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DATA ASSIMILATION AND NULL CONTROLLABILITY OF DEGENERATE/SINGULAR PARABOLIC PROBLEMS

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Abstract. In this article, we use the variational method in data assimilation to study numerically the null controllability of degenerate/singular parabolic problem

$$\partial_t\psi - \partial_x(x^\alpha\partial_x\psi(x)) - \frac{\lambda}{x^\beta}\psi = f, \quad (x,t) \in]0,1[\times]0,T[,$$
$$\psi(x,0) = \psi_0, \quad \psi\big|_{x=0} = \psi\big|_{x=1} = 0.$$

To do this, we determine the source term f with the aim of obtaining $\psi(\cdot,T) = 0$, for all $\psi_0 \in L^2(]0,1[)$. This problem can be formulated in a least-squares framework, which leads to a non-convex minimization problem that is solved using a regularization approach. Also we present some numerical experiments.

1. Introduction

In this article, we study an inverse problem of identifying the source term in degenerate/singular parabolic equation. This in the aim to study the null controllability, which has important applications in various areas of applied science and engineering.

Controllability properties of degenerate/singular parabolic equations has been widely studied (see [1, 4, 13, 12, 26]) using Carleman estimates. Our main contribution is to study numerically the null controllability of problem (1.1), below, using the variational method in data assimilation.

The problem can be stated as follows: Estimate the source term in the degenerate parabolic equation with singular potential

$$\partial_t\psi - \partial_x(x^\alpha\partial_x\psi(x)) - \frac{\lambda}{x^\beta}\psi = f, \quad (x,t) \in \Omega\times]0,T[\tag{1.1}$$

where $\Omega =]0,1[$, $\alpha \in]0,1[$, $\beta \in]0, 2-\alpha[$, $\lambda \le 0$, and $f \in L^2(\Omega\times]0,T[)$.

The mathematical model leads to a non-convex minimization problem

$$\begin{gathered} \text{find } \hat{f} \in A_{ad} \text{ such that} \\ E(\hat{f}) = \min_{f\in A_{ad}} E(f), \end{gathered} \tag{1.2}$$

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where the cost function E is

$$E(f) = \frac{1}{2}\|\psi(t=T)\|^2_{L^2(\Omega)}, \tag{1.3}$$

subject to ψ being the weak solution of the parabolic problem (1.1) with source term f.

Problem (1.2) is ill-posed in the sense of Hadamard, some regularization technique is needed to guarantee numerical stability of the computational process, maybe with noisy input data. The problem thus consists in minimizing a functional of the form

$$J(f) = \frac{1}{2}\|\psi(t=T)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|f\|^2_{L^2(\Omega\times]0,T[)}. \tag{1.4}$$

The last term in (1.4) stands for the so called Tikhonov-type regularization [8, 11], ε being a small regularizing coefficient that provides extra convexity to the functional J.

First we prove that the functional J is continuous, and G-derivable. Numerical experiments are presented later.

2. Problem statement and main result

Consider the problem

$$\begin{gathered} \partial_t\psi + A(\psi) = f \\ \psi(0,t) = \psi(1,t) = 0 \quad \forall t \in]0,T[\\ \psi(x,0) = \psi_0(x) \quad \forall x \in \Omega \end{gathered} \tag{2.1}$$

where, $\Omega =]0,1[$, $f \in L^2(\Omega\times]0,T[)$, $\psi_0 \in L^2(\Omega)$, and A is the operator defined as

$$A(\psi) = -\partial_x(a(x)\partial_x\psi(x)) - \frac{\lambda}{x^\beta}\psi, \quad a(x) = x^\alpha$$

with $\alpha \in]0,1[$, $\beta \in]0,2-\alpha[$, and $\lambda \le 0$.

The minimization problem with regularization associated to this problem is

$$\begin{gathered} \text{find } \hat{f} \in A_{ad} \text{ such that} \\ J(\hat{f}) = \min_{f \in A_{ad}} J(f), \end{gathered} \tag{2.2}$$

where the cost function J is defined as

$$J(f) = \frac{1}{2}\|\psi(t=T)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|f\|^2_{L^2(\Omega\times]0,T[)}, \tag{2.3}$$

subject to ψ being the weak solution of the parabolic problem (2.1) with source term f,

$$A_{ad} = \{u \in L^2(\Omega\times]0,T[) : \|u\|_{L^2(\Omega\times]0,T[)} \le r\}, \tag{2.4}$$

where r is a real strictly positive constant.

We now specify some notation. Let us introduce the functional spaces (see [1, 3, 9])

$$\begin{gathered} V = \{u \in L^2(\Omega) : u \text{ absolutely continuous on } [0,1]\}, \\ S = \{u \in L^2(\Omega) : \sqrt{a}u_x \in L^2(\Omega) \text{ and } u(0) = u(1) = 0\}, \\ H^1_a(\Omega) = V \cap S, \\ H^2_a(\Omega) = \{u \in H^1_a(\Omega) : au_x \in H^1(\Omega)\}, \end{gathered}$$

$$H^1_{\alpha,0} = \{u \in H^1_\alpha : u(0) = u(1) = 0\},$$
$$H^1_\alpha = \{u \in L^2(\Omega) \cap H^1_{\rm loc}(]0,1]) : x^{\frac{\alpha}{2}} u_x \in L^2(\Omega)\},$$
with
$$\|u\|^2_{H^1_a(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\sqrt{a}u_x\|^2_{L^2(\Omega)},$$
$$\|u\|^2_{H^2_a(\Omega)} = \|u\|^2_{H^1_a(\Omega)} + \|(au_x)_x\|^2_{L^2(\Omega)},$$
$$\langle u, v\rangle_{H^1_\alpha} = \int_\Omega (uv + x^\alpha u_x v_x)\, dx.$$
We recall that (see [9]) H^1_a is an Hilbert space and it is the closure of $C^\infty_c(0,1)$ for the norm $\|\cdot\|_{H^1_a}$. If $\frac{1}{\sqrt{a}} \in L^1(\Omega)$ then the following injections
$$H^1_a(\Omega) \hookrightarrow L^2(\Omega),$$
$$H^2_a(\Omega) \hookrightarrow H^1_a(\Omega),$$
$$H^1(0,T;L^2(\Omega)) \cap L^2(0,T;D(A)) \hookrightarrow L^2(0,T;H^1_a) \cap C(0,T;L^2(\Omega))$$
are compact.

The weak formulation of problem (2.1) is
$$\int_\Omega \partial_t\psi v\, dx + \int_\Omega \Big(a(x)\partial_x\psi\partial_x v - \frac{\lambda}{x^\beta}\psi v)\, dx = \int_\Omega f v\, dx, \quad \forall v \in H^1_0(\Omega). \tag{2.5}$$
Let
$$B[\psi, v] = \int_\Omega \Big(a(x)\partial_x\psi\partial_x v - \frac{\lambda}{x^\beta}\psi v\Big)\, dx. \tag{2.6}$$
We discuss the cases non-coercive and subcritical potential cases separately.

Non-coercive case: $\lambda = 0$**.** In this case the bilinear form B becomes
$$B[\psi, v] = \int_\Omega (a(x)\partial_x\psi\partial_x v)\, dx. \tag{2.7}$$
We have $a(x) = 0$ at $x = 0$, from where the bilinear form B will be non-coercive. We recall the following theorem.

Theorem 2.1 ([1, 13, 12])**.** *For all $f \in L^2(\Omega\times]0,T[)$ and $\psi_0 \in L^2(\Omega)$, there exists a unique weak solution to (2.1) such that*
$$\psi \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_a)$$
and there is a constant C_T such that for any solution of (2.1),
$$\sup_{t\in[0,T]} \|\psi(t)\|^2_{L^2(\Omega)} + \int_0^T \|\sqrt{a}\psi_x(t)\|^2_{L^2(\Omega)} dt \le C_T\Big(\|\psi_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega\times]0,T[)}\Big).$$
Furthermore, if $\psi_0 \in H^1_a(\Omega)$ then
$$\psi \in C([0;T], H^1_a) \cap L^2(0,T;H^2_a) \cap H^1(0,T;L^2(\Omega))$$
and there is a constant C_T such that
$$\begin{aligned}&\sup_{t\in[0,T]} \|\psi(t)\|^2_{H^1_a} + \int_0^T (\|\psi_t\|^2_{L^2(\Omega)} + \|(a\psi_x)_x(t)\|^2_{L^2(\Omega)})dt\\ &\le C_T(\|\psi_0\|^2_{H^1_a} + \|f\|^2_{L^2(\Omega\times]0,T[)}).\end{aligned}$$

The continuity of the functional J is deduced from the continuity of the function $\varphi : f \to \psi$, where ψ is the weak solution of (2.1) with source term f.

Theorem 2.2. *Let ψ be the weak solution of* (2.1). *In the non-coercive case, if $\psi_0 \in H^1_a(\Omega)$, then the functon $\varphi : L^2(\Omega\times]0,T[) \to C([0,T];H^1_a(\Omega)) \cap L^2(0,T;H^2_a(\Omega)) \cap H^1(0,T;L^2(\Omega))$, defined by*

$$\varphi(f) = \psi$$

is continuous.

If $\psi_0 \in L^2(\Omega)$, then $\varphi : L^2(\Omega\times]0,T[) \to C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_a)$, $\varphi(f) = \psi$ is continuous.

The differentiability of the functional J is deduced from the differentiability of the function $\varphi : f \to \psi$.

Theorem 2.3. *Let ψ be the weak solution of* (2.1). *If $\psi_0 \in H^1_a(\Omega)$, then the function $\varphi : L^2(\Omega\times]0,T[) \to C([0,T];H^1_a(\Omega)) \cap L^2(0,T;H^2_a(\Omega)) \cap H^1(0,T;L^2(\Omega))$, $\varphi(f) = \psi$ is G-derivable.*

If $\psi_0 \in L^2(\Omega)$, then $\varphi : L^2(\Omega\times]0,T[) \to C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_a)$, $\varphi(f) = \psi$ is G-derivable.

Sub-critical potential case: $\lambda \neq 0$. (see[26, 4]) In this case the bilinear form B becomes

$$B[\psi,v] = \int_\Omega \Big(a(x)\partial_x\psi\partial_x v - \frac{\lambda}{x^\beta}\psi v\Big)\,dx. \tag{2.8}$$

Since $a(x) = 0$ at $x = 0$ and $\lim_{x\to 0}\frac{\lambda}{x^\beta} = +\infty$, the bilinear form B is non-coercive and is non continuous at $x = 0$.

Consider the unbounded operator $(K, D(K))$ where

$$Ku = (x^\alpha u_x)_x + \frac{\lambda}{x^\beta}u, \tag{2.9}$$

for u in

$$D(k) = [u \in H^1_{\alpha,0} \cap H^2_{\rm loc}(]0,1])\big|(x^\alpha u_x)_x + \frac{\lambda}{x^\beta}u \in L^2(\Omega)].$$

Theorem 2.4 ([3, 26])**.** *If $f = 0$, then for all $\psi_0 \in L^2(\Omega)$, problem* (2.1) *has a unique weak solution*

$$\psi \in C([0,T];L^2(\Omega)) \cap C(]0,T];D(K)) \cap C^1(]0,T];L^2(\Omega))\,. \tag{2.10}$$

If $\psi_0 \in D(K)$ then

$$\psi \in C([0,T];D(K)) \cap C^1([0,T];L^2(\Omega))\,. \tag{2.11}$$

If $f \in L^2(\Omega\times]0,T[)$ then for all $\psi_0 \in L^2(\Omega)$, problem (2.1) *has a unique solution*

$$\psi \in C([0,T];L^2(\Omega)). \tag{2.12}$$

We have the following results.

Theorem 2.5. *Let ψ be the weak solution of* (2.1). *In the sub-critical potential case, the function $\varphi : L^2(\Omega\times]0,T[) \to C([0,T];L^2(\Omega))$, $\varphi(f) = \psi$ is continuous.*

Theorem 2.6. *Let ψ be the weak solution of* (2.1). *Then $\varphi : L^2(\Omega\times]0,T[) \to C([0,T];L^2(\Omega))$, $\varphi(f) = \psi$ is G-derivable.*

3. Proof of main results.

Proof of Theorem 2.2. Let $\psi_0 \in H_a^1(\Omega)$, and δf a small variation such that $f+\delta f \in A_{ad}$.

Consider $\delta\psi = \psi^\delta - \psi$, with ψ is the weak solution of (2.1) with source term f and ψ^δ is the weak solution of (2.1) with source term $f^\delta = f + \delta f$. Consequently, $\delta\psi$ is solution of the variational problem

$$\begin{gathered}\int_\Omega \partial_t \delta\psi v \, dx + \int_\Omega a(x)\partial_x \delta\psi(x)\partial_x v \, dx = \int_\Omega \delta f v dx \\ \delta\psi(0,t) = \delta\psi(1,t) = 0 \quad \forall t \in]0,T[\\ \delta\psi(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{3.1}$$

Hence, $\delta\psi$ is the weak solution of (2.1) with source term δf. We apply the estimate in theorem 2.1, to obtain a constant C_T such that

$$\begin{aligned}&\sup_{t\in[0,T]} \|\delta\psi(t)\|^2_{H_a^1(\Omega)} + \int_0^T (\|\partial_t \delta\psi\|^2_{L^2(\Omega)} + \|\partial_x(a\partial_x\delta\psi)(t)\|^2_{L^2(\Omega)})dt \\ &\le C_T\|\delta f\|^2_{L^2(\Omega\times]0,T[)};\end{aligned} \tag{3.2}$$

therefore,

$$\sup_{t\in[0,T]} \|\delta\psi(t)\|^2_{H_a^1(\Omega)} \le C_T\|\delta f\|^2_{L^2(\Omega\times]0,T[)}, \tag{3.3}$$

$$\|\delta\psi\|^2_{C([0,T];H_a^1(\Omega))} \le C_T\|\delta f\|^2_{L^2(\Omega\times]0,T[)}. \tag{3.4}$$

Then from (3.2) we have

$$\begin{gathered}\|\delta\psi(t)\|^2_{H_a^1(\Omega)} + \int_0^T \|\partial_x(a\partial_x\delta\psi)(t)\|^2_{L^2(\Omega)}dt \le C_T\|\delta f\|^2_{L^2(\Omega\times]0,T[)}, \\ \int_0^T \|\delta\psi(t)\|^2_{H_a^1(\Omega)}dt + T\int_0^T \|\partial_x(a\partial_x\delta\psi)(t)\|^2_{L^2(\Omega)}dt \le TC_T\|\delta f\|^2_{L^2(\Omega\times]0,T[)}, \\ \begin{aligned}&\inf(1,T)(\int_0^T \|\delta\psi(t)\|^2_{H_a^1(\Omega)}dt + \int_0^T \|\partial_x(a\partial_x\delta\psi)(t)\|^2_{L^2(\Omega)}dt) \\ &\le TC_T\|\delta f\|^2_{L^2(\Omega\times]0,T[)},\end{aligned} \\ \begin{aligned}&\int_0^T \|\delta\psi(t)\|^2_{H_a^1(\Omega)}dt + \int_0^T \|\partial_x(a\partial_x\delta\psi)(t)\|^2_{L^2(\Omega)}dt \\ &\le \frac{TC_T}{\inf(1,T)}\|\delta f\|^2_{L^2(\Omega\times]0,T[)}.\end{aligned}\end{gathered}$$

Hence,

$$\|\delta\psi\|^2_{L^2(0,T,H_a^2(\Omega))} \le \frac{TC_T}{\inf(1,T)}\|\delta f\|^2_{L^2(\Omega\times]0,T[)}. \tag{3.5}$$

In addition, from (3.2) we have

$$\begin{gathered}\|\delta\psi(t)\|^2_{H_a^1(\Omega)} + \int_0^T \|\partial_t\delta\psi(t)\|^2_{L^2(\Omega)}dt \le C_T\|\delta f\|^2_{L^2(\Omega\times]0,T[)}, \quad \forall t \in [0,T], \\ \begin{aligned}&\|\delta\psi(t)\|^2_{L^2(\Omega)} + \|\sqrt{a}\partial_x\delta\psi(t)\|^2_{L^2(\Omega)} + \int_0^T \|\partial_t\delta\psi(t)\|^2_{L^2(\Omega)}dt \\ &\le C_T\|\delta f\|^2_{L^2(\Omega\times]0,T[)}, \quad \forall t \in [0,T],\end{aligned}\end{gathered}$$

$$
\begin{gathered}
\|\delta\psi(t)\|^2_{L^2(\Omega)} + \int_0^T \|\partial_t \delta\psi(t)\|^2_{L^2(\Omega)} dt \le C_T \|\delta f\|^2_{L^2(\Omega\times]0,T[)}, \quad \forall t \in [0,T], \\
\int_0^T \|\delta\psi(t)\|^2_{L^2(\Omega)} dt + T\int_0^T \|\partial_t \delta\psi(t)\|^2_{L^2(\Omega)} dt \le TC_T \|\delta f\|^2_{L^2(\Omega\times]0,T[)}, \\
\|\delta\psi\|^2_{H^1(0,T;L^2(\Omega))} \le \frac{TC_T}{\inf(1,T)} \|\delta f\|^2_{L^2(\Omega\times]0,T[)}. \qquad (3.6)
\end{gathered}
$$

Inequalities (3.4), (3.5) and (3.6) imply the continuity of the function $\varphi : L^2(\Omega\times]0,T[) \to C([0,T]; H^1_a(\Omega)) \cap L^2(0,T;H^2_a(\Omega)) \cap H^1(0,T;L^2(\Omega))$, $\varphi(f) = \psi$. In the same way we can prove that if $\psi_0 \in L^2(\Omega)$, then the function $\varphi : L^2(\Omega\times]0,T[) \to C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_a)$, $\varphi(f) = \psi$ is continuous. Hence, the cost J is continuous. □

Proof of Theorem 2.3. Let $\psi_0 \in H^1_a(\Omega)$, and δf a small variation such that $f+\delta f \in A_{ad}$, we define the function

$$
\varphi'(f) : \delta f \in A_{ad} \to \delta\psi, \qquad (3.7)
$$

where $\delta\psi$ is the solution of the variational problem

$$
\begin{gathered}
\int_\Omega \partial_t(\delta\psi) v\,dx + \int_\Omega a(x)\partial_x(\delta\psi)\partial_x v\,dx = \int_\Omega \delta f v dx \quad \forall v \in H^1_0(\Omega) \\
\delta\psi(0,t) = \delta\psi(1,t) = 0 \quad \forall t \in]0,T[\\
\delta\psi(x,0) = 0 \quad \forall x \in \Omega
\end{gathered} \qquad (3.8)
$$

and we set

$$
\phi(f) = \varphi(f+\delta f) - \varphi(f) - \varphi'(f)\delta f. \qquad (3.9)
$$

We want to show that

$$
\phi(f) = o(\delta f). \qquad (3.10)
$$

We easily verify that the function ϕ is solution of following variational problem

$$
\begin{gathered}
\int_\Omega \partial_t\phi v\,dx + \int_\Omega a(x)\partial_x\phi\partial_x v\,dx = \int_\Omega (\delta f - (\delta f)^2) v dx \quad \forall v \in H^1_0(\Omega) \\
\phi(0,t) = \phi(1,t) = 0 \quad \forall t \in]0,T[\\
\phi(x,0) = 0 \quad \forall x \in \Omega.
\end{gathered} \qquad (3.11)
$$

In the same way as in the proof of continuity, we deduce that

$$
\|\phi\|^2_{C([0,T],H^1_a(\Omega))} \le C_T \|\delta f - (\delta f)^2\|^2_{L^2(\Omega\times]0,T[)}, \qquad (3.12)
$$

$$
\|\phi\|^2_{L^2(0,T,H^2_a(\Omega))} \le \frac{TC_T}{\inf(1,T)} \|\delta f - (\delta f)^2\|^2_{L^2(\Omega\times]0,T[)}, \qquad (3.13)
$$

$$
\|\phi\|^2_{H^1(0,T;L^2(\Omega))} \le \frac{TC_T}{\inf(1,T)} \|\delta f - (\delta f)^2\|^2_{L^2(\Omega\times]0,T[)}. \qquad (3.14)
$$

Therefore, the function $\varphi : L^2(\Omega\times]0,T[) \to C([0,T];H^1_a(\Omega)) \cap L^2(0,T;H^2_a(\Omega)) \cap H^1(0,T;L^2(\Omega))$ $\varphi(f) = \psi$ is G-derivable.

In the same way we prove that if $\psi_0 \in L^2(\Omega)$, then the function $\varphi : L^2(\Omega\times]0,T[) \to C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_a)$, $\varphi(f) = \psi$ is G-derivable. Hence, we deduce the existence of the gradient of the functional J. □

Proof of Theorem 2.5. Let δf be a small variation such that $f+\delta f \in A_{ad}$. Consider $\delta\psi = \psi^\delta - \psi$, with ψ a the weak solution of (2.1), with source term f, and consider ψ^δ the weak solution of (2.1) with source term $f^\delta = f + \delta f$. Consequently, $\delta\psi$ is the solution of variational problem

$$\begin{gathered}\int_\Omega \partial_t \delta\psi v\,dx + \int_\Omega \Big(a(x)\partial_x\delta\psi\partial_x v - \frac{\lambda}{x^\beta}\delta\psi v\Big)dx = \int_\Omega \delta f v dx, \quad \forall v \in H_0^1(\Omega) \\ \delta\psi(0,t) = \delta\psi(1,t) = 0 \quad \forall t \in]0,T[\\ \delta\psi(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{3.15}$$

Take $v = \delta\psi$, this gives

$$\int_\Omega \partial_t\delta\psi\delta\psi\,dx + \int_\Omega \Big(a(x)(\partial_x\delta\psi)^2 - \frac{\lambda}{x^\beta}(\delta\psi)^2\Big)dx = \int_\Omega \delta f\delta\psi\,dx, \tag{3.16}$$

Ω is independent of t, which gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\delta\psi)^2 dt + \int_\Omega \Big(a(x)(\partial_x\delta\psi)^2 - \frac{\lambda}{x^\beta}(\delta\psi)^2\Big)dx = \int_\Omega \delta f\delta\psi\,dx, \tag{3.17}$$

recall that $\delta\psi(t=0)=0$, by integrating between 0 and t with $t \in [0,T]$ we obtain

$$\begin{aligned}&\frac{1}{2}\|\delta\psi(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega (a(x)(\partial_x\delta\psi)^2 - \frac{\lambda}{x^\beta}(\delta\psi)^2)\,dx\,ds \\ &= \int_0^t\int_\Omega \delta f\delta\psi\,dx\,ds.\end{aligned} \tag{3.18}$$

We have $2ab \le a^2 + b^2$, for all $(a,b) \in R$, therefore

$$\begin{aligned}&\frac{1}{2}\|\delta\psi(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \Big(a(x)(\partial_x\delta\psi)^2 - \frac{\lambda}{x^\beta}(\delta\psi)^2\Big)\,dx\,ds \\ &\le \frac{1}{2}\int_0^t \|\delta f\|^2_{L^2(\Omega)}dt + \frac{1}{2}\int_0^t \|\delta\psi\|^2_{L^2(\Omega)}ds.\end{aligned} \tag{3.19}$$

Then

$$\begin{aligned}&\frac{1}{2}\|\delta\psi(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega (a(x)(\partial_x\delta\psi)^2 - \frac{\lambda}{x^\beta}(\delta\psi)^2)\,dx\,ds \\ &\le \frac{1}{2}\|\delta f\|^2_{L^2(\Omega\times]0,T[)} + \frac{1}{2}\int_0^t \|\delta\psi\|^2_{L^2(\Omega)}ds.\end{aligned} \tag{3.20}$$

Therefore

$$\|\delta\psi(t)\|^2_{L^2(\Omega)} \le \|\|\delta f\|^2_{L^2(\Omega\times]0,T[)} + \int_0^t \|\delta\psi\|^2_{L^2(\Omega)}ds. \tag{3.21}$$

Gronwall's Lemma gives

$$\begin{gathered}\|\delta\psi(t)\|^2_{L^2(\Omega)} \le \|\delta f\|^2_{L^2(\Omega\times]0,T[)}\exp(\int_0^t ds) \quad \forall t \in [0,T], \\ \|\delta\psi(t)\|^2_{L^2(\Omega)} \le \exp(T)\|\delta f\|^2_{L^2(\Omega\times]0,T[)} \quad \forall t \in [0,T],\end{gathered}$$

from where

$$\|\delta\psi\|^2_{C([0;T],L^2(\Omega))} \le \exp(T)\|\delta f\|^2_{L^2(\Omega\times]0,T[)}. \tag{3.22}$$

Which implies the continuity of the function $\varphi : L^2(\Omega\times]0,T[) \to C([0,T];L^2(\Omega))$, $\varphi(f) = \psi$. Hence, the cost J is continuous. $\square$

Proof of Theorem 2.6. Let δf be a small variation such that $f + \delta f \in A_{ad}$, we define the function

$$\varphi'(f) : \delta f \in A_{ad} \to \delta\psi, \tag{3.23}$$

where $\delta\psi$ is the solution of the variational problem

$$\begin{gathered}\int_\Omega \partial_t(\delta\psi)v\,dx + \int_\Omega (a(x)\partial_x(\delta\psi)\partial_x v - \frac{\lambda}{x^\beta}\delta\psi v)\,dx = \int_\Omega \delta f v dx \quad \forall v \in H_0^1(\Omega)\\ \delta\psi(0,t) = \delta\psi(1,t) = 0 \quad \forall t \in]0,T[\\ \delta\psi(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{3.24}$$

We set

$$\phi(f) = \varphi(f + \delta f) - \varphi(f) - \varphi'(f)\delta f. \tag{3.25}$$

We want to show that

$$\phi(f) = o(\delta f). \tag{3.26}$$

We easily verify that the function ϕ is the solution of variational problem

$$\begin{gathered}\int_\Omega \partial_t\phi v\,dx + \int_\Omega \Big(a(x)\partial_x\phi\partial_x v - \frac{\lambda}{x^\beta}\phi v\Big)\,dx = \int_\Omega (\delta f - (\delta f)^2)v dx \quad \forall v \in H_0^1(\Omega)\\ \phi(0,t) = \phi(1,t) = 0 \quad \forall t \in]0,T[\\ \phi(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{3.27}$$

In the same way as that used in the proof of continuity, we deduce

$$\|\phi\|^2_{C([0,T],L^2(\Omega))} \le \exp(T)\|\ \delta f - (\delta f)^2\|^2_{L^2(\Omega\times]0,T[)}. \tag{3.28}$$

Hence, in all cases, the function $\varphi(f) = \psi$ is G-derivable and we deduce the existence of the gradient of the functional J. $\square$

Now, we compute the gradient of J using the adjoint state method.

4. Gradient of J

We define the Gâteaux derivative of ψ at f in the direction $h \in L^2(\Omega\times]0,T[)$, by

$$\hat\psi = \lim_{s\to 0} \frac{\psi(f + sh) - \psi(f)}{s}, \tag{4.1}$$

$\psi(f + sh)$ is the weak solution of (2.1) with source term $f + sh$, and $\psi(f)$ is the weak solution of (2.1) with source term f.

We compute the Gâteaux (directional) derivative of (2.1) at f in some direction $h \in L^2(\Omega\times]0,T[)$, and we get the so-called tangent linear model:

$$\begin{gathered}\partial_t\hat\psi + A\hat\psi = h\\ \hat\psi(0,t) = \hat\psi(1,t) = 0 \quad \forall t \in]0,T[\\ \hat\psi(x,0) = 0 \quad \forall x \in \Omega.\end{gathered} \tag{4.2}$$

We introduce the adjoint variable P, and we integrate,

$$\int_0^1\int_0^T \partial_t\hat\psi P\,dt\,dx + \int_0^1\int_0^T A\hat\psi P\,dx = \int_0^1\int_0^T hP\,dt\,dx, \tag{4.3}$$

$$\int_0^1 \Big([\hat\psi P]_0^T - \int_0^T \hat\psi\partial_t P\,dt\Big)\,dx + \int_0^T \langle A\hat\psi, P\rangle_{L^2(\Omega)}dt = \langle h, P\rangle_{L^2(\Omega\times]0,T[)}, \tag{4.4}$$

$$
\begin{aligned}
&\int_0^1 [\hat{\psi}(T)P(T) - \hat{\psi}(0)P(0)]dx - \int_0^T \langle \hat{\psi}, \partial_t P\rangle_{L^2(\Omega)}dt + \int_0^T \langle A\hat{\psi}, P\rangle_{L^2(\Omega)}dt \\
&= \langle h, P\rangle_{L^2(\Omega\times]0,T[)}.
\end{aligned} \tag{4.5}
$$

Let us take $P(x=0) = P(x=1) = 0$, then we may write $\langle \hat{\psi}, AP\rangle_{L^2(\Omega)} = \langle A\hat{\psi}, P\rangle_{L^2(\Omega)}$. With $P(T) = 0$ we may now rewrite (4.5) as

$$
\int_0^T \langle \hat{\psi}, \partial_t P - AP\rangle_{L^2(\Omega)}dt = -\langle h, P\rangle_{L^2(\Omega\times]0,T[)}
$$

this gives

$$
\begin{gathered}
\int_0^T \langle \hat{\psi}, \partial_t P - AP\rangle_{L^2(\Omega)}dt = -\langle h, P\rangle_{L^2(\Omega\times]0,T[)} \\
P(x=0) = P(x=1) = 0, \quad P(T) = 0.
\end{gathered} \tag{4.6}
$$

The discretization in time of (4.6), using the Rectangular integration method, gives

$$
\begin{gathered}
\sum_{j=0}^{M+1} \langle \hat{\psi}(t_j), \partial_t P(t_j) - AP(t_j)\rangle_{L^2(\Omega)}\Delta t = \langle -P, h\rangle_{L^2(\Omega\times]0,T[)} \\
P(x=0) = P(x=1) = 0, \quad P(T) = 0.
\end{gathered} \tag{4.7}
$$

With

$$
t_j = j\Delta t, \quad j \in \{0, 1, 2, \dots, M+1\},
$$

where Δt is the step in time and $T = (M+1)\Delta t$.

The Gâteaux derivative of J at f in the direction $h \in L^2(\Omega)$ is given by

$$
\hat{J}(h) = \lim_{s\to 0} \frac{J(f+sh) - J(f)}{s}.
$$

After some computations, we arrive at

$$
\hat{J}(h) = \langle \psi(T), \hat{\psi}(T)\rangle_{L^2(\Omega)} + \langle \varepsilon f, h\rangle_{L^2(\Omega\times]0,T[)}. \tag{4.8}
$$

The adjoint model is

$$
\begin{gathered}
\partial_t P(T) - AP(T) = \frac{1}{\Delta t}\psi(T), \quad \partial_t P(t_j) - AP(t_j) = 0 \quad \forall t_j \neq T \\
P(x=0) = P(x=1) = 0 \quad \forall t_j \in]0;T[\\
P(T) = 0.
\end{gathered} \tag{4.9}
$$

From equations (4.7), (4.8) and (4.9), the gradient of J is given by

$$
\frac{\partial J}{\partial f} = -P + \varepsilon f. \tag{4.10}
$$

Problem (4.9) is retrograde, we make the change of variable $t \longleftrightarrow T - t$.

5. Discretized problem

Step 1. Full discretization.

Discrete approximations of these problems need to be made for numerical implementation. To resolve the Direct problem and adjoint problem, we use the Method θ-schema in time. This method is unconditionally stable for $1 > \theta \geq \frac{1}{2}$.

Let h be the step in space and Δt the step in time. Let

$$
x_i = ih, \quad i \in \{0, 1, 2, \dots, N+1\},
$$

$$
\begin{gathered}
c(x_i) = a(x_i) + \gamma, \\
t_j = j\Delta t, \quad j \in \{0, 1, 2, \dots, M+1\}, \\
f_i^j = f(x_i, t_j).
\end{gathered}
$$

We put

$$
\begin{gathered}
\psi_i^j = \psi(x_i, t_j), \\
da(x_i) = \frac{c(x_{i+1}) - c(x_i)}{h}, \\
b(x) = -\frac{\lambda}{x^\beta}.
\end{gathered}
$$

Therefore

$$
\partial_t \psi + A\psi = f \tag{5.1}
$$

is approximated by

$$
\begin{aligned}
&- \frac{\theta\Delta t}{h^2} c(x_i)\psi_{i-1}^{j+1} + \Big(1 + \frac{2\theta\Delta t}{h^2} c(x_i) + da(x_i)\frac{\theta\Delta t}{h} + b(x_i)\theta\Delta t\Big)\psi_i^{j+1} \\
&- \big(\frac{\theta\Delta t}{h^2} c(x_i) + da(x_i)\frac{\theta\Delta t}{h}\big)\psi_{i+1}^{j+1} \\
&= \frac{(1-\theta)\Delta t}{h^2} c(x_i)\psi_{i-1}^j + \Big(1 - \frac{(1-\theta)\Delta t}{h} da(x_i) - \frac{2(1-\theta)\Delta t}{h^2} c(x_i) \\
&\quad - (1-\theta)b(x_i)\Delta t\Big)\psi_i^j + \Big(\frac{(1-\theta)\Delta t}{h} da(x_i) + \frac{(1-\theta)\Delta t}{h^2} c(x_i)\big)\psi_{i+1}^j \\
&\quad + \Delta t[(1-\theta)f_i^j + \theta f_i^{j+1}].
\end{aligned}
$$

Let us define

$$
\begin{gathered}
g_1(x_i) = -\frac{\theta\Delta t}{h^2} c(x_i), \\
g_2(x_i) = 1 + \frac{2\theta\Delta t}{h^2} c(x_i) + da(x_i)\frac{\theta\Delta t}{h} + b(x_i)\theta\Delta t, \\
g_3(x_i) = -\frac{\theta\Delta t}{h^2} c(x_i) - da(x_i)\frac{\theta\Delta t}{h}, \\
k_1(x_i) = \frac{(1-\theta)\Delta t}{h^2} c(x_i), \\
k_2(x_i) = 1 - \frac{(1-\theta)\Delta t}{h} da(x_i) - \frac{2(1-\theta)\Delta t}{h^2} c(x_i) - (1-\theta)b(x_i)\Delta t, \\
k_3(x_i) = \frac{(1-\theta)\Delta t}{h} da(x_i) + \frac{(1-\theta)\Delta t}{h^2} c(x_i).
\end{gathered}
$$

Let $\psi^j = (\psi_i^j)_{i\in\{1,2,\dots,N\}}$, finally we obtain

$$
\begin{gathered}
D\psi^{j+1} = B\psi^j + V^j \text{ with } j \in \{1, 2, \dots, M\} \\
\psi^0 = (f(ih))_{i\in\{1,2,\dots,N\}},
\end{gathered} \tag{5.2}
$$

where

$$D=\begin{bmatrix} g_2(x_1) & g_3(x_1) & 0 & & & & & 0\\ g_1(x_2) & g_2(x_2) & g_3(x_2) & 0 & & & & \\ 0 & g_1(x_3) & g_2(x_3) & g_3(x_3) & 0 & & & \\ & 0 & g_1(x_4) & g_2(x_4) & g_3(x_4) & 0 & & \\ & & 0 & . & . & . & 0 & \\ & & & . & . & . & . & 0\\ & & & & 0 & g_1(x_{N-1}) & g_2(x_{N-1}) & g_3(x_{N-1})\\ 0 & & & & & 0 & g_1(x_N) & g_2(x_N) \end{bmatrix}$$

$$B=\begin{bmatrix} k_2(x_1) & k_3(x_1) & 0 & & & & & 0\\ k_1(x_2) & k_2(x_2) & k_3(x_2) & 0 & & & & \\ 0 & k_1(x_3) & k_2(x_3) & k_3(x_3) & 0 & & & \\ & 0 & k_1(x_4) & k_2(x_4) & k_3(x_4) & 0 & & \\ & & 0 & . & . & . & 0 & \\ & & & . & . & . & . & 0\\ & & & & 0 & k_1(x_{N-1}) & k_2(x_{N-1}) & k_3(x_{N-1})\\ 0 & & & & & 0 & k_1(x_N) & k_2(x_N) \end{bmatrix}$$

$$V^j=\begin{bmatrix} \Delta t[(1-\theta)f(x_1,t_j)+\theta f(x_1,t_j+\Delta t)]\\ \Delta t[(1-\theta)f(x_2,t_j)+\theta f(x_2,t_j+\Delta t)]\\ \vdots\\ \Delta t[(1-\theta)f(x_{N-1},t_j)+\theta f(x_{N-1},t_j+\Delta t)]\\ \Delta t[(1-\theta)f(x_N,t_j)+\theta f(x_N,t_j+\Delta t)] \end{bmatrix}$$

Step 2. Discretization of the functional

$$J(u)=\frac{\varepsilon}{2}\int_0^1 (u(x))^2dx+\frac{1}{2}\int_0^1(\psi(x,T))^2dx. \tag{5.3}$$

We recall that the Simpson methods for calculate an integral is

$$\int_a^b f(x)\,dx \simeq \frac{h}{2}\Big[f(x_0)+2\sum_{i=1}^{\frac{N+1}{2}-1} f(x_{2i})+4\sum_{i=1}^{\frac{N+1}{2}} f(x_{2i+1})+f(x_{N+1})\Big]$$

with $x_0=a$, $x_{N+1}=b$, $x_i=a+ih$, $i\in[1,\dots,N+1]$.

Let

$$\phi(x)=(u(x))^2 \quad \forall x\in\Omega,$$
$$\varphi(x)=(\psi(x,T))^2 \quad \forall x\in\Omega\,.$$

We have

$$\int_0^1 \phi(x)\,dx \simeq \frac{h}{2}\Big[\phi(0)+2\sum_{i=1}^{\frac{N+1}{2}-1}\phi(x_{2i})+4\sum_{i=1}^{\frac{N+1}{2}}\phi(x_{2i+1})+\phi(1)\Big],$$
$$\int_0^1 \varphi(x)\ ,dx \simeq \frac{h}{2}\Big[\varphi(0)+2\sum_{i=1}^{\frac{N+1}{2}-1}\varphi(x_{2i})+4\sum_{i=1}^{\frac{N+1}{2}}\varphi(x_{2i+1})+\varphi(1)\Big].$$

Therefore,

$$J(u) \simeq \frac{\varepsilon h}{4}\Big[\phi(0) + 2\sum_{i=1}^{\frac{N+1}{2}-1}\phi(x_{2i}) + 4\sum_{i=1}^{\frac{N+1}{2}}\phi(x_{2i+1}) + \phi(1)\Big]$$
$$+ \frac{h}{4}[\varphi(0) + 2\sum_{i=1}^{\frac{N+1}{2}-1}\varphi(x_{2i}) + 4\sum_{i=1}^{\frac{N+1}{2}}\varphi(x_{2i+1}) + \varphi(1)].$$

The main steps for descent method at each iteration are:

- Calculate ψ^k solution of (2.1) with source term f^k
- Calculate P^k solution of the adjoint problem
- Calculate the descent direction $d_k = -\nabla J(f^k)$
- Find $t_k = \underset{t>0}{argmin}\ J(f^k + td_k)$
- Update the variable $f^{k+1} = f^k + t_k d_k$.

The algorithm ends when $\big|J(f)\big| < \mu$, where μ is a given small precision.

The value t_k is chosen by the inaccurate linear search by the Armijo-Goldstein Rule as follows:

Let $\alpha_i, \beta \in [0, 1[$ and $\alpha > 0$

if $J(f^k + \alpha_i d_k) \le J(f^k) + \beta\alpha_i d_k^T d_k$, $t_k = \alpha_i$ and stop.

if not, $\alpha_i = \alpha\alpha_i$.

6. Numerical experiments

We did all tests on a PC with the following configurations: Intel Core i3 CPU 2.27GHz; RAM 4GB (2.93 usable). For all tests, we take number of points in space $N = 100$, number of points in time $M = 100$, and initial state the function $\psi_0 = \frac{x(x-1)}{T}$. In the figures below, ψ_0 is drawn red and the rebuilt function ψ in blue.

Noncoercive case. Let $\alpha = \frac{1}{2}$ and $\lambda = 0$. Figure 1 shows results without regularization. Figures 2 and 3 show results with regularization.

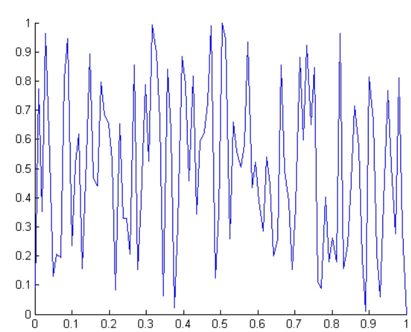

Figure 1. Final temperature without regularization. It shows that we cannot have $\psi(T) \simeq 0$.

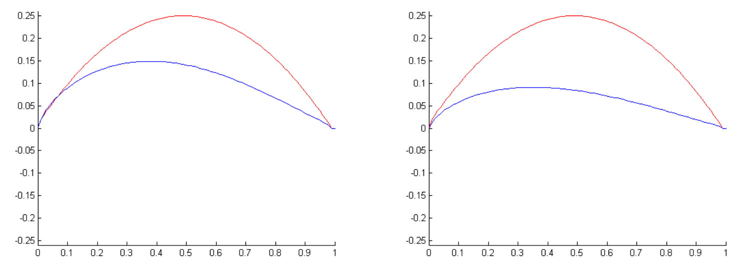

FIGURE 2. Temperature at $t = t_{10}$ (left), and at $t = t_{20}$ (right).

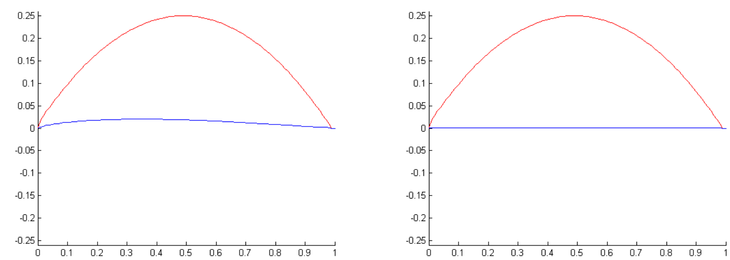

FIGURE 3. Temperature at $t = t_{50}$ which is nearly 0 (left). Final temperature showing that $\psi(T) \simeq 0$ (right).

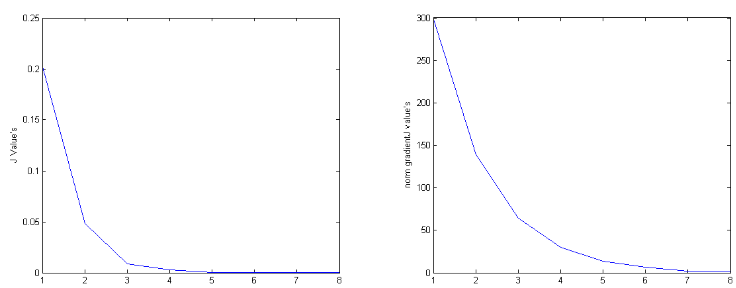

FIGURE 4. Graph of J (left). Norm of gradient (right).

Next we have tests for $\alpha \geq 2$ and $\lambda = 0$. Using the Carleman estimates, in [10] we prove that problem (2.1) is non-null controllable. In this tests we confim numerically this result; see Figures 5 and 6.

6.1. **Sub-critical potential case.** Let $\alpha = \frac{1}{2}$, $\lambda = -\frac{(1-\alpha)^2}{4}$, and $\beta = \frac{2-\alpha}{2}$. Figure 7 shows test without regularization. Figures 8 and 9 have regularization.

Conclusion. This article presents a regularization method for determining the source term. This is done with the aim of studying numerically the null controllability of degenerate/singular parabolic problems.

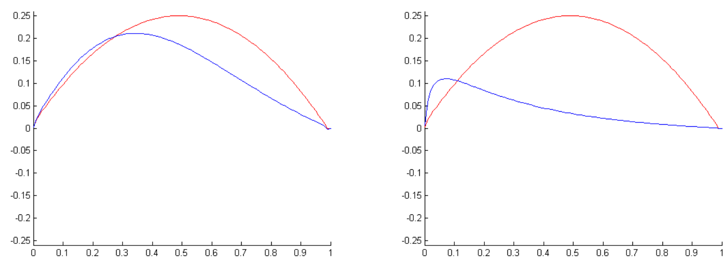

FIGURE 5. Temperature at $t = t_{10}$ with $\alpha = 2$ (left). Final temperature with $\alpha = 2$ which shows the non-null controllability of (2.1) (right).

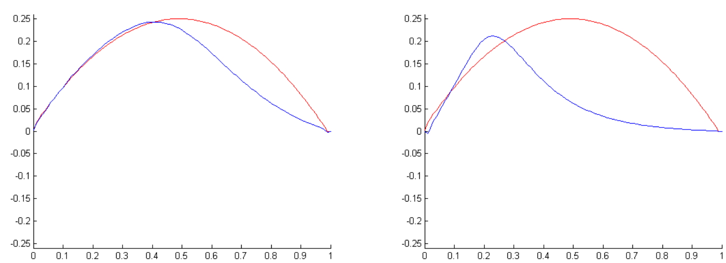

FIGURE 6. Temperature at $t = t_{10}$ with $\alpha = 4$ (left). Final temperature with $\alpha = 4$ which shows the non-null controllability of (2.1) (right).

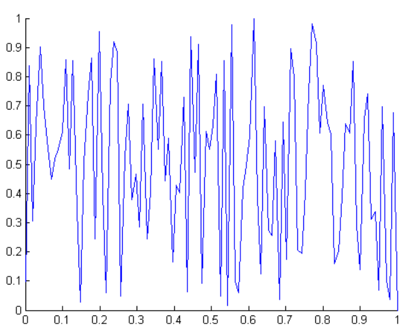

FIGURE 7. Final temperature without regularization which shows that we cannot have $\psi(T) \simeq 0$.

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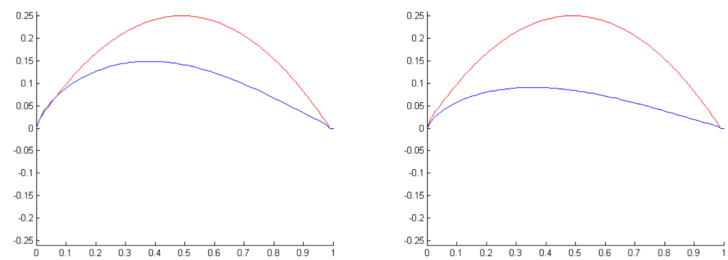

FIGURE 8. Temperature at $t = t_{10}$ (left), and at $t = t_{20}$ (right).

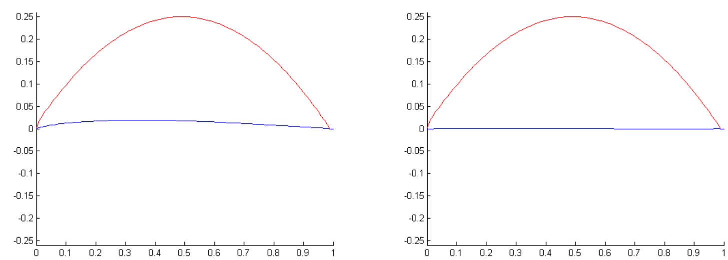

FIGURE 9. Temperature at $t = t_{50}$ which is nearly 0 (left). Final temperature showing that $\psi(T) \simeq 0$ (right).

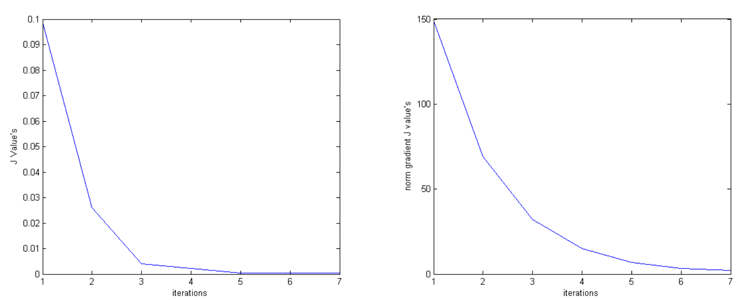

FIGURE 10. Graph of J (left). Norm of gradient (right).

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