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BRESSE SYSTEMS WITH LOCALIZED KELVIN-VOIGT DISSIPATION

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Abstract. We study the effect of localized viscoelastic dissipation for curved beams. We consider a circular beam with three components, two of them viscous with constitutive laws of Kelvin-Voigt type, one continuous and the other discontinuous. The third component is elastic without any dissipative mechanism. Our main result is that the rate of decay depends on the position of each component. More precisely, we prove that the model is exponentially stable if and only if the viscous component with discontinuous constitutive law is not in the center of the beam. We prove that when there is no exponential stability, the solution decays polynomially.

1. Introduction

We study a Bresse system that describes the vibrations of a circular arc beam with three components positioned over intervals $I_1 =]0, \ell_0[$, $I_2 =]\ell_0, \ell_1[$, $I_3 =]\ell_1, \ell[$. We denote by $\widetilde{I} = I_1 \cup I_2 \cup I_3$. More precisely, we consider the system

$$\rho_1 \varphi_{tt} - S_x - lN = 0 \quad \text{in } \widetilde{I} \times (0, +\infty), \tag{1.1}$$

$$\rho_2 \psi_{tt} - M_x + S = 0 \quad \text{in } \widetilde{I} \times (0, +\infty), \tag{1.2}$$

$$\rho_1 w_{tt} - N_x + lS = 0 \quad \text{in } \widetilde{I} \times (0, +\infty), \tag{1.3}$$

where S, M and N stand for the shear force, bending moment, and axial force, respectively. The constitutive law we use here are:

$$S = \kappa(\varphi_x + \psi + lw) + \widetilde{\kappa}(\varphi_{xt} + \psi_t + lw_t),$$

$$M = b\psi_x + \tilde{b}\psi_{xt},$$

$$N = K(w_x - l\varphi) + \widetilde{K}(w_{xt} - l\varphi_t),$$

where w, φ and ψ are the longitudinal, vertical, and shear angle displacements. Here, $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa_0 = EA$, $\kappa = k'GA$, $b = EI$, and $l = R^{-1}$, where ρ is the density of the beam, E the elastic modulus, G the shear modulus, k' the shear factor, A is the cross-sectional area, I the second moment of area of the cross-section and R is the radius of curvature of the beam. Here we assume that all the above coefficients are positive constants. When the curvature l is zero, the model reduces

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to the well-known Timoshenko beam system. Therefore we can consider the Bresse system as an extension of the Timoshenko's model.

The functions $\tilde{\kappa}$, $\tilde{b}$, and $\tilde{K}$ are non negative of the following type:

$$\tilde{\kappa} = \kappa_0 + \kappa_1, \quad \tilde{b} = b_0 + b_1, \quad \tilde{K} = K_0 + K_1,$$

where κ_0, b_0 and K_0 are discontinuous functions over $]0, \ell[$, positive over I_D and vanishing outside I_D. Also κ_1, b_1 and K_1 are C^1 functions over $]0, \ell[$, positive over I_C and vanishing outside I_C. Figures 1–4 depict typical examples of the functions $\tilde{\kappa}$, $\tilde{b}$, and $\tilde{K}$.

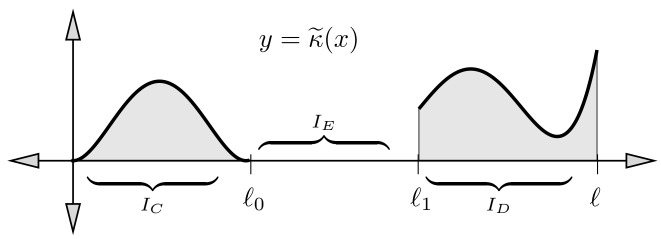

FIGURE 1. The discontinuous component is not centered.

In Figure 1, the discontinuous viscous material is not in the center, so there is only one discontinuity point in ℓ_1. In Figure 2 the discontinuous viscous material is in the center, we have two points of discontinuity, one at ℓ_0 and the second at ℓ_1.

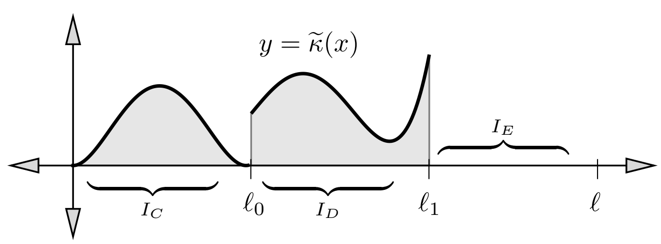

FIGURE 2. The discontinuous component is centered.

Here we assume that there exist positive constants c, C_1, and C_2, such that

$$|b_1'|^2 \le c|b_1|, \quad |\kappa_1'|^2 \le c|\kappa_1|, \quad |K_1'|^2 \le c|K_1|, \tag{1.4}$$

$$C_1\kappa_1 \le b_1 \le C_2\kappa_1. \tag{1.5}$$

The initial conditions

$$\begin{gathered} \varphi(\cdot,0) = \varphi_0, \quad \varphi_t(\cdot,0) = \varphi_1, \quad \psi(\cdot,0) = \psi_0, \\ \psi_t(\cdot,0) = \psi_1, \quad w(\cdot,0) = w_0, \quad w_t(\cdot,0) = w_1 \end{gathered} \tag{1.6}$$

are given over $(0, \ell)$ and we consider the Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(\ell,t) = \psi(0,t) = \psi(\ell,t) = w(0,t) = w(\ell,t) = 0 \quad \text{in } (0, +\infty). \tag{1.7}$$

Additionally, we have the transmission conditions

$$\varphi(\ell_i^-) = \varphi(\ell_i^+), \quad \psi(\ell_i^-) = \psi(\ell_i^+), \quad w(\ell_i^-) = w(\ell_i^+), \tag{1.8}$$

$$S(\ell_i^-) = S(\ell_i^+), \quad M(\ell_i^-) = M(\ell_i^+), \quad N(\ell_i^-) = N(\ell_i^+). \tag{1.9}$$

for $i = 0, 1$. Note that condition (1.9) implies $S, M, N \in H^1(0, \ell)$. If we have more points of discontinuity, the set $\widetilde{I}$ needs to be modified.

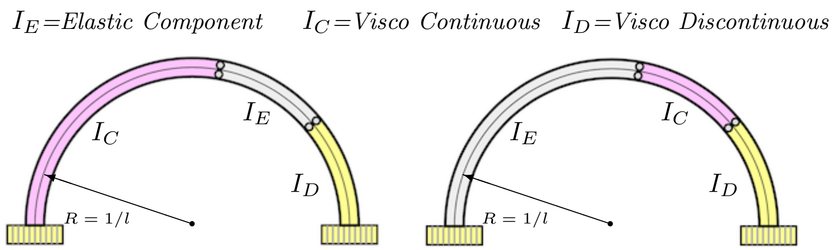

FIGURE 3. Two possible positions for the components of the beam, in which the discontinuous part is not centered.

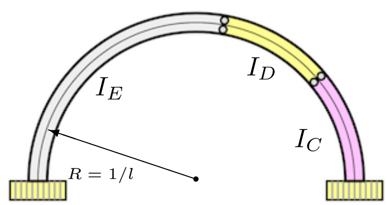

FIGURE 4. In this case, the discontinuous part is in the center of the beam.

It seems to us that the first study about the lack of the exponential stability to localized viscoelastic system is due to Liu and Liu [13], where the authors proved that the wave equation with localized Kelvin-Voigt viscoelastic damping (with discontinuous constitutive law) is not exponentially stable. See also Cheng, Liu, and Liu [6], and Tiang and Zhang [21] for similar results for Timoshenko's model. On the other hand, Liu and Rao in [14] proved that when the localized viscoelastic damping has a C^1-constitutive law, then the corresponding semigroup is exponentially stable. Therefore, the regularity of the constitutive law of localized viscoelastic damping completely changes the asymptotic properties. Concerning Timoshenko system we have the work [16] where the authors consider the transmission problem of Timoshenko's beam composed by N components, each of them being either purely elastic (**E**), or a Kelvin-Voigt viscoelastic material (discontinuous constitutive law **V**), or an elastic material inserted with a frictional damping mechanism (**F**). The authors prove that Timoshenko's model is exponentially stable if and only if all the elastic components are connected with one component with frictional damping. Otherwise, there is no exponential stability, but a polynomial decay of the energy as $1/t^2$. In [2] the authors obtained similar results for Timoshenko's model.

Here we consider two types of localized viscoelastic damping, one with continuous constitutive law and the other with discontinuous constitutive law. We prove that the exponential stability depends on the order of the viscoelastic components of the beam. That is, we show that the semigroup is exponentially stable if and only if the discontinuous component is not in the center of the beam. Furthermore, in case of lack of exponential stability, we show that the semigroup decays polynomially to zero.

The remainder part of this paper is organized as follows. In section 2 we establish the well-posedness of the model. In section 3 we show the exponential stability provided the discontinuous component is not in the center of the beam. Finally in section 4 we show the lack of exponential stability.

2. Semigroup approach

The energy associated with system (1.1)-(1.3) is

$$\begin{aligned} E(t)=\frac{1}{2}\int_0^\ell &\rho_1|\varphi_t|^2+\rho_2|\psi_t|^2+\rho_1|w_t|^2+b|\psi_x|^2+\kappa|\varphi_x+\psi+lw|^2 \\ &+K|w_x-l\varphi|^2\,dx. \end{aligned}\tag{2.1}$$

Multiplying equations (1.1) by φ_t, (1.2) by ψ_t, and (1.3) by w_t, and summing up these products we arrive at

$$\frac{d}{dt}E(t)=-\int_0^\ell b_0|\psi_{xt}|^2\,dx+\kappa_0|\varphi_{xt}+\psi_t+lw_t|^2+K_0|w_{xt}-l\varphi_t|^2\,dx\le 0. \tag{2.2}$$

The definition of the energy motivates us to define the phase space $\mathcal{H}$ as

$$\mathcal{H}=H_0^1(0,\ell)\times L^2(0,\ell)\times H_0^1(0,\ell)\times L^2(0,\ell)\times H_0^1(0,\ell)\times L^2(0,\ell)$$

which is a Hilbert space with the norm

$$\|U\|_{\mathcal{H}}^2=\int_0^\ell \rho_1|\Phi|^2+\rho_2|\Psi|^2+\rho_1|W|^2+b|\psi_x|^2+\kappa|\varphi_x+\psi+lw|^2+K|w_x-l\varphi|^2\,dx \tag{2.3}$$

where $U=(\varphi,\Phi,\psi,\Psi,w,W)^t$. Because of the Dirichlet boundary condition, the above norm is equivalent to the usual norm of $\mathcal{H}$. Denoting by $\varphi_t=\Phi$, $\psi_t=\Psi$, $w_t=W$, system (1.1)-(1.3) can be written as

$$U_t-\mathcal{A}U=0,\quad U(0)=U_0 \tag{2.4}$$

where

$$\mathcal{A}U=\big(\Phi\ ,\ \frac{1}{\rho_1}(S_x+lN)\ ,\ \Psi\ ,\ \frac{1}{\rho_2}(M_x-S)\ ,\ W\ ,\ \frac{1}{\rho_1}(N_x-lS)\big)$$

and

$$D(\mathcal{A})=\big\{U\in\mathcal{H}:\Phi,\Psi,W\in H_0^1,\ \kappa\varphi_x+\tilde\kappa\Phi_x,\ b\psi_x+\tilde b\Psi_x,\ Kw_x+\tilde KW_x\in H^1\big\}. \tag{2.5}$$

Note that the operator $\mathcal{A}$ is dissipative and

$$\operatorname{Re}\langle\mathcal{A}U,U\rangle=-\int_0^\ell\tilde b|\Psi_x|^2\,dx-\int_0^\ell\tilde\kappa|\Phi_x+\Psi+lW|^2\,dx-\int_0^\ell\tilde K|W_x-l\Phi|^2\,dx\le 0.$$

The resolvent system $\lambda U-\mathcal{A}U=F$ with $F=(f_1,f_2,f_3,f_4,f_5,f_6)^t\in\mathcal{H}$ in terms of its components is given by

$$\lambda\varphi-\Phi=f_1,\quad \rho_1\lambda\Phi-S_x-lN=f_2\quad\text{in } \widetilde{I}\times(0,+\infty), \tag{2.6}$$

$$\lambda\psi-\Psi=f_3,\quad \rho_2\lambda\Psi-M_x+S=f_4\quad\text{in } \widetilde{I}\times(0,+\infty), \tag{2.7}$$

$$\lambda w-W=f_5,\quad \rho_1\lambda W-N_x+lS=f_6\quad\text{in } \widetilde{I}\times(0,+\infty). \tag{2.8}$$

Following standard procedures (see [1, 8, 20]) we can show that $0\in\rho(\mathcal{A})$, which implies that $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup.

Theorem 2.1. *Under the above conditions the operator $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contraction. Hence, for any $U_0 \in \mathcal{H}$ there exists only one mild solution $U \in C(0,T;\mathcal{H})$ of problem (2.4). Also if $U_0 \in D(\mathcal{A})$, there exists only one classic solution $U \in C(0,T;D(\mathcal{A})) \cap C^1(0,T;\mathcal{H})$ of problem (2.4).*

3. Asymptotic behavior

The main objective of this section is the characterization of the exponential and polynomial stabilization of the system. The main tool are theorems due to Prüss [18], Huang [10], Gearhart[9], and Borichev and Tomilov [5].

Theorem 3.1. *Let $S(t)$ be a contraction C_0-semigroup, generated by $\mathcal{A}$ over a Hilbert space $\mathcal{H}$. Then, in Prüss [18] is established that there exists $C, \gamma > 0$ satisfying*

$$\|S(t)\| \le Ce^{-\gamma t} \ \Leftrightarrow\ i\mathbb{R} \subset \varrho(\mathcal{A}) \ \textit{and}\ \|(i\lambda\, I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant M, \quad \forall \lambda \in \mathbb{R}. \tag{3.1}$$

For polynomial stability, Borichev and Tomilov [5] result establish that there exists $C > 0$ such that

$$\|\mathcal{S}(t)\mathcal{A}^{-1}\| \leqslant \frac{C}{t^{1/\alpha}} \ \Leftrightarrow\ i\mathbb{R} \subset \varrho(\mathcal{A}) \ \textit{and}\ \|(i\lambda I - \mathcal{A})^{-1}\| \leqslant M|\lambda|^{\alpha}, \quad \forall \lambda \in \mathbb{R}. \tag{3.2}$$

Our first step is to prove that the semigroup associated with system (1.1)-(1.3) is strongly stable. The resolvent equation $i\lambda U - \mathcal{A}U = F$ in terms of its components is given by

$$i\lambda\varphi - \Phi = f_1, \tag{3.3}$$
$$i\rho_1\lambda\Phi - S_x - lN = f_2, \tag{3.4}$$
$$i\lambda\psi - \Psi = f_3, \tag{3.5}$$
$$i\rho_2\lambda\Psi - M_x + S = f_4, \tag{3.6}$$
$$i\lambda w - W = f_5, \tag{3.7}$$
$$i\rho_1\lambda W - N_x + lS = f_6. \tag{3.8}$$

Theorem 3.2. *With the above notation we have $i\mathbb{R} \subseteq \rho(\mathcal{A})$.*

Proof. Let us denote

$$\mathcal{N} = \big\{s \in \mathbb{R}^+ :]-is,\, is[\subset \rho(\mathcal{A})\big\}. \tag{3.9}$$

Since $0 \in \rho(\mathcal{A})$, $\mathcal{N} \neq \varnothing$. Putting $\sigma = \sup\mathcal{N}$ we have two possibilities: $\sigma = +\infty$ which implies that $i\mathbb{R} \subseteq \rho(\mathcal{A})$, and $0 < \sigma$ finite. We will reason by contradiction. Let us suppose that $\sigma < \infty$. Then, exists a sequence $\{\lambda_n\} \subseteq \mathbb{R}$ such that $\lambda_n \to \sigma < +\infty$ and

$$\|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to +\infty$$

Hence, there exists a sequence $\{f_n\} \subseteq \mathcal{H}$ such that $\|f_n\|_{\mathcal{H}} = 1$ and $\|(i\lambda_n I - \mathcal{A})^{-1} f_n\|_{\mathcal{H}} \to +\infty$. Denoting

$$\tilde{U}_n = (i\lambda_n I - \mathcal{A})^{-1} f_n \Longrightarrow f_n = i\lambda_n \tilde{U}_n - \mathcal{A}\tilde{U}_n$$

and $U_n = \frac{\tilde{U}_n}{\|\tilde{U}_n\|}$, $F_n = \frac{f_n}{\|\tilde{U}_n\|}$ we obtain

$$i\lambda_n U_n - \mathcal{A}U_n = F_n \to 0. \tag{3.10}$$

Since $\|\mathcal{A}U_n\| \le C$, it follows that U_n is bounded in $D(\mathcal{A})$. This implies in particular that Φ_n, Ψ_n and W_n are bounded in $H_0^1(0,\ell)$ and φ, ψ, w are bounded in $H^2(I_E)$. Then there exist subsequences such that:

$$\Phi_n \to \Phi, \quad \Psi_n \to \Psi, \quad W_n \to W \quad \text{strongly in } L^2(0,\ell) \tag{3.11}$$

$$\begin{gathered}\varphi_{n,x} + \psi_n + lw_n \to \varphi_x + \psi + lw,\\ \psi_{n,x} \to \psi_x,\ w_{n,x} - l\varphi_n \to w_x - l\varphi, \text{strongly in } L^2(I_E).\end{gathered} \tag{3.12}$$

Taking inner product we obtain

$$i\lambda_n\|U_n\|^2 - \langle \mathcal{A}U_n, U_n\rangle = \langle F_n, U_n\rangle \to 0$$

and taking real part we arrive at

$$-\operatorname{Re}\langle \mathcal{A}U_n, U_n\rangle = \int_0^\ell (b|\Psi_x^n|^2 + \kappa|\Phi_x^n + \Psi^n + lW^n|^2 + K|W_x^n - l\Phi^n|^2)\,dx \to 0.$$

This convergence implies

$$\Psi_x^n, \quad \Phi_x^n + \Psi^n + lW^n, \quad W_x^n - l\Phi^n \to 0 \quad \text{strongly in } L^2(I_C \cup I_D). \tag{3.13}$$

Therefore,

$$\psi_x^n,\ \varphi_x^n + \psi^n + lw^n\ \ w_x^n - l\varphi^n \to 0 \quad \text{strongly in } L^2(I_C \cup I_D). \tag{3.14}$$

From (3.11), (3.12) and (3.14) it follows that $U_n \to U$ strongly in $\mathcal{H}$. Since $\mathcal{A}$ is closed, we conclude that U satisfies

$$i\sigma U - \mathcal{A}U = 0.$$

Moreover, using the convergences (3.13)-(3.14) and the resolvent system, we obtain $U \equiv 0$ over $I_C \cup I_D$. Since $I_E = [\alpha, \beta]$ is linked to I_C or I_D on α or β, we obtain that $U(\alpha) = 0$ or $U(\beta) = 0$. So on $]\alpha, \beta[$ we have

$$\begin{gathered}-\rho_1\sigma^2\varphi + \kappa(\varphi_x + \psi + lw)_x - lK(w_x - l\varphi) = 0,\\ -\rho_2\sigma^2\psi + b\psi_{xx} + \kappa(\varphi_x + \psi + lw) = 0,\\ -\rho_1\sigma^2 w + K(w_x - l\varphi)_x + l\kappa(\varphi_x + \psi + lw) = 0,\end{gathered}$$

with

$$\varphi(\alpha) = \psi(\alpha) = \varphi_x(\alpha) = \psi_x(\alpha) = w(\alpha) = w_x(\alpha) = 0.$$

Looking the above equation as a second order initial value problem, we obtain $\varphi = \psi = w = 0$ over $]\alpha, \beta[$. Hence $U \equiv 0$ on $\mathcal{H}$, which is a contradiction. This completes the proof. □

Remark 3.3. From the dissipativity of the operator $\mathcal{A}$, we have

$$\begin{aligned}&\int_0^\ell b_0|\Psi_x|^2 + \kappa_0|\Phi_x + \Psi + lW|^2 + K_0|W_x - l\Phi|^2\,dx\\ &= \operatorname{Re}(U, F)_{\mathcal{H}} \le \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.\end{aligned} \tag{3.15}$$

The following lemma is crucial to prove the exponential stability of the system.

Lemma 3.4. *Let us suppose that* (1.4) *and* (1.5) *hold. Then, the solution of* (3.3)-(3.8) *satisfies*

$$\int_{I_C} \kappa_1|\lambda\Phi|^2 + b_1|\lambda\Psi|^2\,dx + K_1|\lambda W|^2\,dx \le C_\varepsilon\|U\|\|F\| + C_\varepsilon\|F\|^2 + \varepsilon\|U\|^2.$$

Proof. Multiplying (3.4) by $\overline{i\lambda\kappa_1\Phi}$ and integrating over $[a,b]$,

$$\int_{I_C}\rho_1\kappa_1|\lambda\Phi|^2\,dx=\int_{I_C}(S_x+lN+f_2)\overline{i\lambda\kappa_1\Phi}\,dx.$$

Recalling the definitions of S and N, on I_C we obtain

$$\begin{aligned}&\int_{I_C}\rho_1\kappa_1|\lambda\Phi|^2\,dx\\&=\int_{I_C}[\kappa(\varphi_x+\psi+lw)+\kappa_1(\Phi_x+\Psi+lW)]_x\overline{i\lambda\kappa_1\Phi}\,dx+\int_{I_C}f_2\overline{i\lambda\kappa_1\Phi}\,dx\\&=\mathfrak{G}+\mathfrak{G}_0+\int_{I_C}f_2\overline{i\lambda\kappa_1\Phi}\,dx.\end{aligned}\tag{3.16}$$

Where $\mathfrak{G}=\int_{I_C}[\kappa_1(\Phi_x+\Psi+lW)]i\lambda\overline{(\kappa_1'\Phi+\kappa_1\Phi_x)}\,dx$ and $\mathfrak{G}_0=\int_{I_C}[\kappa(\varphi_x+\psi+lw)]i\lambda\overline{(\kappa_1'\Phi+\kappa_1\Phi_x)}\,dx$. Then

$$\begin{aligned}\mathfrak{G}=&\int_{I_C}[\kappa_1(\Phi_x+\Psi+lW)]i\lambda\overline{(\kappa_1'\Phi+\kappa_1(\Phi_x+\Psi+lW))}\,dx\\&-\int_{I_C}[\kappa_1(\Phi_x+\Psi+lW)]i\lambda\overline{(\kappa_1\Psi+\kappa_1 lW)}\,dx.\end{aligned}\tag{3.17}$$

Taking the real part of the above relation and using (3.15) we obtain

$$\begin{aligned}\operatorname{Re}\mathfrak{G}=&\operatorname{Re}\int_{I_C}[\kappa_1(\Phi_x+\Psi+lW)]i\lambda\overline{(\kappa_1'\Phi)}\,dx\\&-\operatorname{Re}\int_{I_C}[\kappa_1(\Phi_x+\Psi+lW)]i\lambda\overline{(\kappa_1\Psi+\kappa_1 lW)}\,dx\\&\le\epsilon\|\lambda\Phi\|^2+\epsilon\|\lambda\Psi\|^2+\epsilon\|\lambda W\|^2+C_\epsilon\|U\|\|F\|.\end{aligned}\tag{3.18}$$

Similarly, using (3.3), (3.5), (3.7), and (3.15), we obtain

$$\begin{aligned}\operatorname{Re}\mathfrak{G}_0&=\operatorname{Re}\int_{I_C}[\kappa(\varphi_x+\psi+lw)]i\lambda\overline{(\kappa_1'\Phi+\kappa_1\Phi_x)}\,dx\\&\le\epsilon\int_{I_C}|\Phi|^2+|\Psi|^2+|W|^2\,dx+R.\end{aligned}\tag{3.19}$$

Thus, substituting (3.18) and (3.19) in (3.16) yields

$$\int_{I_C}\kappa_1|\lambda\Phi|^2\,dx\le\epsilon\|\lambda\Phi\|^2+\epsilon\|\lambda\Psi\|^2+C_\epsilon\|U\|\|F\|,\tag{3.20}$$

for $|\lambda|>1$. Multiplying (3.6) by $\overline{i\lambda b_1\Psi}$ and multiplying (3.8) by $\overline{i\lambda K_1 W}$ and using the same above procedure we obtain

$$\begin{gathered}\int_{I_C}\rho_2 b_1|\lambda\Psi|^2\,dx\le\epsilon\|\lambda\Psi\|^2+\epsilon\|\lambda W\|^2+C_\epsilon\|U\|\|F\|,\\\int_{I_C}\rho_1 K_1|\lambda W|^2\,dx\le\epsilon\|\lambda\Phi\|^2+\epsilon\|\lambda W\|^2+C_\epsilon\|U\|\|F\|.\end{gathered}$$

From the last three inequalities our conclusion follows. □

We introduce the notation

$$\mathcal{E}_\varphi=\frac{(\kappa q\rho_1)'}{2}|\Phi|^2+\frac{q'}{2}|S|^2,\quad \mathcal{I}_\varphi=\frac{\kappa q\rho_1}{2}|\Phi|^2+\frac{q}{2}|S|^2,\tag{3.21}$$

$$\mathcal{E}_\psi = \frac{(bq\rho_2)'}{2}|\Psi|^2 + \frac{q'}{2}|M|^2, \quad \mathcal{I}_\psi = \frac{bq\rho_2}{2}|\Phi|^2 + \frac{q}{2}|M|^2, \tag{3.22}$$

$$\mathcal{E}_w = \frac{(Kq\rho_1)'}{2}|W|^2 + \frac{q'}{2}|N|^2, \quad \mathcal{I}_w = \frac{Kq\rho_2}{2}|W|^2 + \frac{q}{2}|N|^2 \tag{3.23}$$

$$\mathcal{E} = \mathcal{E}_\varphi + \mathcal{E}_\psi + \mathcal{E}_w, \quad \mathcal{I} = \mathcal{I}_\varphi + \mathcal{I}_\psi + \mathcal{I}_w \tag{3.24}$$

and

$$\begin{aligned}\mathcal{L} = \int_a^b \mathcal{E}(s)\,ds - \int_a^b (\rho_1\kappa q\Phi\overline{\Psi} + \rho_1\kappa ql\Phi\overline{W} + \rho_1 KqlW\overline{\Phi})dx \\ + \int_a^b (qlS\bar{N} + ql\bar{S}N + qS\bar{M})dx.\end{aligned} \tag{3.25}$$

Taking $q(x) = \frac{e^{nx}-e^{na}}{n}$ we have $q'(x) = e^{nx} \gg q(x)$, for n large, hence we obtain

$$C_0\int_a^b \mathcal{E}\,dx \le \mathcal{L} \le C_1\int_a^b \mathcal{E}\,dx \tag{3.26}$$

Remark 3.5. Recalling the definition of S and M we obtain

$$\int_a^b |S|^2\,dx \le C\int_a^b \kappa|\varphi_x + \psi + lw|^2\,dx + \int_a^b |\widetilde{\kappa}(\Phi_x + \Psi + lW)|^2\,dx.$$

Using the dissipative properties, we have

$$\int_a^b |S|^2\,dx \le C\int_a^b |\varphi_x + \psi + lw|^2\,dx + C\|U\|\|F\|.$$

Similarly we have

$$\int_a^b |M|^2\,dx \le C\int_a^b |\psi_x|^2\,dx + C\|U\|\|F\|,$$

$$\int_a^b |N|^2\,dx \le C\int_a^b |w_x - l\psi|^2\,dx + C\|U\|\|F\|.$$

Therefore, for n large we have

$$\int_a^b |\Phi|^2 + |\varphi_x + \psi + lw|^2 + |\Psi|^2 + |\psi_x|^2 + |W|^2 + |w_x - l\psi|^2\,dx \le \int_a^b \mathcal{E}\,dx + C\|U\|\|F\|$$

and

$$\int_a^b \mathcal{E}\,dx \le C\int_a^b |\Phi|^2 + |\varphi_x + \psi + lw|^2 + |\Psi|^2 + |\psi_x|^2 + |W|^2 + |w_x - l\psi|^2\,dx + C\|U\|\|F\|.$$

Lemma 3.6. *On $[a,b] \subset I_C \cup I_E$ we have*

$$\Big|\mathcal{L}(s) - \mathcal{I}(s)\Big|_a^b\Big| \le C_\varepsilon\|U\|\|F\| + C_\varepsilon\|F\|^2 + \varepsilon\|U\|^2$$

Also on $I_D = [a,b]$ we have

$$\Big|\mathcal{L} - \mathcal{I}(s)\Big|_a^b\Big| \le \varepsilon\|U\|^2 + C_\varepsilon|\lambda|^2\|U\|\|F\|^2 + C\|F\|^2.$$

Proof. Multiplying (3.4) by $q\bar{S}$, (3.6) by $q\bar{M}$, and (3.8) by $q\bar{N}$, we obtain

$$\begin{aligned}&-\frac{\rho_1\kappa q}{2}\frac{d}{dx}|\Phi|^2 - \frac{q}{2}\frac{d}{dx}|S|^2 \\ &= R_1 + \rho_1\kappa q\Phi\overline{\Psi} + \rho_1\kappa ql\Phi\overline{W} - i\lambda\rho_1 q\tilde{\kappa}\Phi\overline{(\Phi_x + \Psi + lW)} - ql\bar{S}N,\end{aligned} \tag{3.27}$$

$$-\frac{\rho_2 bq}{2}\frac{d}{dx}|\Psi|^2, -\frac{q}{2}\frac{d}{dx}|M|^2 = R_2 - i\lambda\rho_2 q\tilde{b}\Psi\overline{\Psi_x} - qS\bar{M}, \tag{3.28}$$

$$\begin{aligned} &-\frac{\rho_1 Kq}{2}\frac{d}{dx}|W|^2 - \frac{q}{2}\frac{d}{dx}|N|^2 \\ &= R_3 - \rho_1 Kql W\overline{\Phi} - i\lambda\rho_1 q\tilde{K}W\overline{(W_x - l\Phi)} - qlS\bar{N}. \end{aligned} \tag{3.29}$$

Summing these equations we arrive at

$$\begin{aligned} &-\frac{\rho_1\kappa q}{2}\frac{d}{dx}|\Phi|^2 - \frac{\rho_2 bq}{2}\frac{d}{dx}|\Psi|^2 - \frac{\rho_1 Kq}{2}\frac{d}{dx}|W|^2 - \frac{q}{2}\frac{d}{dx}|S|^2 - \frac{q}{2}\frac{d}{dx}|M|^2 - \frac{q}{2}\frac{d}{dx}|N|^2 \\ &= R_4 + \rho_1\kappa q\Phi\overline{\Psi} + \rho_1\kappa ql\Phi\overline{W} - \rho_1 Kql W\overline{\Phi} - qlS\bar{N} - ql\bar{S}N - qS\bar{M} + J(x) \end{aligned}$$

where

$$J(x) = -i\lambda\rho_1 q\tilde{\kappa}\Phi\overline{(\Phi_x + \Psi + lW)} - i\lambda\rho_2 q\tilde{b}\Psi\overline{\Psi_x} - i\lambda\rho_1 q\tilde{K}W\overline{(W_x - l\Phi)}$$

and using the notation introduced above,

$$\begin{aligned} &-\frac{d}{dx}(\mathcal{I}(x)) + \mathcal{E}(x) \\ &= R_4 + \rho_1\kappa q\Phi\overline{\Psi} + \rho_1\kappa ql\Phi\overline{W} - \rho_1 Kql W\overline{\Phi} - qlS\bar{N} - ql\bar{S}N - qS\bar{M} + J(x). \end{aligned}$$

Note that when $[a,b] \subset I_C \cup I_E$, from Lemma 3.4 we obtain

$$\Big|\int_a^b J(x)\,dx\Big| \le C_\varepsilon\|U\|\|F\| + C_\varepsilon\|F\|^2 + \varepsilon\|U\|^2. \tag{3.30}$$

On I_D we obtain

$$\Big|\int_{I_D} J(x)\,dx\Big| \le \varepsilon\|U\|^2 + C_\varepsilon|\lambda|^2\|U\|\|F\| + \|F\|^2. \tag{3.31}$$

After an integration using the above inequalities our conclusion follows. $\square$

Let us denote

$$\mathbb{E}(s) = \rho_1|\Phi|^2 + \rho_2|\Psi|^2 + \rho_1|W|^2 + b|\psi_x|^2 + \kappa|\varphi_x + \psi + lw|^2 + K|w_x - l\varphi|^2.$$

Theorem 3.7. *The semigroup associated with system* (1.1)–(1.3) *is exponentially stable if the viscous discontinuous part I_D is not in the center of the beam.*

Proof. Since I_D is not in the center then $0 \in I_D$ or $\ell \in I_D$, hence because of the boundary conditions, Poincaré inequality is valid for Φ, Ψ and W. So, we have

$$\int_{I_D}|\Psi|^2\,dx \le C_p\int_{I_D}|\Psi_x|^2\,dx \le C\|U\|\|F\|. \tag{3.32}$$

Using $i\lambda\psi = \Psi + f_3$ and taking λ large we obtain

$$\int_{\ell_1}^{\ell}|\psi_x|^2 + |\Psi|^2\,dx \le C\|U\|\|F\| + C\|F\|^2$$

Using Poincaré's and the triangular inequality we obtain

$$\int_{I_D}|\Phi|^2\,dx \le C\int_{I_D}|\Phi_x|^2\,dx \le C\int_{I_D}\kappa|\Phi_x+\Psi+lW|^2+|\Psi|^2+|lW|^2\,dx \le C\|U\|\|F\|$$

Similarly for W we have

$$\int_{I_D}|W|^2\,dx \le C\int_{I_D}|W_x|^2\,dx \le C\|U\|\|F\|.$$

Hence

$$\int_{I_D} \mathbb{E}(x)\,dx \le C\|U\|\|F\| + C\|F\|^2.$$

On the other hand, integrating (3.4), (3.6), and (3.8) over $[a,b] \subset I_C$ we obtain

$$i\lambda\rho_1 \int_a^b \Phi\,dx - S(b^-) + S(a^+) + l\int_a^b N\,dx = \int_a^b f_2\,dx, \tag{3.33}$$

$$i\lambda\rho_2 \int_a^b \Psi\,dx - M(b^-) + M(a^+) + \int_a^b S\,dx = \int_a^b f_4\,dx, \tag{3.34}$$

$$i\lambda\rho_1 \int_a^b W\,dx - N(b^-) + N(a^+) + l\int_a^b S\,dx = \int_a^b f_6\,dx. \tag{3.35}$$

From Lemma 3.6 we obtain

$$\begin{aligned}
\Big|\int_{I_C} \Phi\,dx\Big| &\le \frac{C}{|\lambda|}\Big([S(b^-) - S(a^+)] + \Big|\int_a^b lN\,dx\Big| + \Big|\int_a^b f_2\,dx\Big|\Big) \\
&\le \frac{C}{|\lambda|}\Big(\|U\|^{1/2}\|F\|^{1/2} + \|U\| + \|F\|\Big).
\end{aligned}$$

Similarly we arrive at

$$\Big|\int_{I_C} \Psi\,dx\Big| \le \frac{C}{|\lambda|}\Big(\|U\|^{1/2}\|F\|^{1/2} + \|U\| + \|F\|\Big), \tag{3.36}$$

$$\Big|\int_{I_C} W\,dx\Big| \le \frac{C}{|\lambda|}\Big(\|U\|^{1/2}\|F\|^{1/2} + \|U\| + \|F\|\Big). \tag{3.37}$$

So we have

$$\begin{aligned}
\int_{I_C} |\Psi|^2\,dx &\le C\Big|\int_a^b \Psi\,dx\Big|^2 + C\int_a^b b_1|\Psi_x|^2\,dx \\
&\le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + \frac{C}{|\lambda|^2}\|F\|^2, \\
\int_{I_C} |W|^2\,dx &\le C\Big|\int_a^b W\,dx\Big|^2 + C\int_a^b b_1|W_x|^2\,dx \\
&\le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + \frac{C}{|\lambda|^2}\|F\|^2.
\end{aligned}$$

Using (3.3), (3.5), (3.7), and (3.15), we obtain

$$\int_{I_C} \mathbb{E}(x)\,dx \le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + C\|F\|^2. \tag{3.38}$$

Since I_D is not in the center of the beam we have $\overline{I}_C \cup \overline{I}_E = [0,\ell_2]$ or $\overline{I}_C \cup \overline{I}_E = [\ell_0,\ell]$. Let us assume the latter case. Using the observability Lemma 3.6 we obtain

$$\begin{aligned}
\int_{I_C} \mathbb{E}(x)\,dx &\le C\mathcal{I}(a) + C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + C\|F\|^2 \\
&\le C\|U\|\|F\| + \varepsilon\|U\|^2 + C\|F\|^2,
\end{aligned}$$

for λ large. Using the observability over the interval $[a,\ell]$ we arrive at

$$\int_{I_C} \mathbb{E}(x)\,dx \le C\|U\|\|F\| + \varepsilon\|U\|^2 + C\|F\|^2. \tag{3.39}$$

From (3.32), (3.39), and (3.39), we obtain

$$\begin{aligned} \|U\|^2 &= \int_0^\ell |\Phi|^2 + |\varphi_x + \psi + lw|^2 + |\Psi|^2 + |\psi_x|^2 + |W|^2 + |w_x - l\varphi|^2 \, dx \\ &\le C\|U\|\|F\| + \varepsilon\|U\|^2 + C\|F\|^2, \end{aligned} \tag{3.40}$$

from where we obtain that $\|U\| \le C\|F\|$. So our conclusion follows. □

We conclude this section by showing the polynomial decay when the discontinuous viscous part is in the center of the beam. We use the result given in [5].

Theorem 3.8. *If the viscoelastic discontinuous part I_D is in the center of the beam, then the semigroup associated with system* (1.1)–(1.3) *decays polynomially as*

$$\|S(t)U_0\| \le Ct^{-1/2}\|U_0\|_{D(\mathcal{A})}. \tag{3.41}$$

Proof. According to the hypothesis, we denote $I_D =]\ell_0, \ell_1[$. Using Lemma 3.6, (3.33), (3.34), and (3.35), for $a = \ell_0$ and $b = \ell_1$ we have

$$\int_a^b |\Psi|^2 \, dx \le C\Big| \int_a^b \Psi \, dx\Big|^2 + C\int_a^b b_1|\Psi_x|^2 \, dx \le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + \frac{C}{|\lambda|^2}\|F\|^2,$$

similarly

$$\begin{aligned} \int_a^b |\Phi|^2 \, dx &\le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + \frac{C}{|\lambda|^2}\|F\|^2 \\ \int_a^b |W|^2 \, dx &\le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + \frac{C}{|\lambda|^2}\|F\|^2. \end{aligned}$$

Then

$$\int_{I_D} \mathbb{E}(x) \, dx \le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + C\|F\|^2. \tag{3.42}$$

On I_C, using the same process as in Theorem 3.7 we obtain that estimate (3.38) is still valid. Let us suppose that $\ell_1 \in \overline{I}_E$. Using Lemma 3.6 on $I_D =]\ell_0, \ell_1[$ we have

$$\mathcal{I}(\ell_1^+) \le \int_{I_D} \mathbb{E}(x) \, dx + \varepsilon\|U\|^2 + C_\varepsilon|\lambda|^2\|U\|\|F\| + C\|F\|^2. \tag{3.43}$$

Since $S(\ell_1^-) = S(\ell_1^+)$, $M(\ell_1^-) = M(\ell_1^+)$, and $N(\ell_1^-) = N(\ell_1^+)$, we have

$$\int_{I_E} \mathbb{E}(x) \, dx \le \mathcal{I}(\ell_1^-) + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \le \varepsilon\|U\|^2 + C_\varepsilon|\lambda|^4\|F\|^2. \tag{3.44}$$

From the above inequality we obtain

$$\|U\|^2 \le C_\varepsilon|\lambda|^4\|F\|^2 + \varepsilon\|U\|^2,$$

Hence from Borichev-Tomilov Theorem (Theorem 3.1, (3.2)) the polynomial decay follows. □

4. Lack of exponential stability

Here we assume that I_D is in the middle of the beam. The main tool use is the following theorem due to Rivera et al. [17].

Theorem 4.1. *Let H be a Hilbert space and H_0 a closed subspace of H. Let $S(t)$ be a contraction semigroup on H, and $S_0(t)$ be a unitary group on H_0. If the difference $S(t) - S_0(t)$ is a compact operator from H_0 to H, then $S(t)$ is not exponentially stable.*

Another key result for our purposes is the following lemma (see [12]).

Lemma 4.2 (Lions-Aubin). *Let be V, H, V_0 Banach spaces such that $V \subseteq V_0 \subseteq H$, where the first embedding is compact. Let*

$$W = \{\varphi \in L^p([a,b); V) \colon \varphi_t \in L^p([a,b]; H\}.$$

Then the embedding $W \subseteq L^p([a,b]; V_0)$ is compact.

Theorem 4.3. *If the viscoelastic discontinuous part I_D is in the center of the beam, then the semigroup associated with system* (1.1)–(1.3) *is not exponentially stable.*

Proof. Let us define the spaces:

$$\mathbb{L}_0 = \{f \in L^2(0,\ell) : f\big|_{[\ell_0,\ell]} = 0\}, \quad V_0 = H_0^1(0,\ell) \cap \mathbb{L}_0,$$
$$H_0 = V_0 \times \mathbb{L}_0 \times V_0 \times \mathbb{L}_0 \times V_0 \times \mathbb{L}_0.$$

Let us consider the model on $[0, \ell_0]$:

$$\begin{gathered}
\rho_1 \tilde{\varphi}_{tt} - \kappa(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w})_x - lK(\tilde{w}_x - l\tilde{\varphi}) = 0, \\
\rho_2 \tilde{\psi}_{tt} - b\tilde{\psi}_{xx} + \kappa(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w}) = 0, \\
\rho_1 \tilde{w}_{tt} - lK(\tilde{w}_x - l\tilde{\varphi})_x + l\kappa(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w}) = 0, \\
\tilde{\varphi}(0,t) = \tilde{\varphi}(\ell_0,t) = \tilde{\psi}(0,t) = \tilde{\psi}(\ell_0,t) = \tilde{w}(0,t) = \tilde{w}(\ell_0,t) = 0.
\end{gathered} \tag{4.1}$$

Let S_0 be the semigroup on H_0 (null extensions on $[\ell_0, \ell]$) associated with (4.1). So we have

$$\|S_0(t)U_0\|^2 = \|U_0\|^2, \quad \forall U_0 \in H_0. \tag{4.2}$$

We will prove that $S(t) - S_0(t) \colon H_0 \to H$ is a compact operator, where

$$\begin{gathered}
S(t)U_0^m = (\varphi^m, \varphi_t^m, \psi^m, \psi_t^m, w^m,\, w_t^m) \in H, \\
S_0(t)U_0^m = (\tilde{\varphi}^m,\, \tilde{\varphi}_t^m, \tilde{\psi}^m, \tilde{\psi}_t^m, \tilde{w}^m, \tilde{w}_t^m) \in H_0.
\end{gathered}$$

Let $v^m := \varphi^m - \tilde{\varphi}^m$, $y^m := \psi^m - \tilde{\psi}^m$, $z^m := w^m - \tilde{w}^m$. By definition we have

$$v^m(x,t) = \begin{cases} \varphi^m - \tilde{\varphi}^m, & \text{if } x \in [0,\ell_0], \\ \varphi^m, & \text{if } x \notin [0,\ell_0], \end{cases} \quad y^m(x,t) = \begin{cases} \psi^m - \tilde{\psi}^m, & \text{if } x \in [0,\ell_0] \\ \psi^m, & \text{if } x \notin [0,\ell_0], \end{cases}$$
$$z^m(x,t) = \begin{cases} w^m - \tilde{w}^m, & \text{if } x \in [0,\ell_0], \\ w^m, & \text{if } x \notin [0,\ell_0]. \end{cases}$$

Moreover v, y, and z verify

$$\begin{aligned}
&\rho_1 v_{tt} - \kappa(v_x + y + lz)_x - \tilde{\kappa}(v_{xt} + y_t + lz_t)_x - lK(z_x - lv) \\
&- l\tilde{K}(z_{xt} - lv_t) = 0,
\end{aligned} \tag{4.3}$$

$$\rho_2 y_{tt} - by_{xx} - \tilde{b}y_{xxt} + \kappa(v_x + y + lz) + \tilde{\kappa}(v_{xt} + y_t + lz_t) = 0, \tag{4.4}$$

$$\begin{aligned}
&\rho_1 z_{tt} - -lK(z_x - lv)_x - lK(z_{xt} - lv_t)_x + l\kappa(v_x + y + lz) \\
&+ l\tilde{\kappa}(v_{xt} + y_t + lz_t) = 0.
\end{aligned} \tag{4.5}$$

Multiplying (4.3) by v_t, and (4.4) by w_t, and integrating on $[0,\ell]$, we obtain

$$\begin{aligned} &\int_0^{\ell} \big(\rho_1|v_t|^2+\rho_2|y_t|^2+\rho_1|z_t|^2+b|y_x|^2+\kappa|v_x+y+lz|^2+K|z_x-lv|^2\big)\,dx \\ &= \kappa v_x v_t\big|_0^{\ell_0} + b y_x y_t\big|_0^{\ell_0} + \kappa z_x z_t\big|_0^{\ell_0} - \int_{\ell_0}^{L} \tilde\kappa|v_{xt}+y_t+lz_t|^2\,dx \\ &\quad - \int_{\ell_0}^{L} \tilde b|y_{xt}|^2\,dx - \int_{\ell_0}^{L} \tilde K|z_{xt}-lv_t|^2\,dx. \end{aligned} \tag{4.6}$$

Using the boundary conditions we obtain

$$\begin{aligned} &\kappa v_x v_t\big|_0^{\ell_0} + b w_x w_t\big|_0^{\ell_0} + \kappa z_x z_t\big|_0^{\ell_0} \\ &= -\kappa\tilde\varphi_x(\ell_0^-,t)\varphi_t(\ell_0^-,t) - b\tilde\psi_x(\ell_0^-,t)\psi_t(\ell_0^-,t) - K\tilde w_x(\ell_0^-,t)w_t(\ell_0^-,t). \end{aligned} \tag{4.7}$$

Now, we denote $\mathfrak{U}^m(t)=[S(t)-S_0(t)]U_0^m=(v^m,v_t^m,y^m,y_t^m,z^m,z_t^m)$. Note that the left-hand side in (4.6) is $\|\mathfrak{U}^m(t)\|_{\mathcal H}^2$. Thus, integrating (4.6) over $[0,\,t]$, we obtain

$$\begin{aligned} &\int_0^t \|\mathfrak{U}^m(t)\|_{\mathcal H}^2\,dt + \int_0^t\int_{\ell_0}^{\ell} \tilde\kappa|v_{xt}^m+y_t^m+lz_t^m|^2+\tilde b|y_{xt}^m|^2+\tilde K|z_{xt}^m-lv_t^m|^2\,dx\,dt \\ &= -\int_0^t (\kappa\tilde\varphi_x^m(\ell_0^-,t)\varphi_t^m(\ell_0^-,t)+b\tilde\psi_x^m(\ell_0^-,t)\psi_t^m(\ell_0^-,t)) + K\tilde w_x^m(\ell_0^-,t)w_t^m(\ell_0^-,t)\,dt. \end{aligned} \tag{4.8}$$

From the observability, we have

$$\begin{gathered} \tilde\varphi_x^m(\ell_0^-,t)\to\tilde\varphi_x(\ell_0^-,t)\quad\text{weakly in } L^2(0,\,T),\\ \tilde\psi_x^m(\ell_0^-,t)\to\tilde\psi_x(\ell_0^-,t)\quad\text{weakly in } L^2(0,\,T),\\ \tilde w_x^m(\ell_0^-,t)\to\tilde w_x(\ell_0^-,t)\quad\text{weakly in } L^2(0,\,T). \end{gathered}$$

We only need to prove that

$$\big(\varphi_x^m(\ell_0^-,t),\,\psi_x^m(\ell_0^-,t),\,w_x^m(\ell_0^-,t)\big)\to\big(\varphi_x(\ell_0^-,t),\,\psi_x(\ell_0^-,t),\,w_x(\ell_0^-,t)\big) \tag{4.9}$$

strongly in $[L^2(0,\,T)]^3$, which implies the norm convergence in (4.8). To do that we use (3.15) and system (1.1)–(1.3) and obtain

$$\varphi_t^m,\psi_t^m,w_t^m\in L^2(0,T;H^1(I_D)),\quad \varphi_{tt}^m,\psi_{tt}^m,w_{tt}^m\in L^2(0,T;H^{-1}(I_D)).$$

Since $H^1\subset H^{1-\delta}\subset H^{-1}$ for $0<\delta<1/2$, where the first inclusion is a compact embedding, the inclusion $H^{1-\delta}\subset C(\overline{I_D})$ is also compact. From Lemma 4.2, there exists a subsequence (we still denote with the same symbol) such that

$$(\varphi_t^m,\psi_t^m,w_t^m)\to(\varphi_t,\psi_t,w_t)$$

strong in $L^2(0,T;H^{1-\delta}(I_D)\times H^{1-\delta}(I_D)\times H^{1-\delta}(I_D))$. and since the embedding $H^{1-\delta}(I_D)\subset C(\overline{I_D})$ is compact, we have

$$(\varphi_t^m,\psi_t^m,w_t^m)\to(\varphi_t,\psi_t,w_t)\quad\text{strongly in } L^2(0,T;C(I_D)\times C(I_D)\times C(I_D))$$

This implies (4.9). Hence inequality (4.8) implies the convergence in norm of $\mathfrak{U}^m$. So, $S(t)-\tilde S_0(t)$ is a compact operator. Then our conclusion follows. $\square$

In summary, we have established the following result.

Theorem 4.4. *The Bresse system* (1.1)-(1.3) *is exponentially stable if and only if the viscoelastic discontinuous part is not in the center of the beam. Otherwise, the system only has polynomial rate of decay.*

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