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EXISTENCE AND GLOBAL ATTRACTIVITY POSITIVE PERIODIC SOLUTIONS FOR A DISCRETE MODEL

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Abstract. Using a fixed point theorem in cones, we obtain conditions that guarantee the existence and attractivity of the unique positive periodic solution for a discrete Lasota-Wazewska model.

1. Introduction

Wazewska-Czyzewska and Lasota [10] investigated the delay differential equation

$$x'(t) = -\alpha x(t) + \beta e^{-\gamma x(t-\tau)}, \quad t \geq 0.$$

as a model for the survival of red blood cells in an animal. The oscillation and global attractivity of this equation have been studied by Kulenovic and Ladas [9]. A few similar generalized model were investigated by many authors, see Xu and Li [12], Graef et al. [4], Jiang and Wei [8], Gopalsamy and Trofimchuk [3]. Recently, Liu [2] studied the existence and global attractivity of unique positive periodic solution for the Lasota-Wazewska model

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{m} p_i(t)e^{-q_i(t)x(t-\tau_i(t))},$$

by using a fixed point theorem, and got some brief conditions to guarantee the conclusions. In [7], the existence of one positive periodic solution was proved by Mawhin's continuation theorem. In [13], the existence of multiple positive periodic solutions was studied by employing Krasnoselskii fixed point theorem in cones.

Though most models are described with differential equations, the discrete-time models are more appropriate than the continuous ones when the size of the population is rarely small or the population has non-overlapping generations [1]. To our knowledge, studies on discrete models by using fixed point theorem are scarce, see [13]. In this paper, we consider the Lasota-Wazewska difference equation

$$\Delta x(k) = -a(k)x(k) + \sum_{i=1}^{m} p_i(k)e^{-q_i(k)x(k-\tau_i(k))}. \tag{1.1}$$

We will use the following hypotheses:

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(H1) $a : Z \to (0,1)$ is continuous and ω-periodic function. i.e., $a(k) = a(k+\omega)$, such that $a(k) \not\equiv 0$, where $\omega$is a positive constant denoting the common period of the system.

(H2) p_i and q_i are positive continuous ω-periodic functions, τ_i are continuous ω-periodic functions $(i = 1, 2, \dots)$.

For convenience, we shall use the notation:

$$\bar{h} = \max_{0\le k\le\omega}\{h(k)\}, \quad \underline{h} = \min_{0\le k\le\omega}\{h(k)\}.$$

where h is a continuous ω-periodic function. Also, we use

$$q = \max_{1\le i\le m}\{\bar{q}_i\}, \quad \tau = \max_{1\le i\le m}\{\bar{\tau}_i\}, \quad p = \omega\sum_{i=1}^{m} p_i(s), \quad (k \le s \le k+\omega-1),$$

$$A = \frac{\prod_{s=0}^{\omega-1}(1-a(s))}{1-\prod_{s=0}^{\omega-1}(1-a(s))},$$

$$B = \frac{1}{1-\prod_{s=0}^{\omega-1}(1-a(s))}, \quad \sigma = \prod_{s=0}^{\omega-1}(1-a(s)) = \frac{A}{B} < 1.$$

Considering the actual applications, we assume the solutions of (1.1) with initial condition

$$x(k) = \phi(k) > 0 \quad \text{for } -\tau \le k \le 0.$$

To prove our result, we state the following concepts and lemmas.

Definition. Let X be Banach space and P be a closed, nonempty subset, P is said to be a cone if

(i) $\lambda x \in P$ for all $x \in P$ and $\lambda \ge 0$

(ii) $x \in P, -x \in P$ implies $x = \theta$.

The semi-order induced by the cone P is denoted by " $\le$ ". That is, $x \le y$ if and $y - x \in P$.

Definition. A cone P of X is said to be normal if there exists a positive constant δ, such that $\|x + y\| \ge \delta$ for any $x, y \in P$. $\|x\| = \|y\| = 1$.

Definition. Let P be a cone of X and $T : P \to P$ an operator. T is called decreasing, if $\theta \le x \le y$ implies $Tx \ge Ty$.

Lemma 1.1 (Guo [5, 6]). *Suppose that*

(i) *P is normal cone of a real Banach space X and $T : P \to P$ is decreasing and completely continuous;*

(ii) *$T\theta > \theta$, $T^2 \ge \varepsilon_0 T\theta$, where $\varepsilon_0 > 0$;*

(iii) *For any $\theta < x \le T\theta$ and $0 < \lambda < 1$, there exists $\eta = \eta(x, \lambda) > 0$ such that*

$$T(\lambda x) \le [\lambda(1+\eta)]^{-1}Tx. \tag{1.2}$$

Then T has exactly one positive fixed point $\tilde{x} > \theta$. Moreover, constructing the sequence $x_n = Tx_{n-1}$ $(n = 1, 2, 3, \dots)$ for any initial $x_0 \in P$, it follows that $\|x_n - \tilde{x}\| \to 0$ as $n \to \infty$.

2. Positive periodic solutions

To apply Lemma 1.1, let $X = \{x(k) : x(k) = x(k+\omega)\}$, $\|x\| = \max\{| x(k) | : x \in X\}$. Then X is a Banach space endowed with the norm $\|\cdot\|$.

Define the cone

$$P = \{x \in X : x(k) \geq 0,\ x(k) \geq \sigma\|x\|\}.$$

Lemma 2.1. *If $x(k)$ is a positive ω-periodic solution of* (1.1)*,then $x(k) \geq \sigma\|x\|$.*

Proof. It is clear that (1.1) is equivalent to

$$x(k+1) = (1-a(k))x(k) + \sum_{i=1}^{m} p_i(k)e^{-q_i(k)x(k-\tau_i(k))}.$$

Multiplying the two sides by $\prod_{s=0}^{k}(1-a(s))^{-1}$, we have

$$\Delta\Big(x(k)\prod_{s=0}^{k-1}\frac{1}{1-a(s)}\Big) = \prod_{s=0}^{k}\frac{1}{1-a(s)}\sum_{i=1}^{m} p_i(k)e^{-q_i(k)x(k-\tau_i(k))}.$$

Summing the two sides from k to $k+\omega-1$,

$$x(k) = \sum_{s=k}^{k+\omega-1} G(k,s)\sum_{i=1}^{m} p_i(s)e^{-q_i(s)x(k-\tau_i(k))}. \tag{2.1}$$

where

$$G(k,s) = \frac{\prod_{r=s+1}^{k+\omega-1}(1-a(r))}{1-\prod_{r=0}^{\omega-1}(1-a(r))}, \quad k \leq s \leq k+\omega-1.$$

Then, $x(k)$ is an ω-periodic solution of (1.1) if and only if $x(k)$ is ω-periodic solution of difference equation (2.1). It is easy to calculate that

$$A := \frac{\prod_{s=0}^{\omega-1}(1-a(s))}{1-\prod_{s=0}^{\omega-1}(1-a(s))} \leq G(k,s) \leq \frac{1}{1-\prod_{s=0}^{\omega-1}(1-a(s))} =: B,$$
$$A = \frac{\sigma}{1-\sigma}, \quad B = \frac{1}{1-\sigma}, \quad \sigma = \frac{A}{B} < 1,$$
$$\|x\| \leq B\sum_{S=k}^{k+\omega-1}\sum_{i=1}^{m} p_i(s)e^{-q_i(s)x(s-\tau_i(s))},$$
$$x(t) \geq A\sum_{S=k}^{k+\omega-1}\sum_{i=1}^{m} p_i(s)e^{-q_i(s)x(s-\tau_i(s))}).$$

Therefore, $x(k) \geq \frac{A}{B}\|x\| = \sigma\|x\|$. $\square$

Define the mapping $T : X \to X$ by

$$(Tx)(k) = \sum_{s=k}^{k+\omega-1} G(k,s)\sum_{i=1}^{m} p_i(s)e^{-q_i(s)x(k-\tau_i(k))}, \tag{2.2}$$

for $x \in X$, $k \in Z$. It is not difficult to see that T is a completely continuous operator on X, and a periodic solution of (1.1) is the fixed point of operator T.

Lemma 2.2. *Under the conditions above, $TP \subset P$.*

Proof. For each $x \in P$, we have

$$\|Tx\| \leq B\sum_{s=k}^{k+\omega-1} G(k,s)\sum_{i=1}^{m} p_i(s)e^{-q_i(s)x(k-\tau_i(k))}$$

From (2.2), we obtain

$$Tx \geq A \sum_{s=k}^{k+\omega-1} G(k,s) \sum_{i=1}^{m} p_i(s) e^{-q_i(s)x(k-\tau_i(k))} \geq \frac{A}{B}\|Tx\| = \sigma\|Tx\|.$$

Therefore, $Tx \in P$, thus $TP \subset P$. $\square$

Lemma 2.3. *$x(k)$ is positive and bounded on $[0, \infty)$.*

Proof. Obviously, $x(k)$ is defined on $[-\tau, +\infty)$ and positive on $[0, +\infty)$. Now, we prove that every solution of (1.1) is bounded, otherwise, there exists an unbounded solution $x(k)$. Thus, for arbitrary $M > Bm\omega\overline{p}/e^{\underline{q}M}$, there exists $N = N(M)$, when $k > N, x(k) > M$. From (2.1), we have

$$x(k) \leq B \sum_{s=k}^{k+\omega-1} \sum_{i=1}^{m} \overline{p} e^{-\underline{q}M} = Bm\omega\overline{p}/e^{\underline{q}M} < M.$$

where

$$\underline{q} = \min_{1\leq i\leq m}\{\underline{q_i}\}, \quad \overline{p} = \max_{1\leq i\leq m}\{\bar{p_i}\},$$

which is a contradiction. Consequently, $x(k)$ is bounded. $\square$

Now, we are in position to state the main results in this section.

Theorem 2.4. *Assume that (H1)-(H2) hold and $Bpq \leq 1$. Then (1.1) has a unique ω-periodic positive solution $\tilde{x}(t)$. Moreover,*

$$\|x(k) - \tilde{x}\| \to 0 (k \to \infty) m$$

where $x(k) = Tx(k-1)(k = 1, 2, \dots)$ for any initial $x_0 \in P$.

Proof. Firstly, it is clear that the cone P is normal. By an easy calculation, we know that T is decreasing, in fact

$$\begin{aligned}
&(Tx)(k) - (Ty)(k) \\
&= \sum_{s=k}^{k+\omega-1} G(k,s) \sum_{i=1}^{m} p_i(s)(e^{-q_i(s)x(s-\tau_i(s))} - e^{-q_i(s)y(s-\tau_i(s))}) \\
&= \sum_{s=k}^{k+\omega-1} G(k,s) \sum_{i=1}^{m} p_i(s) e^{-q_i(s)x(s-\tau_i(s))}[1 - e^{-q_i(s)(y(s-\tau_i(s))-x(s-\tau_i(s)))}] \geq 0
\end{aligned}$$

when $\theta \leq x \leq y$, i.e., $y(s - \tau_i(s)) - x(s - \tau_i(s)) \geq 0$.

Secondly, we will show that the condition (ii) of Lemma 1.1 is satisfied. From (2.2), we have

$$Bp \geq (T\theta)(k) = \sum_{s=k}^{k+\omega-1} G(k,s) \sum_{i=1}^{m} p_i(s) \geq Ap > 0.$$

Thus, $T\theta > \theta$, and

$$\begin{aligned}(T^2\theta)(k) &= \sum_{s=k}^{k+\omega-1} G(k,s)\sum_{i=1}^{m} p_i(s)e^{-q_i(s)(T\theta)(s-\tau_i(s))}\\ &\geq e^{-Bpq}\sum_{s=k}^{k+\omega-1} G(k,s)\sum_{i=1}^{m} p_i(s)\\ &= e^{-Bpq}(T\theta)(k).\end{aligned}$$

So that $T^2\theta \geq \varepsilon_0 T\theta$, where $\varepsilon_0 = e^{-Bpq} > 0$.

Finally, we prove that the condition (iii) of Lemma 1.1 is also satisfied. For any $\theta < x < T\theta$ and $0 < \lambda < 1$, we have $\|x\| \leq \|T\theta\| \leq Bp$ and

$$\begin{aligned}T(\lambda x)(k) &= \sum_{s=k}^{k+\omega-1} G(k,s)\sum_{i=1}^{m} p_i(s)e^{-\lambda q_i(s)x(s-\tau_i(s))}\\ &= \sum_{s=k}^{k+\omega-1} G(k,s)\sum_{i=1}^{m} p_i(s)e^{-q_i(s)x(s-\tau_i(s))}e^{(1-\lambda)q_i(s)x(s-\tau_i(s))} \qquad (2.3)\\ &\leq e^{(1-\lambda)Bpq}(Tx)(k)\\ &= \lambda^{-1}\lambda e^{(1-\lambda)Bpq}(Tx)(k).\end{aligned}$$

Set $f(\lambda) = \lambda e^{Bpq(1-\lambda)}$; therefore, $f'(\lambda) = (1-Bpq\lambda)e^{Bpq(1-\lambda)} > 0$ for $\lambda \in (0,1)$. Thus $0 < f(\lambda) < f(1) = 1$. so set $f(\lambda) = (1+\eta)^{-1}$, where $\eta = \eta(\lambda) > 0$. From (2.3), we have

$$T(\lambda x) \leq \lambda^{-1} f(\lambda)Tx = \lambda^{-1}(1+\eta)^{-1}Tx = [\lambda(1+\eta)]^{-1}Tx.$$

By Lemma 1.1, we see that T has exactly one positive fixed point $\tilde{x} > \theta$. Moreover, $\|x(k) - \tilde{x}\| \to 0(n \to \infty)$, where $x(k) = Tx(k-1)(k = 1,2,\dots)$ for any initial $x_0 \in P$ for $k \in N$. □

Remark 2.5. Theorem 2.4 not only gives the sufficient conditions for the existence of unique positive periodic solution of (1.1), but also contains the conclusion of convergence of $x(k)$ to $\tilde{x}$.

Remark 2.6. From the statements above, we have

$$\tilde{x}(k) = (T\tilde{x})(k) = \sum_{s=k}^{k+\omega-1} G(k,s)\sum_{i=1}^{m} p_i(s)e^{-q_i(s)\tilde{x}(s-\tau_i(s))} \geq Ape^{-q\|\tilde{x}\|} > 0, \qquad (2.4)$$

$$Ape^{-Bpq} \leq \tilde{x}(k) \leq Bp \geq 0. \qquad (2.5)$$

which will be used in the following section.

3. Global attractivity of the solution to (1.1)

Theorem 3.1. *Assume that (H1)-(H2) hold and $Bpq \leq 1$. Then the unique ω-periodic solution $\tilde{x}(k)$ of* (1.1) *is a global attractor of all other positive solutions of* (1.1).

Proof. Let $y(k) = x(k) - \tilde{x}(k)$, where $x(k)$ is arbitrary solution of (1.1), Then it is easy to obtain

$$\begin{aligned} \triangle y(k) &= \triangle(x(k) - \tilde{x}(k)) \\ &= \triangle x(k) - \triangle \tilde{x}(k) \\ &= -a(k)y(k) + \sum_{i=1}^{m} p_i(s) e^{-q_i(s)\tilde{x}(s-\tau_i(s))}(e^{-q_i(s)y(s-\tau_i(s))} - 1). \end{aligned} \tag{3.1}$$

Now, we shall prove $\lim_{k\to\infty} y(k) = 0$ in the following three cases:
Case 1. Suppose that $y(t)$ is eventually positive solution of (3.1). It is easy to see that $\triangle y(k) < 0$ for all sufficiently large k, so $\lim_{k\to\infty} y(k) = l \geq 0$. we claim that $l = 0$. If $l > 0$, then there exists $N > 0$ such that $\triangle y(k) < -la(k), k \geq N$. Summing the two sides of the inequality from N to ∞, we have

$$l - y(N) = \sum_{k=N}^{\infty} \triangle y(k) < -l \sum_{k=N}^{\infty} a(k) = -\infty.$$

which is a contradiction, so $l = 0$.
Case 2. Suppose that $y(k)$ is eventually negative. By similar proof as above we obtain that $l = 0$.
Case 3. Suppose that $y(k)$ is oscillatory, from Lemma 2.3, we know $y(k)$ is bounded. We set

$$\lim_{k\to\infty} \sup y(k) = c \geq 0 \quad \text{and} \quad \lim_{k\to\infty} \inf y(k) = d \leq 0. \tag{3.2}$$

For arbitrarily small positive constant ϵ, $d - \epsilon < 0$ and $c + \epsilon > 0$. In view of (3.2), there exists $N_\epsilon > 0$, such that

$$d - \epsilon < y(k) < c + \epsilon \quad \text{for all } k \geq N_\epsilon - \tau. \tag{3.3}$$

From (3.1)and (3.3),we have

$$y(k+1) - (1 - a(k))y(k) = \sum_{i=1}^{m} p_i(s) e^{-q_i(s)\tilde{x}(s-\tau_i(s))}(e^{-q_i(s)y(s-\tau_i(s))} - 1).$$

Multiplying the two sides by $\prod_{s=0}^{k}(1 - a(s))^{-1}$, we have

$$\begin{aligned} &\triangle(y(k) \prod_{s=0}^{k-1} \frac{1}{1 - a(s)}) \\ &= \prod_{s=0}^{k} \frac{1}{1 - a(s)} \sum_{i=1}^{m} p_i(s) e^{-q_i(s)\tilde{x}(s-\tau_i(s))}(e^{-q_i(s)y(s-\tau_i(s))} - 1) \\ &\leq (e^{-q(d-\epsilon)} - 1) \prod_{s=0}^{k} \frac{1}{1 - a(s)} \sum_{i=1}^{m} p_i(s) e^{-q_i(s)\tilde{x}(s-\tau_i(s))} \\ &= (e^{-q(d-\epsilon)} - 1) \triangle(\tilde{x}(k) \prod_{s=0}^{k-1} \frac{1}{1 - a(s)}). \end{aligned} \tag{3.4}$$

Summing the two sides from N_ϵ to∞, for $k \geq N_\epsilon$, we have

$$\begin{aligned} &y(k+1)\prod_{s=0}^{k}\frac{1}{1-a(s)} - y(N_\epsilon)\prod_{s=0}^{N_\epsilon-1}\frac{1}{1-a(s)} \\ &\leq (e^{-q(d-\epsilon)}-1)[\tilde{x}(k+1)\prod_{s=0}^{k}\frac{1}{1-a(s)} - \tilde{x}(N_\epsilon)\prod_{s=0}^{N_\epsilon-1}\frac{1}{1-a(s)}]. \end{aligned} \tag{3.5}$$

Thus

$$y(k+1) \leq y(N_\epsilon)\prod_{s=N_\epsilon}^{k}(1-a(s)) + (e^{-q(d-\epsilon)}-1)[\tilde{x}(k+1) - \tilde{x}(N_\epsilon)\prod_{s=N_\epsilon}^{k}(1-a(s))]. \tag{3.6}$$

From (3.2), (3.6) and Remark 2.6, we have

$$c \leq Bp(e^{-q(d-\epsilon)}-1).$$

As ϵ is arbitrary small, we have that

$$c \leq Bp(e^{-qd}-1). \tag{3.7}$$

By the similar method as above, we obtain

$$d \geq Bp(e^{-qc}-1). \tag{3.8}$$

From results in [2, 12], $Bpq \leq 1$ implies that (3.7), (3.8) have a unique solution $c = d = 0$. Therefore,

$$\lim_{k\to\infty} y(k) = \lim_{k\to\infty}[x(k) - \tilde{x}(k)] = 0.$$

The proof is complete. □

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