligned} K_b^- f &= \frac{1}{2} f + K_b f, \\ K_b^+ f &= -\frac{1}{2} f + K_b f, \\ f &= K_b^- f - K_b^+ f, \end{aligned} \tag{2.2}$$

almost everywhere on $b\Omega$ and

$$\|K_b f\|^2 \lesssim \|f\|^2.$$

3. A priori estimates for the the $\overline{\partial}$-Neumann operator

In this section, we find a priori estimates that we need in the later sections.

Lemma 3.1 ([43]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary and ρ be a C^2 defining function of Ω. Let σ be a real-valued function that is twice continuously differentiable on $\overline{\Omega}$, with $\sigma \geq 0$. Then, for $f \in C^\infty_{r,s}(\overline{\Omega}) \cap \operatorname{dom} \overline{\partial}^\star_\phi$ with $1 \leq s \leq n-1$, one obtains*

$$\begin{aligned}
&\|\sqrt{\sigma}\,\overline{\partial} f\|^2_\phi + \|\sqrt{\sigma}\,\overline{\partial}^*_\phi f\|^2_\phi \\
&= \sum_{I,K}\sum_{j,\,k=1}^n \int_{b\Omega} \sigma \frac{\partial^2 \rho}{\partial z^j \partial \overline{z}^k} f_{I,jK} \overline{f}_{I,kK} e^{-\phi}\, dS \\
&\quad + \sum_{I,J}\sum_{k=1}^n \int_\Omega \sigma \Big|\frac{\partial f_{I,J}}{\partial \overline{z}^k}\Big|^2 e^{-\phi} dV \\
&\quad + 2\operatorname{Re}\Big\langle \sum_{I,K}\sum_{j=1}^n \frac{\partial \sigma}{\partial z^j} f_{I,jK}\, dz^I \wedge d\overline{z}^K, \overline{\partial}^*_\phi f \Big\rangle_\phi \\
&\quad + \sum_{I,K}\sum_{j,\,k=1}^n \int_\Omega \Big(\sigma \frac{\partial^2 \phi}{\partial z^j \partial \overline{z}^k} - \frac{\partial^2 \sigma}{\partial z^j \partial \overline{z}^k}\Big) f_{I,jK} \overline{f}_{I,kK} e^{-\phi}\, dV.
\end{aligned} \tag{3.1}$$

The case $\sigma \equiv 1$ and $\phi \equiv 0$ is the classical Kohn-Morrey formula.

Proposition 3.2 ([39]). *Let $\Omega \subset \mathbb{C}^n$ be a q-pseudoconvex domain and let $1 \leq q \leq n$. Thus, for any $s \geq q$, there exists a bounded linear operator $N : L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega)$ satisfies the following properties:*

- (i) $\operatorname{range} N \subset dom\,\square$, $N\square = I$ *on* $\operatorname{dom}\square$;
- (ii) *for any $f \in L^2_{r,s}(\Omega)$, one obtains $f = \overline{\partial}\,\overline{\partial}^* N f \oplus \overline{\partial}^* \overline{\partial} N f$;*
- (iii) $\overline{\partial} N = N \overline{\partial}$ *on* $\operatorname{dom}\overline{\partial}$, $q \leq s \leq n-1$, $n \geq 2$;
- (iv) $\overline{\partial}^* N = N \overline{\partial}^*$ *on* $\operatorname{dom}\overline{\partial}^*$, $q+1 \leq s \leq n$;
- (v) *N, $\overline{\partial} N$ and $\overline{\partial}^* N$ are bounded operators with respect to the L^2-norms. That is*

$$\|Nf\| \leq \Big(\frac{e\,d^2}{s}\Big)\|f\|,$$

$$\|\overline{\partial} N f\| + \|\overline{\partial}^* N f\| \leq 2\sqrt{\frac{e\,d^2}{s}}\,\|f\|\,;$$

- (vi) *the Bergmann projection B is given by*

$$B = Id - \overline{\partial}^* N \overline{\partial}.$$

Corollary 3.3. *For every $f \in L^2_{r,s}(\Omega) \cap \ker\overline{\partial}$ and for $s \geq q$. Thus $u = \overline{\partial}^* N f$ satisfying $\overline{\partial} u = f$ in the distribution sense in $b\Omega$ with*

$$\|u\| \leq C\|f\|,$$

where C depends only on the Lipschitz constant and the diameter of Ω, but is independent of f. u is the unique solution to $\overline{\partial} u = f$ that is orthogonal to $\ker\overline{\partial}$, $u = \overline{\partial}^* N f = Sf$ is called the canonical solution operator for the $\overline{\partial}$-equation.

Lemma 3.4 ([23]). *Let $\phi \in C(0,\infty)$ such that $\phi(x) > 0$ for all $x > 0$ and*

$$\lim_{x \to 0^+} \phi(x) = \infty.$$

Thus there exists $\tilde{\phi} \in C^1(0, \infty)$ *such that*

(i) $\inf_{(0,\infty)} \phi(x) \le \tilde{\phi}(x) < \phi(x)$ *for all* $x > 0$,
(ii) $\lim_{x\to 0^+} \tilde{\phi}(x) = +\infty$,
(iii) $\lim_{x\to 0^+} \tilde{\phi}'(x) = -\infty$,
(iv) $\lim_{x\to 0^+} x\tilde{\phi}'(x) = 0$.

Lemma 3.5 ([33, Lemma 1.1])**.** *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$*. Thus* Ω *has a Lipschitz defining function* ρ*. Furthermore, the distance function to the boundary is comparable to* $|d\rho|$ *for any Lipschitz defining function* ρ *near the boundary.*

Proposition 3.6 ([23, Prop. 3])**.** *Let* $\Omega \subset \mathbb{C}^n$ *be a* C^2*-domain with a defining function* ρ *such that* $|d\rho|_{b\Omega} = 1$ *and a weight function* φ *such that* $e^{-\varphi} \in C^2(\overline{\Omega})$*. Thus for any* $g \in C^2_{r,s}(\overline{\Omega})$*,* $1 \le s \le n$*, one obtains*

$$\begin{aligned}
&\|\overline{\partial} g\|^2_\varphi + \langle \overline{\partial}\varphi \vee g, \overline{\partial}\rho \vee g\rangle_{b\varphi} \\
&= \|\vartheta g\|^2_\varphi - 2\operatorname{Re}\langle \overline{\partial}\vartheta g, g\rangle_\varphi + \|\overline{\nabla} g\|^2_\varphi + \|g\|^{*2}_{\partial\overline{\partial}\varphi,\varphi} - \|\overline{\partial}\varphi \vee g\|^2_\varphi + \|g\|^{*2}_{b\partial\overline{\partial}\rho,\varphi} \\
&\quad + \langle \overline{\partial}\rho \vee g, \overline{\partial}^* g\rangle_{b\varphi} - \langle \overline{\partial}(\overline{\partial}\rho \vee g, g\rangle_{b\varphi}
\end{aligned} \tag{3.2}$$

Lemma 3.7 ([39])**.** *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded Lipschitz* q*-pseudoconvex domain. There exists an exhaustion* $\{\Omega_\nu\}$ *of* Ω *such that*

(i) *there exists a Lipschitz function* $\rho : \mathbb{C}^n \to \mathbb{R}$ *such that* $\rho < 0$ *in* Ω*,* $\rho > 0$ *outside* $\overline{\Omega}$ *and satisfies* $C_1 < |d\rho| < C_2$ *a.e. on* $b\Omega$*;*
(ii) $\{\Omega_\nu\}$ *is an increasing sequence of relatively compact subsets of* Ω *and* $\Omega = \cup_\nu \Omega_\nu$*;*
(iii) *each* Ω_ν*,* $\nu = 1, 2, \dots,$ *is strictly* q*-pseudoconvex domains, i.e., each* Ω_ν *has a* C^∞ *strictly* q*-subharmonic defining function* ρ_ν *on a neighbourhood of* $\overline{\Omega}$*, such that*
$$\sideset{}{'}\sum_{I,K} \sum_{j,\,k} \frac{\partial^2 \rho_\nu}{\partial z^j \partial \overline{z}^k} f_{I,jK} \overline{f}_{I,kK} \ge C_0 |f|^2,$$
for $f \in C^\infty_{r,s}(\overline{\Omega}_\nu) \cap \operatorname{dom} \overline{\partial}^*_\nu$ *with* $s \ge q$ *and* $C_0 > 0$ *is independent of* ν*;*
(iv) *there exist positive constants* C_1*,* C_2 *such that* $C_1 \le |\nabla \eta_\nu| \le C_2$ *on* $b\Omega_\nu$*, where* C_1*,* C_2 *are independent of* ν*.*

The proof of the following proposition follows the ideas in Bonami-Charpentier [6] (see also [23, Theorem 3.5.1]).

Proposition 3.8. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded Lipschitz* q*-pseudoconvex domain and let* $1 \le q \le n$*. Let* ρ *be a defining function of* Ω *satisfying*
$$i\partial\overline{\partial}\rho \ge i(-\rho)\phi(-\rho)\partial\overline{\partial}|z|^2$$
on Ω*, for some positive function* $\phi \in C(0, \infty)$ *satisfying*
$$\lim_{x\to 0^+} \phi(x) = +\infty.$$
Thus, for $q + 1 \le s \le n - 1$ *and for all* $f \in W^{1/2}_{r,s}(\Omega) \cap (\ker \overline{\partial})^\perp$ *such that* $\|\overline{\partial} f\|^2_{W^{1/2}(\Omega)} < \infty$*, one obtains*
$$\|\overline{\partial}^* N f\|^2_{W^{1/2}(\Omega)} \lesssim \varepsilon \|f\|^2_{W^{1/2}(\Omega)} + C_\varepsilon \|f\|^2_{W^{-1}(\Omega)}. \tag{3.3}$$

Proof. Let Ω be a strictly q-pseudoconvex domain with smooth boundary. Let δ be the distance function of Ω. As in [10, Lemma 4.3] (see also [6]), a special extension operator on $b\Omega$ is constructed as follows. Let $f \in W^{1/2}_{r,s}(b\Omega)$ be any form on $b\Omega$ with $q+1 \le s \le n-1$, and let $\tilde f \in W^1_{r,s}(\overline{\Omega})$ be any extension of f to the interior of Ω (i.e. f is the boundary trace of $\tilde f$). One can define $T : W^{1/2}_{r,s}(b\Omega) \to L^2_{r,s+1}(\Omega)$ by

$$Tf = -2\overline{\partial}[\vartheta, N](\overline{\partial}\delta \wedge \tilde f).$$

This definition does not depend on the choice of $\tilde f$, since when $f \equiv 0$, we have $\overline{\partial}\delta \wedge \tilde f \in \operatorname{dom} \overline{\partial}^*$ and hence $[\vartheta, N](\overline{\partial}\delta \wedge \tilde f) = 0$. Clearly $\overline{\partial} Tf = 0$, and

$$\overline{\partial}\vartheta Tf = -2(\overline{\partial}\vartheta \Box N - \overline{\partial}\Box N \vartheta)(\overline{\partial}\delta \wedge \tilde f) = 0.$$

Together, these imply that $\Box Tf = 0$, so Tf must have harmonic coefficients. Using the boundary conditions for $\operatorname{dom}\Box = \operatorname{range} N$, one can also see that

$$-\overline{\partial}\delta \vee Tf|_{b\Omega} = 2\overline{\partial}\delta \vee \Box N(\overline{\partial}\delta \wedge \tilde f)|_{b\Omega} = 2\overline{\partial}\delta \vee \overline{\partial}\delta \wedge f$$

so the boundary value of $-\overline{\partial}\delta \vee Tf$ is identical to the tangential component of f. The adjoint $T^* : L^2_{r,s+1}(\Omega) \to W^{-1/2}_{r,s}(b\Omega)$ is precisely the restriction of $\overline{\partial}^* N$ to the boundary of Ω. The adjoint of the trace of the Bergman projection B is precisely $-\vartheta T$ on functions, while on forms $-\vartheta T$ will be the adjoint of the trace of $2\overline{\partial}\delta \vee \overline{\partial}\delta \wedge Bf$. The properties of T immediately give us $-\vartheta Tf \in \ker \overline{\partial} \cap \ker \vartheta$. Assume that $\overline{\partial}\rho = -|d\rho|\overline{\partial}\delta$. Then, for $g \in L^2_{r,s}(\Omega)$ and by applying (3.2) with $\rho/|d\rho|$ as our defining function and $\varphi = -\log(-\rho)$ to obtain

$$\begin{aligned} &\|\overline{\partial} g\|^2_{W^{(-1/2)}(\Omega)} + \|\,|d\rho|^{-1/2}\overline{\partial}\rho \vee g\|^2_{L^2(b\Omega)} \\ &\ge \|\vartheta g\|^2_{W^{(-1/2)}(\Omega)} - 2\operatorname{Re}\langle \overline{\partial}\vartheta g, g\rangle_{W^{(-1/2)}(\Omega)} + \|\sqrt{\varphi(-\rho)}\, g\|^2_{W^{(-1/2)}(\Omega)}. \end{aligned} \tag{3.4}$$

Applying to $g = Tf$ gives us

$$\|\,|d\rho|^{-1/2}\overline{\partial}\rho \wedge f\|^2_{L^2(b\Omega)} \ge \|\vartheta Tf\|^2_{W^{(-1/2)}(\Omega)} + \|\sqrt{\varphi(-\rho)}Tf\|^2_{W^{(-1/2)}(\Omega)}. \tag{3.5}$$

To prove (3.3), we approximate Ω as Lemma 3.7 by a sequence of subdomains $\Omega_\nu = \{\rho < -\epsilon_\nu\}$ such that each Ω_ν is strictly q-pseudoconvex domains with C^∞ smooth boundary, i.e., each Ω_ν has a C^∞ strictly q-subharmonic defining function ρ_ν such that (ii) and (iii) in Lemma 3.4. Thus, we can apply (3.4) and (3.5) on each Ω_ν. We use T_ν, T^*_ν and N_ν, to denote the corresponding operators on each Ω_ν. Then, from (3.5), one obtains

$$\|\,|d\rho_\nu|^{-1/2}\overline{\partial}\rho_\nu \wedge f\|^2_{L^2(b\Omega)} \ge \|\vartheta_\nu T_\nu f\|^2_{W^{(-1/2)}(\Omega_\nu)} + \|\sqrt{\varphi(-\rho_\nu)}\, T_\nu f\|^2_{W^{(-1/2)}(\Omega_\nu)}. \tag{3.6}$$

Passing to the limit, one obtains from (3.6) that

$$\|\,|d\rho|^{-1/2}\overline{\partial}\rho \wedge f\|^2_{L^2(b\Omega)} \ge \|\vartheta Tf\|^2_{W^{(-1/2)}(\Omega)} + \|\sqrt{\varphi(-\rho)}Tf\|^2_{W^{(-1/2)}(\Omega)}. \tag{3.7}$$

Using that for harmonic function h,

$$\|h\|^2_{W^{-1/2}(\Omega)} \gtrsim \|h\|^2_{W^{(-1/2)}(\Omega)},$$

for a proof see [10, Lemma 2.2], or [11]. Given $\varepsilon > 0$, set

$$U_\varepsilon := \{z \in \Omega : \varphi(-\rho) > \varepsilon^{-1}\}.$$

Since Tf has harmonic coefficients, we may use estimate (3.7) and interior regularity for harmonic functions to obtain

$$\|\overline{\partial}\rho \wedge f\|^2_{L^2(b\Omega)} \geq \varepsilon^{-1}\|Tf|_{U_\varepsilon}\|^2_{W^{-1/2}(\Omega)} + C_\varepsilon^{-1}\|Tf|_{U\setminus U_\varepsilon}\|^2_{W^1(\Omega)}.$$

By duality, one obtains

$$\varepsilon\|f\|^2_{W^{1/2}(\Omega)} + C_\varepsilon\|f\|^2_{W^{-1}(\Omega)} \gtrsim \|\overline{\partial}^* Nf\|^2_{L^2(b\Omega)}.$$

A result of Dahlberg (see [12]) tells us that for harmonic function h,

$$\|h\|^2_{W^{(1,-\frac{1}{2})}(\Omega)} \gtrsim \|h\|^2_{L^2(b\Omega)} \gtrsim \|\nabla h\|^2_{W^{(-1/2)}(\Omega)}.$$

Combining this with Lemma 2.4, one can show that

$$\varepsilon\|f\|^2_{W^{1/2}(\Omega)} + C_\varepsilon\|f\|^2_{W^{-1}(\Omega)} \gtrsim \|\overline{\partial}^* Nf\|^2_{W^{1/2}(\Omega)}.$$

□

4. Proof of Theorem 1.1

In this section, we use the estimates in Section 3 to construct a compact solution operator to the $\overline{\partial}_b$ operator. When the domain satisfies the additional conditions of Proposition 3.8, one can use the new jump formula for $K(\zeta, z)$, to show that we have a compact solution operator.

Let $f \in L^2_{r,s}(b\Omega) \cap \ker \overline{\partial}_b$. Choose a ball D so that $\overline{\Omega} \subset D$. Set $\Omega^+ = D \setminus \overline{\Omega}$. By [11, Lemma 9.3.5], (see also [40, Lemma 4.1]), there exist $\overline{\partial}$-closed forms

$$\begin{gathered} f^+(z) = K^+ f(z), \quad f^+(z) \in C^1_{r,s}(\overline{\Omega}^+) \subset W^1_{r,s}(\Omega^+), \\ f^-(z) = K^- f(z), \quad f^-(z) \in C^1_{r,s}(\overline{\Omega}) \subset W^1_{r,s}(\Omega), \end{gathered}$$

such that $f = f^- - f^+$ on $b\Omega$ (in the sense of traces of the coefficients, but also in the sense of restrictions of forms: i.e. the normal components of f^+ and f^- cancel each other out at points of $b\Omega$). Moreover,

$$\begin{gathered} \|f^+\|_{W^{1/2}(\Omega^+)} \leq C\|f\|_{L^2(b\Omega)}, \\ \|f^-\|_{W^{1/2}(\Omega)} \leq C\|f\|_{L^2(b\Omega)}. \end{gathered}$$

Furthermore, f^- and f^+ have harmonic coefficients with boundary values in $L^2(b\Omega)$, so they are both in $W^{1/2}$.

On Ω, one can set $u^- = \overline{\partial}^* Nf^-$, and for any $\varepsilon > 0$ we have $C_\varepsilon > 0$ such that

$$\begin{aligned} \|u^-\|^2_{W^{1/2}(\Omega)} &\leq \varepsilon\|f^-\|^2_{W^{1/2}(\Omega)} + C_\varepsilon\|f^-\|^2_{W^{-1}(\Omega)} \\ &\leq \varepsilon\|f\|^2_{L^2(b\Omega)} + C_\varepsilon\|f^-\|^2_{W^{-1}(\Omega)}, \end{aligned}$$

where we have used Proposition 3.8. Since Ω^+ is a bounded Lipschitz domain, there exists a continuous linear operator E from $W^k(\Omega^+)$ into $W^k(\mathbb{C}^n)$, for any $k \geq 0$, such that for any $g \in W^k(\Omega^+)$,

$$Eg|_{\Omega^+} = g.$$

First extend f^+ from Ω^+ to Ef^+ componentwise on D such that the following estimate holds,

$$\|Ef^+\|^2_{W^{1/2}(D)} \leq C\|f^+\|^2_{W^{1/2}(\Omega^+)}$$

(such an extension exists using [21, Theorem 1.4.3.1]). In fact, one can choose Ef^+ so that

$$\|Ef^+\|^2_{W^k(D)} \le C\|f^+\|^2_{W^k(\Omega^+)}$$

for all k. For our purposes, it suffices to know that

$$V = \begin{cases} -\star\overline{\partial}N\star\overline{\partial}Ef^+ & \text{on } \Omega,\\ 0 & \text{on } D\backslash\overline{\Omega}, \end{cases}$$

defines a form satisfying $\overline{\partial}V = \overline{\partial}Ef^+$ on $\mathbb{C}^n$ and V is supported in $\overline{\Omega}$. Because the Cauchy-Riemann equations are not affected by forms involving dz, the estimate in Proposition 3.8 is easily applied to (n,s)-forms. By applying the dual forms of these estimates, one obtains

$$\begin{aligned} \|V\|_{W^{-1/2}(\Omega)} &\le \varepsilon\|\overline{\partial}Ef^+\|^2_{W^{-1/2}(\Omega)} + C_\varepsilon\|\overline{\partial}Ef^+\|^2_{W^{-1}(\Omega)}\\ &\le \varepsilon\|f^+\|^2_{W^{1/2}(\Omega)} + C_\varepsilon\|f^+\|^2. \end{aligned}$$

Let $\tilde{f}^+ = Ef^+ - V$ so that we have a $\overline{\partial}$-closed form on all of $\mathbb{C}^n$ that satisfies $\tilde{f}^+|_{D\backslash\overline{\Omega}} = f^+$ and

$$\|\tilde{f}^+\|^2_{W^{-1/2}(\Omega)} \lesssim \varepsilon\|f\|^2_{L^2(b\Omega)} + C_\varepsilon\|f^+\|^2.$$

Set $u^+ = \overline{\partial}^* N^D\tilde{f}^+$, where N^D denotes the $\overline{\partial}$-Neumann operator for the ball D. If we pick $\chi \in C_0^\infty(D)$ such that $\chi \equiv 1$ on some neighborhood of Ω, we may use interior regularity to obtain

$$\|\chi u^+\|^2_{W^{1/2}(\Omega)} \lesssim \|\tilde{f}^+\|^2_{W^{-1/2}(\Omega)}.$$

On $b\Omega$, one defines $u = u^- - u^+$. Thus $\overline{\partial}_b u = f$ and

$$\begin{aligned} \|u\|^2_{L^2(b\Omega)} &\lesssim \|\chi u^+\|^2_{W^{1/2}(\Omega)} + \|u^-\|^2_{W^{1/2}(\Omega)}\\ &\le \varepsilon\|f\|^2_{L^2(b\Omega)} + C_\varepsilon\|f^+\|^2 + C_\varepsilon\|f^-\|^2. \end{aligned}$$

Since $\|f^+\|_{W^{1/2}(\Omega)}$ and $\|f^-\|_{W^{1/2}(\Omega)}$ are both bounded by $\|f\|_{L^2(b\Omega)}$ and $\|.\|$ is compact with respect to $\|\cdot\|_{W^{1/2}(\Omega)}$ by the Rellich lemma, the result follows.

5. Proof of Theorems 1.2 and 1.3

The proof of the regularity in the Sobolev space $W^k_{r,s}(\Omega)$ of the Bergman projection B and the canonical solution operator $\overline{\partial}^* N$ for the $\overline{\partial}$-equation is the same as in Berndtsson-Charpentier [3].

Lemma 5.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded Lipschitz q-pseudoconvex domain and let $1 \le q \le n$. Let $\delta(z) = -\rho(z)$, where ρ is C^2-defining function for Ω. Then, if we taking $\phi_\beta = -\beta\log\delta$, where $\beta \in (0,1)$ and u is any form which is orthogonal to $L^2_{r,s-1}(\Omega, e^{-\phi_\beta}) \cap \ker\overline{\partial}$, $q+1 \le s \le n-1$, one obtains u such that*

$$\int_\Omega |u|^2 e^{-\phi_\beta}\,dV \le \int_\Omega |\overline{\partial}u|^2_{i\partial\overline{\partial}\phi_\beta} e^{-\phi_\beta}\,dV. \tag{5.1}$$

Proof. By using (1.1) and by taking $\phi = -k\log\delta$, where k is a positive constant, there exists $\alpha \in (0,1)$ such that $(-\delta^\alpha)$ is strictly plurisubharmonic in Ω and

$$i\partial\phi \wedge \overline{\partial}\phi < \Big(\frac{k}{\alpha}\Big) i\partial\overline{\partial}\phi, \quad \text{on } \Omega.$$

Consequently, for $\sigma \equiv 1$, one obtains from (3.1),

$$\|u\|_\phi^2 \le \|\overline{\partial}\, u\|_\phi^2 + \|\overline{\partial}^*_\phi u\|_\phi^2,$$

for any $u \in C^\infty_{r,s}(\overline{\Omega}) \cap \operatorname{dom} \overline{\partial}^*_\phi$. Thus, by the same argument of [11, Theorem 4.3.4], for $q+1 \le s \le n-1$, for every $f \in L^2_{r,s}(\Omega, \phi)$ with $\overline{\partial} f = 0$, one can find $u \in L^2_{r,s-1}(\Omega, \phi)$ satisfies $\overline{\partial} u = f$ and

$$\int_\Omega |u|^2 e^{-\phi}\, dV \le c \int_\Omega |\overline{\partial} u|^2 e^{-\phi}\, dV. \tag{5.2}$$

One can always select the solution u of (5.2) satisfying the additional property $u \in L^2_{r,s-1}(\Omega, e^{-\phi}) \cap (\ker \overline{\partial})^\perp$, i.e., satisfies

$$\int_\Omega e^{-\phi}\, {}^t u \wedge \star\overline{\upsilon} = 0, \tag{5.3}$$

for any $\overline{\partial}$-closed form $\upsilon \in L^2_{r,s-1}(\Omega, e^{-\phi})$. Hence, if we taking $\phi_\beta = -\beta \log \delta$, where $\beta \in (0,1)$ and u is any form which is orthogonal to $L^2_{r,s-1}(\Omega, e^{-\phi_\beta}) \cap \ker \overline{\partial}$, one obtains u such that

$$\int_\Omega |u|^2 e^{-\phi_\beta}\, dV \le \int_\Omega |\overline{\partial} u|^2_{i\partial\overline{\partial}\phi_\beta} e^{-\phi_\beta}\, dV.$$

□

Proposition 5.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded Lipschitz q-pseudoconvex domain and let $1 \le q \le n$. Let $u = \overline{\partial}^*_\beta N^\beta f$ be the solution to the equation $\overline{\partial} u = f$ in $L^2_{r,s}(\Omega, \delta^\beta)$. Then, by taking $\psi_k = -k \log \delta$, $k \in (0,1)$, for $f \in L^2_{r,s}(\Omega, \delta^{\beta-k})$, $q+1 \le s \le n-1$, with $\overline{\partial} f = 0$, there exists a constant $C_1 > 0$ such that*

$$\int_\Omega |u|^2 \delta^{\beta-k} dV \le C_1 \int_\Omega |f|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)} \delta^{\beta-k} dV. \tag{5.4}$$

Proof. Since $f \in L^2_{r,s}(\Omega, \delta^\beta)$, thus by (5.2) there is a solution $u \in L^2_{r,s-1}(\Omega, \delta^\beta) \cap (\ker \overline{\partial})^\perp$. Put $g = u e^{\psi_k} = u\, \delta^{-k}$. Then

$$\int_\Omega |u|^2 \delta^{\beta-k} dV = \int_\Omega |g|^2 \delta^{\beta+k} dV. \tag{5.5}$$

Thus, from (5.3), one obtains

$$\begin{aligned} 0 = \int_\Omega e^{-\phi_\beta}\ {}^t u \wedge \star\overline{\upsilon} &= \int_\Omega e^{-(\psi_k+\phi_\beta)}\ {}^t g \wedge \star\overline{\upsilon} \\ &= \int_\Omega \delta^{\beta+k}\ {}^t g \wedge \star\overline{\upsilon}. \end{aligned}$$

Thus, g is orthogonal to all $\overline{\partial}$-closed forms of $L^2_{r,s-1}(\Omega, \delta^{\beta+k})$, so by (5.1) one obtains

$$\int_\Omega |g|^2 \delta^{\beta+k} dV \le \int_\Omega |\overline{\partial} g|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)} \delta^{\beta+k} dV.$$

Thus, from (5.5), one obtains

$$\int_\Omega |u|^2 \delta^{\beta-k} dV \le \int_\Omega |\overline{\partial} g|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)} \delta^{\beta+k} dV. \tag{5.6}$$

Since, for any two real numbers a and b, and for every $\varepsilon > 0$, one obtains

$$2|a|\,|b| \le \varepsilon|a|^2 + \frac{1}{\varepsilon}|b|^2,$$

and since $\overline{\partial} g = \delta^{-k}\overline{\partial} u + \delta^{-k}\overline{\partial}\psi_k \wedge u$. Thus, from (5.6), one obtains

$$\begin{aligned}
\int_\Omega |u|^2\delta^{\beta-k}dV &\le \int_\Omega |\overline{\partial} u + \overline{\partial}\psi_k \wedge u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV\\
&\le \int_\Omega |\overline{\partial} u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV + \int_\Omega |\overline{\partial}\psi_k \wedge u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV\\
&\quad + 2\int_\Omega |\overline{\partial} u|_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}|\overline{\partial}\psi_k \wedge u|_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV\\
&\le \Big(1+\frac{1}{\varepsilon}\Big)\int_\Omega |f|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV\\
&\quad + (1+\varepsilon)\int_\Omega |\overline{\partial}\psi_k \wedge u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV.
\end{aligned}$$

Since

$$i\partial\psi_k \wedge \overline{\partial}\psi_k < t\, i\,\partial\overline{\partial}\psi_k$$

is valid for $0 < t < 1$, the norm of the form $\overline{\partial}\psi_k$, measured in the metric with Kähler form $i\partial\overline{\partial}\psi_k$ is smaller than t at any point. Also, we can improve the estimate (5.1) by replacing $|f|_{i\partial\overline{\partial}\phi_\beta}e^{-\phi_\beta}$ by $|f|_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}e^{-\phi_\beta}$ without having to change the weight function from ϕ_β to $\psi_k + \phi_\beta$. Thus

$$|\overline{\partial}\psi_k \wedge u|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)} \le |\overline{\partial}\psi_k|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}|u|^2 \le |\overline{\partial}\psi_k|^2_{i\partial\overline{\partial}\psi_k}\,|u|^2 \le t|u|^2. \tag{5.7}$$

By choosing ε small such that $(1+\varepsilon)t < 1$, one obtains

$$\int_\Omega |u|^2\delta^{\beta-k}dV \le C_1\int_\Omega |f|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)}\delta^{\beta-k}dV$$

with $C_1 = (1+\frac{1}{\varepsilon})/[1-(1+\varepsilon)t]$. □

Proposition 5.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded Lipschitz q-pseudoconvex domain and let $1 \le q \le n$. Then, for $q+1 \le s \le n-1$, the Bergman projection B^β maps $L^2_{r,s}(\Omega, \delta^{\beta-k})$ boundedly to itself, and the operator $\overline{\partial}^*_\beta N^\beta$ maps $L^2_{r,s}(\Omega, \delta^{\beta-k})$ boundedly to itself.*

Proof. From the Kohn's formula, one obtains

$$B^\beta = Id - \overline{\partial}^*_\beta N^\beta_{r,s+1}\overline{\partial}. \tag{5.8}$$

Then, for $u \in L^2_{r,s}(\Omega, \delta^{\beta-k})$ and for $f \in L^2_{r,s}(\Omega, \delta^{\beta-k}) \cap \ker\overline{\partial}$, one obtains

$$\begin{aligned}
\langle B^\beta u, f\rangle_{\beta,\Omega} &= \langle u - \overline{\partial}^*_\beta N^\beta\overline{\partial} u, f\rangle_{\beta,\Omega}\\
&= \langle u, f\rangle_{\beta,\Omega} - \langle\overline{\partial}^*_\beta N^\beta\overline{\partial} u, f\rangle_{\beta,\Omega}\\
&= \langle \delta^{-k}u, f\rangle_{\beta+k,\Omega}\\
&= \langle \delta^{-k}u, f\rangle_{\beta+k,\Omega} - \langle\overline{\partial}^*_{\beta+k}N^{\beta+k}\overline{\partial}(\delta^{-k}u), f\rangle_{\beta+k,\Omega}\\
&= \langle (I - \overline{\partial}^*_{\beta+k}N^{\beta+k}\overline{\partial})(\delta^{-k}u), f\rangle_{\beta+k,\Omega}\\
&= \langle B^{\beta+k}(\delta^{-k}u), f\rangle_{\beta+k,\Omega}\\
&= \langle \delta^k B^{\beta+k}(\delta^{-k}u), f\rangle_{\beta,\Omega}.
\end{aligned}$$

Thus

$$B^\beta(\delta^k B^{\beta+k}(\delta^{-k}u)) = B^\beta u.$$

Using (5.8), one obtains

$$\begin{aligned} B^\beta u &= B^\beta(\delta^k B^{\beta+k}(\delta^{-k}u)) \\ &= (I - \overline{\partial}^*_\beta N^\beta \overline{\partial})\delta^k B^{\beta+k}(\delta^{-k}u) \\ &= \delta^k B^{\beta+k}(\delta^{-k}u) - \overline{\partial}^*_\beta N^\beta(\overline{\partial}\delta^k \wedge B^{\beta+k}(\delta^{-k}u)) \\ &= \delta^k B^{\beta+k}(\delta^{-k}u) - k\,\overline{\partial}^*_\beta N^\beta\Big(\frac{\overline{\partial}\delta}{\delta} \wedge \delta^k B^{\beta+k}(\delta^{-k}u)\Big), \end{aligned} \tag{5.9}$$

because $\overline{\partial}\, B^{\beta+k} = 0$.

For simplicity, write $\xi = \delta^k B^{\beta+k}(\delta^{-k}\, u)$, for $u \in L^2_{r,s}(\Omega, \delta^{\beta-k})$. Then, one obtains

$$\begin{aligned} \int_\Omega |\xi|^2 \delta^{\beta-k}\, dV &= \int_\Omega |\delta^k\, B^{\beta+k}(\delta^{-k}\, u)|^2 \delta^{\beta-k}\, dV \\ &= \int_\Omega |B^{\beta+k}(\delta^{-k}\, u)|^2 \delta^{\beta+k}\, dV \\ &\le \int_\Omega |\delta^{-k}u|^2 \delta^{\beta+k}\, dV \\ &= \int_\Omega |u|^2 \delta^{\beta-k}\, dV. \end{aligned} \tag{5.10}$$

Thus, from (5.4), one obtains

$$\int_\Omega \big|\overline{\partial}^*_\beta N^\beta(\overline{\partial}\psi_k \wedge \xi)\big|^2 \delta^{\beta-k}\, dV \le C_1 \int_\Omega |\overline{\partial}\psi_k \wedge \xi|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)} \delta^{\beta-k}\, dV. \tag{5.11}$$

From (5.7), one obtains

$$|\overline{\partial}\psi_k \wedge \xi|^2_{i\partial\overline{\partial}(\psi_k+\phi_\beta)} \le |\overline{\partial}\psi_k \wedge \xi|^2_{i\partial\overline{\partial}\psi_k} \le t|\xi|^2. \tag{5.12}$$

Substituting (5.10) and (5.12) into (5.11), one obtains

$$\int_\Omega |\overline{\partial}^*_\beta N^\beta(\overline{\partial}\psi_k \wedge \xi)|^2 \delta^{\beta-k}\, dV \le C_1 t \int_\Omega |u|^2 \delta^{\beta-k}\, dV. \tag{5.13}$$

Thus, by using (5.9), (5.10) and (5.13), one obtains

$$\|B^\beta\, u\|^2_{\beta-k,\Omega} \le C_2 \|u\|^2_{\beta-k,\Omega}. \tag{5.14}$$

Thus, the Bergman projection B^β maps $L^2_{r,s}(\Omega, \delta^{\beta-k})$ boundedly to itself. Since $B^\beta u = (I - \overline{\partial}^*_\beta N^\beta \overline{\partial})u$ and $\overline{\partial}^*_\beta N^\beta u = N^\beta \overline{\partial}^*_\beta u$, then $\overline{\partial}^*_\beta N^\beta u = \overline{\partial}^*_\beta N^\beta B^\beta u$ and we already know that B^β is bounded on $L^2_{r,s}(\Omega, \delta^{\beta-k})$ we may as well assume from the start that $\overline{\partial} f = 0$. Then, by using (5.4) and (5.14), one obtains

$$\|\overline{\partial}^*_\beta N^\beta u\|^2_{\beta-k,\Omega} = \|\overline{\partial}^*_\beta N^\beta B^\beta u\|^2_{\beta-k,\Omega} \le C_1 \|B^\beta u\|^2_{\beta-k,\Omega} \le C_1 C_2 \|u\|^2_{\beta-k,\Omega}.$$

Thus, the operator $\overline{\partial}^*_\beta N^\beta$ maps $L^2_{r,s}(\Omega, \delta^{\beta-k})$ boundedly to itself. □

Proposition 5.4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded Lipschitz q-pseudoconvex domain and let $1 \le q \le n$. Then, for $k_0 \in (0,1)$, the Bergman projection B and the operator $\overline{\partial}^* N$ are exact regular in $W^k_{r,s}(\Omega)$ for $0 < k < k_0/2$ and for $q+1 \le s \le n-1$.*

Proof. By Theorem [21, 1.4.4.3], the space $W^k_{r,s}(\Omega)$ is continuously embedded into $L^2_{r,s}(\Omega,\delta^{-2k})$. Since any harmonic function in $L^2_{r,s}(\Omega,\delta^{-2k})$ also lies in $W^k_{r,s}(\Omega)$, under the same assumptions (see [21, Theorem 4.2] together with [14, Lemma 1]). Consider the case of the Bergman projection B on a holomorphic function. Let f be a harmonic function in $W^k(\Omega)$. Then, by the embedding result, f belongs to $L^2(\Omega,\delta^{-2k})$, so by applying Proposition 5.3 with $\beta=0$, Bf belongs to $L^2(\Omega,\delta^{-2k})$. Since Bf is holomorphic, hence harmonic, it follows that Bf belongs to $W^k(\Omega)$. Next, let f be a (r,s)-form in $W^k_{r,s}(\Omega)$, with $q+1\le s\le n-1$. Thus, by the embedding result $f\in L^2_{r,s}(\Omega,\delta^{-2k})$, so by applying Proposition 5.3 with $\beta=0$, $Bf\in L^2_{r,s}(\Omega,\delta^{-2k})$. Note that

$$\overline{\partial}Bf=0 \quad\text{and}\quad \overline{\partial}^*Bf=\overline{\partial}^*f.$$

Hence $\Box Bf$, which as a differential operator is the Laplacian on each component of f satisfies

$$\Box Bf=\overline{\partial\partial}^*f.$$

Since $f\in W^k_{r,s}(\Omega)$, $f=\Box g$ with $g\in W^{k+2}_{r,s}(\Omega)$. (This follows since by [21, Theorem 1.4.3.1] f can be extended to a form with compact support in $W^k_{r,s}(\Omega)$ so we may take g to be the Newtonian potential of this extension.) Hence

$$\Box Bf=\overline{\partial}\,\overline{\partial}^*f=\Box v$$

with $v\in W^k_{r,s}(\Omega)$. Let $w=Bf-v$ so that w is a form with harmonic coefficients. Since both Bf and v lie in $L^2_{r,s}(\Omega,\delta^{-2k})$ by the embedding theorem, so does w. Since w has harmonic coefficients, then w lies in $W^k_{r,s}(\Omega)$, so Bf also belongs to $W^k_{r,s}(\Omega)$ in any degree.

It is only remains to prove that if f is a (r,s)-form in $W^k_{r,s}(\Omega)$ then $u=\overline{\partial}^*Nf$ is also in $W^k_{r,s}(\Omega)$. Since $\overline{\partial}u=f$ and $\overline{\partial}^*u=0$. Thus

$$\Box u=(\overline{\partial}\,\overline{\partial}^*+\overline{\partial}^*\overline{\partial})u=\overline{\partial}^*f\in W^{k-1}_{r,s}(\Omega).$$

By [30, Theorem 0.5] this implies that one can solve $\Box g=\Box u$ with $g\in W^{k+1}_{r,s}(\Omega)\subset W^k_{r,s}(\Omega)$. By the embedding theorem both g and f into $L^2_{r,s}(\Omega,\delta^{-2k})$, so by applying Proposition 5.3 with $\beta=0$, u and $u-g$ also belongs to $L^2_{r,s}(\Omega,\delta^{-2k})$. Since $u-g$ has harmonic coefficients, it follows that $u-g$ lies in $W^k_{r,s}(\Omega)$ and so u lies in $W^k_{r,s}(\Omega)$. □

Corollary 5.5. *For $k_0\in(0,1)$, the $\overline{\partial}$-Neumann operator N is exact regular in the Sobolev space $W^k_{r,s}(\Omega)$ for $0<k<k_0/2$ and for $q+1\le s\le n-1$.*

Proof. By a result of Boas-Straube [4], the $\overline{\partial}$-Neumann operator N is regular if and only if the Bergman projection B is. Thus the exact regularity of N follows. □

Proposition 5.6. *Let $\Omega\subset\mathbb{C}^n$ be a bounded Lipschitz q-pseudoconvex domain and let $1\le q\le n$. Then, for $q-1\le s\le n-1$, the operators N, $\overline{\partial}^*N$ and B are exact regular in the Sobolev space $W^{\pm k}_{r,s}(\Omega)$ for $0<k<k_0/2$ and $s\ge q$.*

Proof. If $\mathcal{S}^*$ is the adjoint map of $\mathcal{S}$ with respect to the L^2-norm, then

$$\|\mathcal{S}f\|_{W^{k/2}_{r,s}(\Omega)}=\sup_{g\in L^2}\frac{\langle\mathcal{S}f,g\rangle_\Omega}{\|g\|_{W^{k/2}_{r,s}(\Omega)}}$$

$$= \sup_{g\in L^2} \frac{\langle f, \mathcal{S}^* g\rangle_\Omega}{\|g\|_{W^{-k/2}_{r,s}(\Omega)}}$$
$$\le \|\mathcal{S}^*\|_{W^{-k/2}_{r,s}(\Omega)} \|f\|_{W^{k/2}_{r,s}(\Omega)}\,.$$

Then, using Corollary 5.5, the proof follows. □

6. Proof of Theorem 1.4 and some consequences

In this section, we shall provide sufficient conditions for compactness of the $\overline{\partial}$-Neumann problem. As in [37], one can prove the following result.

Proposition 6.1. *Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded q-pseudoconvex domain. Let $\psi, \varphi \in C^2(\overline{\Omega})$ with $\psi \ge 0$. Thus, for $f \in C^\infty_{r,s}(\overline{\Omega}) \cap \operatorname{dom} \overline{\partial}^*_\varphi$ with $q \le s \le n$, we have*

$$\begin{aligned} &\|\sqrt{\psi}\,\overline{\partial} f\|^2_\varphi + \big(1+\frac{1}{\tau}\big)\|\sqrt{\psi}\,\overline{\partial}^*_\varphi f\|^2_\varphi \\ &\ge \sum_{I,J}{}' \sum_{k=1}^n \int_\Omega \psi \Big|\frac{\partial f_{I,J}}{\partial \overline{z}^k}\Big|^2 e^{-\varphi}\, dV - \sum_{I,K}{}' \int_\Omega \tau \Big|\frac{1}{\sqrt{\psi}} \sum_{j=1}^n \frac{\partial \psi}{\partial z^j} f_{I,jK}\Big| e^{-\varphi} \\ &\quad + \sum_{I,K}{}' \sum_{j,k=1}^n \int_\Omega \Big(\psi \frac{\partial^2 \varphi}{\partial z^j \partial \overline{z}^k} - \frac{\partial^2 \psi}{\partial z^j \partial \overline{z}^k}\Big) f_{I,jK} \overline{f}_{I,kK} e^{-\varphi}\, dV, \end{aligned} \tag{6.1}$$

for any positive number τ.

Proposition 6.2. *Let Ω be a smooth bounded q-pseudoconvex domain in $\mathbb{C}^n$ and let $1 \le q \le n$. If Ω satisfies a McNeal's Property $(\tilde{P})$, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\|f\|^2 \le \varepsilon(\|\overline{\partial} f\| + \|\overline{\partial}^* f\|) + C_\varepsilon \|f\|^2_{W^{-1}(\Omega)}, \tag{6.2}$$

for $f \in \operatorname{dom} \overline{\partial} \cap \operatorname{dom} \overline{\partial}^*$.

Proof. As in [32, Theorem 4.1], let $\epsilon > 0$ and choose $M \ge \frac{24}{\epsilon}$: For λ_M given by Definition 2.2, set $\varphi = \lambda_M$, $\psi = e^{-\lambda_M}$ and $\tau = \frac{1}{2}$ in (6.1). It follows that

$$\frac{1}{2}\int_\Omega i\partial\overline{\partial}\lambda(f,f) e^{-2\lambda} \le \|\overline{\partial} f\|^2_{2\lambda} + 3\|\overline{\partial}^*_\lambda f\|^2_{2\lambda}, \tag{6.3}$$

for $f \in \mathcal{D}(\Omega)$. Let $G_\mu = \{z \in \mathbb{C}^n : -\mu < \rho(z) \le 0\}$ be a strip near $b\Omega$, with $M > 0$ chosen small enough so that

$$i\partial\overline{\partial}\lambda(z)(f,f) \ge \frac{M}{2}\|f\|^2, \quad z \in G_\mu.$$

It follows, from (6.3), that

$$\frac{M}{2}\int_{G_\mu} |f|^2 e^{-\lambda} \le \|\overline{\partial} f\|^2_\lambda + \|\overline{\partial}^*_\lambda f\|^2_\lambda,$$

when f is supported in the strip G_μ. Since λ is continuous, $\|\cdot\|_{2\lambda}$ is equivalent to the L^2-norm and it follows that

$$\frac{M}{2}\int_{G_\mu} |f|^2 \le \|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2, \tag{6.4}$$

when f is supported in the strip G_μ.

Estimate the integral over $\Omega\backslash G_\mu$ and choose $\gamma_\mu \in \mathcal{D}(\Omega)$ so that $\gamma_\mu(z) = 1$ whenever $\rho(z) \le -\mu$ and $z \in \Omega\backslash G_\mu$. By an interpolation theorem in Sobolev space, we have for a constant $m > 0$ still to be determined the inequality

$$\|\gamma_\mu f\|^2 \le m\|\gamma_\mu f\|^2_{W^1(\Omega)} + \frac{1}{m}\|\gamma_\mu f\|^2_{W^{-1}(\Omega)}. \tag{6.5}$$

Also, since Q is elliptic, by Gårding's inequality, one obtains

$$\begin{aligned}\|\gamma_\mu f\|^2_{W^1(\Omega)} &\le Q(\gamma_\mu f, \gamma_\mu f)\\ &\le \Big(\|\gamma_\mu(\overline{\partial} f)\|^2 + \|\gamma_\mu(\overline{\partial}^* f)\|^2 + \|[\gamma_\mu, \overline{\partial}]\, f\|^2 + \|[\gamma_\mu, \overline{\partial}^*] f\|^2 + \|\gamma_\mu f\|^2\Big)\\ &\le \|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2 + C_\mu\|f\|^2.\end{aligned} \tag{6.6}$$

Because the sum of the commutator terms is bounded by $C_\mu\|f\|^2$ for some constant C_μ dependent of μ, then from (6.5) and (6.6), for a suitable choice of b small, one obtains

$$\|\gamma_\mu f\|^2 - \frac{1}{2}\|f\|^2 \le b(\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2) + \frac{1}{b}\|\gamma_\mu f\|^2_{W^{-1}(\Omega)}. \tag{6.7}$$

By combining (6.4) and (6.7), one obtains

$$\begin{aligned}\frac{1}{2}\|f\|^2 &\le \int_{G_\mu} |f|^2\, dV + \|\gamma_\mu f\|^2 - \frac{1}{2}\|f\|^2\\ &\le \Big(\frac{1}{M} + b\Big)Q(f,f) + \frac{1}{M}\|f\|^2 + \frac{1}{b}\|\gamma_\mu f\|^2_{W^{-1}(\Omega)}.\end{aligned}$$

For M large enough, we obtain

$$\|f\|^2 \le 3\big(\frac{1}{M} + b\big)Q(f,f) + \frac{3}{b}\|\gamma_\mu f\|^2_{W^{-1}(\Omega)}.$$

For any $\epsilon > 0$, if we choose M and b so that $(\frac{1}{M} + b) < \epsilon$ and set $C_\epsilon = \sqrt{\frac{3}{b}}\,\gamma_\mu$, one gets (6.2). $\square$

We will refer to (6.2) as a global compactness estimate. Compactness of the $\overline{\partial}$-Neumann problem can be formulated in several useful ways.

Proposition 6.3. *Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded q-pseudoconvex domain and let $1 \le q \le n$. Thus, for $s \ge q$, the following statements are equivalent:*

- (i) *the $\overline{\partial}$-Neumann operators N, is compact from $L^2_{r,s}(\Omega)$ to itself;*
- (ii) *the embedding of the space* $\operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}^*$*, provided with the graph norm $\|f\| + \|\overline{\partial} f\| + \|\overline{\partial}^* f\|$, into $L^2_{r,s}(\Omega)$ is compact;*
- (iii) *the validity of global compactness estimate* (6.2)*;*
- (iv) *the canonical solution operators to $\overline{\partial}$ given by $\overline{\partial}^* N : L^2_{r,s}(\Omega) \to L^2_{r,s-1}(\Omega)$ and $N\overline{\partial}^* : L^2_{r,s+1}(\Omega) \to L^2_{r,s}(\Omega)$ are compact.*

Proof. The equivalence of (ii) and (iii) is a result of [31, Lemma 1.1]. The general L^2-theory and the fact that $L^2_{r,s}(\Omega)$ embeds compactly into $W^{-1}_{r,s}(\Omega)$ shows that (i) is equivalent to (ii) and (iii). Finally, the equivalence of (i) and (iv) follows from the formula

$$N = (\overline{\partial}^* N)^* \overline{\partial}^* N + \overline{\partial}^* N(\overline{\partial}^* N)^*.$$

(see [17], [34, p.55], [32]). $\square$

Lemma 6.4. *Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded q-pseudoconvex domain and let $1 \le q \le n$. Let $\{U_j\}_{j=1}^N$ be a finite covering of $b\Omega$ by a local patching. If compactness estimates hold in each U_j:*

$$\|f\|^2 \le c\,Q(f,f) + C\|f\|^2_{W^{-1}},$$

for $f \in C^\infty_{r,s}(\overline{\Omega} \cap U_j) \cap \operatorname{dom} \overline{\partial}^*$. Thus we have global compactness estimate (6.2).

As in [31], one can prove the following theorem.

Theorem 6.5. *Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded q-pseudoconvex domain and let $1 \le q \le n$. If N is compact on $L^2_{r,s}(\Omega)$ and for $s \ge q$, N is compact (in particular, continuous) as an operator from $W^k_{r,s}(\Omega)$ to itself, for all $k \ge 0$.*

Remark 6.6. If N is a compact operator on $W^k_{r,s}(\Omega)$ for some $k \ge 0$, thus N is compact in $L^2_{r,s}(\Omega)$.

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References

[1] H. Ahn, N. Q. Dieu; *The Donnelly-Fefferman theorem on q-pseudoconvex domains*, Osaka J. Math., 46 (2009), 599-610.

[2] David E. Barrett; *Behavior of the Bergman projection on the Diederich-Fornæss worm*, Acta Math., 168 (1992), 1-10.

[3] B. Berndtsson, Ph. Charpentier; *A Sobolev mapping property of the Bergman kernel*, Math. Zeitschrift, 235 (2000), 1-10.

[4] H. P. Boas, E. J. Straube; *Equivalence of regularity for the Bergman projection and the $\overline{\partial}$-Neumann operator*, Manuscripta Math., 67 (1990), 25-33.

[5] H. P. Boas, E. J. Straube; *Sobolev estimates for the $\overline{\partial}$-Neumann operator on domains in $\mathbb{C}^n$ admitting a defining function that is plurisubharmonic on the boundary*, Math. Z., 206 (1991), 81-88.

[6] A. Bonami, P. Charpentier; *Boundary values for the canonical solution to $\overline{\partial}$-equation and $W^{1/2}$ estimates*, preprint, Bordeaux, (1990).

[7] J. Cao, M. C. Shaw, L. Wang; *Estimates for the $\overline{\partial}$-Neumann problem and nonexistence of C^2 Levi-flat hypersurfaces in $\mathbb{P}^n$*, Math. Z., 248 (2004), 183-221.

[8] D. Catlin; *Global regularity of the $\overline{\partial}$-Neumann problem*, Complex analysis of several variables, (Madison, Wis., 1982), Proc. Sympos. Pure Math., 41, Amer. Math. Soc., Providence, RI, (1984), 39-49.

[9] D. Catlin; *Subelliptic estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains*, Annals of Math.. 126 (1987), 131-191.

[10] P. Charpentier; *Sur les valeurs au bord de solutions de l'equation $\overline{\partial}u = f$.* in Analyse Complexe Multivariable: récents développements (Guadeloupe, 1988), volume 5 of Sem. Conf., pages 51-81. Edit El, Rende, (1991).

[11] S.-C. Chen, M.-C. Shaw; *Partial differential Equations in several complex variables*, AMS/IP Stud. Adv. Math. 19, Amer. Math. Soc., Providence, RI, (2001).

[12] B. E. J. Dahlberg; *Weighted norm inequalities for the Lusin area integral and the nontangential maximal functions for functions harmonic in a Lipschitz domain*, Studia Math., 67 (1980), 297-314.

[13] J.-P. Demailly; *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe sémi-positif*, Ann. of Scient. Ec. Norm. Sup, 15 (1982), 457-511.

[14] J. Detraz; *Classes de Bergman de fonctions harmoniques*, Bull. Soc. Math. France, 109 (1981), 259-268.

[15] K. Diederich, Fornaess; *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math., 39 (1977), 129-141.

[16] K. Diederich, Fornaess; *Pseudoconvex domains: an example with nontrivial Nebenhülle*, Math. Ann., 225 (1977), 275-292.

[17] G. B. Folland, J. J. Kohn; *The Neumann Problem for the Cauchy-Riemann Complex*, Princeton University Press, Princeton, NJ, (1972), Annals of Mathematics Studies, No. 75.

[18] S. Fu, E. J. Straube; *Compactness of the $\overline{\partial}$-Neumann problem on convex domains*, J. Funct. Anal., 159, no. 2 (1998), 629-641.

[19] S. Fu, E. J. Straube; *Compactness in the $\overline{\partial}$-Neumann problem*, Complex Analysis and Geometry (Columbus, OH, 1999), Ohio State Univ. Math. Res. Inst. Publ., vol. 9, de Gruyter, Berlin, 2001, 141-160.

[20] S. Fu, E. J. Straube; *Semi-classical analysis of Schrödinger operators and compactness in the $\overline{\partial}$-Neumann problem*, J. Math. Anal. Appl. 271(1) (2002), 267-282.

[21] P. Grisvard; *Elliptic problems in nonsmooth domains*, Pitman, Boston 1985.

[22] L. M. Hai, N. Q. Dieu, N. X. Hong; *L^2-Approximation of differential forms by $\overline{\partial}$-closed ones on smooth hypersurfaces*, J. Math. Anal. Appl., 383 (2011), 379-390.

[23] P. S. Harrington; *Compactness and subellipticity for the $\overline{\partial}$-Neumann operator on domains with minimal smoothness*, Ph.D. thesis, University of Notre Dame, (2004).

[24] Phillip S. Harrington; *Global regularity for the $\overline{\partial}$-Neumann operator and bounded plurisubharmonic exhaustion functions*, Advances in Mathematics, 228 (2011), 2522–2551.

[25] Phillip S. Harrington; *The order of plurisubharmonicity on pseudoconvex domains with Lipschitz boundaries*, Math. Res. Lett., 15(3) (2008), 485–490.

[26] Phillip S. Harrington, Yunus E. Zeytuncu; *L^p mapping properties for the Cauchy-Riemann equations on Lipschitz domains admitting subelliptic estimates*, Complex Variables and Elliptic Equations An International Journal, 64(3) (2019), 369–385.

[27] R. Harvey, J. Polking; *Fundamental solutions in complex analysis, I. the Cauchy-Riemann operator*, Duke Math. J., 46 (1979), 253–300.

[28] A.-K. Herbig, J. D. McNeal; *Regularity of the Bergman projection on forms and plurisubharmonicity Conditions*, Vienna, Preprint ESI 1731 (2005), 1-21.

[29] L. Ho; *$\overline{\partial}$-problem on weakly q-pseudoconvex domains*, Mathematische Annalen, 290 (1991), 3-18.

[30] D. Jerison, C. E. Kenig; *The inhomogeneous Dirichlet problem in Lipschitz domains*, Journal of Functional Anal., 130 (1995), 161-219.

[31] J. J. Kohn, L. Nirenberg; *Non-coercive boundary value problems*, Comm. Pure Appl. Math., 18 (1965), 443-492.

[32] J. D. McNeal; *A sufficient condition for compactness of the $\overline{\partial}$-Neumann operator*, J. Funct. Anal., 195 (1) (2002), 190-205.

[33] J. Michel and M. Shaw; *The $\overline{\partial}$-Neumann operator on Lipschitz pseudoconvex domains with plurisubharmonic defining functions*, Duke Math. J. 108, 3 (2001), 421-447.

[34] R. M. Range; *The $\overline{\partial}$-Neumann operator on the unit ball in $\mathbb{C}^n$*, Math. Ann., 266(4) (1984), 449-456.

[35] S. Saber; *Solution to $\overline{\partial}$ problem with exact support and regularity for the $\overline{\partial}$-Neumann operator on weakly q-convex domains*, International Journal of Geometric Methods in Modern Physics, 7(1) (2010), 135-142.

[36] S. Saber; *Sobolev regularity of the Bergman projection on certain pseudoconvex domains*, Transactions of A. Razmadze Mathematical Institute, 171 (2017), 90-102.

[37] S. Saber; *Compactness of the commutators of Toeplitz operators on q-pseudoconvex domains*, Electron. J. Differential Equations, Vol. 2018 no. 100 (2018), 1-12.

[38] S. Saber; *Compactness of the weighted dbar-Neumann operator and commutators of the Bergman projection with continuous functions*, Journal of Geometry and Physics, 138 (2019), 194-205.

[39] S. Saber; *L^2 estimates and existence theorems for $\overline{\partial}_b$ on boundaries of Lipschitz q-pseudoconvex domains*, Submitted.

[40] M.-C. Shaw; *L^2 estimates and existence theorems for $\overline{\partial}_b$ on Lipschitz boundaries*, Math. Z., 244, (2003), 91-123.

[41] E. Straube; *Plurisubharmonic functions and subellipticity of the $\overline{\partial}$-Neumann problem on nonsmooth domains*, Math. Res. Lett., 4 (1997), 459-467.

[42] E. Straube; *Geometric conditions which imply compactness of the $\overline{\partial}$-Neumann operator*, Ann. Inst. Fourier (Grenoble), 54 (3) (2004), 699-710.

[43] E. J. Straube; *Lectures on the L^2-Sobolev Theory of the $\overline{\partial}$-Neumann Problem*, ESI Lectures in Mathematics and Physics, February (2010).

Sayed Saber
Department of Mathematics, Faculty of Science and Arts in Baljurashi, Albaha University, Albaha, Saudi Arabia.
Mathematics Department, Faculty of Science, Beni-Suef University, Egypt
Email address: sayedkay@yahoo.com