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AMBARZUMIAN'S THEOREM FOR TREES

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Abstract. The classical Ambarzumian's Theorem for Schrödinger operators $-D^2+q$ on an interval, with Neumann conditions at the endpoints, says that if the spectrum of $(-D^2+q)$ is the same as the spectrum of $(-D^2)$ then $q=0$. This theorem is generalized to Schrödinger operators on metric trees with Neumann conditions at the boundary vertices.

1. Introduction

In 1929, Ambarzumian [1] published the following foundational result in inverse spectral theory.

Theorem 1.1 (Ambarzumian). *If $q(x)$ is real-valued and continuous function, and the spectrum of the boundary problem*

$$-y'' + q(x)y = \lambda y,$$
$$y'(0) = y'(1) = 0,$$

is $k^2\pi^2$, for $k = 0, 1, 2, \dots,$ then $q(x) = 0$.

As Borg [2, 3] established, the case $q = 0$ turns out to be exceptional. In general the spectra of two Sturm-Liouville problems are needed to determine the potential. However, some generalizations of Ambarzumian's theorem were obtained in [6, 7, 8, 9]. In particular, it is known [9] that one may relax the original hypotheses, assuming that $q \in L^1[0,1]$, and

$$\lim_{k\to\infty}(\lambda_k - k^2\pi^2) = 0.$$

This work establishes an extension of Ambarzumian's theorem to trees, where each (directed) graph edge $e_1,\dots,e_n$ is identified with the interval $[0,1]$. Functions on the graph may then be identified with an ordered n-tuple of functions on $[0,1]$, With these identifications, the eigenvalue equation $-Y''+Q(x)Y = \lambda Y$ for the tree with the real integrable diagonal matrix potential $Q = \mathrm{diag}[q_1,\dots,q_n]$ is a system of n scalar equations

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad q_j \in L^1[0,1]. \tag{1.1}$$

Eigenfunctions for $-Y'' + Q(x)Y = \lambda Y$ on a graph are (nonzero) solutions of (1.1) which satisfy vertex (boundary) conditions. For trees, the Neumann condition

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$y'(v) = 0$ is assumed to hold at the boundary vertices, which have one incident edge. At interior vertices, with more than one incident edge, solutions of (1.1) are required to be continuous and satisfy the Kirchhoff condition, that the derivatives sum to zero in outward pointing local coordinates. For a star shaped graph a version of Ambarzumian's theorem was obtained in [14]. Our generalization is as follows.

Theorem 1.2. *Suppose T is a finite tree with all edges of length 1. For $r = 1, 2, \dots$, let $\{m_r\}$ be a sequence of integers with $\lim_{r\to\infty} m_r = \infty$. If the set of eigenvalues for* (1.1) *satisfying the tree vertex conditions is nonnegative, and contains a subsequence $\{\lambda_r\}$ with*

$$\lim_{r\to\infty} \big(\lambda_r - (\pi m_r)^2\big) = 0, \tag{1.2}$$

then $q_j(x) = 0$ a.e., for each $j = 1, 2, \dots, n$.

The following observations may provide motivation for the subsequent arguments. Consider the equation $-Y'' = \lambda Y$ on a tree T whose edges have length 1. If v is a boundary vertex, and x is the distance in T to v, then for $m = 0, 1, 2, \dots$ the function $\cos(m\pi x)$ will be continuous with derivative 0 at all the vertices. Thus for the case $Q(x) = 0$ the spectrum contains a subsequence $(m\pi)^2$, and these eigenvalues are easily seen to be simple.

If $Q \neq 0$, a number of arguments [9], [15, p. 33-35] suggest that one should expect a spectral shift given by $\int_T Q$. The hypothesized behavior (1.2) suggests that $\int_T Q = 0$. A previously known argument [9, 12] would then show that positivity of all eigenvalues forces Q to be 0.

The proof of Theorem 1.2 uses an analysis of the asymptotic behavior of eigenfunctions for (1.1) when the eigenvalues are close to $(\pi m_r)^2$. This material, along with a review of differential equations on graphs, is provided in Section 2. Section 3 contains three results, which together comprise the proof of Theorem 1.2. The expected result $\int_T Q = 0$ is first established. Second, a version of an argument of [12] is used to complete the proof, except for a technical issue about the form domain for $-D^2 + Q$ when some q_j is integrable, but not square integrable. The third result is included to resolve this technical issue.

2. Background Material

Let T be a metric tree with n edges, all of length 1, and an interior vertex chosen as the root. Local coordinates for the edges identify each edge with $[0, 1]$ so that the local coordinate increases as the distance to the root decreases. This means that each boundary vertex has local coordinate 0, the root has local coordinate 1 on each of its incident edges, and all other interior vertices v have one outgoing edge, with local coordinate 0 for v, while the local coordinate for v is 1 on the remaining incoming incident edges.

The given choice of local coordinates for T provides a convenient form for the vertex conditions. For an edge e_j incident on a boundary vertex, the Neumann condition is

$$y_j'(0) = 0.$$

For the root we impose the continuity conditions

$$y_j(1) = y_k(1)$$

for all incident edges e_j and e_k and the Kirchhoff condition

$$\sum_j y_j'(1) = 0,$$

the sum taken over all edges e_j incident on the root. For all other interior vertices v with incoming edges e_j and outgoing edge e_k the continuity conditions are

$$y_j(1) = y_k(0),$$

and the Kirchhoff condition is

$$y_k'(0) = \sum_j y_j'(1).$$

Denote by $c_j(x,\lambda)$ the solution of (1.1) which satisfies the conditions $c_j(0,\lambda) - 1 = c_j'(0,\lambda) = 0$ and by $s_j(x,\lambda)$ the solution of (1.1) which satisfies the conditions $s_j(0,\lambda) = s_j'(0,\lambda) - 1 = 0$. These solutions and their x derivatives satisfy the integral equations

$$\begin{aligned}
c(x,\lambda) &= \cos(\sqrt{\lambda}x) + \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} q(t)c(t,\lambda)\,dt, \\
c'(x,\lambda) &= -\sqrt{\lambda}\sin(\sqrt{\lambda}x) + \int_0^x \cos(\sqrt{\lambda}(x-t))q(t)c(t,\lambda)\,dt, \\
s(x,\lambda) &= \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} + \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} q(t)s(t,\lambda)\,dt, \\
s'(x,\lambda) &= \cos(\sqrt{\lambda}x) + \int_0^x \cos(\sqrt{\lambda}(x-t))q(t)s(t,\lambda)\,dt.
\end{aligned}$$

As a consequence, for $\lambda > 0$ one obtains the well know estimates

$$c(x,\lambda) = \cos(\sqrt{\lambda}x) + O(1/\sqrt{\lambda}), \quad s(x,\lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} + O(1/\lambda). \tag{2.1}$$

These results are established in [15, p. 13] for $q \in L^2[0,1]$, but the same ideas work for $q \in L^1[0,1]$.

Suppose that $Y(x,\lambda)$ is a vector function whose components y_j satisfy (1.1), and which is given the graph L^2 norm,

$$\|Y(x,\lambda)\|^2 = \sum_{j=1}^n \int_0^1 |y_j(x,\lambda)|^2\,dx.$$

Each $y_j(x,\lambda)$ may be written as a linear combination

$$y_j(x,\lambda) = A_j(\lambda)c_j(x,\lambda) + B_j(\lambda)s_j(x,\lambda).$$

Lemma 2.1. *Suppose $q_j \in L^1[0,1]$ and $\|Y(x,\lambda)\| = 1$. Then there is a $\lambda_0 > 0$ and a constant C such that*

$$|A_j(\lambda)| \le C, \quad |B_j(\lambda)/\sqrt{\lambda}| \le C, \quad \lambda \ge \lambda_0.$$

Proof. The bound on $\|Y(x,\lambda)\|$ gives a bound for each $y_j(x,\lambda)$, so

$$\begin{aligned}
&\int_0^1 |A_j(\lambda)c_j(x,\lambda) + B_j(\lambda)s_j(x,\lambda)|^2\,dx \\
&= \int_0^1 |A_j|^2 c_j^2 + \frac{|B_j|^2}{\lambda}\lambda s_j^2 + (A_j\frac{\overline{B_j}}{\sqrt{\lambda}} + \overline{A_j}\frac{B_j}{\sqrt{\lambda}})c_j\sqrt{\lambda}s_j \le 1.
\end{aligned}$$

Simple trigonometric integrals and (2.1) give

$$\int_0^1 c_j^2 = \frac{1}{2} + O(\lambda^{-1/2}), \quad \int_0^1 \lambda s_j^2 = \frac{1}{2} + O(\lambda^{-1/2}), \quad \int_0^1 \sqrt{\lambda} s_j c_j = O(\lambda^{-1/2}).$$

Since

$$2|\overline{A_j}\frac{B_j}{\sqrt{\lambda}}| \le |A_j|^2 + \frac{|B_j|^2}{\lambda},$$

we have

$$|A_j|^2[\frac{1}{2} - O(\lambda^{-1/2})] + \frac{|B_j|^2}{\lambda}[\frac{1}{2} - O(\lambda^{-1/2})] \le 1,$$

which provides the desired estimate. □

Some refinements of the estimates (2.1) are available if $x = 1$ and λ is close to $m^2\pi^2$. A recent reference is [5], but the ideas have a long history, [2, p. 11]. Define

$$[q_j] = \int_0^1 q_j(t)\,dt, \quad \text{and} \quad \omega_j = \sqrt{\lambda - [q_j]}.$$

Lemma 2.2. *Suppose that for positive integers m we have $\lambda = m^2\pi^2 + O(1)$. For edge indices $j = 1, \dots, n$, the following estimates hold uniformly if $\int_0^1 |q_j(t)|\,dt$ is bounded:*

$$\begin{gathered}
c_j(1,\lambda) = \cos(\omega_j) - (-1)^m 2^{-1} m^{-1} \int_0^1 \sin(2m\pi t) q_j(t)\,dt + O(m^{-2}), \\
c_j'(1,\lambda) = -\omega_j \sin(\omega_j) + (-1)^m 2^{-1} \int_0^1 \cos(2m\pi t) q_j(t)\,dt + O(m^{-1}), \\
s_j(1,\lambda) = \omega_j^{-1} \sin(\omega_j) + (-1)^m 2^{-1} m^{-2} \int_0^1 \cos(2m\pi t) q_j(t)\,dt + O(m^{-3}), \\
s_j'(1,\lambda) = \cos(\omega_j) + (-1)^m 2^{-1} m^{-1} \int_0^1 \sin(2m\pi t) q_j(t)\,dt + O(m^{-2}).
\end{gathered} \tag{2.2}$$

Suppose λ has the form $\lambda = m^2\pi^2 + O(1)$, and $m \to \infty$. Using (2.2), the Riemann-Lebesgue lemma and elementary trigonometric identities give

$$\begin{gathered}
c_j((1,\lambda) = (-1)^m + o(1), \\
c_j'(1,\lambda) = (-1)^m \frac{1}{2} \int_0^1 q_j(x)\,dx + o(1), \\
s_j(1,\lambda) = o(m^{-1}), \\
s_j'(1,\lambda) = (-1)^m + o(1).
\end{gathered} \tag{2.3}$$

At the end of the proof we will use some Hilbert space operator theory, so it will be helpful to recall [11] that on finite graphs the formal Schrödinger operators $-D^2 + Q$ with real-valued edge potentials $q_j(x)$ in $L^1[0,1]$, together with the interior and boundary vertex conditions described above, are associated with a self adjoint operator $\mathcal{L}$, with compact resolvent, acting componentwise on the Hilbert space $\oplus_j L^2(e_j)$ of square integrable functions on the edges. These facts may be established by using the variation of parameters formula to construct the resolvent. The assumption that all eigenvalues are nonnegative in Theorem 1.2 then implies that $\mathcal{L}$ is a nonnegative operator.

3. Proof of Theorem 1.2

This proof will be split into several parts.

Lemma 3.1. *With the hypotheses of Theorem 1.2,*

$$\sum_j [q_j] = \sum_j \int_0^1 q_j(t)\,dt = 0. \tag{3.1}$$

Proof. The sequence $\{m_r\}$ has either an infinite subsequence of even integers, or an infinite subsequence of odd integers. We will assume the first case holds, and use this subsequence instead of the original sequence with no change of notation. If an odd subsequence were used instead, the leading terms in (2.3) would change sign, but this has no significant impact on the proof.

Suppose that $\{Y(x,\lambda_r)\}$ is a sequence of eigenfunctions for (1.1) with norm 1. Write the components $y_j(x,\lambda_r)$ as a linear combination

$$y_j(x,\lambda_r) = A_j(\lambda_r)c_j(x,\lambda_r) + B_j(\lambda_r)s_j(x,\lambda_r).$$

Recall that the coefficients $A_j(\lambda_r)$ and $B_j(\lambda_r)/\sqrt{\lambda_r}$ are bounded sequences by Lemma 2.1. The proof now proceeds in three steps.

First, consider the value of y_j for edges incident on a vertex v. If j is the index of an outgoing edge, then

$$y_j(0,\lambda_r) = A_j(\lambda_r).$$

If j is the index of an incoming edge, then by (2.3)

$$y_j(1,\lambda_r) = A_j(\lambda_r)c_j(1,\lambda_r) + B_j(\lambda_r)s_j(1,\lambda_r) = A_j(\lambda_r) + o(1).$$

The continuity of Y at the vertex v thus implies

$$A_j(\lambda_r) = A_k(\lambda_r) + o(1), \quad r \to \infty, \tag{3.2}$$

for all edges j,k incident on v. Since the tree T is connected, (3.2) extends to all edges j,k.

Second, considering the root vertex to be at the top of the tree T, say that an edge e_j is below an edge $e_k \ne e_j$ if a path from e_j to the root passes through e_k. Label each vertex v with the combinatorial distance from the root, and label the edges with the larger of the vertex labels on the edge. Let M be the maximum label. If the vertex v has label $M-1$, then all its incoming edges e_k join v to a boundary vertex. If e_j is the outgoing edge for v, then the derivative condition at v gives

$$B_j(\lambda_r) = \sum_k A_k(\lambda_r)c_k'(1,\lambda_r) = \frac{1}{2}\sum_k A_k(\lambda_r)[q_k] + o(1).$$

For vertices v with label $M-2$, and outgoing edge e_j, the derivative condition at v and (2.3) give

$$B_j(\lambda_r) = \sum_k [A_k(\lambda_r)c_k'(1,\lambda_r) + B_k s_k'(1,\lambda_r)] = \frac{1}{2}\sum_l A_l(\lambda_r)[q_l] + o(1), \tag{3.3}$$

where the last sum is over edges e_l which are below e_j. This pattern continues as we reduce the vertex label until we reach the root. The derivative condition at the root now gives

$$0 = \sum_j A_l(\lambda_r)[q_j] + o(1) = A_1(\lambda_r)\sum_j [q_j] + o(1),$$

the sum taken over all edges e_j.

Third, from (3.3) and the extension of (3.2) to all edges j,k, we conclude that $\liminf_{r\to\infty} |A_j(\lambda_r)| > 0$ for all j, since otherwise we would have

$$\liminf_{r\to\infty} |y_j(x,\lambda_r)| = 0$$

uniformly in x, and the condition $\|Y(\lambda_r)\| = 1$ would be violated. In particular, $A_1(\lambda_r)$ is bounded away from zero as $r \to \infty$, so we obtain $\sum_j [q_j] = 0$. □

The next step in the proof of Theorem 1.2 uses an argument from [12], which is extended in [9]. Recall [10, pp. 318] that the nonnegative operator $\mathcal{L}$ has an associated nonnegative form

$$\mathbf{t}[f,g] = \langle \mathcal{L}f, g\rangle = \sum_j \int_0^1 (\mathcal{L}f_j)\overline{g_j} = \sum_j \int_0^1 (f_j'\overline{g_j'} + q_j f_j \overline{g_j}),$$

defined for f,g in the domain of $\mathcal{L}$. The closure of this form is

$$\mathbf{t}[f,g] = \langle \mathcal{L}^{1/2}f, \mathcal{L}^{1/2}g\rangle,$$

with the form domain equal to the domain of $\mathcal{L}^{1/2}$, the nonnegative square root of $\mathcal{L}$.

Lemma 3.2. *With the hypotheses of Theorem 1.2, if the function* 1 *is in the domain of* $\mathcal{L}^{1/2}$*, then* $q_j(x) = 0$ *almost everywhere for all edges* e_j*.*

Proof. Recalling that the lowest eigenvalue λ_0 of $\mathcal{L}$ is nonnegative, we apply the minimax principle for our differential operator $\mathcal{L}$. For Y in the domain of $\mathcal{L}^{1/2}$ we have

$$\begin{aligned} 0 \le \lambda_0 &= \min_{||Y||=1} \langle \mathcal{L}^{1/2}Y, \mathcal{L}^{1/2}Y\rangle \\ &= \min_{||Y||=1} \Big(\sum_{j=1}^n \int_0^1 y_j'\overline{y_j'}dx + \sum_{j=1}^n \int_0^1 q_j(x)|y_j|^2 dx \Big). \end{aligned} \tag{3.4}$$

The constant function $Y_0 = (1,1,\dots,1)^t$ is in the domain of $\mathcal{L}^{1/2}$, and the combination of (3.4) and (3.1) shows that 0 is an eigenvalue of $\mathcal{L}$ with eigenfunction Y_0. The proof is completed by noticing that the equation $\mathcal{L}Y_0 = 0$ gives $q_j(x) = 0$ a.e., for all edges e_j. □

Let $\mathcal{D}_0$ denote the domain of the self adjoint operator $\mathcal{L}_0 = -D^2$ associated with the continuity and Kirchhoff boundary conditions described above. That is [4], $\mathcal{D}_0$ consists of those vector functions $G = (g_1,\dots,g_n)$ with g_j, g_j' absolutely continuous, with $g_j'' \in L^2[0,1]$, and which satisfy the continuity and Kirchhoff boundary conditions. The function 1 is in the domain of $\mathcal{L}_0$. Since the domain of $\mathcal{L}^{1/2}$ includes the domain of $\mathcal{L}$, the function 1 will be in the domain of $\mathcal{L}^{1/2}$, thus yielding the conclusion of Theorem 1.2, if the domain of $\mathcal{L}$ is $\mathcal{D}_0$, which will hold under the stronger hypothesis $q_j \in L^2[0,1]$.

A more elaborate argument is required for $q_j \in L^1[0,1]$. Define the symmetric form $\mathbf{t}_1$ with domain $\mathcal{D}_0$,

$$\mathbf{t}_1[F,G] = \sum_{j=1}^n \int_0^1 \Big(f_j'\overline{g_j'} + q_j f_j \overline{g_j}\Big).$$

The proof of Theorem 1.2 will be completed with the next result, whose proof closely follows the argument in [10, pp. 345-346], where additional details are provided.

Theorem 3.3. *The symmetric form $\mathbf{t}_1$ is closable. The self adjoint operator associated with the closure of $\mathbf{t}_1$ is $\mathcal{L} = -D^2 + Q$, with a domain $\mathcal{D}$ consisting of vector functions $F : [0,1] \to \mathbb{C}^n$ whose components f_j satisfy the following conditions:*

(i) *f_j and f_j' are absolutely continuous, and $-f_j'' + q_j(x)f_j \in L^2[0,1]$,*

(ii) *f_j satisfies the Neumann boundary conditions $f_j'(v) = 0$ at boundary vertices, and the continuity conditions and Kirchhoff conditions at interior vertices.*

Proof. In addition to $\mathbf{t}_1$, define a second symmetric form with domain $\mathcal{D}_0$,

$$\mathbf{t}_0[F,G] = \sum_{j=1}^{n} \int_0^1 f_j'(x)\overline{g_j'(x)}\, dx.$$

The form $\mathbf{t}_0$ is nonnegative, and so closable. For any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f_j(x)|^2 \le \epsilon \int_0^1 |f_j'(x)|^2\, dx + \delta \int_0^1 |f_j(x)|^2\, dx.$$

This implies that $\mathbf{t}_1$ is the sum of $\mathbf{t}_0$ and a form with $\mathbf{t}_0$ bound 0, so that [10, p. 320] $\mathbf{t}_1$ is bounded below and closable, and the closures of $\mathbf{t}_0$ and $\mathbf{t}_1$ have the same domain $\mathcal{D}_1$.

Let x and y lie on edges incident on the vertex v. Integrating from x to v and then from v to y on the respective edges, the continuity of $F \in \mathcal{D}_0$ at v gives

$$F(y) - F(x) = \int_x^y F'(t)\, dt.$$

This formula, and the implied continuity at v, extends to functions $F \in \mathcal{D}_1$.

The closure of $\mathbf{t}_1$ has [10, p. 331] an associated self adjoint operator $\mathcal{L}$, with domain $\mathcal{D}$. If $F \in \mathcal{D}$ and $\mathcal{L}F = H$, then

$$\mathbf{t}_1[F,G] = \langle \mathcal{L}F, G\rangle = \langle H, G\rangle, \quad G \in \mathcal{D}_1.$$

For functions z_j such that

$$z_j'(x) = h_j(x) - q_j(x)f_j(x),$$

a rearrangement of terms from $\mathbf{t}_1[F,G] = \langle H, G\rangle$ gives

$$\begin{aligned} \sum_{j=1}^{n} \int_0^1 f_j'(x)\overline{g_j'(x)}\, dx &= \sum_{j=1}^{n} \int_0^1 z_j'(x)\overline{g_j(x)}\, dx \\ &= \sum_{j=1}^{n} \Big(z_j(1)\overline{g_j}(1) - z_j(0)\overline{g_j}(0) \Big) - \sum_{j=1}^{n} \int_0^1 z_j(x)\overline{g_j'(x)}\, dx. \end{aligned} \tag{3.5}$$

If the components of G satisfy $g_j(0) = g_j(1) = 0$, then

$$g_j(x) = \int_0^x g_j'(t)\, dt, \quad \int_0^1 g_j'(t)\, dt = 0.$$

The functions g_j' have dense span in the orthogonal complement of the constants in $L^2[0,1]$, so (3.5) gives

$$f_j' + z_j = c_j. \tag{3.6}$$

This equation shows that f_j' is absolutely continuous, and

$$-f_j'' + q_j f_j = h_j.$$

Since $\mathcal{D} \subset \mathcal{D}_1$, functions $F \in \mathcal{D}$ are continuous at all vertices v, as noted above. Relaxing the constraints on functions $G \in \mathcal{D}_1$, equation (3.5) gives

$$\sum_{j=1}^{n} [c_j - z_j(1)]\overline{g_j}(1) - \sum_{j=1}^{n} [c_j - z_j(0)]\overline{g_j}(0) = 0,$$

or from (3.6)

$$\sum_{j=1}^{n} f_j'(1)\overline{g_j}(1) - \sum_{j=1}^{n} f_j'(0)\overline{g_j}(0) = 0,$$

If G vanishes at each vertex except v, the continuity of G at v gives the desired derivative conditions. For instance, if v has one outgoing edge e_k and incoming edges e_j, we get

$$f_k'(0) = \sum_j f_j'(1).$$

This shows that every function in $\mathcal{D}$ has the desired properties, and one may show as in [10, p. 328] that these properties characterize $\mathcal{D}$. $\square$

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